Refined Characterizations of Approval-Based Committee Scoring Rules

Chris Dong
Technische Universität München
München, Germany
chris.dong@tum.de

Patrick Lederer
Technische Universität München
München, Germany
ledererp@in.tum.de

ABSTRACT

In approval-based committee (ABC) elections, the goal is to elect a fixed size subset of the candidates, a so-called committee, based on the voters' approval ballots over the candidates. One of the most popular classes of ABC voting rules are ABC scoring rules, which have recently been characterized by Lackner and Skowron [23]. However, this characterization relies on a model where the output is a ranking of committees instead of a set of winning committees and no full characterization of ABC scoring rules exists in the latter standard setting. We address this issue by characterizing two important subclasses of ABC scoring rules in the standard ABC election model, thereby refining the result of Lackner and Skowron [23] and extending it to the standard ABC election model. In more detail, we characterize *(i)* the prominent class of Thiele rules and *(ii)* a new class of ABC voting rules called ballot-size weighted approval voting. Both of these results are driven by a consistency notion analogous to the one of Young [32], but rely on different auxiliary conditions. Based on these two theorems, we also derive characterizations of three well-known ABC voting rules, namely multiwinner approval voting, proportional approval voting, and satisfaction approval voting.

1 INTRODUCTION

An important problem for multi-agent systems is collective decision making: given the voters' preferences over a set of alternatives, a common decision has to be made. This problem has traditionally been studied by economists for settings where a single candidate is elected (see, e.g., [1]), but there is also a multitude of applications where a fixed number of the candidates, a so-called committee, needs to be elected. The archetypal example for this is the elections of a city council, but there are also numerous technical applications such as recommender systems [19, 29] or medical diagnostic support systems [18]. This type of elections is typically called *approval-based committee (ABC) elections* and has recently attracted significant attention [e.g., 2, 14, 16, 24]. In more detail, this problem studies *ABC voting rules* which are functions that choose a set of winning committees (i.e., subsets of the candidates of a fixed size) based on the voters' approval ballots (i.e., the subsets of candidates that the voters find acceptable).

Maybe the most prominent class of ABC voting rules are ABC scoring rules. These rules rely on a scoring function s to compute the winning committees and each voter assigns $s(x, y)$ points to a committee if she approves x candidates in the committee and y in total. An ABC scoring rule then chooses the committees with maximal total score. There are many well-known ABC scoring rules, e.g., multi-winner approval voting (AV), satisfaction approval voting (SAV), Chamberlin-Courant approval voting (CCAV), and proportional approval voting (PAV). While these rules have rather different behavior, they are all consistent: if some common committees are chosen for two disjoint elections, precisely these common committees are chosen in a joint election. Indeed, all ABC scoring rules satisfy this axiom and can thus be seen as an equivalent to single-winner scoring rules, which have been characterized by Young [32] based on an analogous consistency condition.

In a recent breakthrough result, Lackner and Skowron [23] have managed to formalize the relation between ABC scoring rules and single-winner scoring rules by characterizing ABC scoring rules with almost the same axioms as Young [32] uses for his characterization of single-winner scoring rules. In more detail, Lackner and Skowron [23] show that ABC scoring rules are the only ABC ranking rules that satisfy anonymity, neutrality, continuity, weak efficiency, and consistency. However, this result discusses ABC ranking rules, which return transitive rankings of committees, whereas the literature on ABC elections typically focuses on sets of committees as output. While Lackner and Skowron [22] also present a result for the latter setting, it is no full characterization of ABC scoring rules as it requires a technical axiom called 2-non-imposition.[1] Since, e.g., AV and SAV fail this condition, this result does not allow to characterize all important ABC scoring rules. Lackner and Skowron also acknowledge this shortcoming by writing that "a full characterization of ABC scoring rules within the class of ABC choice rules remains as important future work" [22, p. 16].

Our contribution. We address this problem by characterizing two subclasses of ABC scoring rules, namely Thiele rules and ballot size weighted approval voting (BSWAV) rules, in the standard ABC election setting, thereby refining the result of Lackner and Skowron [23] while avoiding technical auxiliary conditions. *Thiele rules* are ABC scoring rules that do not depend on the ballot size and have attracted significant attention [e.g., 2, 10, 21, 29]. On the other hand, *BSWAV rules* are a new generalization of multi-winner approval voting where the voters are weighted depending on the size of their ballots. For example, AV, PAV, and CCAV are Thiele rules, whereas SAV is a BSWAV rule as it weights each voter who approves a total of x candidates by a factor of $\frac{1}{x}$. We note that every ABC scoring rule that has been studied before is in one of our two classes

For our characterization of Thiele rules, we rely on the axioms of Lackner and Skowron [23] and additionally require *independence*

[1]We believe that the proof of the main result of Lackner and Skowron [22] is incomplete. Roughly, the proof works by constructing an ABC ranking rule g based on an ABC voting rule f that satisfies the given axioms. Then, Lackner and Skowron [22] show that the axioms of f inherit to g, so g must be an ABC scoring rule (in the ranking setting). This implies that f is an ABC scoring rule (in the choice setting). However, the authors never show that the rankings returned by g are transitive, which is required by definition of ABC ranking rules, and proving this seems surprisingly difficult.

of losers. This axiom demands that a winning committee W stays winning if some voters change their ballot by disapproving candidates outside of W as, intuitively, the quality of W should only depend on its members. We note that this condition is well-known in single-winner elections and choice theory [e.g., 7, 8] and has recently been adapted to ABC elections by Dong and Lederer [12]. Based on this axiom, we show the following theorem: *an ABC voting rule is a Thiele rule if and only if it satisfies anonymity, neutrality, consistency, continuity, and independence of losers (Theorem 1).*

Similarly, for our characterization of BSWAV rules, we introduce a new axiom called *choice set convexity*. This condition requires that if two committees are chosen, then all committees "in between" those committees are chosen, too: if W and W' are chosen, then all committees W'' with $W \cap W' \subseteq W'' \subseteq W \cup W'$ are also chosen. We believe that this axiom is reasonable for elections where only the quality of candidates is of interest because a tie between committees indicates that they are equally good and the candidates in $W \setminus W'$ and $W' \setminus W$ are thus exchangeable. Based on this axiom, we show that *an ABC voting rule is a BSWAV rule if and only if it satisfies anonymity, neutrality, consistency, continuity, weak efficiency, and choice set convexity (Theorem 2).*

While these two characterization are intuitively related to the result of Lackner and Skowron [22], they are logically independent as *all* BSWAV rules (including AV) fail 2-non-imposition. In particular, Theorem 2 allows, in contrast to the result of Lackner and Skowron [22], to characterize AV and SAV. On the other hand, most Thiele rules can be characterized by the result of Lackner and Skowron [22]. We nevertheless believe that Theorem 1 is important as it turns many characterizations for ABC voting rules within the class of Thiele rules into full characterizations.

We demonstrate these points also by characterizing three well-known ABC voting rules in Section 4. In more detail, we first obtain a characterization of AV from our theorems as it is essentially the only rule that is both a BSWAV rule and a Thiele rule. Secondly, we also derive characterizations of PAV and SAV by considering party-list profiles (where candidates are partitioned into parties and each voter supports a single party by approving all of its candidates) and investigating whether it is advantageous for candidates to compete by themselves or to form a party. To the best of our knowledge, the result on SAV is the first full characterizations of this rule. An overview of our results is shown in Figure 1.

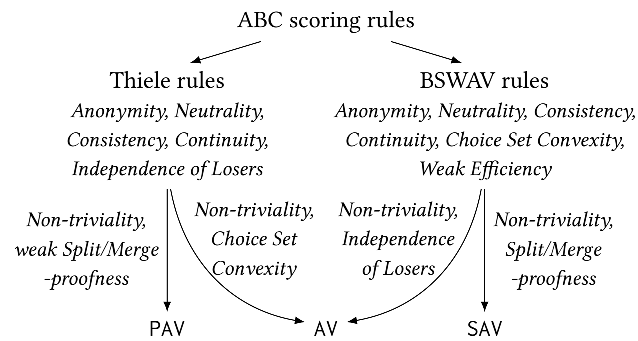

Figure 1: Overview of our results. An arrow from X to Y means that Y is a subset or an element of X. The axioms written on an arrow from X to Y characterize the rule Y within the class X. The axioms written below Thiele rules and BSWAV rules characterize these classes of ABC voting rules.

Related work. Our results are closely connected to those by Lackner and Skowron [23] and Skowron et al. [28]. However, these papers characterize ABC scoring rules in a model where the output is a ranking over the committees, while we characterize subclasses of ABC scoring rules in the standard ABC election framework. Lackner and Skowron [22] also provide a partial axiomatization of ABC scoring rules that allows to characterize specific Thiele rules such as CCAV and PAV. However, this result does not allow to characterize AV or SAV as it relies on a technical auxiliary condition. Except of these papers, we are not aware of any characterization of ABC scoring rules. Indeed, full characterizations of many important ABC voting rules are still missing and there is only one other such result: Dong and Lederer [12] have characterized classes of sequential ABC voting rules such as sequential Thiele rules.

There is also a large amount of papers studying axiomatic properties of ABC scoring rules [e.g., 2, 10, 20, 21, 27]. For instance, Lackner and Skowron [20] have characterized multiwinner approval voting as the only non-trivial ABC scoring rule that is immune to strategic manipulation, and Aziz et al. [2] investigate Thiele rules with respect to how fair they represent groups of voters with similar preferences. Another important aspect of these rules is their computational complexity. In particular, it is known that all Thiele rules but AV are NP-hard to compute on the full domain [3, 29]. There is thus significant work on how to compute these rules by, e.g., restricting the domain of preference profiles [15, 25, 30], considering approximation algorithms for Thiele rules [4, 13], or designing FPT algorithms [9]. For a more detailed overview of results on ABC scoring rules, we refer to the survey by Lackner and Skowron [24].

Finally, in the broader realm of social choice, there are many results that are conceptually similar to ours as they rely on consistency properties: Young [32] has characterized scoring rules for single-winner elections, numerous characterizations of single-winner approval voting rely on consistency [7, 17], Young and Levenglick [33] have characterized Kemeny's rule with the help of this axiom in a setting where sets of linear orders over the candidates are returned, and Brandl et al. [6] characterize a randomized voting rule called maximal lotteries based on this axiom.

2 PRELIMINARIES

Let $\mathbb{N} = \{1, 2, \dots\}$ denote an infinite set of voters and let $C = \{c_1, \dots, c_m\}$ denote a set of $m \geq 2$ candidates. Intuitively, we interpret $\mathbb{N}$ as the set of all possible voters, and a concrete electorate N is a finite and non-empty subset of $\mathbb{N}$. To this end, we define $\mathcal{F}(\mathbb{N}) = \{N \subseteq \mathbb{N} \colon N \textit{ is non-empty and finite}\}$ as the set of all possible electorates. Given an electorate $N \in \mathcal{F}(\mathbb{N})$, we assume that each voter $i \in N$ reports her preferences over the candidates as *approval ballot* A_i, i.e., as a non-empty subset of C which indicates the candidates she finds acceptable. $\mathcal{A}$ is the set of all possible approval ballots. An *approval profile* A is a mapping from N to $\mathcal{A}$, i.e., it assigns an approval ballot to every voter in the given electorate.

Moreover, we define $\mathcal{A}^* = \bigcup_{N \in \mathcal{F}(\mathbb{N})} \mathcal{A}^N$ as the set of all possible approval profiles. For every profile $A \in \mathcal{A}^*$, N_A denotes the set of voters that submit a ballot in A. Furthermore, two approval profiles A, A' are called *disjoint* if $N_A \cap N_{A'} = \emptyset$ and for disjoint profiles $A, A' \in \mathcal{A}^*$, we define the profile $A'' = A + A'$ by $N_{A''} = N_A \cup N_{A'}$, $A''_i = A_i$ for $i \in N_A$, and $A''_i = A'_i$ for $i \in N_{A'}$.

Given an approval profile, our aim is to elect a *committee*, i.e., a subset of the candidates of predefined size. We denote the target committee size by $k \in \{1, \dots, m-1\}$ and the set of all size k committees by $\mathcal{W}_k = \{W \subseteq C : |W| = k\}$. For determining the winning committees for a given preference profile, we use *approval-based committee (ABC) voting rules* which are mappings from $\mathcal{A}^*$ to $2^{\mathcal{W}_k} \setminus \{\emptyset\}$. Note that ABC voting rules may return multiple committees, which indicates that these committees are tied for the win. This is necessary to satisfy basic fairness conditions; e.g., if all voters approve all candidates, all committees are equally acceptable and a fair voting rule cannot distinguish between them.

2.1 ABC Voting Rules

We focus in this paper on three specific types of ABC voting rules: ABC scoring rules, Thiele rules, and BSWAV rules.

ABC scoring rules. ABC scoring rules rely on a scoring function according to which voters assign points to committees and choose the committees with maximal score. Formally, a *scoring function* $s(x, y)$ is a mapping from $\{0, \dots, k\} \times \{1, \dots, m\}$ to $\mathbb{R}$ such that $s(x, y) \geq s(x', y)$ for all $x, x' \in \{\max(0, k+y-m), \dots, \min(k, y)\}$ with $x \geq x'$. We define the score of a committee W in a profile A as $\hat{s}(A, W) = \sum_{i \in N_A} s(|A_i \cap W|, |A_i|)$. Then, an ABC voting rule f is an *ABC scoring rule* if there is a scoring function s such that $f(A) = \{W \in \mathcal{W}_k : \forall W' \in \mathcal{W}_k : \hat{s}(A, W) \geq \hat{s}(A, W')\}$. Note that the set $\{\max(0, k+y-m), \dots, \min(k, y)\}$ contains all "active" intersection sizes: a committee of size k and a ballot of size y intersect at least in $\max(0, k+y-m)$ candidates and at most in $\min(k, y)$ candidates.

Thiele rules. Arguably the most prominent subclass of ABC scoring rules are Thiele rules. Their namesake Thiele [31] proposed them with a simple argument: if the elected committee contains x of the approved candidates of a voter, she should have some benefit $s(x)$ from the committee. Hence, Thiele rules are defined by a non-decreasing *Thiele scoring function* $s : \{0, \dots, k\} \to \mathbb{R}$ with $s(0) = 0$, and choose the committees that maximize the total score. More formally, an ABC voting rule f is a *Thiele rule* if there is a Thiele scoring function s such that $f(A) = \{W \in \mathcal{W}_k : \forall W' \in \mathcal{W}_k : \hat{s}(A, W) \geq \hat{s}(A, W')\}$, where $\hat{s}(A, W) = \sum_{i \in N_A} s(|A_i \cap W|)$. There are numerous important Thiele rules such as multiwinner approval voting (AV; defined by $s_{AV}(x) = x$), proportional approval voting (PAV; defined by $s_{\mathsf{PAV}}(x) = \sum_{z=1}^{x} \frac{1}{z}$ for $x > 0$), and Chamberlin-Courant approval voting (CCAV; defined by $s_{CCAV}(x) = 1$ for $x > 0$).

BSWAV rules. Ballot size weighted approval voting rules form a new subclass of ABC scoring rules which generalize AV by weighting voters based on their ballot size. Formally, a *ballot size weighted approval voting (BSWAV) rule* f is defined by a weight vector $\alpha \in \mathbb{R}^m_{\geq 0}$ and chooses for a profile A the committees W that maximize $\hat{s}(A, W) = \sum_{i \in N_A} \alpha_{|A_i|} |A_i \cap W|$. We note that the score of a committee W for a BSWAV rule can be represented as the sum of the scores of individual candidates $c \in W$ since $\sum_{i \in N_A} \alpha_{|A_i|} |A_i \cap W| = \sum_{c \in W} \sum_{i \in N_A : c \in A_i} \alpha_{|A_i|}$. AV is clearly part of this class by setting $\alpha_i = 1$ for all $i \leq m$. Another well-known BSWAV rule is satisfaction approval voting (SAV) defined by $\alpha_i = \frac{1}{i}$ for $i \in \{1, \dots, m\}$. SAV has been suggested by Brams and Kilgour [5] and can be motivated by the "one man, one vote" principle as every voter distributes a budget of 1 to her approved candidates.

We note that Thiele rules and BSWAV rules are diametrically opposing subclasses of ABC scoring rules: Thiele rules do not depend on the ballot size at all, whereas BSWAV rules only depend on this aspect. Consequently, if $k < m-1$, the sets of BSWAV rules and Thiele rules only intersect in AV and the trivial rule TRIV (which always chooses all size k committees). So, AV is the only non-trivial ABC voting rule that is in both classes; *non-triviality* means that there is a profile A such that $f(A) \neq \mathsf{TRIV}(A)$. Moreover, both classes are proper subsets of the set of ABC scoring rules if $1 < k < m-1$. On the other hand, if $k = 1$ or $k = m-1$, the set of BSWAV rules is equivalent to the set of ABC scoring rules. For instance, if $k = 1$, we can represent an ABC scoring rule as BSWAV rule by setting $\alpha_i = s(1, i) - s(0, i)$ for $i < m$ and $\alpha_m = 0$.

2.2 Basic Axioms

Next, we introduce the axioms used for our characterizations.

Anonymity. Anonymity is one of the most basic fairness properties and requires that all voters should be treated equally. Formally, we define these concepts based on permutations: an ABC voting rule f is *anonymous* if $f(A) = f(\pi(A))$ for all profiles $A \in \mathcal{A}^*$ and permutations $\pi : \mathbb{N} \to \mathbb{N}$. Here, we denote by $A' = \pi(A)$ the profile with $N_{A'} = \{\pi(i) : i \in N_A\}$ and $A'_{\pi(i)} = A_i$ for all $i \in N_A$.

Neutrality. Similar to anonymity, *neutrality* is a fairness property for the candidates. This axiom requires of an ABC voting rule f that $f(\tau(A)) = \{\tau(W) : W \in f(A)\}$ for all profiles $A \in \mathcal{A}^*$ and permutations $\tau : C \to C$. This time, $A' = \tau(A)$ denotes the profile with $N_{A'} = N_A$ and $A'_i = \tau(A_i)$ for all $i \in N_A$.

Weak Efficiency. Weak efficiency requires that unanimously unapproved candidates can never be "better" than approved ones. Formally, we say an ABC voting rule f is *weakly efficient* if $W \in f(A)$ for a committee $W \in \mathcal{W}_k$ with $c \in W \setminus (\bigcup_{i \in N_A} A_i)$ implies that $(W \cup \{c'\}) \setminus \{c\} \in f(A)$ for all candidates $c' \in C \setminus W$.

Continuity. As our fourth axiom, we define continuity. The intuition behind this axiom is that a large group of voters should be able to enforce that some of its desired outcomes are chosen. Hence, an ABC voting rule f is *continuous* if for all profiles $A, A' \in \mathcal{A}^*$, there is $\lambda \in \mathbb{N}$ such that $f(\lambda A + A') \subseteq f(A)$. Here, λA denotes the profile consisting of λ disjoint copies of A; the names of the voters in $N_{\lambda A}$ will not matter as we will focus on anonymous rules. Note that anonymity, neutrality, weak efficiency, and continuity are all very mild axioms which are satisfied by essentially all commonly considered ABC voting rules.

Consistency. The central axiom for our results is consistency. This condition states that if some committees are chosen for two disjoint profiles, then precisely those committees are chosen in the joint profile. Formally, an ABC voting rule f is *consistent* if $f(A + A') = f(A) \cap f(A')$ for all disjoint profiles $A, A' \in \mathcal{A}^*$ with

$f(A) \cap f(A') \neq \emptyset$. Consistency and the previous four axioms have been introduced by Lackner and Skowron [22] for ABC elections.

Independence of Losers. Our next axiom, independence of losers, has been adapted to ABC elections by Dong and Lederer [12] and requires of an ABC voting rule f that a winning committee W should still be a winning committee if voters disapprove candidates outside of W. Formally, we say that f is *independent of losers* if $W \in f(A)$ implies $W \in f(A')$ for all profiles $A, A' \in \mathcal{A}^*$ and committees $W \in \mathcal{W}_k$ such that $N_A = N_{A'}$ and $W \cap A_i = W \cap A'_i$, $A'_i \subseteq A_i$ for all $i \in N_A$. The motivation behind this concept is that the quality of W should only depend on the candidates in W. So, if the voters disapprove candidates $x \notin W$, this does not affect the quality of W and a chosen committee W should thus stay chosen. All commonly studied ABC voting rules that are independent of the ballot size (e.g., Thiele rules, sequential Thiele rules, and Phragmen's rule) satisfy this axiom, whereas most BSWAV rules (e.g. SAV) fail this condition.

Choice Set Convexity. Finally, we introduce a new condition called choice set convexity: an ABC voting rule f is *choice set convex* if $W, W' \in f(A)$ implies that $W'' \in f(A)$ for all committees $W, W', W'' \in \mathcal{W}_k$ and profiles $A \in \mathcal{A}^*$ such that $W \cap W' \subseteq W'' \subseteq W \cup W'$. More informally, this axiom states that if a rule chooses two committees W and W', then all committees "between" W and W' should also be chosen. We believe that choice set convexity is reasonable in elections in which only the individual quality of the candidates matters. For instance, if we want to hire 3 applicants for a job based on the interviewer's preferences, it seems unreasonable that the sets $\{c_1, c_2, c_3\}$ and $\{c_1, c_4, c_5\}$ are good enough to be hired, but $\{c_1, c_2, c_4\}$ is not. More generally, we can interpret the membership of a candidate in a chosen committee as certificate for its quality and all candidates $c \in (W \setminus W') \cup (W' \setminus W)$ are then equally good. Many commonly considered voting rules fail this axiom, but it is always possible to compute the "convex hull" of a choice set.

3 CHARACTERIZATIONS OF CLASSES OF ABC VOTING RULES

We now turn to our characterizations of Thiele rules and BSWAV rules, which are discussed in Section 3.2 and Section 3.3, respectively. Unfortunately, the proofs of these results are very involved, so we defer them to the appendix. Since we nevertheless want to showcase our proof technique, we revisit the result of Lackner and Skowron [22] in Section 3.1 for the case that $k \in \{1, m-1\}$ as this allows us to explain our ideas while avoiding challenging technical details.

3.1 ABC Scoring Rules

As mentioned in the introduction, Lackner and Skowron [22] have partially characterized ABC scoring rules: they have shown that a 2-non-imposing ABC voting rule is an ABC scoring rule if and only if it satisfies anonymity, neutrality, consistency, continuity, and weak efficiency. However, this result focuses on 2-*non-imposing* ABC voting rules, which means that for all committees $W, W' \in \mathcal{W}_k$, there is a profile $A \in \mathcal{A}^*$ with $f(A) = \{W, W'\}$. In this section, we revisit this theorem for the case that $k \in \{1, m-1\}$ to showcase our proof idea for deriving Theorems 1 and 2. In more detail, we will reprove the result of Lackner and Skowron [22] without 2-non-imposition for $k \in \{1, m-1\}$ as it is easy to see that this axiom is not required in this case.

Our techniques are inspired by those of Young [32] and Skowron et al. [28] as we will use the separating hyperplane theorem for convex sets to find the scoring vectors of our rules. To further explain our approach, let f denote an ABC voting rule that satisfies anonymity, neutrality, consistency, and *non-imposition.* The last condition means that for every committee $W \in \mathcal{W}_k$, there is a profile $A \in \mathcal{A}^*$ such that $f(A) = \{W\}$. This axiom is no restriction for our analysis as all non-trivial ABC voting rules that we consider are non-imposing. As first step, we will change the domain of f from approval profiles to a numerical space. For this, we use that f is anonymous and thus only depends on the number of voters who submit a specific ballot. To this end, let $B\colon \{1, \dots, |\mathcal{A}|\} \to \mathcal{A}$ denote a bijection that enumerates all approval ballots and define $v(A)$ as the vector whose ℓ-th entry states how often the ballot $B(\ell)$ appears in the profile A. By anonymity, there is a function $g : \mathbb{N}^{|\mathcal{A}|} \to 2^{\mathcal{W}_k} \setminus \{\emptyset\}$ such that $f(A) = g(v(A))$ for all profiles $A \in \mathcal{A}^*$. Moreover, this function inherits neutrality ($g(\tau(v)) = \{\tau(W)\colon W \in g(v)\}$ for all vectors v and permutations $\tau : C \to C$) and consistency ($g(v + v') = g(v) \cap g(v')$ for all vectors v, v' with $g(v) \cap g(v') \neq \emptyset$) from f. Here, $\tau(v)$ denotes the vector such that $\tau(v)_i = v_j$ for all i, j with $B(i) = \tau(B(j))$. Next, we extend the domain of g from $\mathbb{N}^{|\mathcal{A}|}$ to $\mathbb{Q}^{|\mathcal{A}|}$ while preserving all desirable properties. The proof of this claim can be found in the appendix.

Lemma 1. *Let f denote a non-imposing ABC voting rule that satisfies anonymity, neutrality, and consistency. There is a function $\hat{g} : \mathbb{Q}^{|\mathcal{A}|} \to 2^{\mathcal{W}_k} \setminus \{\emptyset\}$ that satisfies neutrality, consistency, and $\hat{g}(v(A)) = f(A)$ for all $A \in \mathcal{A}^*$.*

Since $\hat{g}$ fully describes f, we aim to represent $\hat{g}$ by a scoring function. For this, we define an arbitrary order $W^1, \dots, W^{|\mathcal{W}_k|}$ over the committees and let $R_i^f = \{v \in \mathbb{Q}^{|\mathcal{A}|}\colon W^i \in \hat{g}(v)\}$. Moreover, $\bar{R}_i^f$ is the closure of R_i^f with respect to $\mathbb{R}^{|\mathcal{A}|}$. It is easy to see that the sets $\bar{R}_i^f$ are convex and their interiors are disjoint because of the properties of $\hat{g}$. We can thus apply the separating hyperplane theorem for convex set to derive a non-zero vector $u^{i,j}$ such that $u^{i,j}v \geq 0$ if $v \in \bar{R}_i^f$ and $u^{i,j}v \leq 0$ if $v \in \bar{R}_j^f$ for every pair of sets $\bar{R}_i^f, \bar{R}_j^f$. Our next lemma shows that there are symmetric non-zero vectors that fully describe the sets $\bar{R}_i^f$.

Lemma 2. *Let f denote a non-imposing ABC voting rule that satisfies anonymity, neutrality, and consistency. There are non-zero vectors $\hat{u}^{i,j}$ that satisfy the following conditions for all $W^i, W^j \in \mathcal{W}_k$:*

(1) $\bar{R}_i^f = \{v \in \mathbb{R}^{|\mathcal{A}|}\colon \forall j' \in \{1, \dots, |\mathcal{W}_k|\} \setminus \{i\}\colon \hat{u}^{i,j'}v \geq 0\}$.
(2) $\hat{u}^{i,j} = -\hat{u}^{j,i}$.
(3) $\hat{u}^{i',j'} = \tau(\hat{u}^{i,j})$ if $\tau(W^i) = W^{i'}$ and $\tau(W^j) = W^{j'}$.

Based on the vectors $\hat{u}^{i,j}$ derived in Lemma 2, we will next aim to derive the score function. We explain our approach for this in the next proposition.

Proposition 1. *Assume $k = 1$ or $k = m - 1$. An ABC voting rule is an ABC scoring rule if and only if it satisfies anonymity, neutrality, consistency, continuity, and weak efficiency.*

Proof. It is easy to check that ABC scoring rules satisfy all given axioms. So, we focus on the converse and let f denote an ABC voting rule that satisfies anonymity, neutrality, consistency, continuity, and weak efficiency for committee size $k = 1$; the case that $k = m - 1$ follows from similar arguments. First, if f is the trivial rule, it is the ABC scoring rule induced by the score function $s(x, y) = 0$. Hence, suppose that f is non-trivial. Our first goal is to show that f is non-imposing. By non-triviality and consistency, there is a ballot $A \in \mathcal{A}$ such that $f(A) \neq \mathcal{W}_k$; otherwise consistency requires that $f(A) = \mathcal{W}_k$ for all profiles $A \in \mathcal{A}^*$. Now, let $c, d \in C$ denote candidates such that $\{c\} \in f(A)$, $\{d\} \notin f(A)$ and consider any permutation $\tau : C \to C$ with $\tau(c) = c$. By neutrality, $\{c\} \in f(\tau(A))$, $\{\tau(d)\} \notin f(\tau(A))$. We next consider the profile A^* that consists of the ballots $\tau(A)$ for every permutation τ with $\tau(c) = c$. By consistency, we infer that $f(A^*) = \bigcap_{\tau: C \to C\colon \tau(c)=c} f(\tau(A)) = \{\{c\}\}$. This means that f is non-imposing due to neutrality.

Next, we use Lemma 1 to obtain the function $\hat{g} : \mathbb{Q}^{|\mathcal{A}|} \to 2^{\mathcal{W}_k} \setminus \{\emptyset\}$ and define the sets $R_i^f = \{v \in \mathbb{Q}^{|\mathcal{A}|} : W^i \in \hat{g}(v)\}$. In turn, we derive from Lemma 2 the existence of symmetric non-zero vectors $\hat{u}^{i,j}$ such that $\bar{R}_i^f = \{v \in \mathbb{R}^{|\mathcal{A}|} : \forall j \in \{1, \dots, |\mathcal{W}_k|\} \setminus \{i\} : \hat{u}^{i,j} v \geq 0\}$. Now, consider committees $W^i, W^j, W^{i'}, W^{j'} \in \mathcal{W}_k$ with $W^i \neq W^j$ and $W^{i'} \neq W^{j'}$. Since $k = 1$, this means that $|W^i \setminus W^j| = |W^{i'} \setminus W^{j'}| = 1$. Moreover, let $B(\ell)$, $B(\ell')$ denote two ballots such that $|B(\ell)| = |B(\ell')|$, $|B(\ell) \cap W^i| = |B(\ell') \cap W^{i'}|$, and $|B(\ell) \cap W^j| = |B(\ell') \cap W^{j'}|$. These assumptions imply that there is a permutation $\tau : C \to C$ such that $\tau(B(\ell)) = B(\ell')$, $\tau(W^i) = W^{i'}$, and $\tau(W^j) = W^{j'}$. Condition (3) of Lemma 2 then shows that $\hat{u}^{i',j'}_{\ell'} = \tau(\hat{u}^{i,j})_{\ell'} = \hat{u}^{i,j}_{\ell}$. From this insight, we infer that there is a function $s^1(x, y, z)$ such that $\hat{u}^{i,j}_{\ell} = s^1(|W^i \cap B(\ell)|, |W^j \cap B(\ell)|, |B(\ell)|)$ for all committees $W^i, W^j \in \mathcal{W}_k$ and ballots $B(\ell) \in \mathcal{A}$. Next, consider two committees W^i and W^j and a permutation τ such that $\tau(W^i) = W^j$, $\tau(W^j) = W^i$, and $\tau(x) = x$ for all candidates $x \in C \setminus (W^i \cup W^j)$. By our previous analysis and Condition (2) of Lemma 2, it holds for all ballots $B(\ell)$ and $B(\ell') = \tau(B(\ell))$ that $-\hat{u}^{i,j}_{\ell'} = \hat{u}^{j,i}_{\ell'} = \tau(\hat{u}^{i,j})_{\ell'} = \hat{u}^{i,j}_{\ell}$ by choosing $W^{i'} = W^j$, $W^{j'} = W^i$. If $W^i \cup W^j \subseteq B(\ell)$ or $B(\ell) \cap (W^i \cup W^j) = \emptyset$, then $\tau(B(\ell)) = B(\ell)$ and this inequality simplifies to $-\hat{u}^{i,j}_{\ell} = \hat{u}^{i,j}_{\ell}$. This is only possible if $\hat{u}^{i,j}_{\ell} = 0$, so $s^1(x, x, z) = 0$ for all $x \in \{0, 1\}$, $z \in \{1, \dots, m\}$. On the other hand, if $|W^i \cap B(\ell)| = 1 > 0 = |W^j \cap B(\ell)|$, then $|W^i \cap B(\ell')| = 0 < 1 = |W^j \cap B(\ell')|$ and we infer that $s^1(1, 0, z) = -s^1(0, 1, z)$ for all $z \in \{1, \dots, m\}$.

We can now infer the score function $s(x, z)$ from $s^1(x, y, z)$: we define $s(0, z) = 0$ and $s(1, z) = s^1(1, 0, z)$ for all $z \in \{1, \dots, m\}$. It is now easy to check that $\hat{u}^{i,j} v = \sum_{\ell \in \{1,\dots,|\mathcal{A}|\}} v_\ell s^1(|W^i \cap B(\ell)|, |W^j \cap B(\ell)|, |B(\ell)|) = \sum_{\ell \in \{1,\dots,|\mathcal{A}|\}} v_\ell (s(|W^i \cap B(\ell)|, |B(\ell)|) - s(|W^j \cap B(\ell)|, |B(\ell)|))$ for all committees W^i, $W^j \in \mathcal{W}_k$ and vectors $v \in \mathbb{R}^{|\mathcal{A}|}$. From this, we infer that $\bar{R}_i^f = \{v \in \mathbb{R}^{|\mathcal{A}|} : \forall j \in \{1, \dots, |\mathcal{W}_k|\} : \sum_{\ell \in \{1,\dots,|\mathcal{A}|\}} v_\ell s(|W^i \cap B(\ell)|, |B(\ell)|) \geq \sum_{\ell \in \{1,\dots,|\mathcal{A}|\}} v_\ell s(|W^j \cap B(\ell)|, |B(\ell)|)\}$. Hence, $f(A) = \hat{g}(v(A)) \subseteq \{W \in \mathcal{W}_k : \forall W' \in \mathcal{W}_k : \hat{s}(A, W) \geq \hat{s}(A, W')\} := f'(A)$ for all $A \in \mathcal{A}^*$.

Next, we will show that this subset relation is an equality. Suppose for this that there is a profile A such that $f(A) \subsetneq f'(A)$ and let $\{d\} \in f'(A) \setminus f(A)$. We note that f' is consistent and non-trivial, so an analogous argument as for f shows that it is non-imposing. Thus, there is a profile A' such that $f'(A') = \{\{d\}\}$. By the consistency of f' and the above subset relation, we have that $f(\lambda A + A') = f'(\lambda A + A') = \{\{d\}\}$ for all $\lambda \in \mathbb{N}$. However, this contradicts the continuity of f, which requires that there is $\lambda \in \mathbb{N}$ such that $f(\lambda A + A') \subseteq f(A)$. So, f is the ABC scoring rule induced by s. Finally, we show that s is non-decreasing. Otherwise, there is a ballot size $y \in \{1, \dots, m-1\}$ such that $0 = s(0, y) > s(1, y)$. Now, consider a single ballot A of size y. By definition of s and f, $f(A) = \{W \in \mathcal{W}_k : W \nsubseteq A\}$. However, this outcome violates weak efficiency, so s needs to be non-decreasing in its first argument. □

3.2 Characterization of Thiele Rules

We now turn to our first full characterization: Thiele rules are the only ABC voting rules that satisfy anonymity, neutrality, consistency, continuity, and independence of losers. We note here that, compared to the results of Lackner and Skowron [23], we only need to replace weak efficiency with independence of losers.

Theorem 1. *An ABC voting rule is a Thiele rule if and only if it satisfies anonymity, neutrality, consistency, continuity, and independence of losers.*

Proof Sketch. We prove here only the implication from left to right and give an informal proof sketch for the other direction. Hence, suppose that f is a Thiele rule and let $s(x)$ denote its Thiele scoring function. By its definition, it is obvious that f satisfies anonymity, neutrality, and continuity. For consistency, we note the scores $\hat{s}(A, W)$ are additive. Hence, if $\hat{s}(A, W) \geq \hat{s}(A, W')$ and $\hat{s}(A', W) \geq \hat{s}(A', W')$ for all W', then $\hat{s}(A + A', W) \geq \hat{s}(A + A', W')$ for all committees $W' \in \mathcal{W}_k$. This entails that f is consistent.

Next, we show that f is independent of losers. For this, consider two profiles $A, A' \in \mathcal{W}_k$ and a committee $W \in \mathcal{W}_k$ such that $W \in f(A)$, $N_A = N_{A'}$, and $A'_i \subseteq A_i$ and $W \cap A'_i = W \cap A_i$ for all $i \in N_A$. By the definition of Thiele scoring functions, it holds that $\hat{s}(A', W) = \hat{s}(A, W)$ since $W \cap A'_i = W \cap A_i$ for all $i \in N_A$. On the other hand, $\hat{s}(A, W') \geq \hat{s}(A', W')$ for all $W' \in \mathcal{W}_k$ as $s(x)$ is non-decreasing. Finally, since $W \in f(A)$, $\hat{s}(A, W) \geq \hat{s}(A, W')$ for all $W' \in \mathcal{W}_k$ and we conclude that $\hat{s}(A', W) = \hat{s}(A, W) \geq \hat{s}(A, W') \geq \hat{s}(A', W')$ for all committees $W' \in \mathcal{W}_k$. So, $W \in f(A')$ and f satisfies independence of losers.

For the other direction, suppose that f is an ABC voting rule satisfying anonymity, neutrality, consistency, continuity, and independence of losers. If f is trivial, it is the Thiele rule defined by $s(x) = 0$ for all x. Hence, suppose that f is non-trivial. As the first step, we then show that f is non-imposing and hence, we can use Lemmas 1 and 2 to derive that f (resp. the function $\hat{g}$) can be described by non-zero vectors $\hat{u}^{i,j}$. Moreover, due to independence of losers, we get that $\hat{u}^{i,j}_{\ell} = \hat{u}^{i,j}_{\ell'}$ for all committees $W^i, W^j \in \mathcal{W}_k$ and ballots $B(\ell), B(\ell') \in \mathcal{A}$ with $|X \cap B(\ell)| = |X \cap B(\ell')|$ for $X \in \{W^i \cap W^j, W^i \setminus W^j, W^j \setminus W^i\}$, regardless of $|B(\ell)|$ and $|B(\ell')|$. Now, if $k = 1$ or $k = m - 1$, we can infer the claim with an analogous reasoning as in the proof of Proposition 1. In contrast, if $1 < k < m - 1$, we need to relate the vectors $\hat{u}^{i,j}$ and $\hat{u}^{i',j'}$ for committees $W^i, W^j, W^{i'}, W^{j'} \in \mathcal{W}_k$ with $|W^i \setminus W^j| \neq |W^{i'} \setminus W^{j'}|$.

For doing so, consider two arbitrary committees W^i and W^j and suppose that $|W^i \setminus W^j| = t > 1$. Next, we construct a sequence of committees $W^{i_0}, \dots, W^{i_t}$ by replacing the candidates in $W^i \setminus W^j$ one after another with those in $W^j \setminus W^i$. Hence, $W^i = W^{i_0}$, $W^j = W^{i_t}$, and $|W^{i_{x-1}} \setminus W^{i_x}| = 1$ for all $x \in \{1, \dots, t\}$. Our main goal is to show that $\hat{u}^{i,j} = \delta \sum_{x=1}^{t-t} \hat{u}^{i_{x-1},i_x}$ for some $\delta > 0$. For proving this, we investigate the linear independence of the vectors $\hat{u}^{i,j}$ and $\hat{u}^{i_{x-1},i_x}$ for $x \in \{1, \dots, t\}$, and prove that the set $\{\hat{u}^{i_0,i_1}, \dots, \hat{u}^{i_{t-1},i_t}\}$ is linearly independent but the set $\{\hat{u}^{i_0,i_1}, \dots, \hat{u}^{i_{t-1},i_t}, \hat{u}^{i,j}\}$ is not. So, $\hat{u}^{i,j}$ is a linear combination of the vectors $\hat{u}^{i_x,i_{x+1}}$ and we only need to derive the coefficients to show our claim.

Based on this insight, we now define our scoring vector. Firstly, we define $s^1(x, y) = \hat{u}^{i,j}_\ell$ for two arbitrary committees W^i, W^j with $|W^i \setminus W^j| = 1$ and a ballot $B(\ell)$ such that $|B(\ell) \cap W^i| = x$ and $|B(\ell) \cap W^j| = y$. Note that s^1 is defined in the same way as in the proof of Proposition 1. Finally, we define the score function $s(x)$ by $s(0) = 0$ and $s(x) = s(x-1) + s^1(x, x-1)$ for $x \geq 1$. By the additivity of the vectors $\hat{u}^{i,j}$, it follows that $\hat{u}^{i,j}_\ell = \delta(s(|W^i \cap B(\ell)|) - s(|W^j \cap B(\ell)|))$, so $\bar{R}^f_i = \{v \in \mathbb{R}^{|\mathcal{A}|} : \forall W^j \in \mathcal{W}_k : \hat{s}(v, W^i) \geq \hat{s}(v, W^j)\}$. Next, the definition of these sets entails that $f(A) = \hat{g}(v(A)) = \{W^i \in \mathcal{W}_k : v(A) \in R^f_i\} \subseteq \{W^i \in \mathcal{W}_k : v(A) \in \bar{R}^f_i\}$. Finally, continuity shows that the subset relation is an equality and independence of losers that s is non-decreasing. Thus, f is a Thiele rule. ◻

Remark 1. Based on Theorem 1, it is simple to prove full characterizations of specific Thiele rules. For instance, it is known that AV is essentially the only Thiele rule that satisfies committee monotonicity (the winning committees of size k are derived from the winning committees of size $k-1$ by only adding candidates) and based on Theorem 1, it is simple to formalize this observation. Another example is a characterization of CCAV by Delemazure et al. [11] within the class of Thiele rules based on mild proportionality and strategyproofness conditions, which can be turned into a full characterization based on Theorem 1. In particular, we note that these characterizations only hold within the class of Thiele rules and not within the class of ABC scoring rules, so analogous claims based on the result by Lackner and Skowron [22] fail.

Remark 2. All axioms are required for Theorem 1. If we omit independence of losers, SAV satisfies all remaining axioms. If we omit continuity, we can define composed Thiele rules analogous to the composed scoring rules of Young [32]: these rules refine Thiele rules by applying another Thiele rule as tie-breaker in case of multiple chosen committees. If we only omit consistency, sequential Thiele rules satisfy all given axioms. These rules compute the winning committees iteratively by always adding the candidate to a winning committee which increases the score the most. If we omit neutrality, we can introduce a bias to certain candidates: for instance, Thiele rules that doubles the points of every committee that contains a satisfy all remaining axioms. Finally, if we omit anonymity, we can treat voters differently and for example double the points that a specific voter assigns to the committees.

3.3 Characterization of BSWAV rules

Next, we discuss the characterization of BSWAV rules: these are the only ABC voting rules that satisfy anonymity, neutrality, consistency, continuity, choice set convexity, and weak efficiency. The central axiom for this characterization (aside of consistency) is choice set convexity as it enforces that candidates become exchangeable.

Theorem 2. *An ABC voting rule is a BSWAV rule if and only if it satisfies anonymity, neutrality, consistency, continuity, choice set convexity, and weak efficiency.*

Proof Sketch. Just as for Theorem 1, we only prove the implication from left to right and give a proof sketch for the converse direction. So, we first assume that f is a BSWAV rule and let $\alpha = (\alpha_1, \dots, \alpha_m)$ denote its weight vector. It is simple to verify that f is neutral, anonymous, continuous, and consistent. Moreover, f is weakly efficient as the weights α_i are all non-negative. Finally, we show that f is choice set convex. For this, we consider a profile A and two committees $W, W' \in f(A)$ with $|W \setminus W'| = t > 0$. Moreover, we choose two candidates $a \in W \setminus W'$ and $b \in W' \setminus W$ and let $W'' = (W \setminus \{a\} \cup) \{b\}$. The central observation is now that BSWAV scores are additive, i.e., $\hat{s}(A, W) = \sum_{x \in W} \hat{s}(A, x)$ for $\hat{s}(A, x) = \sum_{i \in N_A : x \in A_i} \alpha_{|A_i|}$. Hence, $0 \leq \hat{s}(A, W) - \hat{s}(A, W'') = \hat{s}(A, a) - \hat{s}(A, b)$ as $W \in f(A)$. By applying this argument also to W' and $W''' = (W' \setminus \{b\} \cup) \{a\}$, we obtain $0 \leq \hat{s}(A, b) - \hat{s}(A, a)$, so $\hat{s}(A, a) = \hat{s}(A, b)$ and $\hat{s}(A, W) = \hat{s}(A, W'')$. This proves that $W'' \in f(A)$ and by repeating the argument, we infer that $\bar{W} \in f(A)$ for all $\bar{W}$ with $W \cap W' \subseteq \bar{W} \subseteq W \cup W'$.

For the converse direction, we suppose that f is an ABC voting rule which satisfies all given axioms and note that the proof outline for this case is essentially the same as for Theorem 1 but the technical details differ. In particular, if f is the trivial rule, it is the BSWAV rule defined by $\alpha_i = 0$ for $i \in \{1, \dots, m\}$. On the other hand, if f is non-trivial, we show that it is non-imposing and then apply Lemmas 1 and 2. If $k = 1$ or $k = m - 1$, choice set convexity has no consequences and Proposition 1 shows the claim as the set of BSWAV rules coincides with the set of ABC scoring rules in this case. Hence, suppose that $1 < k < m - 1$. We again need to relate normal vectors of committee pairs with different intersection sizes. We show for this that the vectors $\hat{u}^{i,j}$ for $|W^i \setminus W^j| > 1$ can be represented as scaled sum of vectors of $\hat{u}^{i_x,i_{x+1}}$ where W^{i_x} differs only in a single element from $W^{i_{x+1}}$. However, this time, we heavily rely on choice set convexity as this axiom implies that there are α_x such that $\hat{u}^{i,j}_\ell = \alpha_x$ for all committees W^i, W^j and ballots $B(\ell)$ with $|W^i \setminus W^j| = 1$, $|W^i \cap B(\ell)| > |W^j \cap B(\ell)|$, and $|B(\ell)| = x$. Based on this insight, we can derive the score function with an analogous approach as for Thiele rules and see that the score of committee W and a ballot $B(\ell)$ can be represented as $s(|W \cap B(\ell)|, |B(\ell)|) = \alpha_{|B(\ell)|} |W \cap B(\ell)|$. Finally, we can complete the proof by showing that this score function is non-negative and completely describes f. ◻

Remark 3. All axioms are required for Theorem 2. For anonymity, neutrality, and continuity, we can define examples similar to the ones given for Thiele rules. For instance, to only violate anonymity, we can count the vote of a specific agent twice, and to only violate continuity, we can apply a second BSWAV rule as tie-breaker. When omitting consistency, the "convex hull" of Phragmen's rule satisfies all remaining axioms. Theorem 2 of Peters and Skowron [26] then shows that this rule cannot be represented as ABC scoring rule and therefore also not as BSWAV rule. When only omitting weak efficiency, "inverse" AV, which chooses the committees with minimal approval scores, satisfies all given axioms. Finally, every Thiele rule other than AV only fails choice set convexity.

Remark 4. Choice set convexity conflicts with 2-non-imposition if $1 < k < m-1$ as no rule that satisfies choice set convexity can choose precisely two committees that differ in at least two candidates. Hence, our characterization of BSWAV rules is completely independent from the main result of Lackner and Skowron [22]. Moreover, this axiom ensures several desirable properties. For instance, all BSWAV rules are easy to compute and satisfy committee monotonicity.

Remark 5. We define ABC voting rules for a fixed committee size k, but in the literature k is often considered as part of the input. For such rules, Theorems 1 and 2 imply that for every $k \in \{1, \dots, m-1\}$, $f(A, k)$ is a Thiele rule or a BSWAV rule, respectively, if it satisfies the required axioms. However, none of our conditions enforce consistency with respect to k, so we can, e.g., use AV for $k = 2$ and PAV for $k = 3$. To eliminate such behavior, we can use the following condition, which one might call committee size consistency: suppose there are a profile A and a candidate c such that $f(A, k) = \{W \in \mathcal{W}_k : c \in W\}$ for some k. Then, it holds that *(i)* there is a profile A' such that $f(A', k+1) = \{W \in \mathcal{W}_{k+1} : c \in W\}$ and *(ii)* if $f(A+A', k) \subseteq f(A, k)$ for some profile A' such that $c \notin A'_i$ for all $i \in N_{A'}$, then $f(A+A', k) = \{\{c\} \cup W : W \in f(A', k-1) \wedge c \notin W\}$. The intuition of this axiom lies in the profile A: this profile contains no information except that c needs to be chosen (e.g., for many voting rules the profile where a single voter approves c satisfies this). Then, *(i)* requires that such a profile also exist for the next larger committee size and *(ii)* states that if we add a profile A' in which c is never approved but still guaranteed to be in the chosen committee for $A + A'$, we can determine the remaining members of the winning committees by only looking at A'.

4 CHARACTERIZATIONS OF AV, PAV, AND SAV

Finally, we use Theorems 1 and 2 to characterize three specific ABC voting rules, namely AV, SAV, and PAV. Note that, while AV and PAV can also be characterized by combining Theorem 1 with known results from the literature, we prefer to give own characterizations of these rules. For AV, we do so because the characterization follows naturally from our results and for PAV because our characterization highlights a new aspect of this rule. Moreover, to keep the theorems short, we characterize these rules only within the class of Thiele rules or BSWAV rules; our results in Section 3 then generalize them to full characterizations. Due to space restrictions we defer all proofs to the appendix.

For the characterization of AV, we note that this rule and the trivial rule are the only ABC voting rules that are both Thiele rules and BSWAV rules if $k < m-1$. On the other hand, we observe that this result fails if $k = m-1$ because choice set convexity then becomes trivial and all Thiele rules satisfy the given axioms. These insights entail the following theorem.

Theorem 3. *Assume $k \leq m-2$. AV is the only non-trivial Thiele rule that satisfies choice set convexity and the only non-trivial BSWAV rule that satisfies independence of losers.*

Proof Sketch. The direction from left to right is straightforward as AV is both a Thiele rule and a BSWAV rule and thus satisfies independence of losers and choice set convexity by Theorems 1 and 2. For the converse direction, we first note that a Thiele rule that satisfies choice set convexity is by these theorems also a BSWAV rule that satisfies independence of losers. Hence, suppose that f is a non-trivial BSWAV rule satisfying independence of losers and let α denote its scoring vector. By non-triviality, there is $i \in \{1, \dots, m-1\}$ such that $\alpha_i > 0$. As the next step, we consider two committees W, W' with $|W \setminus W'| = 1$ and construct a profile $\bar{A}$ such that $f(\bar{A}) = \{W, W'\}$. Furthermore, we let A denote a profile consisting of two ballots A_i, A_j of size $\ell \in \{2, \dots, m-1\}$ such that A_i differs from A_j by only replacing the candidate $x \in W' \setminus W$ with the candidate $y \in W \setminus W'$ and there is $z \in A_i \setminus (W \cup W')$. By continuity, there is λ such that $f(\lambda\bar{A} + A) \subseteq \{W, W'\}$ and since both committees are symmetric in A and $\bar{A}$, this must be an equality. Finally, let A' be the profile derived from A by replacing A_i with $A_i \setminus \{z\}$. Since $z \notin W \cup W'$, independence of losers and the choice of λ imply that $f(\lambda\bar{A} + A') = \{W, W'\}$. From this, we infer that $\alpha_{\ell-1} = \alpha_\ell$ for all $\ell \in \{2, \dots, m-1\}$, which shows that f is AV. □

Next, we turn to the characterizations of SAV and PAV, for which we focus on party-list profiles. In these profiles, the candidates are partitioned into parties and every voter supports a single party by approving all of its members. Formally, an approval profile A is a *party-list profile* if there is a partition of the candidates $C = P_1 \cup \dots \cup P_\ell$ such that for every voter $i \in N_A$, there is an index $j \in \{1, \dots, \ell\}$ with $A_i = P_j$. In particular, we consider the following question for these profiles: does it make sense for individual candidates to form a party or are they better off if they compete by themselves? Clearly, the answer to this question depends on the voting rule at hand. For instance, consider the profiles A^1, A^2, and A^3 shown below and assume that $k = 3$. Moreover, we assume that the candidates in $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$, respectively, present rather similar positions. In A^1, where all candidates compete by themselves, most voting rules will elect the committee $\{a_1, a_2, a_3\}$. In contrast, AV chooses the committee $\{b_1, b_2, b_3\}$ for A^2 and it thus makes sense for the candidates $\{b_1, b_2, b_3\}$ to form a party if the voters do not change their support. On the other hand, CCAV will choose every committee W with $|W \cap \{a_1, a_2, a_3\}| = 1$ and $|W \cap \{b_1, b_2, b_3\}| = 2$ for A^3 and it thus makes sense for the candidates in A to compete by themselves.

A^1:	$2: \{a_1\}$	$2: \{a_2\}$	$2: \{a_3\}$	$1: \{b_1\}$	$1: \{b_2\}$	$1: \{b_3\}$
A^2:	$2: \{a_1\}$	$2: \{a_2\}$	$2: \{a_3\}$		$3: \{b_1, b_2, b_3\}$	
A^3:		$6: \{a_1, a_2, a_3\}$		$1: \{b_1\}$	$1: \{b_2\}$	$1: \{b_3\}$

We believe that such strategic considerations of candidates about whether to compete as a group or individually are undesirable. Hence, we introduce next the concept of split/merge-proofness which aims to prohibit this behavior. Informally, this axiom requires that it should not matter whether there are j candidates that are each approved by ℓ voters or a party of j candidates that is approved by $j\ell$ voters. To formalize this idea, we define the profile A^X given a party-list profile A and a set of parties $X = \{P_1, \dots, P_j\}$ as follows: $N_{A^X} = N_A$ and $A^X_i = A_i$ if $A_i \notin X$ and $A^X_i = \bigcup X$ if $A_i \in X$. We then define this axiom as follows.

Split/Merge-proofness. An ABC voting rule f is *split/merge-proof* if $f(A) = f(A^X)$ for all party-list profiles $A \in \mathcal{A}^*$ with parties $\mathcal{P} = \{P_1, \dots, P_\ell\}$ and all sets $X \subseteq \mathcal{P}$ such that $|P_i| = |P_{i'}| = 1$ and $|\{j \in N_A : A_j = P_i\}| = |\{j \in N_A : A_j = P_{i'}\}|$ for all $P_i, P'_i \in X$.

As we show next, this axiom characterizes SAV within the class of non-trivial BSWAV rules.

Theorem 4. SAV *is the only non-trivial BSWAV rule that satisfies split/merge-proofness.*

Proof Sketch. First, we note that SAV is split/merge-proof as we can separate the scores of committees into the scores of candidates. Moreover, if a candidate is uniquely approved by ℓ voters, it has a score of ℓ and if a candidate is approved by ℓj voters who approve a total of j candidates, its score is also $\frac{\ell j}{j} = \ell$. Hence, merging candidates into a party does not change their scores and thus also not the final outcome. For the other direction, we consider an arbitrary non-trivial and split/merge-proof BSWAV rule f and its weight vector $\alpha = (\alpha_1, \dots, \alpha_m)$. Since f is non-trivial, there is an index $\ell \in \{1, \dots, m-1\}$ such that $\alpha_\ell > 0$. Hence, if there are z voters that report the same ballot A_i of size z in A, then $f(A) \neq \mathcal{W}_k$. By split/merge-proofness, the same must be true for the profile A' in which each candidate $x \in A_i$ is uniquely approved by a single voter, so $\alpha_1 > 0$. Since f is invariant under scaling α, we assume next that $\alpha_1 = 1$. Moreover, suppose for contradiction that f is not SAV, i.e., that there is $\ell \in \{1, \dots, m-1\}$ with $\alpha_\ell \neq \frac{1}{\ell}$. We derive a contradiction to this by constructing a profile in which split/merge-proofness is violated. For instance, if $\alpha_\ell < \frac{1}{\ell}$, we define $\Delta = \frac{1}{\ell} - \alpha_\ell$, $B = \{c_1, \dots, c_\ell\}$, and choose $t \geq 2$ such that $t\ell\Delta > 1$. Then, we consider the profile A in which each candidate $c_i \in B$ is uniquely approved by t voters and all candidates $c_j \in C \setminus B$ are uniquely approved by $t-1$ voters. In this profile, f chooses the committees W that maximize $|W \cap B|$, but if the voters $i \in N_A$ with $A_i \subseteq B$ change their ballot to B, this is no longer true and split/merge-proofness is violated. □

Finally, for the characterization of PAV, we observe that no Thiele rule satisfies split/merge-proofness. We thus consider a weakening of this axiom.

Weak Split/Merge-proofness. An ABC voting rule f is *weakly split/merge-proof* if $f(A) = f(A^X)$ for all party-list profiles A with parties $\mathcal{P} = \{P_i, \dots, P_\ell\}$ and sets of parties $X \subseteq \mathcal{P}$ such that $|P_i| = |P_{i'}| = 1$ and $|\{j \in N_A \colon A_j = P_i\}| = |\{j \in N_A \colon A_j = P_{i'}\}|$ for all $P_i, P_{i'} \in X$, and $\bigcup X \subseteq W$ for all $W \in f(A)$ or $\bigcup X \subseteq W$ for all $W \in f(A^X)$. Less formally, weak split/merge-proofness has the same consequence as full split/merge-proofness, but only applies if all candidates in the split/merged parties are guaranteed to be elected in one of the considered profiles.

As we show next, PAV is characterized by weak split/merge-proofness within the class of non-trivial Thiele rules.

Theorem 5. PAV *is the only non-trivial Thiele rule that satisfies weak split/merge-proofness.*

Proof Sketch. First, for showing that PAV satisfies weak split/merge-proofness, consider a party-list profile A with parties $\mathcal{P} = \{P_1, \dots, P_j\}$ and let $X \subseteq \mathcal{P}$ denote a set of singleton parties with $|\{i \in N_A \colon A_i = P_{j_1}\}| = |\{i \in N_A \colon A_i = P_{j_2}\}| = c$ for all $P_{j_1}, P_{j_2} \in X$. Moreover, let $\hat{X} = \bigcup_{P_i \in X} P_i$ and consider the profile A^X derived from A by merging the parties in X. As a first step, we show that $\hat{X} \subseteq W$ for all $W \in \mathrm{PAV}(A)$ if and only if $\hat{X} \subseteq W$ for all $W \in \mathrm{PAV}(A^X)$. The central idea here is that in A, each party $P_i \in X$ increases the score of a committee by c, and in A^X, the last member in the new party $\hat{X}$ increases the score of a committee by $\frac{c|\hat{X}|}{|\hat{X}|}$. Hence, if there is for instance a committee $W \in \mathrm{PAV}(A)$ with $\hat{X} \not\subseteq W$, there is a party $P_i \in \mathcal{P} \setminus X$ which also increases the score by at least c, so we can also assign a seat to P_i instead of $\hat{X}$ in A^X without decreasing the score. Based on this insight, it is easy to show that PAV is weakly split/merge-proof as parties in $\mathcal{P} \setminus X$ get the same score in A and A^X. For the converse direction, we suppose for contradiction that f is a non-trivial Thiele rule other than PAV that satisfies weak split/merge-proofness. Similar to the proof of Theorem 4, we can then construct profiles in which weak split/merge-proofness is violated as there is a minimal index ℓ such that $s(\ell) \neq \sum_{x=1}^{\ell} \frac{1}{x}$. Hence, if we merge ℓ parties of size j, the contribution of the last member in the new party changes, which can be used to derive a contradiction. □

5 CONCLUSION

In this paper, we axiomatically characterize two important classes of approval-based committee (ABC) voting rules, namely Thiele rules and BSWAV rules. Thiele rules choose the committees that maximize the total score according to a score function that only depends on the intersection size of the considered committee and the ballots of the voters. On the other hand, BSWAV rules are a new generalization of multi-winner approval voting which weight voters depending on the size of their ballot. For both results, the crucial ingredient is consistency across disjoint profiles, which has famously been used by Young [32] for a characterization of single-winner scoring rules or by Lackner and Skowron [23] for a characterization of ABC scoring rules in a model where the output consists of rankings. In particular, our results allow for simple characterizations of all important ABC scoring rules as all such rules are either Thiele rules or BSWAV rules. We also demonstrate this point by deriving characterizations of the well-known ABC voting rules AV, SAV, and PAV from these results. In particular, the result SAV is, to the best of our knowledge, the first full characterization of this rule. We refer to Figure 1 for a more detailed overview of our results.

Finally, our paper offers several options for future work. Firstly, we believe that many ideas in this paper can turn out useful for characterizing ABC voting rules other than Thiele rules and BSWAV rules. For instance, there are still no characterization of Phragmén's rule, the method of equal shares, or Monroe's rule, and some of our ideas might be helpful for deriving such results. Secondly, we want to point out that both Thiele rules and BSWAV rules are subsets of the class of ABC scoring rules [see 23]. While all commonly studied ABC scoring rules belong to one of our classes, it seems nevertheless interesting to fully characterize this set of rules.

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A OMITTED PROOFS FROM SECTION 3

In this section, we will prove Theorem 1 and Theorem 2 in Appendix A.3 and Appendix A.2, respectively. Since the proofs of both of these results rely on the hyperplane argument explained in Section 3.1, we first discuss Lemmas 1 and 2 in Appendix A.1.

For all of our proofs, we use the notation introduced in Section 3.1. In particular, we suppose that there is a bijection $B : \{1, \dots, |\mathcal{A}|\} \to \mathcal{A}$ that enumerates all our ballots. This function allows us to represent profiles A by vectors v such that the ℓ-th entry of v states how often the ballot $B(\ell)$ is reported in A. When specifying the vector of a specific profile A, we typically write $v(A)$, but we also often consider arbitrary vectors $v \in \mathbb{R}^{|\mathcal{A}|}$ which have usually the same interpretation. Finally, we define the permutation of vectors as follows: $\tau(v)_{\ell_1} = v_{\ell_2}$ for all permutations $\tau : C \to C$, vectors v, and indices $\ell_1, \ell_2 \in \{1, \dots, |\mathcal{A}|\}$ such that $B(\ell_1) = \tau(B(\ell_2))$. Put differently, if there are v_2 ballots of type $B(\ell_2)$ in v, then there are $\tau(v)_{\ell_1} = v_{\ell_2}$ ballots of type $B(\ell_1) = \tau(B(\ell_2))$ in $\tau(v)$.

A.1 Proofs of Lemmas 1 and 2

In this subsection, we will show the hyperplane argument sketched in Section 3.1 and additionally investigate one of its central consequences. Note that our subsequent arguments do not rely on independence of losers, weak efficiency, or choice set convexity and thus form the basis of the proofs of both Theorems 1 and 2.

As the first step, we will prove Lemma 1. For this result, we recall that, since f is anonymous, there is a function $g : \mathbb{N}^{|\mathcal{A}|}$ such that $f(A) = g(v(A))$ for all profiles $A \in \mathcal{A}^*$. We show next how to extend this function to $\mathbb{Q}^{|\mathcal{A}|}$ while preserving its desirable properties.

Lemma 1. *Let f denote a non-imposing ABC voting rule that satisfies anonymity, neutrality, and consistency. There is a function $\hat{g} : \mathbb{Q}^{|\mathcal{A}|} \to 2^{\mathcal{W}_k} \setminus \{\emptyset\}$ that satisfies neutrality, consistency, and $\hat{g}(v(A)) = f(A)$ for all $A \in \mathcal{A}^*$.*

Proof. Let f denote a non-imposing ABC voting rule satisfying anonymity, neutrality, and consistency. Moreover, let $g : \mathbb{N}^{|\mathcal{A}|} \to \mathcal{W}_k$ denote a neutral and consistent function such that $g(v(A)) = f(A)$ for all $A \in \mathcal{A}^*$; f uniquely defines such a function since it is anonymous. We will subsequently extend the domain of g. For doing so, we will heavily rely on the profile A^* in which every ballot is reported once. Moreover, let $v^* = v(A^*)$ and observe that $v^*_\ell = 1$ for all $\ell \in \{1, \dots, |\mathcal{A}|\}$. Clearly, anonymity and neutrality require that $f(A^*) = g(v^*) = \mathcal{W}_k$ as all committees are symmetric to each other with respect to A^*.

Step 1: Extension to $\mathbb{Z}^{|\mathcal{A}|}$

First, we define a function $\bar{g} : \mathbb{Z}^{|\mathcal{A}|} \to \mathcal{W}_k$ that extends g to negative numbers: $\bar{g}(v - \ell v^*) = g(v)$ for all $v \in \mathbb{N}^{|\mathcal{A}|}$ and $\ell \in \mathbb{N}_0$. First, note that $\bar{g}$ is well-defined: for every two integers $\ell, \ell' \in \mathbb{N}_0$ and vectors $v, v' \in \mathbb{N}^{|\mathcal{A}|}$ such that $v - \ell v^* = v' - \ell' v^*$, it holds that $v' = v + (\ell' - \ell)v^*$ and $v = v' + (\ell - \ell')v^*$. Assuming that $\ell > \ell'$, we thus infer from consistency that $g(v) = g(v') \cap g((\ell - \ell')v^*) = g(v')$ as $g(v^*) = g((\ell - \ell')v^*) = \mathcal{W}_k$. Because $g(v) = g(v')$, we have by definition that $\bar{g}(v - \ell v^*) = \bar{g}(v' - \ell' v^*)$. Moreover, $\bar{g}$ is defined for all $v \in \mathbb{Z}^{|\mathcal{A}|}$ since we can always find $v' \in \mathbb{N}^{|\mathcal{A}|}$ and $\ell \in \mathbb{N}_0$ with $v = v' - \ell v^*$. Finally, note that $\bar{g}(v(A)) = \bar{g}(v(A) - 0v^*) = g(v(A)) = f(v(A))$ for all profiles $A \in \mathcal{A}^*$.

Next, it is easy to verify that $\bar{g}$ inherits neutrality and consistency from g. For showing the neutrality of $\bar{g}$, consider a vector $v \in \mathbb{Z}^{|\mathcal{A}|}$ and let $W \in \bar{g}(v)$. By the definition of $\bar{g}$, there are $v' \in \mathbb{N}^{|\mathcal{A}|}$ and $\ell \in \mathbb{N}_0$ such that $v = v' - \ell v^*$ and $\bar{g}(v) = g(v + \ell v^*) = g(v')$. Since $\tau(v^*) = v^*$, it is easy to see that $\tau(v) + \ell v^* = \tau(v')$ for all permutations $\tau : C \to C$. Hence, $\tau(W) \in \bar{g}(\tau(v)) = g(\tau(v) + \ell v^*) = g(\tau(v'))$ due to the neutrality of g.

Finally, for proving that $\bar{g}$ is consistent, consider two vectors $v^1, v^2 \in \mathbb{Z}^{|\mathcal{A}|}$. By definition of $\bar{g}$, there are $\bar{v}^1, \bar{v}^2 \in \mathbb{N}^{|\mathcal{A}|}$ and $\ell_1, \ell_2 \in \mathbb{N}_0$ such that $v^1 = \bar{v}^1 - \ell_1 v^*$, $v^2 = \bar{v}^2 - \ell_2 v^*$, $\bar{g}(v^1) = g(v^1 + \ell_1 v^*) = g(\bar{v}^1)$, and $\bar{g}(v^2) = g(v^2 + \ell_2 v^*) = g(\bar{v}^2)$. Clearly, this implies that $\bar{g}(v^1 + v^2) = g(v^1 + v^2 + \ell_1 v^* + \ell_2 v^*) = g(\bar{v}^1 + \bar{v}^2)$. Hence, if $\bar{g}(v^1) \cap \bar{g}(v^2) = g(\bar{v}^1) \cap g(\bar{v}^2) \neq \emptyset$, then $\bar{g}(v^1 + v^2) = g(\bar{v}^1 + \bar{v}^2) = g(\bar{v}^1) \cap g(\bar{v}^2) = \bar{g}(v^1) \cap \bar{g}(v^2)$ because g is consistent.

Step 2: Extension to $\mathbb{Q}^{|\mathcal{A}|}$

As second step, we extend $\bar{g}$ to the rational numbers. For doing so, we define $\hat{g}(\frac{v}{\ell}) = \bar{g}(v)$ for all $v \in \mathcal{Z}^{|\mathcal{A}|}$ and $\ell \in \mathbb{N}$. Clearly, $\hat{g}$ is defined for all $v \in \mathbb{Q}^{|\mathcal{A}|}$. Next, the consistency of $\bar{g}$ shows that $\hat{g}$ is well-defined: if there are $v, v' \in \mathbb{Z}^{|\mathcal{A}|}$ and $\ell, \ell' \in \mathbb{N}$ such that $\frac{v}{\ell} = \frac{v'}{\ell'}$, then $\ell' v = \ell v'$. By consistency of $\bar{g}$, we hence infer that $\bar{g}(v) = \bar{g}(\ell' v) = \bar{g}(\ell v') = \bar{g}(v')$, proving that $\hat{g}$ is well-defined. Moreover, observe that $\hat{g}(v(A)) = \hat{g}(\frac{v(A)}{1}) = \bar{g}(v(A)) = f(A)$ for all $A \in \mathcal{A}^*$.

Next, it is simple to show that $\hat{g}$ is neutral and consistent. For proving neutrality, let $v \in \mathbb{Q}^{|\mathcal{A}|}$ be an arbitrary vector and $W \in \hat{g}(W)$. By definition, there are $v' \in \mathbb{Z}^{|\mathcal{A}|}$ and $\ell \in \mathbb{N}$ such that $v = \frac{v'}{\ell}$ and $\hat{g}(v) = \bar{g}(v')$. It holds for every permutation τ that $\tau(v) = \frac{\tau(v')}{\ell}$ and thus, we have that $\tau(W) \in \hat{g}(\tau(v)) = \bar{g}(\tau(v'))$ because $\bar{g}$ is neutral.

Similarly, for showing that $\hat{g}$ is consistent, consider two vectors $v^1, v^2 \in \mathbb{Q}^{|\mathcal{A}|}$ such that $\hat{g}(v^1) \cap \hat{g}(v^2) \neq \emptyset$. By definition of $\hat{g}$, there are $\hat{v}^1, \hat{v}^2 \in \mathbb{Z}^{|\mathcal{A}|}$ and ℓ_1, ℓ_2 such that $v^1 = \frac{\hat{v}^1}{\ell_1}$, $v^2 = \frac{\hat{v}^2}{\ell_2}$, $\hat{g}(v^1) = \bar{g}(\hat{v}^1)$ and $\hat{g}(v^2) = \bar{g}(\hat{v}^2)$. Moreover, it holds by definition of $\hat{g}$ that $\hat{g}(v^1 + v^2) = \hat{g}(\frac{\ell_2 \hat{v}^1 + \ell_1 \hat{v}^2}{\ell_1 \ell_2}) = \bar{g}(\ell_2 \hat{v}^1 + \ell_1 \hat{v}^2)$. Since $\bar{g}$ is consistent, we thus infer that $\hat{g}(v^1 + v^2) = \bar{g}(\ell_2 \hat{v}^1 + \ell_1 \hat{v}^2) = \bar{g}(\ell_2 \hat{v}^1) \cap \bar{g}(\ell_1 \hat{v}^2) = \bar{g}(\hat{v}^1) \cap \bar{g}(\hat{v}^2) = \hat{g}(v^1) \cap \hat{g}(v^2)$. This proves that $\hat{g}$ is consistent. □

Next, we turn to the proof of Lemma 2 which states that the function $\hat{g}$ can be described by symmetric hyperplanes. Unfortunately, the proof of this claim is quite involved, so we discuss several auxiliary lemmas before hand (Lemmas 3 to 5). Now, for these lemmas, we first recall some concepts introduced in the main body. In particular, we suppose that the committees in $\mathcal{W}_k$ are arranged in an arbitrary order $W^1, \dots, W^{|\mathcal{W}_k|}$ and define $R_i^f = \{v \in \mathbb{Q}^{|\mathcal{A}|} : W_i \in \hat{g}(v)\}$ as the set of vectors for which $\hat{g}$ chooses W^i. Moreover, note that the sets R_i^f are symmetric: if a permutation $\tau : C \to C$ maps W_i to W_j (i.e., $W_j = \tau(W_i)$), then $v \in R_i^f$ if and only if $\tau(v) \in R_j^f$ because $W_i \in \hat{g}(v)$ if and only if $\tau(W_i) \in \hat{g}(\tau(v))$. Moreover, since $\hat{g}$ is consistent, all R_i^f are $\mathbb{Q}$-convex (i.e., if $v, v' \in R_i^f$, then $\lambda v + (1 - \lambda)v' \in R_i^f$ for all $\lambda \in \mathbb{Q} \cap [0, 1]$). Consequently, the closure of R_i^f with respect to $\mathbb{R}^{|\mathcal{A}|}$, $\bar{R}_i^f$, is a convex cone. As our first auxiliary lemma, we show that these sets can be separated by hyperplanes.

Lemma 3. *Let f denote a non-imposing ABC voting rule that satisfies anonymity, neutrality, and consistency. Furthermore, consider two distinct committees $W^i, W^j \in \mathcal{W}_k$. There is a non-zero vector $u^{i,j} \in \mathbb{R}^{|\mathcal{A}|}$ such that $vu^{i,j} \geq 0$ for all $v \in \bar{R}_i^f$ and $vu^{i,j} \leq 0$ for all $v \in \bar{R}_j^f$.*

Proof. Let f denote a non-imposing ABC voting rule satisfying anonymity, neutrality, and consistency. Furthermore, let $\hat{g}$ denote the extension of f to $\mathbb{Q}^{|\mathcal{A}|}$ as defined in Lemma 1. Finally, we consider two arbitrary committees $W^i, W^j \in \mathcal{W}_k$.

We first show that the interiors of the sets $\bar{R}_i^f$ and $\bar{R}_j^f$ are disjoint, i.e., $\text{int}\bar{R}_i^f \cap \text{int}\bar{R}_j^f = \emptyset$. Assume for contradiction that this is not the case, which means that there is $v \in \text{int}\bar{R}_i^f \cap \text{int}\bar{R}_j^f \cap \mathbb{Q}$. By the definition of $\bar{R}_i^f$ and $\bar{R}_j^f$, this means that $W^i \in \hat{g}(v)$, $W^j \in \hat{g}(v)$. On the other hand, f is non-imposing, so there is a profile A such that $f(A) = \{W^i\}$. By the definition of $\hat{g}$, $\hat{g}(v(A)) = \{W^i\}$. Finally, since v is in the interior of $\bar{R}_j^f$, there must be $\lambda \in (0,1) \cap \mathbb{Q}$ such that $(1-\lambda v) + \lambda v(A) \in \bar{R}_j^f$. However, by consistency, we have that $\hat{g}((1-\lambda)v + \lambda v(A)) = (1-\lambda)\hat{g}(v) \cap \lambda\hat{g}(v(A)) = \{W^i\}$. This is a contradiction and thus, the interiors of $\bar{R}_i^f$ and $\bar{R}_j^f$ must be disjoint.

Next, we observe that the interiors of $\bar{R}_i^f$ and $\bar{R}_j^f$ are non-empty. This follows from the observation that the sets R_i^f, and thus also their closures $\bar{R}_i^f$, are symmetric and that $\mathbb{R}^{|\mathcal{A}|} = \bigcup_{\ell \in \{1,\dots,|\mathcal{W}_k|\}} \bar{R}_\ell^f$. Since there is only a finite number of committees, this entails that the sets $\bar{R}_i^f$ have full dimension and thus have indeed non-empty interiors. Finally, we can now use the separating hyperplane theorem for convex sets to derive that there is a non-zero vector $u^{i,j} \in \mathbb{R}^{|\mathcal{A}|}$ that satisfies the conditions of the lemma. □

For an easy notation, we say that a non-zero vector u separates $\bar{R}_i^f$ from $\bar{R}_j^f$ if $vu \geq 0$ for all $v \in \bar{R}_i^f$ and $vu \leq 0$ for all $v \in \bar{R}_j^f$. In particular, the vectors derived in Lemma 3 are such separating vectors. We show next that the sets $\bar{R}_i^f$ are fully described by *every* set of such separating vectors.

Lemma 4. *Let f denote a non-imposing ABC voting rule that satisfies anonymity, neutrality, and consistency. For all distinct $i, j \in \{1, \dots, |\mathcal{W}_k|\}$, let $u^{i,j} \in \mathbb{R}^{|\mathcal{A}|}$ denote non-zero vectors such that $u^{i,j}$ separates $\bar{R}_i^f$ from $\bar{R}_j^f$. It holds that $\bar{R}_i^f = S_i^f = \{x \in \mathcal{R}^{|\mathcal{A}|} : xu^{i,j} \geq 0 \text{ for all } j \in \{1,\dots,|\mathcal{W}_k|\} \setminus \{i\}\}$ for all $i \in \{1, \dots, |\mathcal{W}_k|\}$.*

Proof. Let f denote a non-imposing ABC voting rule that satisfies all given axioms, let the vectors $u^{i,j}$ be defined as in the lemma, and fix an index $i \in \{1, \dots, |\mathcal{W}_k|\}$. By definition, it holds that $vu^{i,j} \geq 0$ for all $j \in \{1,\dots,|\mathcal{W}_k|\} \setminus \{i\}$ if $v \in \bar{R}_i^f$, so $v \in S_i^f$. This proves that $\bar{R}_i^f \subseteq S_i^f$. For the other direction, note that the sets $\bar{R}_i^f$ are fully dimensional since they are symmetric and $\mathbb{R}^{|\mathcal{A}|} = \bigcup_{j \in \{1,\dots,|\mathcal{W}_k|\}} \bar{R}_j^f$. Since $\bar{R}_i \subseteq S_i$, we thus also have that $\text{int}S_i \neq \emptyset$. Now, let $v \in \text{int}S_i^f$, which means that $vu^{i,j} > 0$ for all $j \in \{1,\dots,|\mathcal{W}_k|\}$, $j \neq i$. In turn, this means that $v \notin \bar{R}_j^f$ for all $j \in \{1,\dots,|\mathcal{W}_k|\} \setminus \{i\}$ because otherwise $vu^{i,j} \leq 0$ by definition. Since $\mathbb{R}^{|\mathcal{A}|} = \bigcup_{j \in \{1,\dots,|\mathcal{W}_k|\}} \bar{R}_j^f$ and all $\bar{R}_j^f$ are closed and convex, this means that $v \in \text{int}\bar{R}_i^f$. Hence, $\text{int}S_i^f \subseteq \text{int}\bar{R}_i^f$, so we deduce that $S_i^f \subseteq \bar{R}_i^f$. □

We note that Lemma 4 proves Claim (1) of Lemma 2 for every set of separating vectors. We thus work now to derive such vectors that are additionally symmetric. For this, we start with the simple but helpful observation that the symmetry of the sets $\bar{R}_i^f$ entails some symmetry for the hyperplanes.

Lemma 5. *Let f denote a non-imposing ABC voting rule that satisfies anonymity, neutrality, and consistency. Moreover, consider committees $W^i, W^j, W^{i'}, W^{j'} \in \mathcal{W}_k$ such that $W^i \neq W^j$, $W^{i'} \neq W^{j'}$ and $|W^i \cap W^j| = |W^{i'} \cap W^{j'}|$, and a permutation $\tau : C \to C$ such that $\tau(W^i \cap W^j) = W^{i'} \cap W^{j'}$, $\tau(W^i \setminus W^j) = W^{i'} \setminus W^{j'}$, and $\tau(W^j \setminus W^i) = W^{j'} \setminus W^{i'}$. If a vector u separates $\bar{R}_i^f$ from $\bar{R}_j^f$, then $\tau(u)$ separates $\bar{R}_{i'}^f$ from $\bar{R}_{j'}^f$.*

Proof. Let f denote an ABC voting rule satisfying all given axioms, and consider committees $W^i, W^j, W^{i'}, W^{j'} \in \mathcal{W}_k$ as defined in the lemma. Moreover, let u denote a vector that separates $\bar{R}_i^f$ from $\bar{R}_j^f$, and let $\tau : C \to C$ be a permutation that satisfies the conditions of the lemma. Now, consider a vector $v' \in \bar{R}_{i'}^f$. By the neutrality of $\hat{g}$, there is a vector $v \in \bar{R}_i^f$ such that $\tau(v) = v'$ because $\tau(W^i) = W^{i'}$. Since u separates $\bar{R}_i^f$ and $\bar{R}_j^f$, we have $vu \geq 0$. Now, it is straightforward that $v'\tau(u) = \tau(v)\tau(u) = vu \geq 0$ because the scalar product does not change if we permute both vectors. An analogous argument also works for vectors $v' \in \bar{R}_{j'}^f$ and thus, $\tau(u)$ separates $\bar{R}_{i'}^f$ from $\bar{R}_{j'}^f$. □

Based on Lemmas 3 to 5, we can finally prove Lemma 2.

Lemma 2. *Let f denote a non-imposing ABC voting rule that satisfies anonymity, neutrality, and consistency. There are non-zero vectors $\hat{u}^{i,j}$ that satisfy the following conditions for all $W^i, W^j \in \mathcal{W}_k$:*

(1) $\bar{R}_i^f = \{v \in \mathbb{R}^{|\mathcal{A}|} : \forall j' \in \{1,\dots,|\mathcal{W}_k|\} \setminus \{i\} : \hat{u}^{i,j'} v \geq 0\}$.
(2) $\hat{u}^{i,j} = -\hat{u}^{j,i}$.
(3) $\hat{u}^{i',j'} = \tau(\hat{u}^{i,j})$ *if* $\tau(W^i) = W^{i'}$ *and* $\tau(W^j) = W^{j'}$.

Proof. Let f denote a non-imposing ABC voting rule that satisfies all given axioms. By Lemma 3, there are non-zero vectors $u^{i,j}$ that separate $\bar{R}_i^f$ from $\bar{R}_j^f$ for all pairs of committees $W^i, W^j \in \mathcal{W}_k$. Our main goal is to make these vectors symmetric and we will heavily rely on Lemma 5 for this. To this end, we define $z = \max_{W^i, W^j \in \mathcal{W}_k} |W^i \setminus W^j|$ as the maximal distance between two committees. Moreover, we fix $z+1$ committees $W^{i_0}, \dots, W^{i_z}$ such that $|W^{i_0} \setminus W^{i_x}| = x$ for all of them.

Next, we will derive the symmetric separating vectors $\hat{u}^{i,j}$. For this, consider an arbitrary index $x \in \{1,\dots,z\}$ and let u^{i_0,i_x} be the vector that separates $\bar{R}_{i_0}^f$ from $\bar{R}_{i_x}^f$. Moreover, we define the sets $X^{i_0 \setminus i_x} = W^{i_0} \setminus W^{i_x}$, $X^{i_0 \cap i_x} = W^{i_0} \cap W^{i_x}$, $X^{i_x \setminus i_0} = W^{i_x} \setminus W^{i_0}$, and $\mathcal{T} = \{\tau \in C^C : \tau(X^{i_0 \cap i_x}) = X^{i_0 \cap i_x}, \tau(X^{i_0 \setminus i_x}) = X^{i_0 \setminus i_x}, \tau(X^{i_x \setminus i_0}) = X^{i_x \setminus i_0}\}$. In particular, it holds for every $\tau \in \mathcal{T}$ that $\tau(W^{i_0}) = W^{i_0}$ and $\tau(W^{i_x}) = W^{i_x}$. Consequently, Lemma 5 shows that $\tau(u^{i_0,i_x})$ also separates $\bar{R}_{i_0}^f$ from $\bar{R}_{i_x}^f$ and the same follows for $\tilde{u}^{i_0,i_x} =$

$\sum_{\tau\in\mathcal{T}}\tau(u^{i_0,i_x})$. This also means that the vector $\tilde{u}^{i_x,i_0} = -\tilde{u}^{i_0,i_x}$ separates $\bar{R}^f_{i_x}$ from $\bar{R}^f_{i_0}$. Next, let τ^* denote a permutation such that $\tau^*(X^{i_0\cap i_x}) = X^{i_0\cap i_x}$, $\tau^*(X^{i_0\setminus i_x}) = X^{i_x\setminus i_0}$, $\tau^*(X^{i_x\setminus i_0}) = X^{i_0\setminus i_x}$, and $\tau^*(\tau^*(c)) = c$ for all candidates $c \in C$. It is easy to verify that $\tau^*(W^{i_0}) = W^{i_x}$ and $\tau^*(W^{i_x}) = W^{i_0}$ and Lemma 5 thus shows that $\tau^*(\tilde{u}^{i_x,i_0})$ separates $\bar{R}^f_{i_0}$ from $\bar{R}^f_{i_x}$. Finally, we define the vector $\hat{u}^{i_0,i_x}$ by $\hat{u}^{i_0,i_x} = \tilde{u}^{i_0,i_x} + \tau^*(\tilde{u}^{i_x,i_0})$ and note that this vector separates $\bar{R}^f_{i_0}$ from $\bar{R}^f_{i_x}$. Moreover, we generalize these vectors to arbitrary committees $W^i, W^j \in \mathcal{W}_k$ as follows: we first determine $x = |W^i \setminus W^j|$ and choose a permutation τ such that $\tau(W^{i_0}) = W^i$ and $\tau(W^{i_x}) = W^j$. Then, $\hat{u}^{i,j} = \tau(\hat{u}^{i_0,i_x})$. This vector separates $\bar{R}^f_i$ from $\bar{R}^f_j$ by Lemma 5.

It remains to show that these vectors satisfy our conditions. In more detail, we first show Claim (1) by investigating the interior of the sets $\bar{R}^f_i$. Then, we prove an auxiliary claim establishing some symmetry properties of the vectors $\hat{u}^{i_0,i_x}$ for every $x \in \{1,\dots,z\}$. Based on this auxiliary claim, we finally show Claims (2) and (3).

Claim (1): $\bar{R}^f_i = \{v \in \mathbb{R}^{|\mathcal{A}|} : \forall j \in \{1,\dots,|\mathcal{W}_k|\}\setminus\{i\} : v\hat{u}^{i,j} \geq 0\}$

For proving this claim, we show that the vectors $\hat{u}^{i,j}$ are non-zero and separate $\bar{R}^f_i$ from $\bar{R}^f_j$ because Lemma 4 then implies the statement. Hence, consider two committees $W^i, W^j \in \mathcal{W}_k$ and let $x = |W^i \setminus W^j|$. First, we note that, by construction, $\hat{u}^{i_0,i_x}$ separates $\bar{R}^f_{i_0}$ from $\bar{R}^f_{i_x}$. Since we derive $\hat{u}^{i,j}$ from $\hat{u}^{i_0,i_x}$ by permuting the vector, Lemma 5 shows that $\hat{u}^{i,j}$ indeed separates $\bar{R}^f_i$ from $\bar{R}^f_j$. Moreover, $\hat{u}^{i,j}$ is a non-zero vector if $\hat{u}^{i_0,i_x}$ is non-zero. Hence, it only remains to show that $\hat{u}^{i_0,i_x}$ is a non-zero vector. For this, we use that $\bar{R}^f_{i_0} = \{x \in \mathcal{R}^{|\mathcal{A}|} : \forall j \in \{1,\dots,|\mathcal{W}_k|\} \setminus \{i_0\} : xu^{i_0,\ell} \geq 0\}$ due to Lemma 4, where the vectors $u^{i_0,\ell}$ define our initial hyperplanes given by Lemma 3. Now, let v denote a point in the interior of $\bar{R}^f_{i_0}$; such a point exists as $\bar{R}^f_{i_0}$ is fully dimensional and thus has a non-empty interior. Since $v \in \text{int}\bar{R}^f_{i_0}$, it holds that $vu^{i_0,\ell} > 0$ for all $\ell \in \{1,\dots,|\mathcal{W}_k|\} \setminus \{i\}$, in particular that $vu^{i_0,i_x} > 0$. Next, we note that also the vectors $\tau(u^{i_0,i_x})$ for $\tau \in \mathcal{T}$ are non-zero and separate $\bar{R}^f_{i_0}$ from $\bar{R}^f_{i_x}$. So, we can exchange u^{i_0,i_x} with $\tau(u^{i_0,i_x})$ in the presentation of $R^f_{i_0}$ and infer that $v\tau(u^{i_0,i_x}) > 0$ since v is still in the interior of $\bar{R}^f_{i_0}$. Hence, $v\tilde{u}^{i_0,i_x} = v\sum_{\tau\in\mathcal{T}}\tau(u^{i_0,i_x}) > 0$, so $\tilde{u}^{i_0,i_x}$ is a non-zero vector. Hence, $\tau^*(\tilde{u}^{i_x,i_0})$ also is a non-zero vector and we can thus also represent $\bar{R}^f_{i_0}$ by exchanging u^{i_0,i_x} with $\tau^*(\tilde{u}^{i_x,i_0})$. This implies again that $v\tau^*(\tilde{u}^{i_x,i_0}) > 0$ and we therefore conclude that $v\hat{u}^{i_0,i_x} = v(\tilde{u}^{i_0,i_x} + \tau^*(\tilde{u}^{i_x,i_0})) > 0$, so $\hat{u}^{i_0,i_x}$ is indeed a non-zero vector.

Auxiliary Claim: Symmetry of $\hat{u}^{i_0,i_x}$

Next, we will prove that the vectors $\hat{u}^{i_0,i_x}$ are rather symmetric. In more detail, we will show that $\hat{u}^{i_0,i_x}_{\ell_1} = \hat{u}^{i_0,i_x}_{\ell_2}$ for all ballots $B(\ell_1), B(\ell_2) \in \mathcal{A}$ and all $x \in \{1,\dots,z\}$ such that $|B(\ell_1)| = |B(\ell_2)|$ and $|B(\ell_1) \cap X| = |B(\ell_2) \cap X|$ for all $X \in \{X^{i_0\cap i_x}, X^{i_0\setminus i_x}, X^{i_x\setminus i_0}\}$. For this, we fix such ballots $B(\ell_1), B(\ell_2)$ and an index $x \in \{1,\dots z\}$. By our assumptions, there is a bijection $\tilde{\tau} : C \to C$ such that $B(\ell_1) = \tilde{\tau}(B(\ell_2))$ and $\tilde{\tau}(X) = X$ for all $X \in \{X^{i_0\cap i_x}, X^{i_0\setminus i_x}, X^{i_x\setminus i_0}\}$. By the latter insight, it follows that $\tilde{\tau}\circ\tau \in \mathcal{T}$ for every permutation $\tau \in \mathcal{T}$. Moreover, for distinct permutations $\tau_1, \tau_2 \in \mathcal{T}$, it holds that $\tilde{\tau}\circ\tau_1 \neq \tilde{\tau}\circ\tau_2$ because there is a candidate c such that $\tau_1(c) \neq \tau_2(c)$. This shows that $\{\tilde{\tau}\circ\tau : \tau \in \mathcal{T}\} = \mathcal{T}$. Since $B(\ell_1) = \tilde{\tau}(B(\ell_2))$, we derive for every vector $u \in \mathbb{R}^{|\mathcal{A}|}$ that $\tilde{\tau}(u)_{\ell_1} = u_{\ell_2}$. Consequently, $\tilde{u}^{i_0,i_x}_{\ell_1} = \sum_{\tau\in\mathcal{T}}\tau(u^{i_0,i_x})_{\ell_1} = \sum_{\tau\in\mathcal{T}}\tilde{\tau}(\tau(u^{i_0,i_x}))_{\ell_2} = \sum_{\tau\in\mathcal{T}}\tau(u^{i_0,i_x})_{\ell_2} = \tilde{u}^{i_0,i_x}_{\ell_2}$. This proves that the vector $\tilde{u}^{i_0,i_x}$ satisfies our symmetry condition, and clearly the vector $\tilde{u}^{i_x,i_0} = -\tilde{u}^{i_0,i_x}$ satisfies this condition then, too. Finally, recall that we choose τ^* such that $\tau^*(X^{i_0\cap i_x}) = X^{i_0\cap i_x}$, $\tau^*(X^{i_0\setminus i_x}) = X^{i_x\setminus i_0}$, and $\tau^*(X^{i_x\setminus i_0}) = X^{i_0\setminus i_x}$. This means that $|B(\ell) \cap X^{i_0\cap i_x}| = |\tau^*(B(\ell)) \cap X^{i_0\cap i_x}|$, $|B(\ell) \cap X^{i_0\setminus i_x}| = |\tau^*(B(\ell)) \cap X^{i_x\setminus i_0}|$, and $|B(\ell) \cap X^{i_x\setminus i_0}| = |\tau^*(B(\ell)) \cap X^{i_0\setminus i_x}|$. Hence, since $|B(\ell_1) \cap X| = |B(\ell_2) \cap X|$ for all $X \in \{X^{i_0\cap i_x}, X^{i_0\setminus i_x}, X^{i_x\setminus i_0}\}$, the same holds for $\tau^*(B(\ell_1))$ and $\tau^*(B(\ell_2))$ and we infer that $\tau^*(\tilde{u}^{i_x,i_0})_{\ell_1} = \tau^*(\tilde{u}^{i_x,i_0})_{\ell_2}$. Finally, this means that $\hat{u}^{i_0,i_x}_{\ell_1} = \tilde{u}^{i_0,i_x}_{\ell_1} + \tau^*(\tilde{u}^{i_x,i_0})_{\ell_1} = \tilde{u}^{i_0,i_x}_{\ell_2} + \tau^*(\tilde{u}^{i_x,i_0})_{\ell_2} = \hat{u}^{i_0,i_x}_{\ell_2}$, which proves our auxiliary claim.

Claim (2): $\hat{u}^{i,j} = -\hat{u}^{j,i}$

Consider two committees $W^i, W^j \in \mathcal{W}_k$, let $x = |W^i \setminus W^j| = |W^j \setminus W^i|$, and fix a ballot $B(\ell)$. We will show that $\hat{u}^{i,j}_\ell = -\hat{u}^{j,i}_\ell$ to prove this claim. For this, let τ denote the permutation such that $\tau(W^{i_0}) = W^i$, $\tau(W^{i_x}) = W^j$, and $\hat{u}^{i,j} = \tau(\hat{u}^{i_0,i_x})$. Similarly, we define τ' as the permutation with $\tau(W^{i_0}) = W^j$, $\tau(W^{i_x}) = W^i$, and $\hat{u}^{j,i} = \tau'(\hat{u}^{i_0,i_x})$. By definition, it holds that $\hat{u}^{i,j}_\ell = \hat{u}^{i_0,i_x}_{\ell_1}$ and $\hat{u}^{j,i}_\ell = \hat{u}^{i_0,i_x}_{\ell_2}$ for the indices ℓ_1, ℓ_2 with $B(\ell) = \tau(B(\ell_1))$ and $B(\ell) = \tau'(B(\ell_2))$. Hence, the claim follows by proving that $\hat{u}^{i_0,i_x}_{\ell_1} = -\hat{u}^{i_0,i_x}_{\ell_2}$. For this, we first observe that the condition on τ and τ' require that $\tau(X^{i_0\cap i_x}) = \tau'(X^{i_0\cap i_x}) = X^{i\cap j}$, $\tau(X^{i_0\setminus i_x}) = X^{i\setminus j}$, $\tau(X^{i_x\setminus i_0}) = X^{j\setminus i}$, $\tau'(X^{i_0\setminus i_x}) = X^{j\setminus i}$, and $\tau'(X^{i_x\setminus i_0}) = X^{i\setminus j}$. Hence, we infer that $|B(\ell_1)\cap X^{i_0\cap i_x}| = |B(\ell)\cap X^{i\cap j}| = |B(\ell_2)\cap X^{i_0\cap i_x}|$, $|B(\ell_1)\cap X^{i_0\setminus i_x}| = |B(\ell)\cap X^{i\setminus j}| = |B(\ell_2)\cap X^{i_x\setminus i_0}|$, and $|B(\ell_1)\cap X^{i_x\setminus i_0}| = |B(\ell)\cap X^{j\setminus i}| = |B(\ell_2) \cap X^{i_0\setminus i_x}|$. Moreover, it clearly holds that $|B(\ell_1)| = |B(\ell)| = |B(\ell_2)|$. Now, we consider again the permutation τ^* used in the definition of $\hat{u}^{i_0,i_x}$ and recall that $\tau^*(W^{i_0}) = W^{i_x}$, $\tau^*(W^{i_x}) = W^{i_0}$, and $\tau^*(\tau^*(c)) = c$ for all $c \in C$. Furthermore, let ℓ_3 denote the index such that $B(\ell_3) = \tau^*(B(\ell_1))$ and note that $|B(\ell_2)\cap X| = |B(\ell_3)\cap X|$ for all $X \in \{X^{i_0\cap i_x}, X^{i_0\setminus i_x}, X^{i_x\setminus i_0}, C\}$. Hence, our auxiliary claim entails that $\hat{u}^{i_0,i_x}_{\ell_2} = \hat{u}^{i_0,i_x}_{\ell_3}$. On the other hand, we have by definition that $\tau^*(\tilde{u}^{i_x,i_0})_{\ell_3} = \tilde{u}^{i_x,i_0}_{\ell_1} = -\tilde{u}^{i_0,i_x}_{\ell_1}$ and $\tau^*(\tilde{u}^{i_x,i_0})_{\ell_1} = \tilde{u}^{i_x,i_0}_{\ell_3} = -\tilde{u}^{i_0,i_x}_{\ell_3}$. It is now easy to compute that $\hat{u}^{i_0,i_x}_{\ell_1} = \tilde{u}^{i_0,i_x}_{\ell_1} + \tau^*(\tilde{u}^{i_x,i_0})_{\ell_1} = \tilde{u}^{i_0,i_x}_{\ell_1} - \tilde{u}^{i_0,i_x}_{\ell_3} = -(\tilde{u}^{i_0,i_x}_{\ell_3} - \tilde{u}^{i_0,i_x}_{\ell_1}) = -(\tilde{u}^{i_0,i_x}_{\ell_3} + \tau^*(\tilde{u}^{i_0,i_x})_{\ell_3}) = -\hat{u}^{i_0,i_x}_{\ell_3}$. We therefore conclude that $\hat{u}^{i_0,i_x}_{\ell_1} = -\hat{u}^{i_0,i_x}_{\ell_3} = -\hat{u}^{i_0,i_x}_{\ell_2}$, which proves this claim.

Claim (3): $\hat{u}^{i',j'} = \hat{\tau}(\hat{u}^{i,j})$ **if** $\tau(W^i) = W^{i'}$ **and** $\hat{\tau}(W^j) = W^{j'}$

For our last claim, we consider four committees $W^i, W^j, W^{i'}, W^{j'}$ and a permutation $\hat{\tau}$ such that $\hat{\tau}(W^i) = W^{i'}$ and $\hat{\tau}(W^j) = W^{j'}$. Moreover, consider two ballots $B(\ell_1), B(\ell_2) \in \mathcal{A}$ such that $B(\ell_1) = \hat{\tau}(B(\ell_2))$. We will show that $\hat{u}^{i',j'}_{\ell_1} = \hat{u}^{i,j}_{\ell_2}$, which implies that $\hat{u}^{i',j'} = \hat{\tau}(\hat{u}^{i,j})$. For this, let $x = |W^i \setminus W^j| = |W^{i'} \setminus W^{j'}|$ and let τ and τ' denote the permutations such that $\tau(W^{i_0}) = W^i$, $\tau(W^{i_x}) = W^j$, $\tau'(W^{i_0}) = W^{i'}$, $\tau'(W^{i_x}) = W^{j'}$, $\hat{u}^{i,j} = \tau(\hat{u}^{i_0,i_x})$, and $\hat{u}^{i',j'} = \tau'(\hat{u}^{i_0,i_x})$. Clearly, there are integers ℓ_3, ℓ_4 such that $B(\ell_1) = \tau'(B(\ell_3))$ and $B(\ell_2) = \tau(B(\ell_4))$. By definition, this means that $\hat{u}^{i',j'}_{\ell_1} = \hat{u}^{i_0,i_x}_{\ell_3}$ and $\hat{u}^{i,j}_{\ell_2} = \hat{u}^{i_0,i_x}_{\ell_4}$. Hence, our equality follows by showing that $\hat{u}^{i_0,i_x}_{\ell_3} = \hat{u}^{i_0,i_x}_{\ell_4}$. For this, we note that $\hat{\tau}(X^{i\cap j}) = X^{i'\cap j'}$, $\hat{\tau}(X^{i\setminus j}) = X^{i'\setminus j'}$, and $\hat{\tau}(X^{j\setminus i}) = X^{j'\setminus i'}$. Moreover, analogous claims hold for τ (between W^{i_0}, W^{i_x} and W^i, W^j) and for τ' (between W^{i_0}, W^{i_x} and $W^{i'}, W^{j'}$). Thus, we can derive the following equalities since permuting sets does not change the size of their set intersection.

$$|B(\ell_3)\cap X^{i_0\cap i_x}| = |B(\ell_1)\cap X^{i'\cap j'}| = |B(\ell_2)\cap X^{i\cap j}| = |B(\ell_4)\cap X^{i_0\cap i_x}|$$

$$|B(\ell_3)\cap X^{i_0\setminus i_x}| = |B(\ell_1)\cap X^{i'\setminus j'}| = |B(\ell_2)\cap X^{i\setminus j}| = |B(\ell_4)\cap X^{i_0\setminus i_x}|$$

$$|B(\ell_3)\cap X^{i_x\setminus i_0}| = |B(\ell_1)\cap X^{j'\setminus i'}| = |B(\ell_2)\cap X^{j\setminus i}| = |B(\ell_4)\cap X^{i_x\setminus i_0}|$$

Moreover, we clearly have that $|B(\ell_3)| = |B(\ell_1)| = |B(\ell_2)| = |B(\ell_4)|$, so our auxiliary claim implies that $\hat{u}^{i_0,i_x}_{\ell_3} = \hat{u}^{i_0,i_x}_{\ell_4}$. This concludes the proof of this claim. □

After proving Lemma 2, we will next investigate its consequences as we will heavily rely on this lemma. In more detail, as demonstrated in the proof of Section 3.1, it is quite helpful to consider the hyperplanes $\hat{u}^{i,j}$ for committees W^i, W^j with $|W^i \setminus W^j| = 1$. We thus show in the next lemma that there is a compact representation of these hyperplanes.

Lemma 6. *Let f denote a non-imposing ABC voting rule that satisfies anonymity, neutrality, and consistency. For every ballot size $r \in \{1, \dots, m\}$, there is a functions $s^1_r(x, y)$ that satisfies the following claims for all ballots $B(\ell)$ with $|B(\ell)| = r$ and committees $W^i, W^j \in \mathcal{W}_k$ with $|W^i \setminus W^j| = 1$.*

(1) $\hat{u}^{i,j}_\ell = s^1_r(|B(\ell) \cap W^i|, |B(\ell) \cap W^j|)$.
(2) $s^1_r(|B(\ell)\cap W^i|, |B(\ell)\cap W^j|) = 0$ if $|B(\ell)\cap W^i| = |B(\ell)\cap W^j|$.
(3) $s^1_r(|B(\ell)\cap W^i|, |B(\ell)\cap W^j|) = -s^1_r(|B(\ell)\cap W^j|, |B(\ell)\cap W^i|)$.

Proof. Let f denote an ABC scoring rule that satisfies all given conditions and let $\hat{u}^{i,j}$ denote the non-zero vectors given by Lemma 2. Our main goal is to show that $\hat{u}^{i,j}_\ell = \hat{u}^{i',j'}_{\ell'}$ for all committees $W^i, W^j, W^{i'}, W^{j'} \in \mathcal{W}_k$ and ballots $B(\ell), B(\ell') \in \mathcal{A}$ such that $|W^i \setminus W^j| = |W^{i'} \setminus W^{j'}| = 1$, $|B(\ell)| = |B(\ell')|$, $|B(\ell) \cap W^i| = |B(\ell') \cap W^{i'}|$, and $|B(\ell) \cap W^j| = |B(\ell') \cap W^{j'}|$. Clearly, this implies the existence of the functions s^1_r as we can just define $s^r_1(x, y) = \hat{u}^{i,j}_\ell$ for arbitrary committees $W^i, W^j \in \mathcal{W}_k$ and a ballot $B(\ell)$ with $|W^i \setminus W^j| = 1$, $|B(\ell)| = r$, $|B(\ell) \cap W^i| = x$, and $|B(\ell) \cap W^j| = y$. For proving our claim, we define $\{a\} = W^i \setminus W^j$, $\{b\} = W^j \setminus W^i$, $\{a'\} = W^{i'} \setminus W^{j'}$, and $\{b'\} = W^{j'} \setminus W^{i'}$. Moreover, we use a case distinction with respect to whether $|B(\ell) \cap W^i| = |B(\ell) \cap W^j|$ or not.

First, we suppose that $|B(\ell)\cap W^i| = |B(\ell)\cap W^j|$ and consequently also $|B(\ell') \cap W^{i'}| = |B(\ell') \cap W^{j'}|$. In this case, we claim that $\hat{u}^{i,j}_\ell = \hat{u}^{i',j'}_{\ell'} = 0$ and prove this statement only for $\hat{u}^{i,j}_\ell$ as the argument for $\hat{u}^{i',j'}_{\ell'}$ is symmetric. The key insight here is that if $|B(\ell) \cap W^i| = |B(\ell) \cap W^j|$, then either $\{a, b\} \subseteq B(\ell)$ or $\{a, b\} \cap B(\ell) = \emptyset$. Now, let τ denote the permutation defined by $\tau(a) = b$, $\tau(b) = a$, and $\tau(x) = x$ for all $x \in C \setminus \{a, b\}$. It is easy to see that $\tau(W^i) = W^j$, $\tau(W^j) = W^i$, and $\tau(B(\ell)) = B(\ell)$. Therefore, we can use Claims (2) and (3) of Lemma 2 to infer that $-\hat{u}^{i,j}_\ell = \hat{u}^{j,i}_\ell = \tau(\hat{u}^{i,j})_\ell = \hat{u}^{i,j}_\ell$. Clearly, this is only possible if $\hat{u}^{i,j}_\ell = 0$, so our claim follows.

A second case, suppose that $|B(\ell) \cap W^i| \neq |B(\ell) \cap W^j|$. Without loss of generality, we suppose that $|B(\ell) \cap W^i| > |B(\ell) \cap W^j|$ (and consequently also $|B(\ell') \cap W^{i'}| > |B(\ell') \cap W^{j'}|$). This implies that $|B(\ell)\cap W^i| = |B(\ell')\cap W^{i'}| = |B(\ell')\cap W^{j'}|+1 = |B(\ell)\cap W^j|+1$, so $a \in B(\ell)$, $b \notin B(\ell)$, $a' \in B(\ell')$, and $b' \notin B(\ell')$. Now, let τ denote the permutation such that $\tau(a) = a'$, $\tau(b) = b'$, $\tau(W^i \cap W^j) = W^{i'} \cap W^{j'}$, and $\tau(B(\ell)) = B(\ell')$; by our assumptions such a permutation exists. Clearly, $\tau(W^i) = W^{i'}$, $\tau(W^j) = W^{j'}$, and $\tau(B(\ell)) = B(\ell')$, so Claim (3) of Lemma 2 entails that $\hat{u}^{i',j'}_{\ell'} = \tau(\hat{u}^{i,j})_{\ell'} = \hat{u}^{i,j}_\ell$. This proves the desired equality.

By the insights of the last two paragraphs, it follows that the functions $s^1_r(x, y)$ with $\hat{u}^{i,j}_\ell = s^1_{|B(\ell)|}(|B(\ell) \cap W^i|, |B(\ell) \cap W^j|)$ for all ballots $B(\ell)$ and committees W^i, W^j with $|W^i \setminus W^j| = 1$ indeed exists. Moreover, the analysis in the second paragraph immediately implies that $s^1_r(|B(\ell) \cap W^i|, |B(\ell) \cap W^j|) = 0$ for all committees W^i, W^j and ballots $B(\ell)$ with $|W^i \setminus W^j| = 1$, $|B(\ell)| = r$, and $|B(\ell) \cap W^i| = |B(\ell) \cap W^j|$ because $\hat{u}^{i,j}_\ell = 0$. Hence, our functions s^1_r satisfy Claims (1) and (2). Moreover, Claim (2) implies Claim (3) in the case that $|B(\ell) \cap W^i| = |B(\ell) \cap W^j|$.

Hence, it remains to show that $s^1_r(|B(\ell) \cap W^i|, |B(\ell) \cap W^j|) = -s^1_r(|B(\ell) \cap W^j|, B(\ell) \cap W^i|)$ if $|B(\ell) \cap W^i| \neq |B(\ell) \cap W^j|$. For this, consider again two committees $W^i, W^j \in \mathcal{W}_k$ with $W^i \setminus W^j = \{a\}$, $W^j \setminus W^i = \{b\}$. Moreover, consider a ballot $B(\ell)$ such that $a \in B(\ell)$, $b \notin B(\ell)$ and let τ denote the permutation defined by $\tau(a) = b$, $\tau(b) = a$, and $\tau(x) = x$ for all $x \in C \setminus \{a, b\}$. By Claims (2) and (3) of Lemma 2, we have for the ballot $B(\ell') = \tau(B(\ell))$ that $-\hat{u}^{i,j}_{\ell'} = \hat{u}^{j,i}_{\ell'} = \tau(\hat{u}^{i,j})_{\ell'} = \hat{u}^{i,j}_\ell$. On the other hand, it is easy to see that $|B(\ell) \cap W^i| = |B(\ell') \cap W^j|$ and $|B(\ell) \cap W^j| = |B(\ell') \cap W^i|$. Hence, we infer that $s^1_r(B(\ell) \cap W^i|, |B(\ell) \cap W^j|) = \hat{u}^{i,j}_\ell = -\hat{u}^{i,j}_{\ell'} = -s^1_r(B(\ell') \cap W^i|, |B(\ell') \cap W^j|) = -s^1_r(B(\ell) \cap W^j|, |B(\ell) \cap W^i|)$, which proves this claim. □

A.2 Proof of Theorem 2

We will next turn to the proof of Theorem 2: BSWAV rules are the only ABC voting rules that satisfy anonymity, neutrality, consistency, continuity, choice set convexity, and weak efficiency. Since the proof that every BSWAV rule satisfies these axioms is in the main body, we focus here on the converse direction. Unfortunately, this direction is rather involved and we thus introduce several auxiliary lemmas before proving Theorem 2. In more detail, we first construct several important auxiliary profiles in Lemma 7 to show that every non-trivial ABC voting rule f that satisfies all of our axioms is non-trivial. This allows us to access the vectors $\hat{\imath}^{i,j}$ derived in Lemma 2. By investigating the linear independence of these vectors, we can then show that f is a BSWAV rule.

To ease the outlay of our lemmas, we introduce some additional notation. Firstly, we define $\mathcal{F}^1$ as the set of ABC voting rules that satisfy anonymity, neutrality, consistency, continuity, choice set convexity, and weak efficiency (i.e., all axioms required for Theorem 2). Secondly, we define the *convex hull* of two committees $W^i, W^j \in \mathcal{W}_k$ as $[W_i, W_j] = \{W \in \mathcal{W}_k : W_i \cap W_j \subseteq W \subseteq W_i \cup W_j\}$.

We start the proof of Theorem 2 by constructing profiles in which a single candidate is either guaranteed to be chosen or to be not chosen. In more detail, given a candidate $x \in C$ and a ballot size r, we consider the profile $A^{x,r}$ which contains each ballot A with

$|A| = r$ and $x \in A$ once, and the profile $A^{-x,r}$ which contains each ballot A with $|A| = r$ and $x \notin A$ once.

Lemma 7. *Let $f \in \mathcal{F}^1$ denote a non-imposing ABC voting rule. It holds for all candidates $x \in C$ and ballot sizes $r \in \{1, \dots, m\}$ that $f(A^{x,r}) = \{W \in \mathcal{W}_k : x \in W\}$ and $f(A^{-x,r}) = \{W \in \mathcal{W}_k : x \notin W\}$ if there is a ballot $A \in \mathcal{A}$ with $|A| = r$ and $f(A) \neq \mathcal{W}_k$.*

Proof. Consider an ABC voting rule $f \in \mathcal{F}^1$, a ballot size $r \in \{1, \dots, m\}$, and a candidate $x \in C$. Moreover, suppose that there is ballot A such that $|A| = r$ and $f(A) \neq \mathcal{W}_k$. First, this implies that $r \neq m$ because otherwise $A = C$ and neutrality requires that all committees are chosen. Next, by anonymity and neutrality, there are only three possible outcomes for $f(A^{x,r})$ and $f(A^{-x,r})$: for both of these profiles, either $\{W \in \mathcal{W}_k : x \in W\}$, $\{W \in \mathcal{W}_k : x \notin W\}$, or $\mathcal{W}_k$ has to be chosen as all committees in the first two sets are symmetric to each other. Moreover, weak efficiency excludes that $f(A^{-x,r}) = \{W \in \mathcal{W}_k : x \in W\}$ as this axiom allows us to replace x with any other candidate $y \in C \setminus \{x\}$. Finally, we note that anonymity and neutrality require that $f(A^{x,r} + A^{-x,r}) = \mathcal{W}_k$ because the profile $A^{x,r} + A^{-x,r}$ consists of all ballots of size r. Hence, by consistency and our previous observations, we either have that $f(A^{-x,r}) = f(A^{x,r}) = \mathcal{W}_k$, or $f(A^{-x,r}) = \{W \in \mathcal{W}_k : x \notin W\}$ and $f(A^{x,r}) = \{W \in \mathcal{W}_k : x \in W\}$. Indeed, for all other possible combinations, it holds that $f(A^{x,r}) \cap f(A^{-x,r}) \neq \emptyset$ and $f(A^{x,r}) \cap f(A^{-x,r}) \neq \mathcal{W}_k$, so consistency would be violated.

Now suppose for contradiction that $f(A^{-x,r}) = f(A^{x,r}) = \mathcal{W}_k$. If $r = 1$ or $r = m - 1$, this conflicts with the assumption that there is a ballot A of size r such that $f(A) \neq \mathcal{W}_k$. The reason for this is that either $A^{x,r}$ or $A^{-x,r}$ only consist of a single ballot and neutrality between A and $A^{x,r}$ (resp. $A^{-x,r}$) then requires that not all committees of size k are chosen. Hence, we assume that $1 < r < m - 1$. For this case, let X^+, X^- denote two disjoint and possibly empty sets of candidates. Moreover, we define A^{X^+,X^-} as the profile containing each ballot A with $|A| = r$, $X^+ \subseteq A$, and $X^- \cap A = \emptyset$ once. Note that A^{X^+,X^-} is not the empty profile if $|X^+| \leq r$ and $|X^-| \leq m - r$. Our goal is to prove that $f(A^{X^+,X^-}) = \mathcal{W}_k$ for all disjoint sets X^+, X^- by an induction over $t = |X^+ \cup X^-| \in \{1, \dots, \min(r, m-r)\}$. When $t = \min(r, m-r)$, then A^{X^+,X^-} consists of a single ballot and thus, this insight conflicts again with neutrality and the assumption that there is a ballot A of size r with $f(A) \neq \mathcal{W}_k$.

Now, the induction basis $t = 1$ of our claim follows from our assumptions since $f(A^{-x,r}) = f(A^{x,r}) = \mathcal{W}_k$, and neutrality allows us to rename x to any other candidate. We therefore assume that the induction hypothesis holds up to some $t \in \{1, \dots, \min(r, m-r) - 1\}$ and will prove it for $t + 1$. For this, we will first show an auxiliary claim: given two disjoint sets of candidates X^+, X^- with $|X^+ \cup X^-| = t - 1$ and two candidates $x, y \in C \setminus (X^+ \cup X^-)$, it holds that $f(A^{X^+ \cup \{x,y\},X^-}) = f(A^{X^+,X^- \cup \{x,y\}})$ and $f(A^{X^+ \cup \{x\},X^- \cup \{y\}}) = f(A^{X^+ \cup \{y\},X^- \cup \{x\}})$. We prove here only the first claim, but the second one follows analogously. The central observation for the proof is that $A^{X^+ \cup \{x,y\},X^-} + A^{X^+,X^- \cup \{x\}} + A^{X^+,X^- \cup \{y\}} = A^{X^+,X^-} + A^{X^+,X^- \cup \{x,y\}}$. Moreover, by the induction hypothesis, we know that $f(A^{X^+,X^- \cup \{y\}}) = f(A^{X^+,X^- \cup \{x\}}) = f(A^{X^+,X^-}) = \mathcal{W}_k$. Hence, we infer from consistency that

$$
\begin{aligned}
f(A^{X^+ \cup \{x,y\},X^-}) &= f(A^{X^+ \cup \{x,y\},X^-}) \cap f(A^{X^+,X^- \cup \{x\}}) \\
&\quad \cap f(A^{X^+,X^- \cup \{y\}}) \\
&= f(A^{X^+ \cup \{x,y\},X^-} + A^{X^+,X^- \cup \{x\}} + A^{X^+,X^- \cup \{y\}}) \\
&= f(A^{X^+,X^-} + A^{X^+,X^- \cup \{x,y\}}) \\
&= f(A^{X^+,X^-}) \cap f(A^{X^+,X^- \cup \{x,y\}}) \\
&= f(A^{X^+,X^- \cup \{x,y\}}).
\end{aligned}
$$

Finally, consider an arbitrary set of candidates $X = \{x_1, \dots, x_{t+1}\}$. By weak efficiency, anonymity, and neutrality, we have that $W \in f(A^{\emptyset,X})$ for all committees W that minimize $|X \cap W|$. Now, consider the profile $A^{\{y\},X \setminus \{y\}}$ for $y \in X$. First, by our auxiliary claim, we have that $f(A^{\{y\},X \setminus \{y\}}) = f(A^{z,X \setminus \{z\}})$ for all $y, z \in X$. Now, if there is a committee $W \in f(A^{\{y\},X \setminus \{y\}})$ that minimizes $|W \cap X|$, then $f(A^{\emptyset,X}) \cap f(A^{\{y\},X \setminus \{y\}}) \neq \emptyset$, which means that $f(A^{\emptyset,X}) \cap f(A^{\{y\},X \setminus \{y\}}) = f(A^{\emptyset,X \setminus \{y\}}) = \mathcal{W}_k$ by consistency and the induction hypothesis.

Hence, suppose next that there are only ballots W in $f(A^{\{y\},X \setminus \{y\}})$ that do not minimize $|W \cap X|$. By weak efficiency, we know for every such committee W that we can replace the candidates in $z \in W \cap (X \setminus \{y\})$ with a candidate $z' \in C \setminus (W \cup X)$ and the resulting committee W' must still be chosen for $A^{\{y\},X \setminus \{y\}}$. Now, if $|W \cap X| \geq 1$ for each $W \in \mathcal{W}_k$, this means that f chooses a committee with minimal intersection with X as we can exchange all but one candidate in X with candidates from outside X. Since this contradicts the assumption that f does not choose such a committee, we suppose next that $|W \cap X| = 0$ for some committee. In this case, we can in an committee $W \in f(A^{\{y\},X \setminus \{y\}})$ first replace all candidates but y by weak efficiency. Then, we look at a second candidate $z \in X$ and use the fact that $f(A^{\{y\},X \setminus \{y\}}) = f(A^{z,X \setminus \{z\}})$ to also replace y. Hence, we derive again that a committee minimizing $|W \cap X|$ is chosen for $f(A^{\{y\},X \setminus \{y\}})$. So, we have in both cases that $f(A^{\emptyset,X}) \cap f(A^{\{y\},X \setminus \{y\}}) \neq \emptyset$ and consistency requires therefore that $f(A^{\emptyset,X}) = f(A^{\{y\},X \setminus \{y\}}) = \mathcal{W}_k$. By applying our auxiliary claim to these two profiles, we derive analogous claims for all profiles A^{X^+,X^-} with $X^+ \cap X^- = \emptyset$ and $X^+ \cup X^- = X$. Finally, since X is chosen arbitrarily, this proves the induction step and we can thus infer that $f(A) = \mathcal{W}_k$ for each ballot A of size r. This contradicts our assumptions, so the lemma follows. ◻

As a consequence of Lemma 7, every non-trivial ABC voting rule $f \in \mathcal{F}^1$ is non-imposing. Indeed, for every such voting rule f, there is some ballot A such that $f(A) \neq \mathcal{W}_k$; otherwise, consistency requires that $f(A) = \mathcal{W}_k$ for all profiles $A \in \mathcal{A}^*$. Consequently, we can use Lemma 7 to construct a profile $A^{x,r}$ for some ballot size r and candidate x such that $f(A^{x,r}) = \{W \in \mathcal{W}_k : x \in W\}$. Finally, by consistency, it is easy to infer that $f(A^W) = \{W\}$, where A^W is the profile that consists of all $A^{x,r}$ with $x \in W$. Since the trivial rule is clearly the BSWAV rule induced by $\alpha_r = 0$ for all $r \in \{1, \dots, m\}$, we therefore focus on non-imposing ABC voting rules for the rest of the proof. Furthermore, we will restrict our attention to committee sizes $k \in \{2, \dots, m-2\}$. The reason for this is that all ABC voting rules satisfy choice set convexity and all scoring rules are BSWAV rules if $k \in \{1, m-1\}$. Hence, Proposition 1 implies Theorem 2 in this case.

Since we will focus on non-imposing rules from now, we can access the normal vectors $\hat{u}^{i,j}$ from Lemma 2 and the functions s_r^1

defined in Lemma 6. In particular, we will next investigate these functions s_r^1 in more detail and show that they are actually constant. Note that in the next lemma, the set $Q(k,r) = \{\max(0, m-k-r), \dots, \min(k,r)-1\}$ contains all integers x such that there is a ballot A of size r and two committees committee $W^i, W^j \in \mathcal{W}_k$ such that $|W^i \setminus W^j| = 1$, $|A \cap W^i| = x+1$, and $|A \cap W^j| = x$.

Lemma 8. *Suppose $2 \leq k \leq m-2$ and let $f \in \mathcal{F}^1$ denote a non-imposing ABC voting rule. For all ballot sizes $r \in \{1, \dots, m\}$, there is constant $\alpha_r \in \mathbb{R}$ such that $s_r^1(x+1, x) = \alpha_r$ if $x \in Q(k,r)$. Moreover, if there is a ballot of size r such that f does not choose $\mathcal{W}_k$ on it, then $\alpha_r > 0$.*

Proof. Let $f \in \mathcal{F}^1$ denote a non-imposing ABC voting rule and let s_r^1 denote the functions derived in Lemma 6. Moreover, we consider an arbitrary ballot size $r \in \{1, \dots, m\}$. First, if $f(A) = \mathcal{W}_k$ for all ballots A of size r, then $s_r^1(x+1, x) = 0$ for all $x \in Q(k,r)$. Otherwise, there are two committees W^i, W^j and a ballot $B(\ell)$ such that $|B(\ell)| = r$, $|W^i \setminus W^j| = 1$, $|A \cap W^i| = |A \cap W^j| + 1$ and $s_r^1(|B(\ell) \cap W^i|, |B(\ell) \cap W^j|) \neq 0$. By Claim (1) of Lemma 6, this means that $\hat{u}_\ell^{i,j} \neq 0$, where $\hat{u}^{i,j}$ is one of the vectors derived in Lemma 2. Hence, by Claim (2) of Lemma 2, we either have that $v(A)\hat{u}^{i,j} < 0$ or $v(A)\hat{u}^{j,i} < 0$ for the profile A that only contains ballot $B(\ell)$. We suppose subsequently that $v(A)\hat{u}^{i,j} < 0$. This means that $v(A) \notin \bar{R}_i^f$ by Claim (1) of Lemma 2 and thus $W^i \notin \hat{g}(v(A)) = f(A)$ because of the definition of $\bar{R}_i^f$. However, this contradicts that $f(A) = \mathcal{W}_k$ for all ballots of size r and thus, $s_r^1(x+1, x) = 0$ must hold for all $x \in Q(k,r)$.

Hence, we suppose next that f is non-trivial on ballot size r. In this case, we consider two committees $W^i, W^j \in \mathcal{W}_k$ with $|W^i \cap W^j| = k-2$; such committees exist since $2 \leq k \leq m-2$. For a simple notation, we further define $W^i \setminus W^j = \{a_1, a_2\}$ and $W^j \setminus W^i = \{b_1, b_2\}$. The main goal for our proof is to show that $s_r^1(x+1, x) = s_r^1(x+2, x+1)$ for all $x, x+1 \in Q(k,r)$. By repeatedly applying this argument, it follows that $s_r^1(x+1, x) = s_r^1(y+1, y)$ for all $x, y \in Q(k,r)$.

To prove this claim, fix some index x such that $x, x+1 \in Q(k,r)$. In particular, this means that there is a ballot A of size r such that $|W^i \cap A| = x+2$. Now, since f is non-trivial for ballot size r, Lemma 7 shows that $f(A^{x,r}) = \{W \in \mathcal{W}_k \colon x \in W\}$ and $f(A^{-x,r}) = \{W \in \mathcal{W}_k \colon x \in W\}$ for every candidate $x \in C$. Next, consider the profile A^1 that consists of a copy of $A^{x,r}$ for every $x \in W^i \cap W^j$ and of a copy of $A^{-x,r}$ for every $x \in C \setminus (W^i \setminus W^j)$. By consistency, it is easy to verify that $f(A^1) = [W^i, W^j]$. As third step, consider the profile A^2 which consists of the following two ballots: the first voter in A^2 approves a_1, a_2, x candidates of $W^i \cap W^j$, and $r-x-2$ candidates of $C \setminus (W^i \cup W^j)$, and the second voter has the same ballot except that he replaces a_1 and a_2 with b_1 and b_2; such a ballot exists by the definition of $Q(k,r)$.

Now, by the continuity of f, there is $\lambda \in \mathbb{N}$ such that $f(\lambda A^1 + A^2) \subseteq [W^i, W^j]$. Based on choice set convexity, anonymity, and neutrality, we will show that this subset relation is actually an equality. For this, we note that for every permutation τ with $\tau(\{a_1, a_2\}) = \{b_1, b_2\}$, $\tau(\{b_1, b_2\}) = \{a_1, a_2\}$, and $\tau(x) = x$ for $x \in C \setminus \{a_1, a_2, b_1, b_2\}$, it holds that $\tau(\lambda A^1 + A^2) = \lambda A^1 + A^2$ (possibly after reordering voters). Hence, anonymity and neutrality show that if $W^i \in f(\lambda A^1 + A^2)$, then $W^j \in f(\lambda A^1 + A^2)$, and if $W \in f(\lambda A^1 + A^2)$ for $W \in [W^i, W^j] \setminus \{W^i, W^j\}$, then $[W^i, W^j] \setminus \{W^i, W^j\} \subseteq f(\lambda A^1 + A^2)$. Now, if $W^i, W^j \in f(\lambda A^1 + A^2)$, the choice set convexity immediately shows that $f(\lambda A^1 + A^2) = [W^i, W^j]$. On the other hand, if $[W^i, W^j] \setminus \{W^i, W^j\} \subseteq f(\lambda A^1 + A^2)$, then the committees $W = \{a_1, b_2\} \cup (W^i \cap W^j)$ and $W' = \{b_1, a_2\} \cup (W^i \cap W^j)$ are chosen. By choice set convexity, we thus infer again that $W^i, W^j \in f(\lambda A^1 + A^2)$ because $W^i, W^j \in [W, W']$. Thus, we indeed have $f(\lambda A^1 + A^2) = [W^i, W^j]$.

Now, let $v^1 = v(A^1)$, $v^2 = v(A^2)$, and $v^* = v(\lambda A^1 + A^2)$ denote the vectors corresponding to the profiles A^1, A^2, and $\lambda A^1 + A^2$, respectively. Moreover, consider the committee $W^\ell = \{a_1, b_2\} \cup (W^i \cap W^j)$ which lies strictly between W^i and W^j. Finally, let $\hat{u}^{i',j'}$ denote the hyperplanes constructed in Lemma 2 and note that $\hat{u}^{i,\ell}$ can be described by the the functions s_r^1. In particular, it holds that $v^2\hat{u}^{i,\ell} = s_r^1(x+2, x+1) + s_r^1(x, x+1)$ as the first voter in A^2 approves $x+2$ members of W^i and $x+1$ members of W^ℓ, and the second voter approves x members of W^i and and $x+1$ members of W^ℓ. Our goal is hence to show that $s_r^1(x+2, x+1) + s_r^1(x, x+1) = 0$ since Claim (3) in Lemma 6 then implies that $s_r^1(x+2, x+1) = s_r^1(x+1, x)$. For doing this, we note that $v^*\hat{u}^{i,\ell} = (\lambda v^1 + v^2)\hat{u}^{i,\ell}$, so it is enough to show that $v^1\hat{u}^{i,\ell} = 0$ and $v^*\hat{u}^{i,\ell} = 0$. For the latter, we observe that $v^* \in \bar{R}_i^f$ and $v^* \in \bar{R}_\ell^f$ since $W^i, W^\ell \in f(\lambda A^1 + A^2) = \hat{g}(v^*)$. Since Claim (1) of Lemma 2 shows that $\bar{R}_{i'}^f = \{v \in \mathbb{R}^{|\mathcal{A}|} \colon \forall j' \in \{1, \dots, |\mathcal{W}_k|\} \setminus \{i'\} \colon v\hat{u}^{i',j'} \geq 0\}$ for all i', we derive that $v^*\hat{u}^{i,\ell} = 0$. Finally, to show that $v^1\hat{u}^{i,\ell} = 0$, let τ denote the permutation defined by $\tau(a_2) = b_2$, $\tau(b_2) = a_2$, and $\tau(x) = x$. It can be checked that $\tau(A^{x,r}) = A^{x,r}$ and $\tau(A^{-x,r}) = A^{-x,r}$ (up to renaming voters) for every candidate $x \in C \setminus \{a_2, b_2\}$. Hence, it also holds that $\tau(A^1) = A^1$ (up to renaming voters). On the other hand, we have that $\tau(W^i) = W^\ell$ and $\tau(W^\ell) = W^i$. Hence, we can use Claims (2) and (3) of Lemma 2 to compute that $2v^1\hat{u}^{i,\ell} = v^1\hat{u}^{i,\ell} + \tau(v^1)\tau(\hat{u}^{i,\ell}) = v^1\hat{u}^{i,\ell} + v^1\hat{u}^{\ell,i} = v^1\hat{u}^{i,\ell} - v^1\hat{u}^{i,\ell} = 0$. Clearly, this implies that $v^1\hat{u}^{i,\ell} = 0$, so it indeed holds that $s_r^1(x+2, x+1) = s_r^1(x+1, x)$, which shows that there are constants α_r such that $s_r^1(x+1, x) = \alpha_r$ for all $x \in Q(k,r)$.

Finally, we need to show that $\alpha_r > 0$ if there is a ballot A of size r with $f(A) \neq \mathcal{W}_k$. For doing so, we consider two committees $W^i, W^j \in \mathcal{W}_k$ with $|W^i \setminus W^j| = 1$. Moreover, let A denote the profile that consists of $A^{x,r}$ for every $x \in W^i$. By consistency and Lemma 7, it is easy to derive that $f(A) = \{W^i\}$. Moreover, by continuity and consistency, there is $\lambda \in \mathbb{N}$ such that $f(\lambda A + B(\ell)) = \{W^i\}$ for every ballot $B(\ell) \in \mathcal{A}$. This implies for the vector $v(A)$ that it is in the interior of $\bar{R}_i^f$. By Lemma 2, we hence infer that $v(A)\hat{u}^{i,j} > 0$. Finally, since $|W^i \setminus W^j| = 1$, we can represent $\hat{u}^{i,j}$ by s_r^1. Because all ballots in A have size r and the symmetry properties of s_r^1 identified in Lemma 6, it thus holds that $v\hat{u}^{i,j} = \alpha_r c_1 - \alpha_r c_2$, where c_1 states how many voters in A approve more candidates in W^i then in W^j and c_2 counts how many candidates prefer more candidates in W^j than in W^i. Finally, by the construction of A, it is easy to see that $c_1 > c_2$, so $v(A)\hat{u}^{i,j} > 0$ implies that $\alpha_r > 0$. □

Note that Lemma 8 has strong consequences for the vectors $\hat{u}^{i,j}$ constructed in Lemma 2, in particular if we consider committees $W^i, W^j, W^{i'}, W^{j'} \in \mathcal{W}_k$ such that $W^i \setminus W^j = W^{i'} \setminus W^{j'} = \{a\}$ and $W^j \setminus W^i = W^{j'} \setminus W^{i'} = \{b\}$. For these committees, the lemma shows that $\hat{u}_\ell^{i,j} = s_{|B(\ell)|}(|B(\ell) \cap W^i|, |B(\ell) \cap W^j|) = s_{|B(\ell)|}(|B(\ell) \cap$

$W^{i'}|, |B(\ell) \cap W^{j'}|) = \hat{u}_\ell^{i',j'}$ for every ballot $B(\ell) \in \mathcal{A}$ with $a \in B(\ell)$, $b \notin B(\ell)$. Hence, by Lemma 6, it thus follows that $\hat{u}^{i,j} = \hat{u}^{i',j'}$. To reflect this insight, we change from now on the notation from $\hat{u}^{i,j}$ to $\hat{u}^{a,b}$, where a and b are the candidates such that $W^i \setminus W^j = \{a\}$, $W^j \setminus W^i = \{b\}$.

In the next lemma, we apply this insight to derive some auxiliary claims on the profiles $A^{x,r}$ and $A^{-x,r}$.

Lemma 9. *Suppose $2 \leq k \leq m-2$ and let $f \in \mathcal{F}^1$ denote a non-imposing ABC voting rule. The following claims hold for all distinct candidates $a, b, c \in C$ and ballot sizes $r \in \{1, \dots, m\}$:*

(1) $\hat{u}^{a,b} v(A^{c,r}) = \hat{u}^{a,b} v(A^{-c,r}) = 0$
(2) $\hat{u}^{a,b} v(A^{b,r}) = -\hat{u}^{b,a} v(A^{b,r})$ and $\hat{u}^{a,b} v(A^{-b,r}) = -\hat{u}^{b,a} v(A^{-b,r})$
(3) $\hat{u}^{a,b} v(A^{a,r}) > 0$ and $\hat{u}^{a,b} v(A^{-a,r}) < 0$ if there is a ballot A of size r with $f(A) \neq \mathcal{W}_k$
(4) $\hat{u}^{a,b} + \hat{u}^{b,c} = \hat{u}^{a,c}$

Proof. Let $f \in \mathcal{F}^1$ denote a non-imposing ABC voting rule and fix three candidates a, b, c and a ballot size r. Moreover, we let X denote a set of $k-1 \leq m-3$ candidates with $\{a, b, c\} \cap X = \emptyset$ and define the committees $W^a = X \cup \{a\}$, $W^b = X \cup \{b\}$, and $W^c = X \cup \{c\}$.

Claim (1): The claim follows by considering the permutation τ with $\tau(a) = b$, $\tau(b) = a$, and $\tau(x) = x$ for all $x \in C$. Then, it is easy to see that $\tau(A^{c,r}) = A^{c,r}$ (up to renaming voters) and $\tau(W^a) = W^b$, $\tau(W^b) = W^a$. Hence, we get that $2v(A^{c,r})\hat{u}^{a,b} = v(A^{c,r})\hat{u}^{a,b} v(A^{c,r}) + \tau(v(A^{c,r}))\tau(\hat{u}^{a,b}) = v(A^{c,r})\hat{u}^{a,b} v(A^{c,r}) - v(A^{c,r})\hat{u}^{a,b} = 0$ by using Lemma 6. This implies that $v(A^{c,r})\hat{u}^{a,b} = 0$ and an analogous argument works for $A^{-c,r}$.

Claim (2): The claim follows immediately from Claim (2) in Lemma 2 because $\hat{u}^{a,b} = -\hat{u}^{b,a}$.

Claim (3): We focus on the profile $A^{a,r}$ as the claim for $A^{-a,r}$ can be shown analogously. Hence, note that for every ballot $B(\ell)$ in the profile $A^{a,r}$, either $a, b \in B(\ell)$ or $a \in B(\ell)$, $b \notin B(\ell)$. By Claim (2) of Lemma 6, we infer that $\hat{u}_\ell^{a,b} = 0$ if $a, b \in B(\ell)$ and by Lemma 8, we infer that $u_\ell^{a,b} = \alpha_r$ if $a \in B(\ell)$, $b \notin B(\ell)$ for some constant α_r. Hence, $v(A^{a,r})\hat{u}^{a,b} = n_r \alpha_r$, where $n_r > 0$ states the number of ballots $B(\ell)$ in $A^{A,r}$ with $a \in B(\ell)$, $b \notin B(\ell)$. Finally, if there is a ballot A of size r with $f(A) \neq \mathcal{W}_k$, Lemma 8 shows that α_r and the claim follows.

Claim (4): For this claim, we consider a ballot $B(\ell)$ and note that $\hat{u}_\ell^{x,y} = s^1_{|B(\ell)|}(|B(\ell) \cap W^x|, |B(\ell) \cap W^y|)$ for all distinct $x, y \in \{a, b, c\}$. The statement now follows by considering the 8 cases enumerating whether $a \in B(\ell)$, $b \in B(\ell)$, and $c \in B(\ell)$. For instance, if $a, c \in B(\ell)$, $b \notin B(\ell)$, then $\hat{u}_\ell^{a,c} = 0$ (by Claim (2) of Lemma 6) and $\hat{u}_\ell^{a,b} = s^1_{|B(\ell)|}(|B(\ell) \cap W^a|, |B(\ell) \cap W^b|) = -s^1_{|B(\ell)|}(|B(\ell) \cap W^b|, |B(\ell) \cap W^c|) = -\hat{u}_\ell^{b,c}$ (by Claim (3) of Lemma 6). The remaining cases work similar and we leave the to the reader. ◻

We now turn to the central part of the proof of Theorem 2. For this, consider two committees W^i, W^j with $|W^i \setminus W^j| = t > 1$ and let $\{a_1, \dots, a_t\} = W^i \setminus W^j$ and $\{b_1, \dots, b_t\} = W^j \setminus W^i$. Our main goal is to show that the vector $\hat{u}^{i,j}$ can be represented as the sum of all vectors $\hat{u}^{a_x,b_x}$ for $x \in \{1, \dots, t\}$ as this will allow us to represent the underlying voting rule as BSWAV rule.

To this end, we first show the linear independence of a large set of vectors $\hat{u}^{a,b}$.

Lemma 10. *Suppose $2 \leq k \leq m-2$ and let $f \in \mathcal{F}^1$ denote a non-imposing ABC voting rule. Moreover, consider $2t$ distinct candidates $a_1, b_1, \dots, a_t, b_t$. The set $U = \{\hat{u}^{a_1,b_1}, \hat{u}^{a_2,b_2}, \dots, \hat{u}^{a_t,b_t}\} \cup \{\hat{u}^{a_1,a_2}, \hat{u}^{a_2,a_3}, \dots, \hat{u}^{a_{t-1},a_t}\}$ is linearly independent.*

Proof. Let $f \in \mathcal{F}^1$ denote a non-imposing ABC voting rule. Moreover, consider $2t$ candidates $a_1, b_1, \dots, a_t, b_t$ and let M be defined as in the lemma. For the proof of the lemma, we put the vectors in U as rows into a matrix $M \in \mathbb{R}^{2t-1 \times |\mathcal{A}|}$. In more detail, let row i of M with $0 < i \leq t$ be given by $\hat{u}^{a_i,b_i}$ and let row $t+i$ with $0 < i < t$ be given by $\hat{u}^{a_i,a_{i+1}}$. We want to show that for each $i \leq 2t-1$ there is a vector $v \in \mathbb{R}^{|\mathcal{A}|}$ such that $Mv = w$ satisfies that $w_i \neq 0$ and $w_j = 0$ for all $j \neq i$. Then, the dimension of the image of M is $2t-1$, which is the column-rank of the matrix. Since it is a basic fact in linear algebra that the column rank equals the row rank, this means that the vectors in U are linearly independent.

For showing this claim, we first observe that there is thus a ballot size r such that $f(A^{x,r}) = \{W \in \mathcal{W}_k : x \in W\}$ and $f(A^{-x,r}) = \{W \in \mathcal{W}_k : x \in W\}$ by Lemma 7 and the non-imposition of f. Based on the claims in Lemma 9, we define vectors v^i that satisfy our constraints for the rows $i \in \{1, \dots, t\}$. In more detail, it suffices to consider the profile $A^{-b_i,r}$ and its corresponding vector v^{b_i} for this. Indeed, we note here that for all vectors $\hat{u} \in U$ but $\hat{u}^{a_i,b_i}$, it follows that $v^{-b_i}\hat{u} = 0$ by Claim (1) in Lemma 9. On the other hand, Claims (2) and (3) in this lemma show that $v^{-b_i}\hat{u}^{a_i,b_i} \neq 0$. So v^{-b_i} satisfies indeed our requirements.

For the other $t-1$ rows, we consider a sightly more complicated profile: let A^i denote the profile that consists of $A^{a_j,r}$ and $A^{b_j,r}$ for all $j \leq i$ and let v^i denote the corresponding vector. Using Claim (1) of Lemma 9, we infer for all $\hat{u}^{a_j,b_j}$ with $j \leq i$ that $v^i \hat{u}^{a_j,b_j} = \sum_{\ell=1}^{i}(v(A^{a_\ell,r}) + v(A^{b_\ell,r}))\hat{u}^{a_j,b_j} = (v(A^{a_j,r}) + v(A^{b_j,r}))\hat{u}^{a_j,b_j}$. Now by the choosing τ with $\tau(a_j) = b_j$, $\tau(b_j) = a_j$, and $\tau(x) = x$ for all other candidates, we get that $(v(A^{a_j,r}) + v(A^{b_j,r}))\hat{u}^{a_j,b_j} = v(A^{a_j,r})\hat{u}^{a_j,b_j} + \tau(v(A^{b_j,r}))\tau(\hat{u}^{a_j,b_j}) = v(A^{a_j,r})\hat{u}^{a_j,b_j} + v(A^{a_j,r})\hat{u}^{b_j,a_j}) = 0$, where the last equality uses Claim (2) of Lemma 9. An analogous argument also applies for the vectors $\hat{u}^{a_j,a_{j+1}}$ with $j < i$. Next, by Claim (1) of Lemma 9, we infer also $v^i \hat{u}^{a_j,b_j} = 0$ and $v^i \hat{u}^{a_j,a_{j+1}}$ for $j > i$. Finally, consider the vector $\hat{u}^{a_i,a_{i+1}}$. By Claim (1), we have that $v^1 \hat{u}^{a_i,a_{i+1}} = \sum_{j=1}^{i}(v(A^{a_j,r}) + (A^{b_j,r}))\hat{u}^{a_i,a_{i+1}} = v(A^{a_i,r})\hat{u}^{a_i,a_{i+1}} = n_r \alpha_r$. By the same argument as in the last paragraph, this is non-zero, thus proving the lemma. ◻

As next step, we show that the linear independence observed in Lemma 10 turns into a linear dependence once we add a vector $\hat{u}^{i,j}$ with $W^i \setminus W^j = \{a_1, \dots, a_t\}$ and $W^j \setminus W^i = \{b_1, \dots, b_t\}$

Lemma 11. *Suppose $2 \leq k \leq m-2$ and let $f \in \mathcal{F}^1$ denote a non-imposing ABC voting rule. Moreover, consider $2t$ distinct candidates $a_1, b_1, \dots, a_t, b_t$, and two committees W^i, W^j with $W^i \setminus W^j = \{a_1, \dots, a_t\}$ and $W^j \setminus W^j = \{a_1, \dots, a_t\}$. Then, set $U = \{\hat{u}^{a_1,b_1}, \hat{u}^{a_2,b_2}, \dots, \hat{u}^{a_t,b_t}\} \cup \{\hat{u}^{a_1,a_2}, \hat{u}^{a_2,a_3}, \dots, \hat{u}^{a_{t-1},a_t}\} \cup \{\hat{u}^{i,j}\}$ is linearly dependent.*

Proof. Let $f \in \mathcal{F}^1$ denote an non-imposing ABC voting rule, and consider candidates $a_1, b_1, \dots, a_t, b_t$ and committees W^i, W^j as defined in the lemma. We assume for contradiction that the vectors in U are linearly independent. Our goal is to find a vector v^* such that $v^* \notin \bar{R}_i^f$ for every $i \in \{1, \dots, |\mathcal{W}_k|\}$. This contradicts one of our basic insights, namely that $\bigcup R_i^f = \mathbb{Q}^{|\mathcal{A}|}$ as this requires that $\bigcup \bar{R}_i^f = \mathbb{R}^{|\mathcal{A}|}$.

To his end, we first consider the the matrix M that contains the vectors $u \in U$. Since U is by assumption linear independent, the image of M has full dimension $\mathbb{R}^{2t}$. This means that there is a vector $v \in \mathbb{R}^{|\mathcal{A}|}$ such that $v\hat{u}^{a_x,b_x} = 1$ for all $x \in \{1, \dots, t\}$, $v\hat{u}^{a_x,a_{x+1}} = 0$ for all $x \in \{1, \dots, t-1\}$, and $\hat{u}^{i,j} = -1$. First, we note that, by repeatedly applying Claim (4) of Lemma 9, it is easy to infer that $v\hat{u}^{a_x,a_y} = v\sum_{\ell=x}^{y-1} \hat{u}^{a_\ell,a_{\ell+1}} = 0$ for all $x, y \in \{1, \dots, t\}$ with $x < y$. Moreover, by the symmetry of these vectors (see Lemma 2), the same holds for $\hat{u}^{a_y,a_x}$. By applying again claim (4) of Lemma 9, we thus infer that $v\hat{u}^{a_x,b_y} = v(\hat{u}^{a_x,a_y} + \hat{u}^{a_y,b_y}) = 1$ for all $x, y \in \{1, \dots, t\}$. This insight implies that for every committee $W^{i'} \in [W^i, W^j]$, there is another committee $W^{j'} \in [W^i, W^j]$ such that $v\hat{u}^{i',j'} < 0$. In more detail, if $W^{i'} \neq W^i$, there are candidates $a_x \notin W^{i'}$ and $b_y \in W^{i'}$. Then, it holds for the committee $W^{j'} = (W^{i'} \setminus \{b_y\}) \cup \{a_i\}$ that $v\hat{u}^{i',j'} = v\hat{u}^{b_y,a_x} = -v\hat{u}^{a_x,b_y} = -1$. On the other side, for W^i, it holds by definition of v that $v\hat{u}^{i,j} = -1$.

For the second step, let r denote a ballot size such that $f(A) \neq \mathcal{W}_k$ for some ballot A with $|A| = r$. This means that $f(A^{x,r}) = \{W \in \mathcal{W}_k : x \in W\}$ and $f(A^{-x,r}) = \{W \in \mathcal{W}_k : x \notin W\}$ by Lemma 7. Now, consider the profile A^1 containing one copy $A^{x,r}$ for each $x \in W^i \cap W^j$ and one copy of each $A^{-x,r}$ for $x \notin W^i \cup W^j$. By consistency, $f(A^1) = [W^{j_0}, W^{j_t}]$. For $W^{i'}, W^{j'} \in [W^{j_0}, W^{j_t}]$, Claim (1) of Lemma 2 shows now that $\hat{u}^{i,j}v(A^1) = 0$. On the other hand, we will show that for every committee $W^{i'} \notin [W^i, W^j]$, there is another committee $W^{j'}$ such that $v(A^1)\hat{u}^{i',j'} < 0$. For this, let $a \in (W^i \cup W^j) \setminus W^{i'}$, $b \in W^{i'} \setminus (W^i \cup W^j)$, and define $W^{j'}$ as $W^{j'} = (W^{i'} \setminus \{b\}) \cup \{a\}$. By Claims (1) to (3) of Lemma 9, it follows that $v(A^1)\hat{u}^{i',j'} = v(A^1)\hat{u}^{b,a} = (v(A^{a,r}) + v(A^{-b,r})\hat{u}^{b,a} = v(A^{-b,r})\hat{u}^{b,a} - v(A^{a,r})\hat{u}^{a,b} < 0$.

Now, let $v^n = nv(A^1) + v$, where n is large enough such that still $v^n\hat{u}^{i',j'} < 0$ for all $W^{i'} \notin [W^{j_0}, W^{j_t}]$ and their corresponding $W^{j'}$. It holds that $v^n \notin \bar{R}_i^f$ for all $W^i \in \mathcal{W}_k$. Firstly, by definition of v_n, $v_n \notin \bar{R}_{i'}^f$ for all $W^{i'} \notin [W^i, W^j]$. Next, consider a committee $W^{i'} \in [W^i, W^j]$ and a corresponding $W^{j'} \in [W^i, W^j]$ with $v\hat{u}^{i',j'} < 0$. By construction $v^n\hat{u}^{i',j'} = (nv(A^1) + v)\hat{u}^{i',j'} < 0$ since $v(A^1)\hat{u}^{i',j'} = 0$. In total, $v^n \notin \cup_{i \leq \binom{m}{k}} \bar{R}_i^f = \mathbb{R}^{|\mathcal{A}|}$. Clearly, this is a contradiction, so the vectors in U are linearly dependent. □

As as consequence of Lemma 10 and Lemma 11, there are unique real coefficients $\delta_1, \dots \delta_{2t-1}$ such that $\hat{u}^{i,j} = \delta_{a_1,b_1}\hat{u}^{a_1,b_1} + \dots + \delta_{a_{t-1},a_t}\hat{u}^{a_{t-1},a_t}$. In the next lemma, we determine these coefficient, which will show that $\hat{u}^{i,j}$ can indeed be represented as a (scaled) sum of the $\hat{u}^{a_x,b_x}$.

Lemma 12. *Suppose $2 \leq k \leq m-2$ and let $f \in \mathcal{F}^1$ denote a non-imposing ABC voting rule. Moreover, consider two committees $W^i, W^j \in \mathcal{W}_k$ such that $|W^i \setminus W^j| = t \geq 1$ and let $W^i \setminus W^j = \{a_1, \dots a_t\}$ and $W^i \setminus W^j = \{b_1, \dots b_t\}$. There is $\delta > 0$ such that $\hat{u}^{j_0,j_t} = \delta \sum_{i \leq t} \hat{u}^{a_i,b_i}$.*

Proof. Let $f \in \mathcal{F}^1$ be a non-imposing ABC voting rule, and let $W^i, W^j, a_1, b_1, \dots a_t, b_t$ be defined as in the lemma. First, we observe that the case $t = 1$ is trivial with $\delta = 1$ and hence suppose that $t > 1$. By Lemma 11, the $2t$ vectors in $U = \{\hat{u}^{a_1,b_1}, \hat{u}^{a_2,b_2}, \dots, \hat{u}^{a_t,b_t}\} \cup \{\hat{u}^{a_1,a_2}, \hat{u}^{a_2,a_3}, \dots, \hat{u}^{a_{t-1},a_t}\} \cup \{\hat{u}^{i,j}\}$ are linearly dependent, whereas Lemma 10 shows that the vectors in $U \setminus \{\hat{u}^{i,j}\}$ are linearly independent. Thus, there are unique real coefficients $\delta_{a_1,b_1}, \dots \delta_{a_{t-1},a_t}$ such that $\hat{u}^{i,j} = \delta_1\hat{u}^{a_1,b_1} + \dots + \delta_{a_{t-1},a_t}\hat{u}^{a_{t-1},a_t}$.

Our goal is to determine these coefficients. For this, we let r denote a ballot size such that $f(A) \neq \mathcal{W}_k$ for some ballot A of size r, and define $v^x = v(A^{x,r})$ for every profile $A^{x,r}$. Next, we proceed in three steps to show the lemma.

Step 1: $v^{x_1}\hat{u}^{x_1,y_1} = v^{x_2}\hat{u}^{x_2,y_2} > 0$

First, we show that $v^{x_1}\hat{u}^{x_1,y_1} = v^{x_2}\hat{u}^{x_2,y_2} > 0$ for all $x_1, x_2, y_1, y_2 \in C$. For this, let τ denote the permutation defined by $\tau(x_1) = x_2$, $\tau(x_2) = x_1$, $\tau(y_1) = y_2$, $\tau(y_2) = y_1$, and $\tau(z) = z$ for all $z \in C \setminus \{x_1, x_2, y_1, y_2\}$. By Claim (3) in Lemma 2, it holds that $\tau(\hat{u}^{x_1,y_1}) = \hat{u}^{x_2,y_2}$, and by the symmetry of the profiles $A^{x,r}$, we infer that $\tau(v^{x_1}) = v^{x_2}$. Hence, we can now compute that $v^{x_1}\hat{u}^{x_1,y_1} = \tau(v^{x_1})(\hat{u}^{x_1,y_1}) = v^{x_2}\hat{u}^{x_2,y_2}$. Finally, Claim (3) in Lemma 9 shows that $v^{x_2}\hat{u}^{x_2,y_2}$, thus proving the claim.

Step 2: $\delta_{a_1,b_1} = \dots = \delta_{a_t,b_t} \geq 0$.

For this step, we consider two indices $x, y \in \{1, \dots, t\}$ and the vectors v^{b_x} and v^{b_y}. Moreover, let τ denote the permutation with $\tau(b_x) = b_y$, $\tau(b_y) = b_x$, $\tau(a_x) = a_t$, $\tau(a_y) = a_x$ and $\tau(z) = z$ for all candidates $z \in C \setminus \{a_x, a_y, b_x, b_y\}$. By Claim (1) in Lemma 9, $v^{b_x}\hat{u}^{i,j} = v^{b_x}(\delta_{a_1,b_1}\hat{u}^{a_1,b_1} + \dots + \delta_{a_{t-1},a-t}\hat{u}^{a_{t-1},a_t}) = \delta_{a_x,b_x}v^{b_x}\hat{u}^{a_x,b_x}$ and $v^{b_y}\hat{u}^{i,j} = \delta_{a_y,b_y}v^{b_y}\hat{u}^{a_y,b_y}$ follows from an analogous reasoning. Finally, we note that $\tau(\hat{u}^{i,j}) = \hat{u}^{i,j}$ since $\tau(W^i) = W^i$ and $\tau(W^j) = W^j$. Hence, we can compute that $\delta_{a_x,b_x}v^{b_x}\hat{u}^{a_x,b_x} = v^{b_x}\hat{u}^{i,j} = \tau(v^{b_x})\tau(\hat{u}^{i,j}) = v^{b_j}\hat{u}^{i,j} = \delta_{a_y,b_y}v^{b_y}\hat{u}^{a_y,b_y}$. Since $v^{b_x}\hat{u}^{a_x,b_x} = v^{b_y}\hat{u}^{a_y,b_y} < 0$ (by Step 1 and Claim (2) of Lemma 2, this proves that $\delta_{a_x,b_x} = \delta_{a_y,b_y}$. Moreover, since $v^{b_x} \in \bar{R}_j^f$ (as $W^j \in f(A^{b_x,r})$), we infer from Claim (1) of Lemma 2 $v^{b_x}\hat{u}^{i,j} = -v^{b,x}\hat{u}^{j,i} \leq 0$. Since $v^{b_x}\hat{u}^{a_x,b_x} < 0$, this means that $\delta_{a_x,b_x} \geq 0$.

Step 3: $\delta_{a_1,a_2} = \dots = \delta_{a_{t-1},a_t} = 0$.

For this step, suppose first that $t = 2$. Then, we only need to show that $\hat{u}^{a_1,a_2} = 0$. To do so, we consider the vector v^{a_1}. First, we note that $v^{a_1}\hat{u}^{a_2,b_2} = 0$ by Claim (1) of Lemma 9. Hence, $v^{a_1}\hat{u}^{i,j} = \delta_{a_1,b_1}v^{a_1}\hat{u}^{a_1,b_1} + \delta_{a_1,a_2}v^{a_1}\hat{u}^{a_1,a_2}$. Now, we define $\Delta = v^{a_1}\hat{u}^{a_1,b_1} > 0$ and note that by Step 1, $v^{a_1}\hat{u}^{i,j} = \Delta(\delta_{a_1,b_1} + \delta_{a_1,a_2})$. Analogously, $v^{a_2}\hat{u}^{i,j} = \delta_{a_2b_2}v^{a_2}\hat{u}^{a_2,b_2} + \delta_{a_1,a_2}v^{a_2}\hat{u}^{a_1,a_2} = \Delta(\delta_{a_2b_2} - \delta_{a_1,a_2})$. To derive our claim, we consider the permutation τ with $\tau(a_1) = a_2$, $\tau(a_2) = a_1$, and $\tau(x) = x$ for all candidates. Just as in the last step, we can now infer that $\Delta(\delta_{a_1,b_1} + \delta_{a_1,a_2}) = v^{a_1}\hat{u}^{i,j} = \tau(v^{a_1})\tau(\hat{u}^{i,j}) = v^{a_2}\hat{u}^{i,j} = \Delta(\delta_{a_2,b_2} - \delta_{a_1,a_2})$. Since $\Delta > 0$ and $\delta_{a_1,b_1} = \delta_{a_2,b_2}$, this equality can only be true if $\delta_{a_1,a_2} = 0$, thus proving our claim.

Finally, consider the case that $t > 2$ and consider an index $i \in \{2, \dots, t-1\}$. Moreover, let $\bar{v}^i = (v^{a_i} + v^{b_i})$. Next, we note for all $\hat{u} \in U \setminus \{\hat{u}^{i,j}, \hat{u}^{a_i,b_i}, \hat{u}^{a_{i-1},a_i}, \hat{u}^{a_i,a_{i+1}}\}$ that $\bar{v}^i\hat{u} = 0$ by Claim (1) of Lemma 9. Furthermore, $\bar{v}^i\hat{u}^{a_i,b_i} = v^{a_i}\hat{u}^{a_i,b_i} - v^{b_i}\hat{u}^{b_i,a_i} = 0$ by Step

1. On the other hand, $\bar{v}^i\hat{u}^{a_i,a_{i+1}} = v^{a_i}\hat{u}^{a_i,a_{i+1}} = -v^{a_i}\hat{u}^{a_{i-1},a_i} > 0$ by the symmetry of the $\hat{u}^{x,y}$ and Step 1. Hence, we conclude that $\bar{v}^i\hat{i}^{i,j} = \bar{v}^i(\delta_{a_1,b_1}\hat{u}^{a_1,b_1} + \dots + \delta_{a_{t-1},a_t}\hat{u}^{a_{t-1},a_t} = \delta_{a_i,a_{i+1}}v^{a_i}\hat{u}^{a_i,a_{i+1}} + \delta_{a_{i-1},a_i}v^{a_i}\hat{u}^{a_{i-1},a_i} = \delta_{a_i,a_{i+1}}v^{a_i}\hat{u}^{a_i,a_{i+1}} - \delta_{a_{i-1},a_i}v^{a_i}\hat{u}^{a_i,a_{i+1}}$. On the other hand, $\bar{v}^i\hat{u}^{i,j} = -\tau v^i\hat{u}^{i,j} = 0$ by using the symmetry of $\bar{v}^i$ with respect to a_i and b_i. By using Step 1, we thus infer now that $\delta_{a_{i-1},a_i} = \delta_{a_i,a_{i+1}}$ for all $i \in \{2, \dots, t-1\}$.

Finally, we will show that all δ_{a_{i-1},a_i} are 0. For this consider the vectors v^{a_1} and v^{a_2}. For the permutation τ which mirrors a_1 and a_2 while fixing all remaining candidates, we obtain $v^{a_1}\hat{u}^{i,j} = \tau(v^{a_1})\tau(\hat{u}^{i,j}) = v^{a_2}\hat{u}^{i,j}$. On the other hand, we can derive from Lemma 9 that $v^{a_1}\hat{u}^{i,j} = \delta_{a_1,b_1}v^{a_1}\hat{u}^{a_1,b_1} + \delta_{a_1,a_2}v^{a_1}\hat{u}^{a_1,a_2}$ and $v^{a_2}\hat{u}^{i,j} = \delta_{a_2,b_2}v^{a_2}\hat{u}^{a_2,b_2} + \delta_{a_1,a_2}v^{a_2}\hat{u}^{a_1,a_2} + \delta_{a_2,a_3}v^{a_2}\hat{u}^{a_2,a_3}$. By our previous analysis, $\delta_{a_1,b_1}v^{a_1}\hat{u}^{a_1,b_1} = \delta_{a_2,b_2}v^{a_2}\hat{u}^{a_2,b_2}$ and $\delta_{a_1,a_2}v^{a_1}\hat{u}^{a_1,a_2} = \delta_{a_2,a_3}v^{a_2}\hat{u}^{a_2,a_3}$, so our equations imply that $\delta_{a_1,a_2}v^{a_2}\hat{u}^{a_1,a_2} = 0$. Since $v^{a_2}\hat{u}^{a_1,a_2} \neq 0$, this means that $\delta_{a_1,a_2} = 0$ and thus, all these δ's are 0. Finally, since $\hat{u}^{i,j}$ is a non-zero vector and $\delta_{a_x,b_x} = \delta_{a_y,b_y} \geq 0$ for all $x, y \in \{1, \dots, t\}$, this inequality must be strict and the lemma follows by choosing $\delta = \delta_1$. □

Finally, we are now ready to prove Theorem 2.

Theorem 2. *An ABC voting rule is a BSWAV rule if and only if it satisfies anonymity, neutrality, consistency, continuity, choice set convexity, and weak efficiency.*

Proof. First, the direction from left to right has been shown in the main body. Moreover, if $k \in \{1, m-1\}$, then the set of BSWAV rules is equal to the set of ABC scoring rules and choice set convexity becomes trivial. Hence, the theorem follows from Proposition 1. Next, assume that $f \in \mathcal{F}^1$ for $k \in \{2, \dots, m-2\}$. If f is trivial, it is clearly the BSWAV rule induced by the weights $\alpha_x = 0$ for all x. On the other hand, if f is non-trivial, Lemma 7 holds and consistency therefore entails that f is non-imposing. Hence, we can access all our auxiliary lemmas now. In particular, our goal is to show that f is the BSWAV rule described by the weights α_r constructed in Lemma 8. For doing so, we define the score function $s(|B(\ell) \cap W^i|, |B(\ell)|) = \alpha_{|B(\ell)|}|B(\ell) \cap W^i|$ and extend it to vectors $v \in \mathbb{R}^{|\mathcal{A}|}$ by $\hat{s}(v, W^i) = \sum_{1,\dots,|\mathcal{A}|\}} v_\ell s(|B(\ell) \cap W^i|, |B(\ell)|)$. Departing from here, our proof proceeds in two steps. First, we show that there is for all committees W^i, W^j a constant $\delta > 0$ such that$\delta\hat{u}^{i,j}_\ell = s(|B(\ell) \cap W^i|, |B(\ell)|) - s(|B(\ell) \cap W^j|, |B(\ell)|)$ for all ballots $B(\ell)$. Based on this insight, we show in the second step that $f(A) \subseteq f'(A) := \{W^i \in \mathcal{W}_k : \forall W^j \in \mathcal{W}_k : \hat{s}(v(A), W^i) \geq \hat{s}(v(A), W^j)\}$. Finally, we turn this subset relation in an equality in the last step and prove that f is a BSWAV rule.

Step 1: There is $\delta > 0$ such that $\delta\hat{u}^{i,j}_\ell = s(|B(\ell) \cap W^i|, |B(\ell)|) - s(|B(\ell) \cap W^j|, |B(\ell)|)$ for all $B(\ell)$.

For this step, consider two arbitrary committees $W^i, W^j \in \mathcal{W}_k$ and let $B(\ell) \in \mathcal{A}$ denote a ballot. Moreover, let $r = |B(\ell)|$ denote size of $B(\ell)$ and define $W^i \setminus W^j = \{a_1, \dots, a_t\}$, $W^j \setminus W^i = \{b_1, \dots, b_t\}$. By the definition of s, we have that $s(|B(\ell) \cap W^i|, |B(\ell)|) - s(|B(\ell) \cap W^j|, |B(\ell)|)$. On the other hand, we know by Lemma 12 that $\hat{u}^{i,j} = \delta' \sum_{x=1}^{t} \hat{u}^{a_x,b_x}$ for some $\delta' > 0$. Hence, this step follows by showing that $\sum_{x=1}^{t} \hat{u}^{a_x,b_x}_\ell = \alpha_r(|B(\ell) \cap W^i| - |B(\ell) \cap W^j|)$.

For doing so, we partition the the indices $I = \{1, \dots, t\}$ into four sets: $I_1 = \{x \in I : a_x, b_x \in B(\ell)\}$, $I_2 = \{x \in I : a_x, b_x \notin B(\ell)\}$, $I_3 = \{x \in I : a_x \in B(\ell), b_x \notin B(\ell)\}$, and $I_4 = \{x \in I : a_x \notin B(\ell), b_x \in B(\ell)\}$. Now, by Lemma 6, we know that $\hat{u}^{a_x,b_x}_\ell = 0$ for all $x \in I_1 \cup I_2$ as $\hat{u}^{a_x,b_x}_\ell = \hat{u}^{i',j'}_\ell = s^1_r(|B(\ell) \cap W^{i'}|, |B(\ell) \cap W^{j'}|) = 0$ for two arbitrary committees $W^{i'}, W^{j'} \in \mathcal{W}_k$ with $W^{i'} \setminus W^{j'} = \{a_x\}$ and $W^{j'} \setminus W^{i'} = \{b_x\}$. Moreover, a similar reasoning and Lemma 8 show that $\hat{u}^{a_x,b_x}_\ell = \alpha_r$ for all $x \in I_3$ and $\hat{u}^{a_x,b_x}_\ell = -\alpha_r$ for all $x \in I_4$. We thus have that $\sum_{x=1}^{t} \hat{u}^{a_x,b_x}_\ell = \alpha_r|I_3| - \alpha_r|I_4| = \alpha_r(|B(\ell) \cap \{a_1, \dots, a_t\}| - |B(\ell) \cap \{b_1, \dots, b_t\}|) = \alpha_r(|B(\ell) \cap W^i| - |B(\ell) \cap W^j|)$, thus proving our claim.

Step 2: $f(A) \subseteq f'(A)$ for all $A \in \mathcal{A}^*$.

For proving this claim, we recall the function $\hat{g}$ and the sets $\bar{R}^f_i$ defined in and after Lemma 1. In particular, by the definition of these objects, we have that $f(A) = \hat{g}(v(A)) = \{W^i \in \mathcal{W}_k : v(A) \in \bar{R}^f_i\} \subseteq \{W^i \in \mathcal{W}_k : v(A) \in \bar{R}^f_i\}$. Hence, the claim follows by showing that $f'(A) = \{W^i \in \mathcal{W}_k : \forall W^j \in \mathcal{W}_k : \hat{s}(v(A), W^i) \geq \hat{s}(v(A), W^j)\} = \{W^i \in \mathcal{W}_k : v(A) \in \bar{R}^f_i\}$. For doing this, we note that $\bar{R}^f_i$ can be represented as $\bar{R}^f_i = \{v \in \mathcal{W}_k : \forall j \in \{1, \dots, \mathcal{W} \setminus \{i\} : v\hat{u}^{i,j} \geq 0\}$ (Claim (1) of Lemma 2). Hence, to show our equivalence, we need to prove that $\hat{s}(v(A), W^i) \geq \hat{s}(v(A), W^j)$ if and only if $v(A)\hat{u}^{i,j} \geq 0$ for every profile A and committees W^i, W^j.

To do so, let consider an arbitrary profile $A \in \mathcal{A}^*$ and let W^i, W^j denote two arbitrary committees. By the last step, we have a $\delta > 0$ such that $\delta\hat{u}^{i,j}_\ell = s(|B(\ell) \cap W^i|, |B(\ell)|) - s(|B(\ell) \cap W^j|, |B(\ell)|)$ for all ballots $B(\ell)$. Hence, it is easy to see that $\delta v(A)\hat{u}^{i,j} = \sum_{x \in \{1,\dots,|\mathcal{A}|\}} v(A)_\ell(s(|B(\ell) \cap W^i|, |B(\ell)|) - s(|B(\ell) \cap W^j|, |B(\ell)|)) = \hat{s}(v(A), W^i) - \hat{s}(v(A), W^j)$, which clearly implies our claim.

Step 3: $f(A)$ is a BSWAV rule.

For this step, we show that $f(A) = f'(A)$ and that $f'(A)$ is a BSWAV rule. For the latter point, we only need to observe that all α_r are non-negative. Assume for contradiction that this is not the case. Then, there is a ballot size r such that $\alpha_r < 0$ and $f'(A) = \{W \in \mathcal{W}_k : \forall W' \in \mathcal{W}_k : |A \cap W| \leq |A \cap W'|\}$ for an arbitrary ballot A with $|A| = r$. Since f chooses a subset of $f'(A)$, f violates weak efficiency as we cannot decrease the number of unapproved candidates in any committee. This contradicts our assumptions, so $\alpha_r \geq 0$ and f' is a BSWAV rule by definition.

Next, to show that $f(A) = f'(A)$, we assume for contradiction that there is a profile A' for which this is not the case. This means that there is a committee $W \in f'(A) \setminus f(A)$. Moreover, since f' is a BSWAV rule, it satisfies all preconditions of Lemma 7. So, we can infer analogously as for f that f' is non-imposing and consistent. Now, let A' denote a profile with $f'(A') = \{W\}$. By consistency of f' and the subset relation of the last step, we have that $f(\lambda A + A') = f'(\lambda A + A') = \{W\}$ for all $\lambda \in \mathbb{N}$. However, this contradicts the continuity of f and thus our assumption that $f'(A) \neq f(A')$ is wrong. Hence, f is indeed the BSWAV rule induced by α_r. □

A.3 Proof of Theorem 1

In this section, we will prove our characterization of Thiele rule (Theorem 1). We focus here on showing that every rule that satisfies anonymity, neutrality, consistency, continuity, and independence of losers is indeed a Thiele; the other direction can be found in the main body. To prove this claim, we essentially follow the same steps as for the proof of Theorem 2 and a detailed proof sketch is given in the main body. Moreover for a short notation, we define $\mathcal{F}^2$ as the set of all ABC voting rules that satisfy anonymity, neutrality, consistency, continuity, and independence of losers.

For the first step in our analysis, we consider again the profiles $A^{x,r}$ and $A^{-x,r}$ in which all ballots A of size r with $x \in A$ and $x \notin A$, respectively, are reported once.

Lemma 13. *Let $f \in \mathcal{F}^2$ denote a non-trivial ABC voting rule. There is a ballot size r such that $f(A^{x,r}) = \{W \in \mathcal{W}_k : x \in W\}$ and $f(A^{-x,r}) = \{W \in \mathcal{W}_k : x \notin W\}$ for all $x \in C$.*

Proof. Let $f \in \mathcal{F}^2$ denote a non-imposing ABC voting rule. To show this lemma, we will find a ballot size r such that $f(A^{-x,r}) = \{W \in \mathcal{W}_k : x \notin W\}$ for all $x \in C$. This implies that $f(A^{x,r}) = \{W \in \mathcal{W}_k : x \in W\}$ because of the following reasoning. First, anonymity and neutrality entail that $f(A^{x,r} + A^{-x,r}) = \mathcal{W}_k$ as the profile $A^{x,r}+A^{-x,r}$ contains all ballots of size r. Hence, consistency requires that either $f(A^{x,r}) = f(A^{-x,r}) = \mathcal{W}_k$ or $f(A^{x,r}) \cap f(A^{-x,r}) = \emptyset$. In particular, if $f(A^{-x,r}) = \{W \in \mathcal{W}_k : x \notin W\}$, then $f(A^{x,r}) \subseteq \{W \in \mathcal{W}_k : x \in W\}$. Finally, by applying again anonymity and neutrality to $A^{x,r}$, it is easy to see that this subset relation must be an equality.

Now, for finding the index r such that $f(A^{-x,r}) = \{W \in \mathcal{W}_k : x \notin W\}$, we first investigate the choice of f for single ballots. By neutrality, it holds that $f(C) = \mathcal{W}_k$ for the ballot C in which all candidates are approved. Next, consider an arbitrary candidate $x \in C$. By independence of losers, we know that $\{W \in \mathcal{W}_k : x \notin W\} \subseteq f(C \setminus \{x\})$. Invoking again neutrality, we derive that there are only two possible outcomes for the ballot $C \setminus \{x\}$: either $f(C \setminus \{x\}) = \{W \in \mathcal{W}_k : x \notin W\}$ or $f(C \setminus \{x\}) = \mathcal{W}_k$. If the former is the case, the profile only containing the ballot $C \setminus \{x\}$ constitutes $A^{-x,r}$ for $r = m - 1$ and we are done.

Hence, suppose that $f(C \setminus \{x\}) = \mathcal{W}_k$. By neutrality, the same holds for every ballot of size $m-1$. In this case, we can simply repeat deleting candidates from $C \setminus \{x\}$ until we arrive at a ballot $C \setminus X$ such that $f(C \setminus X) \neq \mathcal{W}_k$; such a ballot must exist as consistency otherwise implies that f is trivial. Moreover, we suppose that the set of deleted candidates X is minimal. By neutrality, this means that $f(C \setminus Y) = \mathcal{W}_k$ for every set of candidates Y with $|Y| < |X|$. Now, let y denote an arbitrary candidate in X and define $X^y = X \setminus \{y\}$. By our previous insight, $f(C \setminus X^y) = \mathcal{W}_k$ and independence of losers then shows that $\{W \in \mathcal{W}_k : y \notin W\} \subseteq f(C \setminus X)$. Since y is chosen arbitrarily, we can apply this argument for every $y \in X$ and derive that $\{W \in \mathcal{W}_k : X \not\subseteq W\} \subseteq f(C \setminus X)$. Moreover, since $f(C \setminus X) \neq \mathcal{W}_k$ and f is neutral, we infer that this subset relation must actually be an equality. Also, since $f(C \setminus X) \neq \mathcal{W}_k$, we get that $|X| \leq k$ because otherwise the set $\{W \in \mathcal{W}_k : X \not\subseteq W\}$ contains all committees.

Now, fix a candidate $x \in C$ and consider $A^{-x,r}$ for $r = m - |X|$. This profile contains each ballot $C \setminus Y$ with $|Y| = |X|$ and $x \in Y$ once. By neutrality, we have for each of these ballots that $f(C \setminus Y) = \{W \in \mathcal{W}_k : Y \not\subseteq W\}$. In particular, since $x \in Y$ for all considered ballots, it holds that $\{W \in \mathcal{W}_k : x \notin W\} \subseteq f(C \setminus Y)$ for all ballots in $A^{-x,r}$. Hence, consistency applies for $A^{-x,r}$ and shows that $f(A^{-x,r}) = \bigcap_{Y \subseteq C : |Y|=|X| \land x \in Y} f(C \setminus Y)$. This means that $W \notin f(A^{-x,r})$ for all committees $W \in \mathcal{W}_k$ with $x \in W$ as there is a set Y with $x \in Y$ and $|Y| = |X|$ such that $Y \subseteq W$. Conversely, it immediately follows from consistency that $W \in f(A^{-x,r})$ for all committees $W \in \mathcal{W}_k$ with $x \notin W$. Hence, we have that $f(A^{-x,r}) = \{W \in \mathcal{W}_k : x \notin W\}$, which concludes the proof of the lemma. □

Since the ballot size r for the profiles $A^{x,r}$ and $A^{-x,r}$ plays no role in our subsequent analysis, we will omit it and mean by A^x and A^{-x} the profiles $A^{x,r}$ and $A^{-x,r}$ for the ballot size r given by Lemma 13.

Moreover, based on Lemma 13, it follows straightforwardly that every non-trivial ABC voting rule $f \in \mathcal{F}^2$ is non-imposing. For this, it suffices to consider the profile A^W that consists of a copy of A^x for every candidate $x \in W$ as consistency then ensures that $f(A^W) = \{W\}$. Since the trivial rule is clearly the Thiele rule defined by $s(x) = 0$ for all x, we therefore focus from now on non-imposing rules. This allows us to use the vectors $\hat{u}^{i,j}$ constructed in Lemma 2 and the functions s_r^1 constructed in Lemma 6. As the next step, we use independence of losers to remove the dependence on the ballot size of the functions s_r^1.

Lemma 14. *Let $f \in \mathcal{F}^2$ denote a non-imposing ABC voting rule. There is a function $s^1(x, y)$ such that $s^1(|W^i \cap B(\ell)|, |W^j \cap B(\ell)|) = \hat{u}_\ell^{i,j}$ for all committees $W^i, W^j \in \mathcal{W}_k$ and ballots $B(\ell) \in \mathcal{A}$ with $|W^i \setminus W^j| = 1$.*

Proof. Let $f \in \mathcal{F}^2$ denote a non-imposing ABC voting rule and consider two committees $W^i, W^j \in \mathcal{W}_k$ with $|W^i \setminus W^j| = 1$. Now, by Lemma 6, there are functions s_r^1 such that $\hat{u}_\ell^{i,j} = s_{|B(\ell)|}(|W^i \cap B(\ell)|, |W^j \cap B(\ell)|)$ for all $B(\ell) \in \mathcal{A}$. For proving this lemma, it thus suffices to show that $s_{|B(\ell)|}(|W^i \cap B(\ell)|, |W^j \cap B(\ell)|) = s_{|B(\ell')|}(|W^i \cap B(\ell')|, |W^j \cap B(\ell')|)$ for any two ballots $B(\ell), B(\ell')$ with $|W^i \cap B(\ell)| = |W^i \cap B(\ell')|$ and $|W^j \cap B(\ell)| = |W^j \cap B(\ell')|$. If $|B(\ell)| = |B(\ell')|$, this follows from the definition of the functions s_r^1. Moreover, if $|W^i \cap B(\ell)| = |W^i \cap B(\ell')| = |W^j \cap B(\ell)| = |W^j \cap B(\ell')|$, then Claim (2) of Lemma 6 shows that $s_{|B(\ell)|}(|W^i \cap B(\ell)|, |W^j \cap B(\ell)|) = s_{|B(\ell')|}(|W^i \cap B(\ell')|, |W^j \cap B(\ell')|) = 0$.

Hence, we suppose that $|B(\ell)| \neq |B(\ell')|$ and $|W^i \cap B(\ell)| \neq |W^j \cap B(\ell)|$. Without loss of generality, we make this more precise by assuming that $|B(\ell)| > |B(\ell')|$ and $|W^i \cap B(\ell)| = |W^j \cap B(\ell)| + 1$. In particular, the latter observation means that $a \in B(\ell) \cap B(\ell')$ and $b \notin B(\ell) \cup B(\ell')$ for the candidates $\{a\} = W^i \setminus W^j$ and $\{b\} = W^j \setminus W^i$. By this insight, it it easy to see that there is a permutation τ such that $B(\ell'') = \tau(B(\ell')) \subseteq B(\ell)$, $|W^i \cap B(\ell)| = |W^i \cap B(\ell'')|$ and $|W^j \cap B(\ell)| = |W^j \cap B(\ell'')|$. Moreover, it holds that $s_{|B(\ell')|}(|W^i \cap B(\ell')|, |W^j \cap B(\ell')|) = s_{|B(\ell'')|}(|W^i \cap B(\ell'')|, |W^j \cap B(\ell'')|)$, so it suffices to show that $s_{|B(\ell'')|}(|W^i \cap B(\ell'')|, |W^j \cap B(\ell'')|) = s_{|B(\ell)|}(|W^i \cap B(\ell)|, |W^j \cap B(\ell)|)$.

For this, consider the profile A in which all ballots are reported once. Clearly, $f(A) = \mathcal{W}_k$ by anonymity and neutrality. Next, let A' denote the profile derived from A by replacing the ballot $B(\ell)$ with $B(\ell'')$. Since $B(\ell'') \subseteq B(\ell)$, $|W^i \cap B(\ell)| = |W^i \cap B(\ell'')|$, and $|W^j \cap B(\ell)| = |W^j \cap B(\ell'')|$, this means that we only disapproved candidates $x \in C \setminus (W^i \cup W^j)$. So, independence of losers implies that $W^i, W^j \in f(A')$. Now, consider the vector $\hat{u}^{i,j}$ given by Lemma 2. By Claim (1) of this lemma, we must have that $v(A)\hat{u}^{i,j} = v(A')\hat{u}^{i,j} = 0$ because $v(A), v(A') \in \bar{R}_i^f$ and $v(A), v(A') \in \bar{R}_j^f$.

Hence, $v(A)\hat{u}^{i,j} - v(A')\hat{u}^{i,j} = \hat{u}^{i,j}_{\ell} - \hat{u}^{i,j}_{\ell''} = 0$. This implies that $s_{|B(\ell)|}(|W^i \cap B(\ell)|, |W^j \cap B(\ell)|) = \hat{u}^{i,j}_{\ell} = \hat{u}^{i,j}_{\ell''} = s_{|B(\ell'')|}(|W^i \cap B(\ell'')|, |W^j \cap B(\ell'')|)$, thus proving the lemma. □

We note that the function s^1 clearly inherits the symmetry properties of the functions s^1_r discussed in Claims (2) and (3) of Lemma 6.

Now, it follows essentially from Lemmas 13 and 14 as well as the proof of Proposition 1 that all rules in $\mathcal{F}^2$ are Thiele rule if $k = 1$ or $k = m - 1$. We thus focus on the case that $2 \leq k \leq m - 2$. To show this case, we need to relate the vectors $\hat{u}^{i,j}$ and $\hat{u}^{i',j'}$ to each other for committees $W^i, W^j, W^{i'}, W^{j'}$ with $|W^i \setminus W^j| \neq |W^{i'} \setminus W^{j'}|$. For doing so, we consider a sequence of sequence of committees $W^{j_0}, \dots, W^{j_t}$ such that $|W^{j_0} \setminus W^{j_t}| = t$ and $|W^{j_{x-1}} \setminus W^{j_x}| = 1$ for every $x \in \{1, \dots, t\}$. Our goal is then to show that $\hat{u}^{j_0,j_t}$ is essentially the sum over the vectors $\hat{u}^{j_{x-1},j_x}$. To this end, we proceed analogously as in Appendix A.2 and first show that the vectors $\{\hat{u}^{j_0,j_1}, \dots, \hat{u}^{j_{t-1},j_t}\}$ are linearly independent.

Lemma 15. *Suppose $2 \leq k \leq m - 2$ and let $f \in \mathcal{F}^2$ denote a non-imposing ABC voting rule. Moreover, consider a sequence of committees $W^{j_0}, \dots, W^{j_t}$ for $t \geq 2$ such that $|W^{j_0} \setminus W^{j_t}| = t$ and $|W^{j_{x-1}} \setminus W^{j_x}| = 1$ for all $x \in \{1, \dots, t\}$. The vectors $\hat{u}^{j_0,j_1}, \hat{u}^{j_1,j_2}, \dots, \hat{u}^{j_{t-1},j_t}$ are linearly independent.*

Proof. Let $f \in \mathcal{F}^2$ denote a non-imposing ABC voting rule and consider a sequence of committees $W^{j_0}, \dots, W^{j_t}$ as specified by the lemma. The conditions that $|W^{j_0} \setminus W^{j_t}| = t$ and $|W^{j_{x-1}} \setminus W^x| = 1$ for $x \in \{1, \dots, t\}$ means that when moving from $W^{j_{x-1}}$ to W^{j_x}, we need to exchange a candidate $a \in W^{j_{x-1}} \cap (W^{j_0} \setminus W^{j_t})$ with a candidate $b \in W^{j_x} \cap (W^{j_t} \setminus W^{j_0})$. Hence, we can write each committee in this sequence as $W^{j_x} = \{b_1, \dots, b_x, a_{x+1}, \dots, a_t, c_{t+1}, \dots, c_k\}$. In particular, $W^{j_0} \setminus W^{j_t} = \{a_1, \dots a_t\}$, $W^{j_t} \setminus W^{j_0} = \{b_1, \dots, b_t\}$, and $W^{j_0} \cap W^{j_t} = \{c_{t+1}, \dots, c_k\}$.

For showing that the vectors $\hat{u}^{j_{x-1},j_x}$ for $x \in \{1, \dots, t\}$ are linearly independent, we consider the matrix M whose x-th row corresponds to the vector $\hat{u}^{j_{x-1},j_x}$ for $x \in \{1, \dots, t\}$. In more detail, we will show that the image of M has full dimension, i.e., $\{w \in \mathbb{R}^t : \exists v \in \mathbb{R}^{|\mathcal{A}|} : Mv = w\} = \mathbb{R}^t$. This suffices to prove the lemma because the image dimension of a matrix is equivalent to its rank, which is equivalent to the number of linearly independent rows. To this end, we consider the profiles A^x constructed in Lemma 13. In more detail, we claim that the vectors $w = Mv(A^{a_x})$ satisfy $w_x \neq 0$ and $w_y = 0$ for $y \neq x$, which implies that the image of M has full dimension.

To prove this claim, we consider first two indices $x, y \in \{1, \dots, t\}$ with $x \neq y$. Now, let τ denote the permutation defined by $\tau(a_y) = b_y$, $\tau(b_y) = a_y$, and $\tau(z) = z$ for all other candidates. It is easy to see $\tau(v(A^{a_x})) = v(A^{a_x})$ due to the symmetry of this profile and $\tau(\hat{u}^{j_{y-1},j_y}) = \hat{u}^{j_y,j_{y-1}} = -\hat{u}^{j_{y-1},j_y}$ because of Lemma 2. Hence, it follows that $v(A^x)\hat{u}^{j_{y-1},j_y} = \tau(v(A^x))\tau(\hat{u}^{j_{y-1},j_y}) = -v(A^x)\hat{u}^{j_{y-1},j_y}$, which is only possible if $v(A^x)\hat{u}^{j_{y-1},j_y} = 0$.

Finally, for showing that $v(A^{a_x})\hat{u}^{j_{x-1},j_x} \neq 0$, we consider a committee W^i with $a_x \in W^i$, $b_x \notin W^i$ and define A as the profile that contains a copy of A^z for every $z \in W^i$. By an analogous argument as in the last paragraph, we infer that $v(A)\hat{u}^{j_{x-1},j_x} = \sum_{z \in W^i} v(A^z)\hat{u}^{j_{x-1},j_x} = v(A^{a_x})\hat{u}^{j_{x-1},j_x}$. On the other hand, consistency and Lemma 13 show that $f(A) = \{W^i\}$. Moreover, continuity implies that there is a $\lambda \in \mathbb{N}$ such that $f(\lambda A + B(\ell)) = \{W^i\}$ for all ballots $B(\ell)$, so we can infer that $v(A) \in \text{int}\bar{R}^f_i$. This means that $v(A)\hat{u}^{j_x,j_{x-1}} > 0$ by Claim (1) of Lemma 2 and thus, $v(A)\hat{u}^{j_x,j_{x-1}} = v(A^{a_x})\hat{u}^{j_{x-1},j_x} < 0$. □

Our next goal is to show that the linear independence observed in Lemma 15 turns into a linear dependence if we add the vector $\hat{u}^{j_0,j_t}$ to the set. For this, we first show an auxiliary claim.

Lemma 16. *Assume $2 \leq k \leq m$ and let $f \in \mathcal{F}^2$ denote a non-imposing ABC voting rule. Moreover, consider an arbitrary sequence of committees $W^{j_0}, \dots, W^{j_t} \in \mathcal{W}_k$ such that $|W^{j_0} \setminus W^{j_t}| = t$ and $|W^{j_{x-1}} \setminus W^{j_x}| = 1$ for all $x \in \{1, \dots, t\}$. Let $B(\ell)$ be any ballot such that $|W^{j_0} \cap B(\ell)| \leq |W^{j_t} \cap B(\ell)|$. It holds that $\sum_{x=1}^t s^1(|B(\ell) \cap W^{j_{x-1}}|, |B(\ell) \cap W^{j_x}|) = \sum_{x=|W^{j_0} \cap B(\ell)|+1}^{|W^{j_t} \cap B(\ell)|} s^1(x-1, x)$.*

Proof. Let $f \in \mathcal{F}^2$ denote an non-imposing ABC voting rule and let the committees $W^{j_0}, \dots, W^{j_t}$ and the ballot $B(\ell)$ be defined as in the lemma. Furthermore, let $i^x = |B(\ell) \cap W^{j_x}|$ be the x-th intersection size and consider the sum $\sum_{x=1}^t s^1(i^{x-1}, i^x)$. If $i^{x-1} = i^x$, then $s^1(i^{x-1}, i^x) = 0$. Hence, we can shorten the sum by removing all such terms without affecting the sum. More rigorously, we define $y_x = 0$ and $\hat{i}^x = i^x$ for $x = 0$, and $\hat{i}^{x+1} = i^{y_{x+1}}$ where y_{x+1} is the smallest integer $y > y_x$ such that $i^y \neq i^{y_x}$. Moreover, let $\hat{t}$ denote the length of this new sequence and observe that $\hat{i}^x = \hat{i}^{x+1} - 1$ or $\hat{i}^x = \hat{i}^{x+1} + 1$ for all $x \in \{0, \dots, \hat{t} - 1\}$. Furthermore, by definition, $\sum_{x=1}^t s^1(i^{x-1}, i^x) = \sum_{x=1}^{\hat{t}} s^1(\hat{i}^{x-1}, \hat{i}^x)$.

Next, we suppose that $\hat{t} > |W^{j_t} \cap B(\ell)| - |W^{j_0} \cap B(\ell)|$. In this case, it is straightforward to see that there must be an index $\hat{i}^x$ such that $\hat{i}^x = \hat{i}^{x+2}$. By Lemma 6, we thus have that $s_1(i^x, i^{x+1}) = -s_1(i^{x+1}, i^x) = -s_1(i^{x+1}, i^{x+2})$ and we can hence remove these two terms from our sum. Clearly, we can then compress our indices again and repeat the argument until we have only $\bar{t} \leq |W^{j_t} \cap B(\ell)| - |W^{j_0} \cap B(\ell)|$ intersection sizes left. Let $\bar{i}^0, \dots, \bar{i}^{\bar{t}}$ denote this reduced set and note that $|\bar{i}^x - \bar{i}^{x+1}| = 1$ for all x. Moreover, it is not difficult to see that $\bar{i}^0 = i^0$ and $\bar{i}^t = i^t$, so we have that $\bar{i}^{x-1} = \bar{i}^x - 1$ for all $x \in \{1, \dots, \bar{t}\}$ and $\bar{t} = |W^{j_t} \cap B(\ell)| - |W^{j_0} \cap B(\ell)|$. Finally, since we only remove terms that sum up to 0, it clearly holds that $\sum_{x=1}^t s^1(i^{x-1}, i^x) = \sum_{x=1}^{\bar{t}} s^1(\bar{i}^{x-1}, i^x)s^1(\bar{i}^{x-1}, i^x) = \sum_{x=|W^{j_0} \cap B(\ell)|+1}^{|W^{j_t} \cap B(\ell)|} s^1(x-1, x)$, thus proving the lemma. □

Based on Lemmas 15 and 16, we will now show that the vector $\hat{u}^{j_0,j_t}$ can be represented as (scaled) sum of the vectors $\hat{u}^{j_0,j_1}, \hat{u}^{j_1,j_2}, \dots, \hat{u}^{j_{t-1},j_t}, \hat{u}^{j_0,j_t}$ are linearly dependent.

Lemma 17. *Suppose $2 \leq k \leq m - 2$ and let $f \in \mathcal{F}^2$ denote a non-imposing ABC voting rule. Moreover, consider an arbitrary sequence of committees $W^{j_0}, \dots, W^{j_t} \in \mathcal{W}_k$ such that $|W^{j_0} \setminus W^{j_t}| = t$ and $|W^{j_{x-1}} \setminus W^{j_x}| = 1$ for all $x \in \{1, \dots, t\}$. There is a $\delta > 0$ such that $\hat{u}^{j_0,j_t} = \delta \sum_{x=1}^t \hat{u}^{j_{x-1},j_x}$.*

Proof. Let $f \in \mathcal{F}^2$ denote an non-imposing ABC voting rule and consider a sequence of committees $W^{j_0}, \dots, W^{j_t} \in \mathcal{W}_k$ as stated by the lemma. Moreover, we define $W^{j_0} \setminus W^{j_t} = \{a_1, \dots, a_t\}$, $W^{j_t} \setminus W^{j_0} = \{b_1, \dots, b_t\}$, and $W^{j_x} = (W^{j_0} \cap W^{j_t}) \cup \{b_1, \dots, b_x, a_{x+1}, \dots, b_t\}$. Next, we consider the function $s^1(x+1, x)$ derived in Lemma 14. First, if this function is constant, then we can use the same arguments as in Appendix A.2. In particular, note here that we show in Lemma 8 an analogous claim and that the subsequent lemmas (Lemmas 9 to 12) do not rely on choice set convexity or weak efficiency.

We hence suppose that there is a index $p \geq 1$ such $s(p+1,p) \neq s(p,p-1)$. Moreover, we define p as minimal such value and let $\alpha = s^1(1,0)$. In particular, we have that $s^1(x+1,x) = \alpha$ for all $x < p$. In this case, we will prove the lemma by an induction on the length of the considered sequence. First, if $t = 1$, the statement is trivial and the induction basis thus holds. Hence, we aim to show the lemma for $t > 1$ and suppose that there is a δ' with $\hat{u}^{i_0,i_{t'}} = \delta' \sum_{x=1}^{t'} \hat{u}^{i_{x-1},i_x}$ for all sequences of committees $W^{i_0}, \dots, W^{i^{t'}}$ with $|W^{i_0} \setminus W^{i_{t'}}| = t' < t$ and $|W^{i_{x-1}} \setminus W^{i_x}| = 1$ for all $x \in \{1, \dots, t'\}$. To prove the induction step for our sequence $W^{j_0}, \dots, W^{j_t}$, we proceed in multiple steps. In our first four steps (Steps 1.1 to 1.4), we will show that the vectors $\{\hat{u}^{j_0,j_1}, \hat{u}^{j_1,j_2}, \dots, \hat{u}^{j_{t-1},j_t}, \hat{u}^{j_0,j_t}\}$ are linearly dependent. Based on Lemma 15, it thus follows that there are coefficients δ_x, not all of which are 0, such that $\hat{u}^{j_0,j_t} = \sum_{x=1}^{t} \delta_x \hat{u}^{j_{x-1},j-x}$. In the last step (Step 2), we then show that all δ_x are the same and greater 0, thus proving the lemma.

Step 1: The vectors $\{\hat{u}^{j_0,j_1}, \hat{u}^{j_1,j_2}, \dots, \hat{u}^{j_{t-1},j_t}, \hat{u}^{j_0,j_t}\}$ are linearly independent.

Assume for contradiction that the given vectors are linearly independent. To derive a contradiction to this assumption, we will construct a vector v^4 that is not contained in $\bar{R}_i^f$ for any $W^i \in \mathcal{W}_k$. Consequently, $\bigcup_{x \in \{1,\dots,|\mathcal{W}_k|\}} \bar{R}_i^f \neq \mathbb{R}^{|\mathcal{A}|}$. Just as for Lemma 11, this contradicts the insight that $\bigcup_{x \in \{1,\dots,|\mathcal{W}_k|\}} R_i^f = \mathbb{Q}^{|\mathcal{A}|}$ and therefore $\bigcup_{x \in \{1,\dots,|\mathcal{W}_k|\}} \bar{R}_i^f = \mathbb{R}^{|\mathcal{A}|}$. So, the assumption that the set $\{\hat{u}^{j_0,j_1}, \hat{u}^{j_1,j_2}, \dots, \hat{u}^{j_{t-1},j_t}, \hat{u}^{j_0,j_t}\}$ is linearly independent must have been wrong. For constructing v^4, we will step by step narrow down the choice set.

Step 1.1: For our first step, let $\mathcal{W}^1 = \{W \in \mathcal{W}^k \colon W^i \cap W^j \subseteq W \subseteq W^i \cup W^j\}$, i.e., $\mathcal{W}^1$ is the convex hull of W^i and W^j. We will construct a vector v^1 such that for every committee $W^i \notin \mathcal{W}^1$, there is another committee W^j such that $v^1\hat{u}^{i,j} < 0$. This shows that $v^1 \notin \bar{R}_i^f$ for these committees by Claim (1) of Lemma 2.

For constructing this vector, we recall the profiles A^x and A^{-x} constructed in Lemma 13. First, we note that $v(A^x)\hat{u}^{i,j} = v(A^{-x})\hat{u}^{',j} = 0$ for all committees W^i, W^j with $|W^i \setminus W^j| = 1$, $\{x\} \neq \{a\} = W^i \setminus W^j$, and $\{x\} \neq \{b\} = W^j \setminus W^i$. For showing this claim, considering the permutation τ with $\tau(a) = b$, $\tau(b) = a$, and $\tau(z) = z$ for all remaining candidates. It is now easy to verify that $v(A^x)\hat{u}^{i,j} = \tau(v(A^x))\tau(\hat{u}^{i,j}) = v(A^x)\hat{u}^{j,i} = -v(A^x)\hat{u}^{i,j}$ due to the symmetry of A^x and Lemma 2. This is only possible if $v(A^x)\hat{u}^{i,j} = 0$ and an analogous argument also shows our claim for A^{-x}. Furthermore, it holds that $v(A^x)\hat{u}^{i,j} > 0$ and $v(A^{-x})\hat{u}^{i,j} < 0$ for all committees W^i, W^j with $W^i \setminus W^j = \{x\}$. For showing this, consider the profile A that consists of a copy of $A^x$for every $x \in W^i$ (the claim for A^{-x} works analogously by considering the profile consisting of A^{-x} for $x \notin W^i$). By consistency, $f(A) = \{W^i\}$, and by continuity, we infer that $v(A) \in \operatorname{int} \bar{R}_i^f$. Hence, by Claim (1) of Lemma 2, $v(A)\hat{u}^{i,j} > 0$. On the other hand, we have that $v(A)\hat{u}^{i,j} = \sum_{z \in W^i} v(A^z)\hat{u}^{i,j} = v(A^x)\hat{u}^{i,j}$ as $v(A^z)\hat{u}^{i,j} = 0$ for all $z \in W^i \cap W^j$. Combining these insights shows that $v(A^x)\hat{u}^{i,j} > 0$.

Now, for completing this step, we define A^1 as the profile that contains a copy of A^x for $x \in W^{j_0} \cap W^{j_t}$ and a copy of A^{-x} for $x \in C \setminus (W^{j_0} \cup W^{j_t})$. By consistency, it is easy to infer that $f(A^1) = \mathcal{W}^1$. Hence, Claim (1) of Lemma 2 shows for $v^1 = v(A^1)$ and $W^i, W^j \in \mathcal{W}^1$ that $v^1\hat{u}^{i,j} = 0$. Next, consider a committee $W^i \notin \mathcal{W}^1$. This means that there is a pair of candidates $a \in W^i, b \notin W^i$ such that $a \in C \setminus (W^{j_0} \cap W^{j_t})$ and $b \in W^{j_0} \cap W^{j_t}$, or $a \in C \setminus (W^{j_0} \cup W^{j_t})$ and $b \in W^{j_0} \cup W^{j_t}$. In both cases, it follows from our previous analysis that $v^1\hat{u}^{i,j} < 0$ for the committee W^j defined by $W^j = (W^i \setminus \{a\}) \cup \{b\}$. For instance, if $a \in C \setminus (W^{j_0} \cup W^{j_t}) \subseteq C \setminus (W^{j_0} \cap W^{j_t})$, $b \in W^{j_0} \cap W^{j_t}$, then $v^1\hat{u}^{i,j} = v(A^{-a})\hat{u}^{i,j} + v(A^b)\hat{u}^{i,j} = v(A^{-a})\hat{u}^{i,j} - v(A^b)\hat{u}^{j,i} < 0$. This completes this step.

Step 1.2: For our second step, let $\mathcal{W}^2 = \{W \in \mathcal{W}^1 \colon \forall x \in \{1, \dots, t\} \colon \{a_x, b_x\}\}$. As second step, we will construct a vector v^2 such that for each $W^i \notin \mathcal{W}^2$, there is a committee W^j such that $v^2\hat{u}^{i,j} < 0$.

For constructing this vector, recall that $s^1(x+1,x) = \alpha$ for all $x < p$ and $s^1(p+1,p) \neq \alpha$. Moreover, consider two arbitrary committees $W^i, W^j \in \mathcal{W}^1$ with $\{W^i, W^j\} \neq \{W^{j_0}, W^{j_t}\}$. By this assumption, it holds that $|W^i \setminus W^j| = t' < t$, so we can use our induction hypothesis to construct $\hat{u}^{i,j}$. For doing so, let $W^{i_0}, \dots W^{i_{t'}}$ denote a sequence of committees from W^i to W^j. By the induction hypothesis, $\hat{u}^{i,j} = \delta \sum_{x=1}^{t'} \hat{u}^{i_{x+1},i_x} =$ for some $\delta > 0$. In turn, Lemma 16 shows that $\hat{u}_\ell^{i,j} = \delta \sum_{x=|B(\ell) \cap W^i|+1}^{|B(\ell) \cap W^j|} s^1(x-1,x)$ for all ballots $B(\ell)$ with $|B(\ell) \cap W^i| \leq |B(\ell) \cap W^j|$. Hence, if additionally $|B(\ell) \cap W^i| \leq p$ and $|B(\ell) \cap W^j| \leq p$, then $\hat{u}_\ell^{i,j} = -\delta\alpha(|B(\ell) \cap W^j| - |B(\ell) \cap W^i|) = \delta\alpha(|B(\ell) \cap W^i| - |B(\ell) \cap W^j|)$. On the other hand, if $|B(\ell) \cap W^i| \leq p$ and $|B(\ell) \cap W^i| = p+1$, then $\hat{u}_\ell^{i,j} = \delta\alpha(|B(\ell) \cap W^i| - p) - \delta s(p+1,p) = \delta\alpha(|B(\ell) \cap W^i| - |B(\ell) \cap W^j|) + \delta\alpha - \delta s(p+1,p)$.

Now for constructing the vector v^2, we consider first the profiles $\bar{A}^x$ that contain ballot A with $|A| = p+1$ and $a_x, b_x \in A$ once. Moreover, we define $\bar{A}$ as the profile that consists of a copy of $\bar{A}^x$ for all $x \in \{1, \dots, t\}$ and let $\bar{v} = v(\bar{A})$.

First, we observe that all candidates in $(W^{j_0} \cup W^{j_t}) \setminus (W^{j_0} \cap W^{j_t})$ are approved by the same number of voters in $\bar{A}$. Hence all committees in $\mathcal{W}^1$ have the same total number of approvals in $\bar{A}$, i.e., $\sum_{\ell \leq |\mathcal{A}|} \bar{v}_\ell |B(\ell) \cap W^i| = \sum_{\ell \leq |\mathcal{A}|} \bar{v}_\ell |B(\ell) \cap W^j|$ for all committees $W^i, W^j \in \mathcal{W}^1$. Moreover, note that for every ballot A in $\bar{A}$ and every committee $W \in \mathcal{W}^2$, it holds that $|W \cap A| \leq p$ because $|A| = p+1$ and $\{b_x, a_x\} \subseteq A$ for some $x \in \{1, \dots, t\}$ but $\{b_x, a_x\} \not\subseteq W$. Hence, for all $W^i, W^j \in \mathcal{W}^2$ with $\{W^i, W^j\} \neq \{W^{j_0}, W^{j_t}\}$, the following equation holds due to our previous analysis. In this equation, we define $I_1 = \{\ell \in \{1, \dots, |\mathcal{A}|\} \colon |B(\ell) \cap W^i| \leq |B(\ell) \cap$

$W^j|\}$ and $I_1 = \{\ell \in \{1, \dots, |\mathcal{A}|\} \colon |B(\ell) \cap W^i| > |B(\ell) \cap W^j|\}$.

$$\begin{aligned}\bar{v}\hat{u}^{i,j} &= \sum_{\ell \in \{1,\dots,|\mathcal{A}|\}} \bar{v}_\ell \hat{u}^{i,j}_\ell \\ &= \sum_{\ell \in I_1|} \bar{v}_\ell \hat{u}^{i,j}_\ell - \sum_{\ell \in I_2} \bar{v}_\ell \hat{u}^{j,i}_\ell \\ &= \sum_{\ell \in I_1} \bar{v}_\ell \delta\alpha(|B(\ell) \cap W^i| - |B(\ell) \cap W^j|) \\ &\quad - \sum_{\ell \in I_2} \bar{v}_\ell \delta\alpha(|B(\ell) \cap W^j| - |B(\ell) \cap W^i|) \\ &= \delta\alpha\left(\sum_{\ell \in \{1,\dots,|\mathcal{A}|\}} \bar{v}_\ell |B(\ell) \cap W^i| - \sum_{\ell \in \{1,\dots,|\mathcal{A}|\}} \bar{v}_\ell |B(\ell) \cap W^j|\right) \\ &= 0.\end{aligned}$$

Moreover, it also holds that $\bar{v}\hat{u}^{j_0,j_t} = 0$. For explaining this, we consider the permutation τ with $\tau(a_x) = b_x$, $\tau(b_x) = a_x$ for all $x \in \{1, \dots, t\}$ and $\tau(z) = z$ for all remaining candidates. It is easy to see that $\tau(v(\bar{A}^x)) = v(\bar{A}^x)$ for all $x \in \{1, \dots, t\}$ and hence $\tau(\bar{v}) = \bar{v}$. By our usual permutation arguments, it thus follows that $\bar{v}\hat{u}^{j_0,j_t} = 0$.

Next, consider a committee $W^i \in \mathcal{W}^1 \setminus \mathcal{W}^2$. Then $a_x, b_x \in W^i$ for some $x \in \{1, \dots, t\}$. Moreover, let γ denote the number of ballots A in $\bar{A}$ such that $|A \cap W^i| = p + 1$. Since $p + 1 \leq k$ and $a_x, b_x \in W^y$, there is at least one ballot A in $\bar{A}^x$ such that $A \subseteq W$ and thus $\gamma \geq 1$. We claim that $\bar{v}\hat{u}^{j_0,i} > 0$ if $\alpha > s^1(p, p-1)$ and $\bar{v}\hat{u}^{j_0,i} < 0$ if $\alpha < s^1(p+1, p)$. Note for this first that $|W^{i_0} \cap W^y| < t$ since $W^i \in \mathcal{W}^1$ and $a_x \in W^i$. Moreover, it holds for every ballot $B(\ell)$ with $\bar{v}_\ell \neq 0$ that $|W^{j_0} \cap B(\ell)| \leq p$ since $|B(\ell)| = p + 1$ and $b_x \in B(\ell) \setminus W^{j_0}$ for some $x \in \{1, \dots, t\}$. On the other hand, $|W^i \cap B(\ell)| \leq p + 1$. Using our initial insights, we thus derive the following equations, where $I_1 = \{\ell \in \{0, \dots, |\mathcal{A}|\} \colon |B(\ell) \cap W^i| = p + 1\}$ and $I_2 = \{\ell \in \{0, \dots, |\mathcal{A}|\} \colon |B(\ell) \cap W^i| \leq p\}$.

$$\begin{aligned}\bar{v}\hat{u}^{j_0,i} &= \sum_{\ell \in I_1} \bar{v}_\ell \hat{u}^{j_0,i}_\ell + \sum_{\ell \in I_2} \bar{v}_\ell \hat{u}^{j_0,i}_\ell \\ &= \sum_{\ell \in I_1} \bar{v}_\ell \delta\alpha(|B(\ell) \cap W^{j_0}| - |B(\ell) \cap W^i|) \\ &\quad + \sum_{\ell \in I_1} \bar{v}_\ell (\delta\alpha - \delta s^1(p+1, p)) \\ &\quad + \sum_{\ell \in I_2} \bar{v}_\ell \delta\alpha(|B(\ell) \cap W^{j_0}| - |B(\ell) \cap W^i|) \\ &= \delta\gamma(\alpha - s^1(p+1, p)) + \delta\alpha \sum_{\ell \in \{1,\dots,|\mathcal{A}|\}} \bar{v}_\ell |B(\ell) \cap W^{j_0}| \\ &\quad - \delta\alpha \sum_{\ell \in \{1,\dots,|\mathcal{A}|\}} \bar{v}_\ell |B(\ell) \cap W^i| \\ &= \delta\gamma(\alpha - s^1(p+1, p)).\end{aligned}$$

In particular, we use in the last steps that all committees in $\mathcal{W}^1$ have the same total number of approvals. Since $\delta > 0$ and $\gamma > 0$, this shows that $\bar{v}\hat{u}^{j_0,i} < 0$ if $\alpha < s(p+1, p)$ and $\bar{v}\hat{u}^{j_0,i} > 0$ if $\alpha > s(p+1, p)$. Hence, with the right sign in front of $\bar{v}$, all $W^y \in \mathcal{W}^1 \setminus \mathcal{W}^2$ are dominated.

Finally, let $v^2 = \lambda v^1 + \bar{v}$ if $\alpha > s(p+1, p)$ and $v^2 = \lambda v^1 - \bar{v}$ if $\alpha < s(p+1, p)$. In this definition, $\lambda > 0$ is so large that for all committees $W^i \in \mathcal{W}_k \setminus \mathcal{W}^1$, there is another committee $W^j \in \mathcal{W}_k$ such that $v^2\hat{u}^{i,j} < 0$. Now, note that for all $W^i \in \mathcal{W}^1 \setminus \mathcal{W}^2$, we have that $v^2\hat{u}^{i,j_0} = -v^2\hat{u}^{j_0,x} < 0$ since $v^1\hat{u}^{j_0,i} = 0$ and we choose the sign of $\bar{v}$ such that $\pm\bar{v}\hat{u}^{j_0,i} > 0$. Following a similar reasoning, it is easy to see that $v^2\hat{u}^{i,j} = 0$ for $W^i, W^j \in \mathcal{W}^2$.

Step 1.3: For constructing our next vector, we define $\mathcal{W}^3 = \{W \in \mathcal{W}^2 \colon \forall x \in \{1, \dots, t-1\} \colon \{a_x, b_{x+1}\} \not\subseteq W\}$. Put differently, $\mathcal{W}^3$ consists all committees W^x such that $W^x = W^{j_0} \cap W^{j_t} \cup \{b_1, \dots, b_x, a_{x+1}, \dots, b_t\}$, i.e., $\mathcal{W}^3 = \{W^{j_0}, \dots, W^{j_t}\}$. We aim to construct a vector v^3 such that all committees outside of $\mathcal{W}^3$ are dominated by some other committee.

To this end, consider the profile $\hat{A}^x$ for $x \in \{1, \dots, t-1\}$ that contains each ballot A with $|A| = p + 1$ and $\{a_x, b_{x+1}\}$ once. Furthermore, define the profile $\hat{A}$ as follows: $\hat{A}$ contains a copy of $\hat{A}^x$ for each $x \in \{1, \dots, t-1\}$ and so many copies of the ballots $\{a_t\}$ and $\{b_1\}$ that every candidate in $(W^{j_0} \cup W^{j_t}) \setminus (W^{j_0} \cap W^{j_t})$ is approved by the same number of voters. Moreover, we define $\hat{v} = v(\hat{A})$. By the definition of $\hat{A}$, it immediately follows that $\sum_{\ell \in \{1,\dots,|\mathcal{A}|\}} \hat{v}_\ell |B(\ell) \cap W| = \sum_{\ell \in \{1,\dots,|\mathcal{A}|\}} \hat{v}_\ell |B(\ell) \cap W'|$ for all $W, W' \in \mathcal{W}^2$. Furthermore, it holds for all committees $W \in \mathcal{W}^3$ and ballots A in $\hat{A}$ that $|W \cap A| \leq p$. Hence, it follows from exactly the same reasoning as in the last step that $\hat{v}\hat{u}^{i,j} = 0$ for all $W^i, W^j \in \mathcal{W}^3$ with $\{W^i, W^j\} \neq \{W^{j_0}, W^{j_t}\}$. Moreover, the permutation τ with $\tau(a_x) = b_{t-x+1}$, $\tau(b_x) = a_{t-x+1}$ for $x \in \{1, \dots, t\}$ and $\tau(z) = z$ for all other candidates maps $\hat{A}^x$ to $\hat{A}^{t-x+1}$. This means that $\tau(\hat{A}) = \hat{A}$ and our permutation argument thus also shows that $\hat{v}\hat{u}^{j_0,j_t} = 0$.

Next, consider a committee $W^i \in \mathcal{W}^2 \setminus \mathcal{W}^3$. In particular, this means that $\{a_x, b_{x+1}\} \subseteq W^i$ for some $x \in \{1, \dots, t-1\}$. Now, let γ denote the number of ballots A in $\hat{A}$ such that $|A \cap W^y| = p + 1$. Since $p + 1 \leq k$ and $\{a_x, b_{x+1}\} \subseteq W$, there is clearly a ballot $B(\ell)$ such that $|B(\ell)| = p + 1$, $\hat{v}_\ell \neq 0$, and $B(\ell) \subseteq W$, so $\gamma \geq 1$. Since $a_x \in W^i$, this means that $|W^{j_0} \setminus W^i| < t$. Again, for all ballots $B(\ell)$ with $\hat{v}_\ell \neq 0$, we have $|B(\ell) \cap W^{j_0}| \leq p$ and $|B(\ell) \cap W^i| \leq p + 1$. So, we can apply the same reasoning as for $\bar{v}$ to infer that $\hat{u}^{j_0,i}\hat{v} > 0$ if $\alpha > s^1(p+1, p)$ and $\hat{u}^{j_0,i}\hat{v} < 0$ if $\alpha < s^1(p+1, p)$.

Finally, let $v^3 = \lambda v^2 + \hat{v}$ if $\alpha > s(p+1, p)$ and $v^3 = \lambda v^2 - \hat{v}$ if $\alpha < s(p+1, p)$. Here, we choose $\lambda > 0$ again so large that for every committee $W^i \in \mathcal{W}_k \setminus \mathcal{W}^2$, there is another committee $W^j \in \mathcal{W}_k$ such that $v^3\hat{u}^{i,j} < 0$. Moreover, note that for every committee $W^i \in \mathcal{W}^2 \setminus \mathcal{W}^3$, we have that $v^3\hat{u}^{i,j_0} < 0$ because $\hat{u}^{i,j_0}v^2 = 0$ and we choose the sign of $\hat{v}$ so that $\pm\hat{v}\hat{u}^{j_0,i} > 0$. Finally, $v^3\hat{u}^{x,y} = 0$ for all $x, y \in \mathcal{W}^3$ because $v^2\hat{u}^{x,y} = 0$ and $\hat{v}\hat{u}^{x,y} = 0$. Hence, for every committee $W^i \in \mathcal{W}_k \setminus \mathcal{W}^3$, there is another committee W^j such that $v^3\hat{u}^{i,j} < 0$, and for all $W^i, W^j \in \mathcal{W}^3$, it holds that $v^3\hat{u}^{i,j} = 0$.

Step 1.4: As last step, we consider the matrix M that contains the vectors $\hat{u}^{j_0,j_1}, \dots, \hat{u}^{j_{x-1},j-x}, \hat{u}^{j_0,j_t}$ as rows. More specifically, we assume that the x-th row of M is $\hat{u}^{j_{x-1},j_x}$ for $x \in \{1, \dots, t\}$ and the $t+1$-th row of M is $\hat{u}^{j_0,j_t}$. Now, by assumption, the rows of M are linearly independent, i.e., the matrix has row rank of $t + 1$. This means equivalently that it has a column rank of $t+1$, which in turn implies that the image of M has full dimension. Thus, there is a vector v^* such that $w = Mv^*$ satisfies $w_x = 1$ for $x \in \{1, \dots, t\}$ and $w_{t+1} = -1$.

Next, just as in the previous steps, we define $v^4 = \lambda v^3 + v^*$, where $\lambda > 0$ is so large that for every committee $W^i \in \mathcal{W}_k \setminus \mathcal{W}^3$, there is another committee $W^j \in \mathcal{W}_k$ with $v^4\hat{u}^{y,z} < 0$. Now, by definition of v^4 and Claim (1) of Lemma 2, $v^4 \notin \bar{R}^f_i$ for every $W^i \in \mathcal{W}_k \setminus \mathcal{W}^3$. On the other hand, we have shown in Step 3 that $v^3\hat{u}^{i,j} = 0$ for all $W^i, W^j \in \mathcal{W}^3$. So $v^4\hat{u}^{i,j} = v^*\hat{u}^{i,j}$ for these committees. This means that $v^4\hat{u}^{j_0,j_t} = -1 < -0$ and $v^4\hat{u}^{j_{x-1}j_x} = -v^4\hat{u}^{j_x,j_{x-1}} = -1 < 0$ for all $x \in \{1,\dots,t\}$. So, $v^4 \notin \bar{R}^f_i$ for any committee $W^i \in \mathcal{W}_k$, which gives us the desired contradiction. Hence, the initial assumption is wrong and the vectors $\hat{u}^{j_0,j_1},\dots,\hat{u}^{j_{t-1},j_t},\hat{u}^{j_0,j_t}$ are linearly independent.

Step 2: There is $\delta > 0$ such that $\hat{u}^{j_0,j_t} = \delta\sum_{x=1}^t \hat{u}^{j_{x-1},j-x}$

By Step 1, we know that the set $\{\hat{u}^{j_0,j_1},\dots,\hat{u}^{j_{t-1},j_t},\hat{u}^{j_0,j_t}\}$ is linearly dependent, whereas the set $\{\hat{u}^{j_0,j_1},\dots,\hat{u}^{j_{t-1},j_t}\}$ is linearly independent (Lemma 15). Consequently, there are unique values δ_x for $x \in \{1,\dots,t\}$, not all of which are 0, such that $\hat{u}^{j_0,j_t} = \sum_{x=1}^t \delta_x \hat{u}^{j_{x-1},j_x}$. Now, consider the profiles A^x constructed in Lemma 13. As discussed before, it holds that $\hat{u}^{i,j}v(A^c) = 0$ for all committees $W^i, W^j \in \mathcal{W}_k$ such that $|W^i \setminus W^j| = 1$ and $c \in W^i \cap W^j$ or $c \notin W^i \cup W^j$. Conversely, $\hat{u}^{i,j}v(A^c) > 0$ for all committees $W^i, W^j \in \mathcal{W}_k$ with $W^i \setminus W^j = \{c\}$. Moreover, observe that $\hat{u}^{y_1,z_1}v(A^{c_1}) = \hat{u}^{y_2,z_2}v(A^{c_2}) \neq 0$ for all committees $W^{y_1}, W^{z_1}, W^{y_2}, W^{z_2} \in \mathcal{W}_k$ with $W^{y_1} \setminus W^{z_1} = \{c_1\}$ and $W^{y_2} \setminus W^{z_2} = \{c_2\}$. This can again be proven by choosing a suitable permutation τ.

Now, let $x \in \{1,\dots,t\}$ be an arbitrary index. By our previous argument, we have that $v(A^{a_x})\hat{u}^{j_{x-1},j_x} > 0$ because $a_x \in W^{j_{x-1}} \setminus W^{j_x}$. On the other side, $a_x \in W^{j_{y-1}} \cap \bar{W}^{j_y}$ for all $1 \leq y < x$, and $a_x \notin \bar{W}^{j_{y-1}} \cup \bar{W}^{j_y}$ for all $y > x$. So, we infer that $\hat{u}^{j_{y-1},j_y}v(A^{a_x}) = 0$ for all $y \in \{1,\dots,t\}$ with $y \neq x$. Hence, it follows that $v(A^{a_x})\hat{u}^{j_0,j_t} = \sum_{y=1}^t \delta_y v(A^{a_x})\hat{u}^{j_{y-1},j_y} = \delta_x v(A^{a_x})\hat{u}^{j_{x-1},j_x}$.

As next step, we consider two distinct indices $x_1, x_2 \in \{1,\dots,t\}$ and the profiles $A^{a_{x_1}}$ and $A^{a_{x_2}}$. Moreover, let $\tau : C \to C$ be a permutation such that $\tau(a_{x_1}) = a_{x_2}$, $\tau(a_{x_2}) = a_{x_1}$, and $\tau(c) = c$ for all other candidates. First, $a_{x_1}, a_{x_2} \in W^{j_0} \setminus W^{j_t}$, so $\tau(W^{j_0}) = W^{j_0}$ and $\tau(W^{j_t}) = W^{j_t}$. Hence, $v(A^{a_{x_1}})\hat{u}^{j_0,j_t} = \tau(v(A^{a_{x_1}}))\tau(\hat{u}^{j_0,j_t}) = v(A^{a_{x_2}})\hat{u}^{j_0,j_t}$ by Claim (3) of Lemma 2. Combining our last two insights thus shows that $\delta_{x_1}\hat{u}^{j_{x_1-1},j-x_1}v(A^{a_{x_1}}) = \delta_{x_2}\hat{u}^{j_{x_2-1},j_{x_2}}v(A^{a_{x_2}})$. Since $v(A^{a_{x_1}})\hat{u}^{j_{x_1-1},j_{x_1}} = v(A^{a_{x_2}})\hat{u}^{j_{x_2-1},j_{x_2}} \neq 0$, this means that $\delta_{x_1} = \delta_{x_2}$. This proves that $\hat{u}^{j_0,j_t} = \delta\sum_{x=1}^t \hat{u}^{j_{x-1},j_x}$ for some $\delta \in \mathbb{R}$. Moreover, since there is at least one non-zero δ_x, it follows that $\delta \neq 0$

Finally, we need to show that $\delta > 0$. For this, we note that $W^{j_0} \in f(A^{a_1})$ as $a_1 \in W^{j_0}$ and therefore $v(A^{a_1}) \in \bar{R}^f_{j_0}$. Now, by Claim (1) of Lemma 2, this means that $v(A^{a_1})\hat{u}^{j_0,j_t} \geq 0$. On the other side, we have already shown that $v(A^{a_1})\hat{u}^{i,j} = \delta v(A^{a_1})\hat{u}^{j_0,j_1}$ and that $v(A^{a_1})\hat{u}^{j_0,j_1}$ and $\delta \neq 0$. Combining these claims then shows that $\delta > 0$ and thus proves the lemma. □

Finally, we are now ready to prove Theorem 1.

Theorem 1. *An ABC voting rule is a Thiele rule if and only if it satisfies anonymity, neutrality, consistency, continuity, and independence of losers.*

Proof. We note that the direction from left to right has been shown in the main body. Hence, we focus on the converse direction and assume that $f \in \mathcal{F}^2$. Now, if f is the trivial rule, it is clearly the Thiele rule defined by $s(0) = 0$. On the other hand, we can assume that f is non-imposing if it is non-trivial by Lemma 13. This allows us to access the vectors $\hat{u}^{i,j}$ constructed in Lemma 2 and the function s^1 constructed in Lemma 14. Now, we define the function $s(x)$ as follows: $s(0) = 0$ and $s(x) = \sum_{y=1}^x s(y, y-1)$ for all $x \in \{1,\dots,k\}$. Moreover, we extend s to vectors by $\hat{s}(v, W) = \sum_{\ell\in\{1,\dots,|\mathcal{A}|\}} v_\ell s(|B(\ell) \cap W|)$. We will show this lemma by proving that $f(A) = f'(A) := \{W \in \mathcal{W}_k : \forall W' \in \mathcal{W}_k : \hat{s}(v(A), W) \geq \hat{s}(v(A), W')\}$ and that f' is a Thiele rule. For doing so, we proceed in multiple steps.

Step 1: There is $\delta > 0$ such that $\delta\hat{u}^{i,j}_\ell = s(|B(\ell) \cap W^i|) - s(|B(\ell) \cap W^i|)$ for all ballots $B(\ell)$.

As first step, we show that the vectors $\hat{u}^{i,j}$ can be represented by the function s. For this, consider two arbitrary committees $W^i, W^j \in \mathcal{W}_k$ and a ballot $B(\ell)$. First, if $|W^i \setminus W^j| = 1$, then this claim follows from Lemma 14. Hence, suppose that $|W^i \setminus W^j| = t \geq 2$, which requires that $2 \leq k \leq m-2$, which means that **??** applies. To use this lemma, let $W^{j_0},\dots,W^{j_t}$ denote a sequence of committees from W^i to W^j. Then, we infer that $\hat{u}^{i,j} = \delta\sum_{x=1}^t \hat{u}^{j_{x-1},j_x}$ for some $\delta > 0$. Now, suppose that $|B(\ell) \cap W^i| \leq |B(\ell) \cap W^j|$. Then, Lemma 16 shows that $\sum_{x=1}^t \hat{u}^{j_{x-1},j_x} = \sum_{x=|B(\ell)\cap W^i|+1}^{|B(\ell)\cap W^j|} s^1(x-1, x)$. By the definition of s and the fact that $s^1(x-1,x) = -s^1(x, x-1)$, we get that $\sum_{x=|B(\ell)\cap W^i|+1}^{|B(\ell)\cap W^j|} s^1(x-1,x) = -\sum_{x=|B(\ell)\cap W^i|+1}^{|B(\ell)\cap W^j|} s^1(x,x-1) = -(s(|B(\ell) \cap W^j|) - s(|B(\ell) \cap W^i|)) = s(|B(\ell) \cap W^i|) - s(|B(\ell) \cap W^j|)$. Hence, the claim is proven in this case.

Next, assume that $|B(\ell) \cap W^i| > |B(\ell) \cap W^j|$. In this case, we can consider the vectors $\hat{u}^{j,i}$ and our previous argument shows that $\hat{u}^{j,i}_\ell = \delta(s(|B(\ell) \cap W^j|) - s(|B(\ell) \cap W^i|))$. Finally, the step follows again since $\hat{u}^{i,j}_\ell = -\hat{u}^{j,i}_\ell$.

Step 2: $f(A) \subseteq f'(A)$ **for all profiles** $A \in \mathcal{A}^*$

For showing this step, consider an arbitrary profile A. By our lemmas, we have that $f(A) = \hat{g}(v(A)) = \{W^i \in \mathcal{W}_k : v(A) \in R^f_i\} \subseteq \{W^i \in \mathcal{W}_k : v(A) \in \bar{R}^f_i\}$. Hence, our goal is to show that $v(A) \in \bar{R}^f_i$ if and only if $\hat{s}(v(A), W^i) \geq s(v(A), W^j)$ for all committees $W^j \in \mathcal{W}_k$. For doing so, we recall that $\bar{R}^f_i = \{v \in \mathbb{R}^{|\mathcal{A}|} : \forall j \in \{1,\dots,|\mathcal{W}_k|\} \setminus \{i\} : v\hat{u}^{i,j} \geq 0\}$. Hence, it clearly suffices to show that $v(A)\hat{u}^{i,j} \geq 0$ if and only if $\hat{s}(v(A), W^i) \geq \hat{s}(v(A), W^j)$. For this, we observe that by Step 1, $v(A)\hat{u}^{i,j} = \sum_{\ell\in\{1,\dots,|\mathcal{A}|\}} v(A)_\ell \delta(s(|B(\ell) \cap W^i|) - s(|B(\ell) \cap W^i|)) = \delta(\hat{s}(v(A), W^i) - \hat{s}(v(A), W^j))$ for some $\delta > 0$. This shows that our claim holds and thus this step follows.

Step 3: $f(A) \subseteq f'(A)$ **for all profiles** $A \in \mathcal{A}^*$ **and** f' **is a Thiele rule**

First, we show that $f'(A)$ is a Thiele rule. For this, we note first that $s(0) = 0$ by definition and it thus only remains to prove that s is non-decreasing. Now, assume for contradicting that there is an index $p \in \{1,\dots,k\}$ such that $s(p) < s(p-1)$. First, suppose that $p > 1$. In this case, consider the profile A in which every ballot of size p is reported once. By anonymity and neutrality, it follows that $f(A) = f'(A) = \mathcal{W}_k$. Next, we consider two arbitrary committees $W, W' \in \mathcal{W}_k$ and let $c \in \mathcal{W} \setminus W'$. Moreover, let $B(\ell)$ denote a ballot such that $B(\ell) \subseteq W$ and $c \in B(\ell)$. Finally, let A' denote the profile in which we replace $B(\ell)$ with $B(\ell) \setminus \{c\}$. It is easy to see that $s(A', W) = s(A, W) - s(p) + s(p-1) > s(A, W) = s(A, W') = s(A', W')$. Hence, $W' \notin f'(A')$ and therefore also $W' \notin f(A')$.

However, this contradicts independence of losers as $W' \in f(A)$ and $c \notin W'$. Hence, we infer that $s(p) \geq s(p-1)$ for all $p \in \{2, \dots, k\}$. As second case suppose that $p = 1$, which means that $s(1) < s(0) = 0$. In this case, let A denote the profile consisting of all ballots of size 2. Now, consider a ballot $\{x, y\}$ and let W and W' denote committees such that $x \in W$, $y \notin W$ and $x \notin W'$, $y \in W'$. Finally, let A' denote the profile derived from A by replacing the ballot $\{x, y\}$ with the ballot $\{y\}$. It is easy to see that $s(A', W) = s(A, W) - s(1) + s(0) > s(A, W) = s(A, W') = s(A', W')$. Hence, $W' \in f(A)$ and $W' \notin f(A)$ as $W' \notin f'(A')$. This contradicts independence of losers as $x \notin W'$. Both cases combined show that s is non-decreasing, so f' is indeed a Thiele rule.

Finally, we show that $f(A) = f'(A)$ for all profiles $A \in \mathcal{A}^*$. Assume that this is not the case, which means that there is a profile A and a committee W such that $W \in f'(A) \setminus f(A)$ because of Step 2. Moreover, note that f' is a Thiele rule and hence satisfies consistency and all the other axioms of Theorem 1. Next, since s is non-zero (as the vectors $\hat{u}^{i,j}$ are non-zero), f' is not the trivial rule. Hence, we can use Lemma 13 to show that f' is non-imposing. In particular, there is a profile A' such that $f(A') = f'(A') = \{W\}$. By consistency of f', this means that $f(\lambda A + A') = f'(\lambda A + A') = \{W\}$ for every $\lambda \in \mathbb{N}$. However, this contradicts the continuity of f and therefore, our initial assumption must have been wrong. This shows that f is the Thiele rule defined by s. □

B OMITTED PROOFS FROM SECTION 4

Finally, we discuss in this sections the proofs of the results in Section 4. We start by discussing the characterization of AV in Theorem 3.

Theorem 3. *Assume $k \leq m - 2$. AV is the only non-trivial Thiele rule that satisfies choice set convexity and the only non-trivial BSWAV rule that satisfies independence of losers.*

Proof. Fix an arbitrary number of candidates m and a committee size $k \in \{1, \dots, m-2\}$. First, we note that AV is both a Thiele rule and a BSWAV rule and therefore satisfies all given axioms because of Theorems 1 and 2.

For the other direction, we first note that these axioms show that every choice set convex Thiele rule f is also a BSWAV rule that satisfies independence of losers because Thiele rules are by definition weakly efficient. Hence, suppose that f is a non-trivial BSWAV rule that satisfies independence of losers. By definition of these rules, there is a weight vector $\alpha = (\alpha_1, \dots, \alpha_m)$ such that $f(A) = \{W \in \mathcal{W}_k \colon \forall W' \in \mathcal{W}_k \colon \sum_{i \in N_A} \alpha_{|A_i|} |A_i \cap W| \geq \sum_{i \in N_A} \alpha_{|A_i|} |A_i \cap W'|\}$. First, we note that the value α_m does not affect the value because for the only ballot of size m, C, it holds for all committees $W \in \mathcal{W}_k$ that $|W \cap C| = |W|$. Hence, we simply ignore α_m from now. Next, since f is non-trivial, there is a ballot size $\ell < m$ such that $\alpha_\ell > 0$ and it is easy to see that $f(A^x) = \{W \in \mathcal{W}_k \colon x \in W\}$ for the profile A^x in which each ballot A with $x \in A$ and $|A| = \ell$ is reported once. Conversely, $f(A^{-x}) = \{W \in \mathcal{W}_k \colon x \notin W\}$ for the profile that contains each ballot A with $x \notin A$ and $|A| = \ell$ once. Now, consider two committees $W, W' \in \mathcal{W}_k$ with $|W \setminus W'| = 1$ and let $\{a\} = W \setminus W'$, $\{b\} = W' \setminus W$, $W \cap W' = \{c_1, \dots, c_{k-1}\}$. Moreover, let $\bar{A}$ denote the committee that consists of the profiles A^x for all $x \in W \cap W'$ and A^{-x} for all $x \in C \setminus (W \cup W')$. By consistency, it holds that $f(\bar{A}) = \{W, W'\}$. Clearly, this implies that $\sum_{i \in N_{\bar{A}}} \alpha_{\bar{A}_i} |W \cap \bar{A}_i| = \sum_{i \in N_{\bar{A}}} \alpha_{\bar{A}_i} |W' \cap \bar{A}_i| > \sum_{i \in N_{\bar{A}}} \alpha_{\bar{A}_i} |W'' \cap \bar{A}_i|$ for all $W'' \in \mathcal{W}_k \setminus \{W, W'\}$. Moreover, by continuity and consistency, there is a $\lambda \in \mathbb{N}$ such that $f(\lambda \bar{A} + A) \subseteq \{W, W'\}$ for all subsequent profiles A.

Next, consider an arbitrary ballot size $\ell \in \{2, \dots, m-1\}$ and let A denote an arbitrary ballot of size $\ell - 1$ such that there is a candidate $x \in A \setminus (W \cup W')$ and $a, b \notin A$. Such a ballot exists since $|C \setminus (W \cup W')| = m - (k+1) \geq 1$ as $k \leq m-2$ and $\ell \leq m-1$. Furthermore, consider the profile A' that consists of the two ballots $A \cup \{a\}$ and $A \cup \{b\}$ and let $A^1 = \lambda \bar{A} + A'$. By the definition of λ, $f(A^1) \subseteq \{W, W'\}$ and since these two committees are completely symmetric in A^1 as we can simply rename a to b and vice versa, it holds that $f(A^1) = \{W, W'\}$. Next, let A'' be the profile derived from A' by letting the voter with ballot $A \cup \{a\}$ disapprove the candidate $x \in A \setminus \{W \cup W'\}$. By independence of losers, it holds for the profile $A^2 = \lambda A + A''$ that $W, W' \in f(A^2)$ and by the choice of λ, $f(A^2) = \{W, W'\}$. This means that $\sum_{i \in N_{A^2}} \alpha_{|A_i^2|} |W \cap A_i^2| = \sum_{i \in N_{A^2}} \alpha_{|A_i^2|} |W' \cap A_i^2|$. Since $\sum_{i \in N_{\bar{A}}} \alpha_{|\bar{A}_i|} |W \cap \bar{A}_i| = \sum_{i \in N_{\bar{A}}} \alpha_{|\bar{A}_i|} |W' \cap \bar{A}_i|$, this implies that $\alpha_{\ell-1} |A^{+a-x} \cap W| + \alpha_\ell |A^{+b} \cap W| = \alpha_{\ell-1} |A^{+a-x} \cap W'| + \alpha_\ell |A^{+b} \cap W'|$, where $A^{+a-x} = A \cup \{a\} \setminus \{x\}$ and $A^{+b} = A \cup \{b\}$. Now, it is easy to see that $|A^{+a-x} \cap W| = |A^{+b} \cap W'| = |A^{+b} \cap W| + 1 = |A^{+a-x} \cap W'| + 1$ since $x \notin W \cup W'$. Hence, we can simplify our equation and derive that $\alpha_{\ell-1} = \alpha_\ell$. Since this holds for every $\ell \in \{2, \dots, m-1\}$, this means that $\alpha_i = \alpha_j$ for all $i, j \in \{1, \dots, m-1\}$. Finally, since non-triviality requires that there is α_i with $\alpha_i > 0$, this means that f is AV. □

Next, we turn to the characterization of SAV.

Theorem 4. SAV *is the only non-trivial BSWAV rule that satisfies split/merge-proofness.*

Proof. First, we will show that SAV is split/merge-proof. For this, we recall that the score of a committee W can be attributed to candidates $x \in W$ for all BSWAV rules: for every committee $W \in \mathcal{W}_k$ and profile A, it holds that $\sum_{i \in N_A} \alpha_{|A_i|} |W \cap A_i| = \sum_{x \in W} \sum_{i \in N_A \colon x \in A_i} \alpha_{|A_i|}$. Hence, we can define $s_{\mathsf{SAV}}(x, A) = \sum_{i \in N_A \colon x \in A_i} \frac{1}{|A_i|}$ for every candidate x and SAV chooses the committee W that maximizes $\sum_{x \in W} s_{\mathsf{SAV}}(x, A)$. Now, consider a party-list profile A and let $X = \{\{c_1\}, \dots, \{c_j\}\}$ denote a set of singleton parties, each of which is approved by ℓ voters in A. Clearly, $s_{\mathsf{SAV}}(c_{j'}, A) = \ell$ for all $j' \in \{1, \dots, j\}$. Next, let A^X denote the profile derived from A by changing the ballot of every voter i with $A_i \in \{\{c_1\}, \dots, \{c_j\}\}$ to $A_i^X = \{c_1, \dots, c_j\}$. In A', every candidate $c_{j'}$ is approved by $j\ell$ voters who approve a total of j candidates. Hence, $s_{\mathsf{SAV}}(c_{j'}, A') = \frac{j\ell}{j} = \ell$. Moreover, $s_{\mathsf{SAV}}(x, A') = s_{\mathsf{SAV}}(x, A)$ for all other candidates $x \in C \setminus \{c_1, \dots, c_j\}$. Hence, the SAV score of each committee W is the same in A and A^X, so $\mathsf{SAV}(A) = \mathsf{SAV}(A^X)$.

For the other direction, consider a non-trivial BSWAV rule f that satisfies split/merge-proofness and let $\alpha = (\alpha_1, \dots, \alpha_m)$ denote its weight vector. By non-triviality, there is an index $z \in \{1, \dots, m-1\}$ such that $\alpha_z > 0$. As first step, we will show that $\alpha_1 > 0$, too. Consider for this the profile A in which z voters report a ballot A_i of size z. Since $\alpha_z > 0$, $f(A) \neq \mathcal{W}_k$. Moreover, by split/merge-proofness, the same must hold for the profile A' in which each candidate $x \in A_i$ is approved by a single voter. This is only possible if $\alpha_1 > 0$. Finally, since BSWAV rules are invariant under scaling the weight vector, we assume from now on that $\alpha_1 = 1$.

Next, suppose that there is an index $\ell \in \{2, \dots, m-1\}$ such that $\alpha_\ell \neq \frac{1}{\ell}$. First, suppose that $\alpha_\ell < \frac{1}{\ell}$ and let $\Delta = \frac{1}{\ell} - \alpha_\ell$. Moreover, we choose $t \geq 2$ as an integer such that $t\ell\Delta > 1$ and we denote our candidates by $C = \{c_1, \dots, c_m\}$. In this case, consider the profile the following profile A: for each candidate c_i with $i \in \{1, \dots, \ell\}$, there are t voters who report $\{c_i\}$, and for each candidate c_i with $i \in \{\ell-1, \dots, m\}$, there are $t-1$ voters who report $\{c_i\}$. Since $\alpha_1 = 1$, it holds that $s_\alpha(c_i, A) = t$ if $i \in \{1, \dots, \ell\}$ and $s_\alpha(c_i, A) = t-1$ otherwise. Hence, it holds that $f(A) = \{W \in \mathcal{W}_k \colon \forall W' \in \mathcal{W}_k \colon |W \cap \{c_1, \dots, c_\ell| \geq |W' \cap \{c_1, \dots, c_\ell\}|\}$. Next, let A' denote the profile in which $t\ell$ voters report $\{c_1, \dots, c_\ell\}$ and for each $i \in \{\ell+1, \dots m\}$, there are $t-1$ voters who approve $\{c_i\}$. We can again compute the scores of the individual candidates and derive that $s_\alpha(c_i, A') = t\ell\alpha_\ell = t\ell(\frac{1}{\ell} - \Delta) < t-1$ for $i \in \{1, \dots, \ell\}$ and $s_\alpha(c_i, A') = t-1$ for all other candidates. Hence, $f(A') = \{W \in \mathcal{W}_k \colon \forall W' \in \mathcal{W}_k \colon |W \cap \{c_1, \dots, c_\ell\}| \leq |W' \cap \{c_1, \dots, c_\ell\}|\}$. So, f chooses for A the committees that maximize the intersection with $\{c_1, \dots, c_\ell\}$ and for A' the committees that minimize this intersection. Since $k \leq m-1$ and $\ell \leq m-1$, this means $f(A) \neq f(A')$, which contradicts split/merge-proofness.

For the second case, assume that $\alpha_\ell > \frac{1}{\ell}$. In this case, we define $\Delta = \alpha_\ell - \frac{1}{\ell}$ and choose $t \geq 1$ such that $t\ell\Delta > 1$. Now, consider the profile A in which t voters approve each candidate in $\{c_1, \dots, c_\ell\}$ and $t+1$ voters approve each candidate in $\{c_{\ell+1}, \dots, c_m\}$. It is easy to see that $f(A) = \{W \in \mathcal{W}_k \colon \forall W' \in \mathcal{W}_k \colon |W \cap \{c_1, \dots, c_\ell| \leq |W' \cap \{c_1, \dots, c_\ell\}|\}$ as $s_\alpha(c_i, A) = t$ for $i \in \{1, \dots, \ell\}$ and $s_\alpha(c_i, A) = t+1$ for $i \in \{\ell+1, \dots, m\}$. Next, let A' denote the profile in which ℓt voters report $\{c_1, \dots, c_\ell\}$ and for each $i \in \{\ell+1, \dots, m\}$, there are $t+1$ voters who report $\{c_i\}$. It is easy to compute that $s_\alpha(c_i, A) = t\ell\alpha_\ell = t\ell(\frac{1}{\ell} + \Delta) > t+1$ for $i \in \{1, \dots, \ell\}$ and $s_\alpha(c_i, A) = t+1$ for $i \in \{\ell+1, \dots, m\}$. Hence, $f(A') = \{W \in \mathcal{W}_k \colon \forall W' \in \mathcal{W} \colon |W \cap \{c_1, \dots, c_\ell| \geq |W' \cap \{c_1, \dots, c_\ell\}|\}$. Clearly, this implies again that $f(A) \neq f(A')$, which contradicts split/merge-proofness.

Finally, we have now that $\alpha_i = \frac{1}{i}$ for all $i \in \{1, \dots, m-1\}$. Moreover, f is independent of the value of α_m since $|W \cap C| = k$ for all committees $W \in \mathcal{W}_k$. This means that f is SAV, thus proving our characterization. □

As last point, we characterize PAV based on weak split/merge-proofness

Theorem 5. PAV *is the only non-trivial Thiele rule that satisfies weak split/merge-proofness.*

Proof. The theorem consists of two claims: first, we will show that PAV is weakly split/merge-proof and then that no other rule satisfies these conditions.

Claim 1: PAV is weakly split/merge-proof.

Consider an arbitrary party-list profile A with parties $\mathcal{P} = \{P_1, \dots, P_\ell\}$ and let X denote a subset of the parties such that $|P_i| = 1$ and $|\{j \in N_A \colon A_j = P_i\}| = c$ for all $P_i \in X$ and an constant $c \in \mathbb{N}$. For simplicity, we define $|X| = j$ and $\hat{X} = \bigcup X$. We will first discuss some general insights and note that, in party-list profiles, the PAV score of a committee can be separated in scores assigned to the parties by it. More formally, let n_A^i denote the number of voters who approve party P_i in A. Then, $\sum_{i \in N_A} s_{\mathsf{PAV}}(|A_i \cap W|) = \sum_{P_i \in \mathcal{P}} n_A^i s_{\mathsf{PAV}}(|P_i \cap W|)$ for all committees $W \in \mathcal{W}_k$. Using the definition of PAV, we can now easily derive Equation (1) for all $W \in \mathsf{PAV}(A)$, $W' \in \mathcal{W}_k$ and Equation (2) for all $W \in \mathsf{PAV}(A^X)$, $W' \in \mathcal{W}_k$. Also, we note that the inequalities are strict if $W' \notin \mathsf{PAV}(A)$ and $W' \notin \mathsf{PAV}(A^X)$, respectively.

$$c|W \cap \hat{X}| \cdot s_{\mathsf{PAV}}(1) + \sum_{P_i \in \mathcal{P} \setminus X} n_A^i s_{\mathsf{PAV}}(|W \cap P_i|) \geq c|W' \cap \hat{X}| \cdot s_{\mathsf{PAV}}(1) + \sum_{P_i \in \mathcal{P} \setminus X} n_A^i s_{\mathsf{PAV}}(|W' \cap P_i|) \tag{1}$$

$$jc \cdot s_{\mathsf{PAV}}(|W \cap \hat{X}|) + \sum_{P_i \in \mathcal{P} \setminus X} n_{A^X}^i s_{\mathsf{PAV}}(|W \cap P_i|) \geq jc \cdot s_{\mathsf{PAV}}(|W' \cap \hat{X}|) + \sum_{P_i \in \mathcal{P} \setminus X} n_{A^X}^i s_{\mathsf{PAV}}(|W' \cap P_i|) \tag{2}$$

We note that we can also omit the terms $s_{\mathsf{PAV}}(1)$ as $s_{\mathsf{PAV}}(1) = 1$.

Next, the following equation holds for all committees $W \in \mathcal{W}_k$ because $A_i = P_j$ if and only if $A_i^X = P_j$ for all voters $i \in N_A$ and parties $P_j \in \mathcal{P} \setminus X$.

$$\sum_{P_i \in \mathcal{P} \setminus X} n_A^i s_{\mathsf{PAV}}(|W \cap P_i|) = \sum_{P_i \in \mathcal{P} \setminus X} n_{A^X}^i s_{\mathsf{PAV}}(|W \cap P_i|) \tag{3}$$

Based on these equations, we next show that $\hat{X} \subseteq W$ for all $W \in \mathsf{PAV}(A)$ if and only if $\hat{X} \subseteq W$ for all $W \in \mathsf{PAV}(A^X)$. First, assume that $\hat{X} \subseteq W$ for all $W \in \mathsf{PAV}(A)$ but there is $W' \in \mathsf{PAV}(A^X)$ with $\hat{X} \not\subseteq W'$. Moreover, fix an arbitrary committee $W \in \mathsf{PAV}(A)$. For this case, we compute that $jc \cdot s_{\mathsf{PAV}}(|W \cap \hat{X}|) - jc \cdot s_{\mathsf{PAV}}(|W' \cap \hat{X}|) = jc \sum_{x=|W' \cap \hat{X}|+1}^{j} \frac{1}{x} \geq c|(W \cap \hat{X}) \setminus (W' \cap \hat{X})| = c|W \cap \hat{X}| - c|W' \cap \hat{X}|$. Consequently, $\sum_{P_i \in \mathcal{P} \setminus X} n_{A^X}^i s(|P_i \cap W'|) - \sum_{P_i \in \mathcal{P} \setminus X} n_{A^X}^i s(|P_i \cap W|) \geq c|W \cap \hat{X}| - c|W' \cap \hat{X}|$ since Equation (2) needs to hold for A^X. By Equation (3), the last insight also holds for A and the subsequent computation thus shows that $W' \in \mathsf{PAV}(A)$ as W' has a higher score than W in A.

$$\begin{aligned}
&\left(c|W \cap \hat{X}| + \sum_{P_i \in \mathcal{P} \setminus X} n_A^i s(|W \cap P_i|)\right) - \\
&\qquad \left(c|W' \cap \hat{X}| + \sum_{P_i \in \mathcal{P} \setminus X} n_A^i s(|W' \cap P_i|)\right) \\
&= c\left(|W \cap \hat{X}| - |W' \cap \hat{X}|\right) \\
&\qquad + \sum_{P_i \in \mathcal{P} \setminus X} n_A^i s(|W \cap P_i|) - n_A^i s(|W' \cap P_i|) \\
&\leq c|\hat{X} \setminus W'| - c|\hat{X} \setminus W'| = 0.
\end{aligned}$$

As second case, suppose that $\hat{X} \subseteq W$ for all $W \in \mathsf{PAV}(A^X)$ but there is $W' \in \mathsf{PAV}(A)$ such that $\hat{X} \not\subseteq W'$. We fix in this case a committee $W \in \mathsf{PAV}(A^X)$. Next, we compute that $c|W \cap \hat{X}| \cdot s_{\mathsf{PAV}}(1) - c|W' \cap \hat{X}| \cdot s_{\mathsf{PAV}}(1) = c|\hat{X} \setminus W'|$ as $\hat{X} \subseteq W$. Furthermore, since $W' \in \mathsf{PAV}(A)$, we can apply Equation (1). For this, let $W'' = (W' \cup \{x\}) \setminus \{y\}$ for $x \in \hat{X} \setminus W'$, $y \in W' \setminus \hat{X}$. We can compute that $c(|W'' \cap \hat{X}| - c|W' \cap \hat{X}|) = c$ and hence, Equation (1) shows that $c \leq \sum_{P_i \in \mathcal{P} \setminus X} n_A^i s(|W' \cap P_i|) - \sum_{P_i \in \mathcal{P} \setminus X} n_A^i s(|W'' \cap P_i|) = n_A^j (s(|W' \cap P_j|) - s(|W'' \cap P_j|)) = \frac{n_A^j}{|W' \cap P_j|}$, where P_j is the party with $y \in P_j$. Since $y \in W' \setminus \hat{X}$ is chosen arbitrarily, we infer that $\frac{n_A^i}{|P_i \cap W'|} \geq c$ for all parties $P_i \in \mathcal{P} \setminus X$ with $|P_i \cap W'| \neq 0$. Now, we consider the committee W in A^X and note that there is a party $P_i \in \mathcal{P} \setminus X$ such that $|P_i \cap W| < |P_i \cap W'|$ because $|W \cap \hat{X}| = j > |W' \cap \hat{X}|$. Finally, consider the committee $\bar{W}$ derived from W by replacing a candidate $x \in W \cap \hat{X}$ with a candidate $y \in P_i$. Removing x from W reduces the score by $jc(s_{\mathsf{PAV}}(j) - s_{\mathsf{PAV}}(j-1)) = c$ and adding y increases the score by $n_A^i(s_{\mathsf{PAV}}(|W \cap P_i| + 1) - s_{\mathsf{PAV}}(|W \cap P_i|)) = \frac{n_A^i}{|W \cap P_i| + 1} \geq \frac{n_A^i}{|W' \cap P_i|} \geq c$. Hence, $\bar{W} \in \mathsf{PAV}(A^X)$, but this contradicts our assumption that $\hat{X} \subseteq W$ for all $W \in \mathsf{PAV}(A^X)$. This shows that $\hat{X} \subseteq W$ for all $W \in \mathsf{PAV}(A)$ if and only if $\hat{X} \subseteq W$ for all $W \in \mathsf{PAV}(A^X)$.

Finally, we will show that $\mathsf{PAV}(A) = \mathsf{PAV}(A^X)$ if $\hat{X} \subseteq W$ for all $W \in \mathsf{PAV}(A)$ or $\hat{X} \subseteq W$ for all $W \in \mathsf{PAV}(A^X)$. First, by our previous analysis, the latter condition is only true if $\hat{X} \subseteq W$ for all $W \in \mathsf{PAV}(A) \cup \mathsf{PAV}(A^X)$. Now, let $W \in \mathsf{PAV}(A)$, $W' \in \mathsf{PAV}(A^X)$. Since $\hat{X} \subseteq W$ and $\hat{X} \subseteq W'$, Equations (1) and (2) simplify to $\sum_{P_i \in \mathcal{P} \setminus X} n_A^i s(|W \cap P_i|) \geq \sum_{P_i \in \mathcal{P} \setminus X} n_A^i s(|W' \cap P_i|)$ and $\sum_{P_i \in \mathcal{P} \setminus X} n_{A^X}^i s(|W' \cap P_i|) \geq \sum_{P_i \in \mathcal{P} \setminus X} n_{A^X}^i s(|W \cap P_i|)$. By Equation (3), this can only be true if both inequalities are tight and hence, $W' \in \mathsf{PAV}(A^X)$ and $W \in \mathsf{PAV}(A)$. Hence, it follows that $\mathsf{PAV}(A) = \mathsf{PAV}(A^X)$ and PAV satisfies weak split/merge-proofness.

Claim 2: PAV is the only non-trivial Thiele rule that satisfies weak split/merge-proofness

For the other direction, let f denote a non-trivial Thiele rule that satisfies weak split/merge-proofness and let s denote the corresponding scoring vector. We will show that f is PAV. First, since f is non-trivial, there is an index $\ell \in \{1, \dots, k\}$ with $s(\ell) > 0$. Moreover, we suppose that ℓ is minimal, i.e., $s(x) = 0$ for all $x < \ell$. As first step, suppose for contradiction that $\ell \neq 1$ and thus $s(1) = 0$. In this case, consider the party-list profile A' in which ℓ voters report the ballot $\{c_1, \dots, c_\ell\}$. By the minimality of ℓ, it holds that $f(A') = \{W \in \mathcal{W}_k : \{c_1, \dots, c_\ell\} \subseteq W\}$. Next, consider the profile A in which each candidate $c_1, \dots, c_\ell$ is approved by a single candidate. Since $s(1) = 0$, it clearly holds that $f(A) = \mathcal{W}_k$. However, weak split/merge-proofness requires that $f(A) = f(A') \neq \mathcal{W}_k$. This is a contradiction and therefore, the assumption that $s(1) = 0$ is wrong. Moreover, since Thiele rules are invariant under scaling and shifting s, we assume from now on that $s(0) = 0$ and $s(1) = 1$.

Next, suppose that there is an index $\ell \in \{2, \dots, k\}$ such that $s(\ell) \neq \sum_{x=1}^{\ell} \frac{1}{x}$. Additionally, we suppose again that ℓ is minimal. We proceed next with a case distinction and suppose first that $s(\ell) < \sum_{x=1}^{\ell} \frac{1}{x}$. In this case, we define $\Delta = \sum_{x=1}^{\ell} \frac{1}{x} - s(\ell)$ and let $t \geq 2$ be an integer such that $t\ell\Delta > 1$. Now, consider the profile A in which each candidate $c_1, \dots, c_\ell$ is approved by t voters, and each candidate $c_{\ell+1}, \dots, c_m$ is approved by $t-1$ voters. Since $s(1) > 0$ and $\ell \leq k$, it is easy to verify that $f(A) = \{W \in \mathcal{W}_k : \{c_1, \dots, c_\ell\} \subseteq W\}$. Next, let $X = \{\{c_1\}, \dots, \{c_\ell\}\}$. In the profile A^X, the score of a committee $W \in \mathcal{W}_k$ is $\sum_{i \in N_A} s(|A_i \cap W|) = t\ell s(|W \cap \bigcup X|) + (t-1)(k - |W \cap \bigcup X|)$. Now, consider two committees $W, W' \in \mathcal{W}_k$ with $|\bigcup X \cap W| = \ell$ and $|\bigcup X \cap W'| = \ell - 1$. The score difference between these two committees is

$$\begin{aligned}
&\sum_{i \in N_A} s(|A_i \cap W|) - \sum_{i \in N_A} s(|A_i \cap W'|) \\
&= t\ell\,(s(\ell) - s(\ell-1)) + (t-1)\,((k-\ell) - (k-\ell+1)) \\
&= t\ell(\sum_{x=1}^{\ell} \frac{1}{x} - \Delta - \sum_{x=1}^{\ell-1} \frac{1}{x}) - (t-1) \\
&= t - t\ell\Delta - (t-1) \\
&< 0.
\end{aligned}$$

Consequently, $W \notin f(A^X)$ as W' has a higher score than W. However, this contradicts weak split/merge-proofness as $\bigcup X \subseteq W$ for all $W \in f(A)$ but $f(A) \neq f(A^X)$. Hence, $s(\ell) \neq \sum_{x=1}^{\ell} \frac{1}{x} - \Delta$ for $\Delta > 0$.

For the second case, suppose that $s(\ell) > \sum_{x=1}^{\ell} \frac{1}{x}$. In this case, we define $\Delta = s(\ell) - \sum_{x=1}^{\ell} \frac{1}{x}$ and let t denote an integer such that $t\Delta > 1$. Next, let A denote the profile in which each candidate $c_1, \dots, c_\ell$ is uniquely approved by t voters, and each candidate $c_{\ell+1}, \dots, c_m$ is uniquely approved $t+1$ voters. It is simple to see that $f(A) = \{W \in \mathcal{W}_k : \forall W' \in \mathcal{W}_k : |W \cap \{c_1, \dots, c_\ell\}| \leq |W' \cap \{c_1, \dots, c_\ell\}|\}$, i.e., f chooses the committees with minimal intersection with $\{c_1, \dots, c_\ell\}$. Next, we let again $X = \{\{c_1\}, \dots, \{c_\ell\}\}$ and consider the profile A^X. The score of a committee $W \in \mathcal{W}_k$ in A^X is $t\ell s(|W \cap \{c_1, \dots, c_\ell\}|) + (t+1)(k - |W \cap \{c_1, \dots, c_\ell\}|)$. Now, consider a $W, W' \in \mathcal{W}_k$ with $\bigcup X \subseteq W$, and $\bigcup X \not\subseteq W'$ and let $j = |\bigcup X \cap W'| < \ell$. Due to the minimality of ℓ, the score difference in these two committees is

$$\begin{aligned}
&\sum_{i \in N_A} s(|A_i \cap W|) - \sum_{i \in N_A} s(|A_i \cap W'|) \\
&= t\ell\,(s(\ell) - s(j)) + (t+1)\,((k-\ell) - (k-j)) \\
&= t\ell\left(\sum_{x=1}^{\ell} \frac{1}{x} + \Delta - \sum_{x=1}^{j} \frac{1}{x}\right) - (t+1)(\ell - j) \\
&\geq t\ell\Delta + t\ell\frac{1}{\ell}(\ell - j) - (t+1)(\ell - j) \\
&> 0.
\end{aligned}$$

Since all committees $W, W' \in \mathcal{W}$ with $\bigcup X \subseteq W \cap W'$ clearly have the same score in A^X, we thus conclude that $f(A^X) = \{W \in \mathcal{W}_k : \bigcup X \subseteq W\}$. Hence, by weak split/merge-proofness, the same needs to be true for $f(A)$, but $f(A)$ consists of the committees with minimal intersection with $\bigcup X$. This is a contradiction as $k < m$ and hence, the assumption that $s(\ell) = \sum_{x=1}^{\ell} \frac{1}{x} + \Delta$ must be wrong, too. Combined with the first case, we infer that $s(\ell) = \sum_{x=1}^{\ell} \frac{1}{x}$, which proves that f is the PAV. □