Technische Universität München

Department of Mathematics

Master's Thesis

Efficiency and Incentives in Randomized Social Choice

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Contents

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1 Introduction

We consider the problem of probabilistic social choice. Given a set of agents with ordinal preferences over alternatives, an outcome is a lottery over alternatives instead of a single alternative or a set of alternatives. We call such a mechanism a *social decision scheme*. This allows to ensure fairness in case of symmetries with respect to alternatives as well as voters. As a special case we study the problem of assigning n objects to n agents where every agent receives a lottery over alternatives. Before reasoning about mechanisms, which return a lottery or random assignment we need to make assumptions on how agents compare lotteries. In particular, we study how to extend preferences over alternatives to preferences over lotteries. Properties of randomized mechanisms depend largely on the underlying lottery extension. A commonly used extension is (first order) *stochastic dominance (SD)*, justified by its connection to von Neumann-Morgenstern-utilities. If one lottery stochastically dominates another, the former yields higher expected utility for every compatible vNM-utility function. However, Cho (2012) questions the unchallenged use of stochastic dominance. He introduces new lottery extension and studies them axiomatically. A generalization of the vNM-utility model was introduced by Fishburn (1982b). *Skew symmetric bilinear-utility (SSB-utility)* weakens the axioms of order and independence vNM-utility is based on. In a similar way as vNM-utility, this utility notion can be extended to a lottery extension.

One key question in choice theory is *efficiency* or not choosing a lottery which can be improved for some agent without making another agent worse off. For an inventory of extensions, we study efficiency in social choice leading to generalizations of characterizations obtained by Cho and Bogomolnaia and Moulin (2001) for assignment problems. Another notion of non-wastefulness is *social welfare maximization*—maximizing the overall sum of utilities. For assignments Bogomolnaia and Moulin show that social welfare maximization for some vNM-utility function implies *SD*-efficiency. McLennan (2002) proves the converse in what is known as the *ordinal efficiency welfare theorem.* One of our main contributions is a generalization of this statement, further generalizing a theorem by Carroll (2010) to SSB-utility. Besides efficiency, a desirable property of social decision schemes and assignment mechanisms is not to provide incentive for agents to misrepresent their preferences, which is referred to as *strategyproofness*. The notion of strategyproofness is based on the underlying lottery extension. We check well-known mechanisms for their efficiency and strategyproofness properties and provide impossibility results for some combinations of these properties.

2 Preliminaries

In Section 4, we are concerned with one agent extending her preferences over deterministic outcomes to preferences over probability distributions on alternatives. We denote by $A = \{1, 2, \ldots, m\}$ the set of alternatives. Alternatives are denoted by x, y, z. Each agent is equipped with a preference relation R over alternatives, which we assume to be complete and transitive unless stated otherwise. For the set of such relations we use $\mathcal{R}(A)$. We denote by P the corresponding strict preference relation, i.e., $x \ P \ y$ if $x \ R \ y$ and not $y \ R \ x$ and indifference by I, i.e., $x \ I \ y$ if $x \ R \ y$ and $y \ R \ x$. As a short notation for a preference relation we use e.g., $x, \{yz\}$, meaning $x \ P \ y \ I \ z$. Note that we allow for weak preferences. Given a relation R over objects, we define $\max_R(B) = \{x \in B \colon x \ R \ y \text{ for all } y \in B\}$ as the set of maximal elements of B. Then the *k-th indifference class* is defined recursively as follows:

$$I^k = \max_R(A \setminus \bigcup_{j=1}^{k-1} I^j)$$

for $k = 1, 2, \ldots$. A *lottery* is a probability distribution over alternatives. The set of lotteries is denoted by ΔA. For elements of ΔA, we use p, q, r, p', and q'. The probability that p assigns to an alternative x is denoted by p_x. The *support* of a lottery is $supp(p) = \{x \in A \colon p_x > 0\}$. A lottery is *degenerate* if $|supp(p)| = 1$. To state concrete instances of lotteries, we write e.g., $p = [x \colon \frac{1}{2}, y \colon \frac{1}{2}]$ for the lottery which gives probability 1/2 each to x and y. In case it is clear from the context which alternatives we refer to, we omit the latter and write a vector of probabilities only, e.g., $p = [\frac{1}{2}, \frac{1}{2}]$. Our goal is to extend preferences over objects to preferences over lotteries. Let $\mathcal{R}(\Delta A)$ be the set of reflexive relations over lotteries. Formally, a *lottery extension* is a mapping $e \colon \mathcal{R}(A) \to \mathcal{R}(\Delta A)$. Let R^e denote the relation over lotteries obtained by extending R. Again P^e and I^e denote the corresponding strict preference and indifference relation. By R^{-1} we denote the inverse of R, i.e., for all $x, y \in A$, $x \ R \ y$ if and only if $y \ R^{-1} \ x$. Similarly, if R^e is a relation over lotteries, we denote by $(R^e)^{-1}$ the relation which reverses the comparison of lotteries, i.e., $p \ R^e \ q$ if and only if $q \ (R^e)^{-1} \ p$. We say that an extension e satisfies a property p if for each relation R, R^e satisfies p.

3 vNM and SSB-utility

A common way to compare lotteries are (linear) *von Neumann-Morgenstern utilities.* An agent assigns an expected utility value to each lottery. Then lotteries are ordered according to the utility value they yield. Formally, a vNM-utility function is a linear mapping $u\colon \Delta A \to \mathbb{R}$. Every vNM-utility function is uniquely determined by its values on degenerate lotteries, since by linearity, the expected utility of a lottery $p \in \Delta A$ is

$$u(p) = \sum_{x \in A} p_x u(x).$$

We say u is *consistent* with $R \in \mathcal{R}(A)$ if for all $x, y \in A$, $u(x) - u(y) \geq 0$ if and only if $x \mathrel{R} y$. For more on linear utilities we refer to von Neumann and Morgenstern (1953). This approach is justified by an axiomatic characterization of vNM-utility. For a relation R on lotteries, we consider the axioms of continuity, order, and independence:

$$p \mathrel{P} q \mathrel{P} r \text{ implies } \exists \alpha \in (0,1)\colon [\alpha p + (1-\alpha) r] \mathrel{I} q \tag{CON}$$

$$R \text{ is transitive} \tag{ORD}$$

$$p \mathrel{P} q \text{ and } \lambda \in (0,1) \text{ implies } [\lambda p + (1-\lambda) r] \mathrel{P} [\lambda q + (1-\lambda) r] \tag{IND}$$

These axioms imply that R can be expressed by a vNM-utility function, i.e., R satisfies the axioms of continuity, order, and independence if and only if there is a vNM-utility function u such that $R = \{(p,q)\colon u(p) - u(q) \geq 0\}$ (see e.g., Fishburn, 1970). The continuity axiom we use here is slightly different from the axiom used by Fishburn. It can be shown however, that in the presence of order and independence, they are equivalent. Despite being the dominant utility concept in economic literature, vNM-utility has been widely criticized. Especially the independence axiom does not hold in many real world examples. One instance of this is the preference reversal phenomenon, which is discussed by Grether and Plott (1979) among others. Experiments show that an agent who prefers p to q might prefer $[\lambda q + (1-\lambda) r]$ to $[\lambda p + (1-\lambda) r]$ for appropriate $\lambda \in (0,1)$ and $r \in \Delta A$, reversing her preferences over p and q. Such an occurrence contradicts independence.

Tackling this shortcoming, Fishburn (1982b) introduced *skew symmetric bilinear utility (SSB-utility)*, which is a generalization of vNM-utility. An SSB-function ϕ is a mapping from $\Delta A \times \Delta A$ into the reals which is skew symmetric and bilinear[1], i.e.,

$$\phi(p,q) = -\phi(q,p),$$
$$\phi(\alpha p + \beta q, r) = \alpha\phi(p,r) + \beta\phi(q,r).$$

The value $\phi(p,q)$ can be seen as a measure of how much p is preferred to q. For a reasonable decision maker, one would expect that p is preferred to q by the same margin than q is less preferred than p, thus, skew symmetry appears reasonable. Bilinearity is similar to linearity in the vNM-utility theory. However, in the SSB-model linearity is only required for the preference margin with respect to a fixed lottery, not in terms of absolute utility.

Fishburn (1982b) introduces the axioms of dominance and symmetry. In the following definition of these axioms, p, q, and r are lotteries and $\lambda \in (0,1)$.

$$p\ P\ q \text{ and } p\ R\ r \text{ implies } p\ P\ [\lambda q + (1-\lambda)r];\ q\ P\ p \text{ and } r\ R\ p \text{ implies } [\lambda q + (1-\lambda)r]\ P\ p;$$
$$p\ I\ q \text{ and } p\ I\ r \text{ implies } p\ I\ [\lambda q + (1-\lambda)r] \qquad \text{(DOM)}$$
$$p\ P\ q\ P\ r,\ p\ P\ r \text{ and } q\ I\ [\frac{1}{2}p + \frac{1}{2}r] \text{ implies } [\lambda p + (1-\lambda)r]\ I\ [\frac{1}{2}p + \frac{1}{2}q]$$
$$\text{if and only if } [\lambda r + (1-\lambda)p]\ I\ [\frac{1}{2}r + \frac{1}{2}q] \qquad \text{(SYM)}$$

It can be checked that the combination of order and independence implies dominance. Roughly spoken, dominance makes for bilinearity of the SSB-function. It can also be looked at as a convexity condition. If p and q are both preferred to z, then any convex combination of p and q is preferred to z. Thus, the set of lotteries dominating z is convex. The symmetry axiom is rather unintuitive, but it is a direct implication of independence. To see this, assume $q\ I\ [\frac{1}{2}p + \frac{1}{2}r]$. By independence $[\frac{1}{2}p + \frac{1}{2}q]\ I\ [\frac{3}{4}p + \frac{1}{4}r]$ and $[\frac{1}{2}r + \frac{1}{2}q]\ I\ [\frac{3}{4}r + \frac{1}{4}q]$. So for $\lambda = \frac{3}{4}$ both indifferences hold, while for other values of λ neither holds. Intuitively, symmetry makes for skew symmetry of the SSB-function. For more explanation on symmetry see Fishburn (1982a).

Fishburn shows that a relation R on lotteries satisfies the axioms of continuity, dominance, and symmetry if and only if there is an SSB-function ϕ such that $R = \{(p,q) \colon \phi(p,q) \geq 0\}$. Moreover, ϕ is unique up to similarity transformations. By linearity, an SSB-function is uniquely defined if its values on degenerate lotteries (or alternatives) are

[1]Operations on lotteries are defined element wise. So for $\lambda \in [0,1]$, $p, q \in \Delta A$, and $x \in A$, $(\lambda p + (1-\lambda)q)_x = \lambda p_x + (1-\lambda)q_x$

fixed. For lotteries p, q, we get

$$\phi(p,q) = \sum_{(x,y)\in A\times A} p_x q_y \phi(x,y).$$

Thus, a skew symmetric function on $A \times A$ can be extended to an SSB-function on $\Delta A \times \Delta A$. ϕ is *consistent* with $R \in \mathcal{R}(A)$ if $\phi(x,y) \geq 0$ if and only if $x \; R \; y$. The set of all SSB-functions will be denoted by Φ. Obviously, every vNM-utility function u can be associated with an SSB-function $\phi_u(p,q) = u(p) - u(q)$, which shows that the SSB-model is more general than vNM-utilities.
One natural subclass of SSB-functions are SSB-functions monotonically increasing in the first argument. Intuitively spoken, if x is preferred to y, it is reasonable to demand that the utility of x compared to z is at least as high as the utility of y compared to z for any alternative z. To make this formal, we define the following subset of SSB-functions:

Definition 1. An SSB-function ϕ is monotonically increasing in the first argument if and only if for all $x, y, z \in A$

$$x \; R \; y \text{ implies } \phi(x,z) \geq \phi(y,z)$$

The set of SSB-functions monotonically increasing in the first argument is denoted by Φ_1.

We define a monotonicity axiom introduced by Cho (2012) to characterize Φ_1. Therefore additional terminology is needed. A lottery p is a *monotonic improvement* over q if p is obtained from q by shifting probability from some alternative to a more preferred alternative. We write $p \; M_R \; q$ if there exist x, y such that $x \; P \; y$ and $p_x \geq q_x$ and $p_z = q_z$ for all $z \in A \setminus \{x, y\}$. Monotonicity is satisfied if monotonically improving a lottery yields a more preferred lottery.

$$p \; M_R \; q \text{ implies } p \; R \; q \tag{MON}$$

If a relation on lotteries satisfies the axioms needed for the SSB-representation and monotonicity in addition, it can be expressed by a monotonically increasing SSB-function.

Theorem 1. *Let $R \in \mathcal{R}(\Delta A)$. Then the following are equivalent:*

1. *R satisfies* (CON), (DOM), (SYM), *and* (MON),
2. $\exists \phi \in \Phi_1 \colon R = \{(p,q) \colon \phi(p,q) \geq 0\}$.

Proof. For the direction from left to right, assume that R satisfies the axioms listed in 1. By Theorem 1 of Fishburn (1982b), there exists $\phi \in \Phi$, such that $\phi(p, q) \geq 0$ if and only if $p\ R\ q$. Now, assume for a contradiction that $\phi \notin \Phi_1$. By definition of Φ_1, there exist $x, y, z \in A$ with $x\ R\ y$ and $\phi(x, z) < \phi(y, z)$. Define $p = [x \colon \epsilon, y \colon 0, z \colon (1-\epsilon)]$ and $q = [x \colon 0, y \colon \epsilon, z \colon (1-\epsilon)]$. Then p is a monotonic improvement over q but $\phi(p, q) = \epsilon^2\phi(x, y) + \epsilon(1-\epsilon)(\phi(x, z) - \phi(y, z)) < 0$ for $\epsilon > 0$ small, which contradicts monotonicity of R.

The opposite direction is immediate and the proof is omitted. □

It is worth noting that the axioms in Theorem 1 do not imply either of the axioms of order and independence. This will be shown formally in the discussion of an instance of a function in Φ_1 later.

4 Lottery Extensions

In this section we discuss the problem of how to compare lotteries over alternatives. In general this can be an arbitrary relation on the set of lotteries. However, we focus on the case where an agent submits ordinal preferences over alternatives and her preferences over lotteries can only depend on those. Preferences are extended by applying an extension operator. Therefore, we first introduce some desirable axioms on extension operators. Afterwards we state instances of operators and conclude the section by checking their properties and examining inclusion relationships of the extended preference relations.

4.1 Axioms on Extensions

When reasoning about preferences over lotteries, it might be desirable to impose certain conditions. Therefore, in the following we study axioms on lottery extensions. For more axioms and an extensive discussion of those we refer to Cho (2012). Throughout this section, e denotes a lottery extension. For some properties frequently considered for relations over alternatives, their definition directly carries over to lotteries.

An extension is *anti-symmetric* if there cannot be indifference between unequal lotteries, i.e., $p\ I^e\ q$ implies $p = q$ for all $R \in \mathcal{R}(A)$. *Completeness* of an extension implies that extending any preference relation yields a complete relation over lotteries. Formally, e is complete if R^e is complete for all $R \in \mathcal{R}(A)$. A *transitive* extension produces transitive relations over lotteries, that is, R^e is transitive for all $R \in \mathcal{R}(A)$.

Monotonicity was already discussed in Section 3. Since this notion is quite strong for incomplete extensions, we introduce weak monotonicity. An extension is *weakly monotonic* if monotonically improving a lottery does not make it a less preferred lottery, i.e., $p\ M_R\ q$ implies not $q\ P\ p$. For complete extensions, both notions of monotonicity coincide. Note that all axioms defined so far do not require an extension operator.

For the last axiom, we introduce the notion of duality. Two operators e and e' are *dual* if $(R^e)^{-1} = (R^{-1})^{e'}$. An extension is *self-dual* if it is its own dual. So extending a relation and reversing preferences over lotteries yields the same as extending the reversed preferences over alternatives. Self-duality can be seen as a symmetry condition for extensions with respect to extending from top down and from bottom up respectively.

4.2 Extension Operators

Given preferences over alternatives, it is not clear how to compare lotteries over alternatives. An extension operator maps preferences over alternatives to (not necessarily complete) preferences over lotteries, allowing to compare them. In this section, we define the lottery extensions relevant for the remainder.

In the following, $R \in \mathcal{R}(A)$ is a preference relation and $p, q \in \Delta A$ are lotteries. The *trivial lottery extension (TRIV)* only allows to compare degenerate lotteries. Preferences over alternatives extend only to degenerate lotteries, i.e.,

$$p\ R^{TRIV}\ q \text{ if } p_x q_y = 1 \text{ for some } x\ R\ y.$$

Next, we consider *deterministic dominance (DD)* (Kelly, 1977). The reasoning behind deterministic dominance is that an agent should be certain to receive a better alternative in p than in q. Formally,

$$p\ R^{DD}\ q \text{ if } p = q \text{ or } x\ P\ y \text{ for all } x \in supp(p), y \in supp(q).$$

The *sure thing (ST)* extension was introduced by Aziz et al. (2013b) and is similar to deterministic dominance, but ignores all alternatives to which p and q assign the same probability.

$$p\ R^{ST}\ q \text{ if for all } x \in supp(p),\ \text{either } x\ P\ y\ \text{ for all } y \in supp(q) \text{ or } p_x = q_x.$$

One of the most frequently used extensions in the literature is *stochastic dominance (SD)*. A large part of the popularity of stochastic dominance is its motivation through expected vNM-utility, which we will discuss in more detail later. The lottery p stochastically dominates q if for every alternative x, the probability that p yields an alternative at least as good as x is higher than the corresponding probability for q.

$$p \; R^{SD} \; q \text{ if for all } x \in A\colon \sum_{y\colon yRx} p_y \geq \sum_{y\colon yRx} q_y.$$

Fishburn (1984b) formally introduced *bilinear dominance (BD)*. It requires that for every pair of alternatives x, y with $x \; R \; y$, the ratio of probabilities $p(x)/p(y)$ is higher than $q(x)/q(y)$. To avoid division by zero,

$$p \; R^{BD} \; q \text{ if for all } x, y \in A, x \; P \; y \text{ implies } p_x q_y - p_y q_x \geq 0.$$

As an instance of a vNM-utility function, we introduce *equidistant utility (EU)*. The associated utility function u^{EU} gives utility $m-k$ to the k-th ranked alternative. Lotteries are ordered according to the utility they yield, i.e.,

$$p \; R^{EU} \; q \text{ if } u^{EU}(p) \geq u^{EU}(q).$$

There are several reasonable ways to extend this function to weak preferences. Since we only consider this extension for comparisons sake, we will not go into more detail here. *Pairwise comparison (PC)* (Brandt) reasons about lotteries without referring to vNM-utility. It is obtained from a natural instance of an SBB-function that is

$$\phi^{PC}(x, y) = \begin{cases} 1 & \text{if } x \; P \; y, \\ 0 & \text{if } x \; I \; y, \\ -1 & \text{if } y \; P \; x. \end{cases}$$

This SSB-function induces a relation on lotteries by comparing SSB-utility, i.e.,

$$p \; R^{PC} \; q \text{ if } \phi^{PC}(p, q) \geq 0.$$

The reasoning behind pairwise comparison is to compare lotteries by preferring those lotteries that yield better alternatives more frequently. This seems especially reasonable if no assumptions about the agents utility profile or more general, the intensity of their pairwise comparisons can be made.

Finally, we define two extensions which have been introduced and extensively discussed by Cho (2012), the *downward lexicographic (DL)* and *upward lexicographic (UL)* extension. They can be looked at as preferences of an optimistic and pessimistic agent respectively. Downward lexicographic ordering prefers the lottery with higher probability on the most preferred alternative, in case this is equal, with higher probability on the second most preferred alternative and so on. Formally,

$$p \; R^{DL} \; q \text{ if } p = q \text{ or there exists } x \in A\colon p_y = q_y \text{ for all } y \; P \; x \text{ and } p_x > q_x.$$

Upward lexicographic ordering is opposed to the latter, starting at the bottom ranked alternative and moving upwards, i.e.,

$$p \; R^{UL} \; q \text{ if } p = q \text{ or there exists } x \in A\colon p_y = q_y \text{ for all } x \; P \; y \text{ and } p_x < q_x.$$

Lexicographic preferences extend straightforwardly to weak preferences. To adjust the definitions, p_x is replaced with the sum of probabilities over all alternatives in the corresponding indifference class. Equality of lotteries is achieved if they assign the same probabilities to all indifference classes.
It is worth noting that some of the extension operators mentioned above can extend intransitive preferences over alternatives without further adjustments. Clearly the trivial lottery extension fulfills this criterion, but more relevant, the extensions based on SSB-utility, namely bilinear dominance and pairwise comparison, since the SSB-representation does not require transitivity. Fishburn (1978) studied how to also extend stochastic dominance to intransitive preferences.

4.3 Characterization of Extensions

Although the comparison of lotteries obtained from a certain extension might make intuitive sense, there should be theoretical justification for it. One approach is to characterize an extension given assumptions about the underlying utility model. We will examine three different possibilities. Agents are equipped either with a vNM-utility function, an SSB-function, or an SSB-function which is monotonically increasing in the first argument. In either case, only the underlying utility model is known and not the concrete utility function.
The popularity of stochastic dominance is explained by its connection to vNM-utility. As shown in the following well-known theorem, a lottery p stochastically dominates q if and only if p yields higher expected utility than q for any vNM-utility function which is

consistent with the agents preferences.

Theorem 2. *Let U be the set of all vNM-utility functions consistent with $R \in \mathcal{R}(A)$. Then*

$$p \; R^{SD} \; q \textit{ if and only if } u(p) - u(q) \geq 0 \textit{ for all } u \in U.$$

In fact, stochastic dominance is the appropriate lottery extension even if we widen the class of an agents possible utility functions. If an agents preferences over lotteries are evaluated with an SSB-function monotonically increasing in the first argument, she will necessarily prefer p over q if p stochastically dominates q as shown by Fishburn (1984b).

Theorem 3. *Let Φ_1 be the set of all SSB-functions monotnoically increasing in the first argument and consistent with $R \in \mathcal{R}(A)$. Then*

$$p \; R^{SD} \; q \textit{ if and only if } \phi(p,q) \geq 0 \textit{ for all } \phi \in \Phi_1.$$

However, stochastic dominance is too weak if an agent can have arbitrary SSB-preferences. To illustrate this matter, we provide an example. Consider an agent with preferences xyz and let $p = [x\colon \frac{1}{3}, y\colon \frac{1}{3}, z\colon \frac{1}{3}]$ and $q = [x\colon \frac{1}{6}, y\colon \frac{1}{2}, z\colon \frac{1}{3}]$. It can be checked that p stochastically dominates q. For an SSB-function with values $\phi(x,y) = 1$, $\phi(x,z) = 1$, and $\phi(y,z) = 4$, we get

$$\phi(p,q) = (\frac{1}{6} - \frac{1}{18}) \cdot \phi(x,y) + (\frac{1}{9} - \frac{1}{18}) \cdot \phi(x,z) + (\frac{1}{9} - \frac{1}{6}) \cdot \phi(y,z) = -\frac{1}{18}$$

Thus, an agent with SSB-preferences according to ϕ would not prefer p to q. Therefore, stochastic dominance cannot serve as a lottery extension in this case. However, bilinear dominance is more restrictive and is linked similarly to SSB-utility as is stochastic dominance to vNM-utility. The next theorem found by Fishburn formalizes this statement.

Theorem 4. *Let Φ be the set of all SSB-functions consistent with $R \in \mathcal{R}(A)$. Then*

$$p \; R^{BD} \; q \textit{ if and only if } \phi(p,q) \geq 0 \textit{ for all } \phi \in \Phi.$$

Proof. First we prove the direction from left to right. Let p, q be two lotteries and ϕ an arbitrary SSB-function. Then

$$\begin{aligned}
\phi(p,q) &= \sum_{x,y} p_x q_y \phi(x,y) && (1)\\
&= \sum_{x} \sum_{y\colon xPy} p_x q_y \phi(x,y) + \sum_{x} \sum_{y\colon yPx} p_x q_y \phi(x,y) && (2)\\
&= \sum_{x} \sum_{y\colon xPy} p_x q_y \phi(x,y) - \sum_{x} \sum_{y\colon yPx} p_x q_y \phi(y,x) && (3)\\
&= \sum_{x} \sum_{y\colon xPy} p_x q_y \phi(x,y) - \sum_{y} \sum_{x\colon xPy} p_y q_x \phi(x,y) && (4)\\
&= \sum_{x} \sum_{y\colon xPy} p_x q_y \phi(x,y) - \sum_{x} \sum_{y\colon xPy} p_y q_x \phi(x,y) && (5)\\
&= \sum_{x} \sum_{y\colon xPy} (p_x q_y - p_y q_x) \phi(x,y) \geq 0. && (6)
\end{aligned}$$

In (1) and (2) we use skew symmetry. In (3) labels of x and y are switched in the second sum and (4) is reordering of the latter. In (6) the definition of $p\ R^{BD}\ q$ is used.
For the opposite direction, let $\phi(p,q) \geq 0$ for all SSB-functions $\phi \in \Phi$ and assume for a contradiction that there exist $a, b \in A$ such that $a\ P\ b$ and $p_a q_b - p_b q_a < 0$. Now define the SSB-function ϕ for $x, y \in A$, $x\ R\ y$ (the rest follows by skew symmetry) as

$$\phi(x,y) = \begin{cases} \epsilon & \text{if } x\ P\ y \text{ and } (x,y) \neq (a,b) \\ 0 & \text{if } x\ I\ y \\ 1 & \text{otherwise} \end{cases}$$

for some $\epsilon > 0$. Then

$$\begin{aligned}
\phi'(p,q) &= \sum_{x,y} p_x q_y \phi(x,y)\\
&\leq 2m^2\epsilon + 2(p_a q_b - p_b q_a) < 0
\end{aligned}$$

for small ϵ, which yields a contradiction. □

We note that the DL and UL extension cannot be characterized as being consistent with a subset of SSB-functions for three or more alternatives. Assume there was a subset Φ' of SSB-functions, such that $p\ R^{DL}\ q$ if and only if $\phi(p,q) \geq 0$ for all $\phi \in \Phi'$ and let ϕ be an element of it. For three alternatives $x\ P\ y\ P\ z$ and the lotteries $p = [x\colon \epsilon, y\colon 0, z\colon 1-\epsilon]$ and $q = [x\colon 0, y\colon 1, z\colon 0]$, we have $p\ R^{DL}\ q$, but $\phi(p,q) < 0$ for ϵ small enough.

For further research it would be interesting to find other natural subsets of SSB-functions and the corresponding lottery extensions. However, this would likely not be subsets of vNM-utility functions, since these seem already too restrictive.

4.4 Properties of and Relations among Extensions

In order to create a clearer picture about the relations introduced in this section, we check some of them for the axioms defined above. We also establish inclusion relationships between various extensions, which will help to compare the strength of the associated efficiency and strategyproofness notions. Of particular interest in this respect are bilinear dominance and pairwise comparison. For more on stochastic dominance and downward and upward lexicographic preferences see e.g., Cho (2012).
We consider bilinear dominance first. By Theorems 2 and 4, it is clear that R^{BD} is as subset of R^{SD}, which already implies incompleteness. From the definitions it can be quickly verified that the sure-thing relation is a subset of the bilinear dominance relation. The following theorem clarifies which of the remaining axioms are satisfied.

Theorem 5. *R^{BD} is anti-symmetric, transitive, self-dual, weakly monotonic, but not monotonic.*

Proof. For anti-symmetry, let $p\ I^{BD}\ q$. This implies $p_x q_y - p_y q_x = 0$ for all x and y by definition. Consequently, $p = q$ since probabilities add up to one in both lotteries.
To prove transitivity, let $p\ R^{BD}\ q\ R^{BD}\ r$. This implies $p_x q_y - p_y q_x \geq 0$ and $q_x r_y - q_y r_x \geq 0$ for all $x\ R\ y$. We have to show $p_x r_y - p_y r_x \geq 0$. If $r_x = 0$ this is obvious. Otherwise we get $q_y \leq q_x \frac{r_y}{r_x}$. Using the assumption, we have $0 \leq p_x q_y - p_y q_x \leq p_x q_x \frac{r_y}{r_x} - p_y q_x = q_x (p_x \frac{r_y}{r_x} - p_y)$, which implies $p_x r_y - p_y r_x \geq 0$.
Self-duality is shown in the following way:

$$\begin{aligned} p\ (R^{BD})^{-1}\ q &\Leftrightarrow q\ R^{BD}\ p \\ &\Leftrightarrow \text{ for all } x, y \in A, y\ P\ x \colon q_y p_x - q_x p_y \geq 0 \\ &\Leftrightarrow \text{ for all } x, y \in A, x\ P^{-1}\ y \colon p_x q_y - p_y q_x \geq 0 \\ &\Leftrightarrow p\ (R^{-1})^{BD}\ q. \end{aligned}$$

For weak monotonicity, assume q is monotonically improved to p by shifting probability from y to x with $x\ P\ y$. It is straightforward to see that $p_x q_y - p_y q_x > 0$, thus, q cannot be SSB-preferred to p.
As an example contradicting monotonicity let $x\ P\ y\ P\ z$ and $p = [x \colon \frac{1}{2}, y \colon 0, z \colon \frac{1}{2}]$,

Axioms on extensions	ST	BD	SD	PC	EU	DL	UL
Anti-symmetry	+	+	+	-	-	+	+
Completeness	-	-	-	+	+	+	+
Transitivity	+	+	+	-	+	+	+
weak Monotonicity	+	+	+	+	+	+	+
Monotonicity	-	-	+	+	+	+	+
Self-duality	+	+	+	+	+	-	-

Table 1: Properties of lottery extensions

$q = [x\colon 0, y\colon \frac{1}{2}, z\colon \frac{1}{2}]$. Then p is a monotonic improvement over q but $p_b q_c - p_c q_b = -\frac{1}{4} < 0$. $\square$

We focus on pairwise comparison next. It is easy to see from the definition that ϕ^{PC} is monotonically increasing in the first argument. Thus, by Theorem 3 $R^{SD} \subset R^{PC}$ follows immediately. Clearly, R^{PC} is complete, since $\phi^{PC}(p, q)$ is well defined for any pair of lotteries. Now we gather the rest of the axioms satisfied by pairwise comparison.

Theorem 6. *R^{PC} is not anti-symmetric, not transitive, self-dual, and monotonic.*

Proof. For anti-symmetry, consider preferences xyz and the lotteries $p = [x\colon \frac{1}{2}, y\colon 0, z\colon \frac{1}{2}]$ and $q = [x\colon 0, y\colon 1, z\colon 0]$. It can be checked that $\phi^{PC}(p, q) = 0$ despite p and q not being equal.
Transitivity even in a weaker sense is violated, since preferences can cycle strictly. For the lotteries $p = [\frac{3}{5}, 0, 0, \frac{2}{5}]$, $q = [0, 1, 0, 0]$, and $r = [\frac{3}{7}, 0, \frac{4}{7}, 0]$, we have $p \; P^{PC} \; q \; P^{PC} \; r \; P^{PC} \; p$. This example also shows a violation of independence. p is preferred to q by pairwise comparison, but $\lambda p + (1-\lambda) r$ is not preferred to $\lambda q + (1-\lambda) r$ for $\lambda < \frac{1}{2}$.
To prove self-duality, we have to show $(R^{PC})^{-1} = (R^{-1})^{PC}$ for any relation R.

$$\begin{aligned} p(R^{PC})^{-1} q &\Leftrightarrow \phi^{PC}(p, q) \leq 0 \Leftrightarrow \sum p_x q_y \phi^{PC}(x, y) \leq 0 \\ &\Leftrightarrow \sum p_x q_y (-\phi^{PC}(x, y)) \geq 0 \Leftrightarrow p(R^{-1})^{PC} q \end{aligned}$$

Monotonicity follows from the characterization of Φ_1 in Theorem 1. $\square$

Properties of the other extensions and their relation are either well established or easy to see. Figure 1 gives an illustration of how the previously defined extensions relate to each other. None of the missing inclusions hold, e.g., there is no relation between R^{DL} and R^{PC}. A table which shows properties of various extension is given in Table 1.

$$
\begin{tikzcd}
 & & TRIV \arrow[d] & & \\
 & & DD \arrow[d] & & \\
 & & ST \arrow[d] & & \\
 & & BD \arrow[d] & & \\
 & & SD \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
DL & UL & & PC & EU
\end{tikzcd}
$$

Figure 1: Inclusion relationship between lottery extensions. An arrow denotes set inclusion between the corresponding relations on lotteries, e.g., $R^{SD} \subset R^{UL}$.

5 Randomized Social Choice

Our objective so far was to reason about how a single agent compares lotteries. This is the foundation for analyzing an economy with multiple agents, who have preferences over a set of alternatives. Early impossibility results by de Condorcet (1785) and Arrow (1951) and a number of follow up results for deterministic social choice suggest that randomized social choice might be a promising escape route. The goal is to map a preference profile to a lottery in a fair, efficient, and non-manipulable manner. Since the set of feasible lotteries is much larger than the set of possible deterministic outcomes, there is hope for such a mapping, a *social decision scheme (SDS)*, to exist for reasonable combinations of these properties.

After introducing the required notation, we will examine different notions of efficiency in detail. Then we show a generalization of the ordinal efficiency welfare theorem by McLennan (2002). In the last part we discuss social decision schemes—both in general and for particular instances—and prove several impossibilities.

As before, the set of alternatives is denoted by A. The set of agents (or voters) is $N = \{1, 2, \ldots, n\}$. We use $1, 2, \ldots$ to reference specific agents. Each of them is equipped with preferences over alternatives R_i, the preference profile including all agents preferences is denoted by R. The notation for preferences over lotteries is adapted in the same way. We say that a lottery p *e-dominates* q at a profile R, if all agents e-prefer the former to the latter, i.e., $p\ R_i^e\ q$ for all $i \in N$ with one strict preference. If a lottery is not e-dominated by any lottery, it is *e-efficient*.

5.1 Efficiency

Our goal in this section is to study the connection between efficiency and the support of a lottery. Whenever for an extension e-efficiency only depends on the support of a lottery, we say e-efficiency is support dependent. In this case, we give a characterization of supports for which lotteries are e-efficient using a set extension which is a mapping from preferences over alternatives to preferences over sets of alternatives. This can be an incomplete relation just as for lottery extensions. If all lotteries which have a certain support are efficient, we call this an efficient support. It can be seen from the definition that efficiency becomes stronger if the relation on lotteries becomes more complete, e.g., $R^{SD} \subset R^{UL}$ implies that *UL*-efficiency is stronger than *SD*-efficiency.

5.1.1 *SD*-efficiency

It has been shown by Bogomolnaia and Moulin (2001) that in the random assignment domain *SD*-efficiency is support dependent. Efficient supports are characterized by a relation on alternatives. We will generalize their statement to social choice. First, we show that if a lottery is *SD*-efficient, every lottery with equal or smaller support is also *SD*-efficient. A further theorem characterizes *SD*-efficient supports using the newly introduced retentive set extension.

Theorem 7. *If p is SD-efficient and $supp(q) \subseteq supp(p)$, then q is SD-efficient.*

Proof. The idea is that if q is *SD*-dominated by some lottery q' we can shift probability shares the same way in p as from q to q' because of contained supports and construct a lottery which dominates p. Thus, assume for a contradiction that p is *SD*-efficient, $supp(q) \subset supp(p)$, and q is stochastically dominated by some lottery q'. Define $\Delta = q' - q$, i.e., for all $x \in A, \Delta(x) = q'(x) - q(x)$. Then by dominance, we have for all $i \in N, x \in A$:

$$\sum_{yR_ix} q(y) + \Delta(y) = \sum_{yR_ix} q'(y) \geq \sum_{yR_ix} q(y).$$

Therefore, for all $i \in N, x \in A$:

$$\sum_{yR_ix} \Delta(y) \geq 0. \tag{7}$$

Now we define $p' = p + \epsilon\Delta$. p' is a proper lottery, because $supp(q) \subseteq supp(p)$, which implies $p'(x) \geq 0$ for all $x \in A$ if $\epsilon > 0$ small. The probabilities in p' sum up to 1, since Δ is the elementwise difference of two lotteries. Now let $i \in N$ and $x \in A$, then

$$\sum_{yR_ix} p'(y) = \sum_{yR_ix} p(y) + \epsilon \sum_{yR_ix} \Delta(y) \geq \sum_{yR_ix} p(y). \tag{8}$$

The inequality in (8) holds because of (7). This shows that p' stochastically dominates p an contradicts the assumption. $\square$

Interestingly, PC-efficiency is not support dependent. Intuitively, this is the case, since the effect on PC of shifting probability also depends on the distribution of probability. A counter example proves this fact.

Remark 1. *Theorem 7 does not hold if SD-efficiency is replaced by PC-efficiency.*

Proof. Consider the following preference profile with 4 agents:

$$\begin{aligned} 1&: a, b, c, d \\ 2&: d, b, c, a \\ 3&: a, b, d, c \\ 4&: c, a, d, b \end{aligned}$$

The lottery $p = [0, \frac{1}{2}, \frac{1}{2}, 0]$ is not PC-efficient, since it is dominated by $p' = [\frac{1}{2}, 0, 0, \frac{1}{2}]$. It can be checked that agents $1, 2, 4$ are indifferent and 3 strictly prefers p' over p according to PC. But $q = [0, \frac{1}{4}, \frac{3}{4}, 0]$ has the same support as p and is PC-efficient. This is shown by solving a system of linear inequalities.

Let $[x_a, x_b, x_c, x_d]$ be a lottery that PC-dominates q. Then we get one equation and one inequality for each agent.

$$x_a, x_b, x_c, x_d \geq 0 \tag{9}$$

$$x_a + x_b + x_c + x_d = 1 \tag{10}$$

$$x_a + \frac{3}{4}x_b - \frac{1}{4}x_c - x_d \geq 0 \tag{11}$$

$$-1x_a + \frac{3}{4}x_b - \frac{1}{4}x_c + x_d \geq 0 \tag{12}$$

$$x_a + \frac{3}{4}x_b - \frac{1}{4}x_c + \frac{1}{2}x_d \geq 0 \tag{13}$$

$$-\frac{1}{2}x_a - \frac{3}{4}x_b + \frac{1}{4}x_c - \frac{1}{2}x_d \geq 0 \tag{14}$$

Adding up (11) and (12) gives $3x_b \geq x_c$. Then we plug $3x_b$ instead of x_c in (14) and get $-\frac{1}{2}x_a - \frac{1}{2}x_d \geq 0$, which implies that $x_a = x_d = 0$. Using $x_a = x_d = 0$ in (14) yields $x_c \geq 3x_b$. Hence, $3x_b = x_c$, since $3x_b \geq x_c$ from before. Thus, putting $3x_b = x_c$ in (10) gives $4x_b = 1$. So finally $x_b = 1/4$ and $x_c = 3/4$. Hence, if a lottery *PC*-dominates q, it is the same lottery. $\square$

In fact the supports which allow for *SD*-efficiency can be characterized by a set extension, which we introduce here. The responsive set extension is defined on multi-sets to make the connection between dominating sets and the *SD*-extension. The underlying idea is that for every alternative in the dominated set there is an alternative in the dominating set which is at least as good.

Definition 2 (Aziz)**.** Consider two multi-sets S and T with elements from A. Then $S\ R_i^{RS}\ T$ i.e., S is weakly preferred to T by i via the *responsive set extension* if there exists a bijection $f_i \colon T \to S$ such that for each $a \in T$, $f(a)\ R_i\ a$.

The responsive set extension naturally induces an efficiency notion for sets of alternatives. A set is responsive set-efficient if and only if it does not contain a dominated set.

Definition 3 (Aziz)**.** A set of alternatives T admits a *RS-dominated sub-support* if there exists a multi-set T' with all elements from T such that there exists a multi-set S' with elements from A for which $|S'| = |T'|$ and $S'\ R_i^{RS}\ T'$ for all $i \in N$ and one strict preference. If T does not contain a dominated sub-support, it is *RS*-efficient.

The goal is to show that *RS*-efficiency of *supp*(p) is equivalent to *SD*-efficiency of p. We prove both directions in the following two lemmas.

Lemma 1 (Aziz)**.** *If $supp(p)$ is not RS-efficient, then p is not SD-efficient.*

Proof. By assumption, $supp(p)$ admits a dominated sub-support. Then there exists some multi-set T with elements from $supp(p)$ for which there is a multi-set S such that $|S| = |T|$ and $S \; R_i^{RS} \; T$ for all $i \in N$ and strict for one.
By the definition of responsiveness, for each $i \in N$ there exists a bijection $f_i \colon T \to S$ such that for each $a \in T$, $f_i(a) \; R_i \; a$. Now for p and some arbitrarily small $\epsilon > 0$, let the lottery q be such that $q(a) = p(a) + \epsilon(S_a - T_a)$, where S_a is the number of copies of a in S. For each agent i, an ϵ decrease in the probability of an alternative a results in an ϵ increase in some alternative $f_i(a) \; R_i \; a$. Therefore, $q \; R_i^{SD} \; p$ for all $i \in N$ and strict for one. $\square$

Lemma 2 (Aziz)**.** *If p is not SD-efficient, then $supp(p)$ is not RS-efficient.*

Proof. Let p be a lottery that is not *SD*-efficient. We assume for now that for each $x \in N$, $p(x)$ is rational.
Since p is not *SD*-efficient, there exists a lottery q such that $q \; R_i^{SD} \; p$ for all $i \in N$ and $q \; P_i^{SD} \; p$ for some $i \in N$. In particular, q can be obtained as a solution of the following linear program:

$$\begin{aligned}
\max \quad & \sum_{i \in N} \sum_{l=1}^{k_i} \sum_{a \in \bigcup_{I_i^1}^{I_i^l}} q(x) \\
\text{subject to:} \quad & \sum_{x \in \bigcup_{I_i^1}^{I_i^l}} q(x) \geq \sum_{x \in \bigcup_{I_i^1}^{I_i^l}} p(x) \quad \text{for each } i \in N \text{ and } l \in \{1, \ldots, k_i\} \\
& \sum_{x \in A} q(x) = 1.
\end{aligned}$$

The linear program has a finite optimum, so for each $x \in A$, $q(x)$ is rational. Since $q \neq p$, there exists a non-empty set of alternatives $C = \{x : q(x) < p(x)\}$. Similarly, there exists a non-empty set of alternatives $B = \{x : q(x) > p(x)\}$. Since $p(x) > 0$ for each $x \in C$, we know that $C \subseteq supp(p)$.
Let $\Delta(x) = q(x) - p(x)$ for all $x \in A$. Since $q(x)$ and $p(x)$ are both rational, hence, $\Delta(x)$ is also rational. Let $\epsilon = GCD\{|\Delta(x)| : x \in B \cup C\}$. We note that ϵ is well-defined because it is the greatest common divisor of a set of rational numbers. Let B' be a multi-set of elements from B where each $x \in B$ features $\Delta(x)/\epsilon$ times. Let C' be a

multi-set of elements from C where each $x \in C$ features $\Delta(x)/\epsilon$ times. Therefore q is a lottery such that

$$q(x) = \begin{cases} p(x) & \text{if } a \in A \setminus (B \cup C) \\ p(x) - (|\Delta(x)|/\epsilon)\epsilon & \text{if } x \in C \\ p(x) + (|\Delta(x)|/\epsilon)\epsilon & \text{if } x \in B. \end{cases}$$

Since q *SD*-dominates p, for each $i \in N$ and $x \in C$ for which there is an ϵ decrease, there is a corresponding $y \in B$ such that $y \; R_i \; x$ for which there is an ϵ increase in probability. This implies that q can be obtained from p by decrementing each alternative in C' by ϵ and incrementing each alternative in B' by ϵ. Therefore, $B' \; R_i^{RS} \; C'$ for all $i \in N$ and strict for one. Hence, $supp(p)$ admits a dominated sub-support.

What we have proved is that if p is rational and is not *SD*-efficient, then $supp(p)$ admits a dominated sub-support. We now prove that the same statement also holds for irrational lotteries. Consider a lottery q such that $supp(q) = supp(p)$ and there exists at least one $x \in A$ such that $q(x)$ is irrational. It is easy to see that q also admits a dominated sub-support since we can use the same multi-sets B' and C' where C' contains elements from $supp(q)$ and $B' \; R_i^{RS} \; C'$ for all $i \in N$ and strict for one. □

Theorem 8 (Aziz)**.** *A lottery p is SD-efficient if and only if $supp(p)$ is RS-efficient.*

Proof. From Lemmas 1 and 2. □

The theorem above implies in particular Theorem 7. Even though *SD*-efficiency is probably the most common efficiency notion in randomized social choice besides Pareto-optimality, it is based on the assumption of vNM or monotonic SSB-utilities. This might be too prohibitive or otherwise undesirable in some cases. Therefore, we will in the following study stronger and weaker notions of efficiency.

5.1.2 *BD*-efficiency

Since *BD*-efficiency has to our best knowledge not been studied before, we will go into more detail here. The basic procedure will be the same as before. First, we show support dependence of *BD*-efficiency and then give a characterization of the supports in question, establishing a connection to the well-known Fishburn set extension.

The idea for the proof of the following theorem is the same as for *SD*-efficiency. If a lottery is not *BD*-efficient we can swap probabilities the same way to make any lottery with equal support inefficient. Notice that we demand for equal supports here in contrast to support inclusion in Theorem 7.

Theorem 9. *If p is BD-efficient and $supp(q) = supp(p)$, then q is BD-efficient.*

Proof. Let p, q be two lotteries such that p is BD-efficient and $supp(p) = supp(q)$. Assume for a contradiction that q is not BD-efficient, i.e., there exists $q' \in \Delta A$ such that $q' \; R_i^{BD} \; q$ for all $i \in N$ with at least one strict preference. Let $\odot$ denote elementwise multiplication. We can write $q' = q \odot \pi + \Delta$ with $\pi, \Delta \in \mathbb{R}^m$ such that $\pi_x = 0$ for $x \notin supp(q)$ and $\Delta_x = 0$ for $x \in supp(q)$. This implies $\pi, \Delta \geq 0$. By the characterization of R^{BD} in Theorem 4, q' dominating q is equivalent to the following: for all $i \in N, x \; P_i \; y$ implies $q'_x q_y - q'_y q_x \geq 0$. Using the representation of q', we get

$$
\begin{aligned}
0 \leq q'_x q_y - q'_y q_x &= (q_x \pi_x + \Delta_x) q_y - (q_y \pi_y + \Delta_y) q_x \\
&= \begin{cases} q_x q_y (\pi_x - \pi_y) & \text{if } x, y \in supp(q), \\ -\Delta_y q_x & \text{if } x \in supp(q), y \notin supp(q), \\ \Delta_x q_y & \text{if } x \notin supp(q), y \in supp(q), \\ 0 & \text{if } x, y \notin supp(q). \end{cases}
\end{aligned} \tag{15}
$$

Note that in the second case $\Delta_y = 0$ necessarily. We define $p' = p \odot \pi \epsilon + \eta \Delta$ such that both $\epsilon, \eta > 0$. This is possible since $\sum \epsilon p_x \pi_x < 1$ for ϵ small enough. Now we show that p' BD-dominates p:

$$
0 \leq p'_x p_y - p'_y p_x = \begin{cases} \epsilon p_x p_y (\pi_x - \pi_y) & \text{if } x, y \in supp(p), \\ -\eta \Delta_y p_x & \text{if } x \in supp(p), y \notin supp(p), \\ \eta \Delta_x p_y & \text{if } x \notin supp(p), y \in supp(p), \\ 0 & \text{if } x, y \notin supp(p). \end{cases} \tag{16}
$$

The fact that the inequalities in (16) hold can easily be seen from the inequalities in (15). If an inequality holds strict in (15), then it also does in (16) since $supp(p) = supp(q)$ and $\epsilon, \eta > 0$. Therefore, p' BD-dominates p, which is a contradiction. □

On our way to characterize the supports which make for BD-efficient lotteries, we introduce the Fishburn set extension due to Gärdenfors (1979). It is a refinement of a set extension going back to Kelly (1977) and allows to compare sets with overlap in case they are comparable by the Kelly extension on alternatives they do not intersect on. The proof will use Theorem 9 and yield a generalization of the latter.

Definition 4. Let $S, T \subset A$ be two sets of alternatives. S Fishburn-dominates T, denoted by $S \; R^F \; T$, if for all

$$
\begin{aligned}
x \in S \setminus T \colon x \; P \; y \quad & \text{for all } y \in T, \\
x \in T \setminus S \colon y \; P \; x \quad & \text{for all } y \in S.
\end{aligned}
$$

Now we are in a position to establish the connection between BD-efficiency and the Fishburn set extension. Efficiency for sets of alternatives with respect to a set extension is defined in the same way as for lotteries and lottery extensions.

Theorem 10. *A lottery p is BD-efficient if and only if $supp(p)$ is Fishburn-efficient.*

Proof. For the direction from left to right, assume that p is BD-efficient and there is $S \subset A$ which Fishburn-dominates $supp(p)$. We define the lottery q such that

$$
\begin{aligned}
q_x = 1 \text{ for some } x \in S \setminus supp(p) \qquad & \text{if } S \setminus supp(p) \neq \emptyset, \\
q_x p_y - q_y p_x = 0 \text{ for all } x, y \in S \text{ and } \sum_{x \in S} q_x = 1 \qquad & \text{if } S \subset supp(p).
\end{aligned}
$$

In the first case, S contains an alternative x which Pareto-dominates all alternatives in $supp(p)$. Therefore, the degenerate lottery with weight one on x BD-dominates p. In the second case, all alternatives in $supp(p) \setminus S$ are Pareto-dominated by all alternatives in S. Thus, putting all probability on alternatives in S in a way that does not change the ratios of probabilities yields a lottery dominating p. In either case we get a contradiction. For the direction from right to left, assume that $supp(p)$ is Fishburn-efficient and p is not BD-efficient. Therefore, by transitivity of R^{BD}, there is some BD-efficient lottery q which BD-dominates p. Using Theorem 9, we get $supp(q) \neq supp(p)$. Now assume there is $x \in supp(q) \setminus supp(p)$, $y \in supp(p)$, $i \in N$, such that $y \; P_i \; x$. Then $q_y p_x - q_x p_y = -q_x p_y < 0$, which is a contradiction. On the other hand, if there is $x \in supp(p) \setminus supp(q)$, $y \in supp(q)$, $i \in N$, such that $x \; P_i \; y$ then $q_x p_y - q_y p_x = -q_y p_x < 0$, which is again a contradiction. Thus, we have $supp(q) \; R_i^F \; supp(p)$ for all $i \in N$ and strict for one. This contradicts $supp(p)$ being Fishburn-efficient. □

Since bilinear dominance is a very incomplete relation, we would like to give some intuition on the strength of BD-efficiency. A natural efficiency notion is Pareto-optimality. A lottery p is *Pareto-optimal*, if $supp(p)$ does not contain a Pareto-dominated alternative. As the following corollary shows, BD-efficiency is weaker than Pareto-optimality.

Corollary 1. *Pareto-optimality implies BD-efficiency, but the converse is not true.*

Proof. For the first statement, assume a lottery p is Pareto-optimal but not BD-efficient. Let q be a BD-efficient lottery which BD-dominates p. By Theorem 10, $supp(q)$ Fishburn dominates $supp(p)$, in particular $supp(q) \neq supp(p)$. We distinguish two cases:
Case 1: $supp(q) \setminus supp(p) \neq \emptyset$. It is clear that $x \; R_i \; y$ for all $i \in N$ for every pair of alternatives $x \in supp(q) \setminus supp(p)$ and $y \in supp(p)$. If $x \; P_i \; y$ for some i, then y is Pareto dominated, which is a contradiction. Otherwise every agent is indifferent between all alternatives in $supp(p)$ and $supp(q)$, which contradicts BD-dominance of q.
Case 2: $supp(p) \setminus supp(q) \neq \emptyset$. Similar to the previous case, $x \; R_i \; y$ for all $i \in N$ for all alternatives $x \in supp(q)$ and $y \in supp(p) \setminus supp(q)$. The rest of the argument is the same as above.
In any case, we get a contradiction, which proves the first statement.
To show the second part, consider two agents with preferences *abc* and *cab* respectively. The set $\{b, c\}$ is Fishburn-efficient, however, b is Pareto dominated. Again by Theorem 10, any lottery with support $\{b, c\}$ is BD-efficient, but supports a Pareto-dominated alternative. □

As noted before, SD-efficient lotteries remain SD-efficient if their support becomes smaller (cf. Theorem 7), while this was not shown for BD-efficiency (cf. Theorem 9). In fact this does not hold and can be viewed as the set of BD-efficient lotteries not being closed. Furthermore, we will show that this set is convex, providing a contrast to the conclusions of Figure 2 for SD, DL, and UL-efficiency.

Corollary 2. *The set of BD-efficient lotteries is convex and not closed.*

Proof. For convexity, it suffices to show by Theorem 10 that for two Fishburn-efficient sets of alternatives X and Y, their union is also Fishburn-efficient. Assume for a contradiction $X \cup Y$ is not Fishburn-efficient. Thus, there exists Z which Fishburn-dominates $X \cup Y$. Clearly, $Z \neq X \cup Y$. We distinguish two cases:
Case 1: $Z \setminus (X \cup Y) \neq \emptyset$. Let z be such an alternative. Then $z \; R_i \; x$ for all $x \in X \cup Y$ and all $i \in N$. Henceforth either X or Y is not Fishburn-efficient.
Case 2: $(X \cup Y) \setminus Z \neq \emptyset$. Let x be such an alternative. Without loss of generality $x \in X$. If $Z \cap X$ is empty, Z Fishburn-dominates X. In the other case $Z \cap X$ Fishburn-dominates X.
In any case we get a contradiction, which proves convexity.

To show the second statement, consider the following preference profile with 2 agents:

$$1\colon a, b, c$$
$$2\colon c, a, b$$

The lottery $p^\epsilon = [a\colon 0, b\colon 1-\epsilon, c\colon \epsilon]$ is *BD*-efficient for any $\epsilon \in (0,1)$, since it the set $\{b, c\}$ is Fishburn-efficient. However p^0 is not *BD*-efficient, because a Pareto-dominates b. □

Although *BD*-efficiency is seemingly very weak compared to established efficiency notions, it is often times a reasonable notion to demand for SDSs when requiring some sort of strategyproofness in addition. Furthermore, Theorem 4 provides the theoretical justification if we assume agents to have unknown SSB-preferences. Especially the nice structure of convexity makes *BD*-efficiency much easier to satisfy for SDSs.

5.1.3 *DL*/*UL*-Efficiency

It was shown by Cho (2012) that for assignment problems *DL*, *UL*, and *SD*-efficiency coincide. So with the characterization of *SD*-efficiency by Bogomolnaia and Moulin, both *UL* and *DL*-efficiency only depend on the support of an assignment. The latter still holds for social choice, however, the efficiency notions differ in this broader domain. We will introduce set extensions which characterize *DL*/*UL*-efficient supports. Since the statements and proofs are very similar for the *DL* and *UL*-extension, we will only prove them for *DL*.

Definition 5. Let $R_i \in \mathcal{R}(A)$ be a preference relation and S, T two multi-sets with elements from A. We say

(a) S *DL*-dominates T, i.e., $S \; R_i^{DL} \; T$ if either

$$|S \cap I_i^j| = |T \cap I_i^j| \quad \text{for } j = 1, 2, \dots$$

or there exists $k \in \{1, 2, \dots\}$ such that

$$\begin{aligned} &|S \cap I_i^j| = |T \cap I_i^j| \quad \text{for } j = 1, \dots, k-1, \\ &|S \cap I_i^k| > |T \cap I_i^k|. \end{aligned}$$

(b) S *UL*-dominates T, i.e., $S\ R_i^{UL}\ T$ if either

$$|S \cap I_i^j| = |T \cap I_i^j| \quad \text{for } j = 1, 2, \dots$$

or there exists $k \in \{1, 2, \dots\}$ such that

$$\begin{aligned}&|S \cap I_i^k| < |T \cap I_i^k|,\\ &|S \cap I_i^j| = |T \cap I_i^j| \quad \text{for } j = k+1, k+2, \dots\end{aligned}$$

As for the retentive set extension, this notion of dominance carries over to non-multi-sets in a natural way. It can be used to define the corresponding notion of efficiency for sets of alternatives rather than lotteries. Note that we use the term *DL*-efficiency both for sets and lotteries. Since Theorem 11 will show that *DL*-efficiency of a lottery and its support are equivalent, this use of terminology is justified.

Definition 6. Let $R \in \mathcal{R}(A)^n$ be a preference profile and $S, T \subset A$. Then S *DL*-dominates (*UL*-dominates) T if there exist multi-sets S', T' with elements from S and T respectively such that $|S'| = |T'|$ and $S'\ R_i^{DL}\ T'$ ($S'\ R_i^{UL}\ T'$) for all $i \in N$ and strict for one. If S is not *DL*-dominated (*UL*-dominated), it is *DL*-efficient (*UL*-efficient).

The proof of the following theorem is similar to the proof of Theorem 8. If $supp(p)$ is not *DL*-efficient, we can shift probabilities in p from alternatives in $supp(p)$ to alternatives in the set which dominates $supp(p)$. On the other hand, if p is dominated by some p', we can construct a set which *DL*-dominates $supp(p)$ or rather a subset of it which suffices.

Theorem 11. *A lottery p is DL-efficient (UL-efficient) with respect to R if and only if $supp(p)$ is DL-efficient (UL-efficient).*

Proof. To prove the direction from left to right, assume for a contradiction that $supp(p)$ is not *DL*-efficient. Then there exist multi-sets S', T' as in Definition 6, where T' consists of elements from $supp(p)$. For a multi-set S', S'_x denotes the number of occurrences of x in S'. Define $\Delta \in \mathbb{N}^m$, $\Delta_x = S'_x - T'_x$ for $x \in A$. Note that $\Delta_x \geq 0$ for $x \notin supp(p)$ and $\sum_{x \in A} \Delta_x = 0$. We aim to show that $p' = p + \epsilon\Delta$ is a lottery which *DL*-dominates p for small $\epsilon > 0$. Let $i \in N$. Since S' *DL*-dominates T', either $S'\ I_i^{DL}\ T'$ or there is k as in Definition 5 and we have

$$\sum_{x\in I_i^j} p'_x = \sum_{x\in I_i^j} p_x + \epsilon(S'_x - T'_x) = \sum_{x\in I_i^j} p_x \quad \text{for } j < k,$$
$$\sum_{x\in I_i^k} p'_x = \sum_{x\in I_i^k} p_x + \epsilon(S'_x - T'_x) > \sum_{x\in I_i^k} p_x.$$

Thus, $p' \ R_i^{DL} \ p$ for all i and strict preference for at least one, which contradicts DL-efficiency of p.

Now we prove the opposite direction. Assume that p is not DL-efficient. So by definition, there is $p' \in \Delta A$ such that $p' \ R_i^{DL} \ p$ for all $i \in N$ and strict for one. We assume without restriction that p_x and p'_x are rational for all $x \in A$ with the same argument as in the proof of Lemma 2. Define $\Delta_x = p'_x - p_x$ and set $\text{gcd} = GCD\{\Delta_x \colon x \in A\}$. Then we add $\frac{\Delta_x}{\text{gcd}}$ copies of x to the multi-set S if $\Delta_x > 0$ and to T otherwise. It is left to show that $S \ R_i^{DL} \ T$ for all $i \in N$ and strict for one. If $p' \ P_i^{DL} \ p$, let k be the smallest index where p' assigns more probability to an indifference class than p.

$$|S \cap I_i^j| = \sum_{\substack{x\in I_i^j \\ \Delta_x > 0}} \frac{\Delta_x}{\text{gcd}} = \sum_{\substack{x\in I_i^j \\ \Delta_x < 0}} -\frac{\Delta_x}{\text{gcd}} = |T \cap I_i^j| \quad \forall j = 1, \dots, k-1, \tag{17}$$
$$|S \cap I_i^k| = \sum_{\substack{x\in I_i^k \\ \Delta_x > 0}} \frac{\Delta_x}{\text{gcd}} > \sum_{\substack{x\in I_i^k \\ \Delta_x < 0}} -\frac{\Delta_x}{\text{gcd}} = |T \cap I_i^k|. \tag{18}$$

The second equality in (17) holds since p and p' assign the same probability to the j-th indifference class for $j < k$ which implies $\sum_{x\in I_i^j} \Delta_x = 0$. The first and third equality hold by the definition of S and T. The inequality in (18) holds, because p' assigns more probability to the k-th indifference class than p. If $p' \ I_i^{DL} \ p$, a similar proof shows $S \ I_i^{DL} \ T$. Together, this implies that $supp(p)$ is not DL-efficient and finishes the proof. □

In particular Theorem 11 shows that DL-efficiency only depends on the support of a lottery. However, as for the characterization of SD-efficiency, it is not clear how to combinatorically check DL-efficiency for sets of alternatives. It can be verified easily that any DL-efficient lottery only puts weight on alternatives which are ranked first at least once. But a UL-efficient lottery may support alternatives which are ranked last for some agent. So even DL and UL seem very symmetric to each other, formally DL is the dual of UL, there is a fundamental difference between both notions. This will show up

1	1	1	1	1
{a,c}	{b,c}	{a,d}	{b,d}	{a,b}
{b,d}	{a,d}	{b,c}	{a,c}	{c,d}

Table 2: The top row of the profile specifies the number of agents with the respective preferences.

in impossibility results later.

Remark 2. *The combination of DL and UL-inefficiency does not imply SD-inefficiency.*

Proof. In the following profile any lottery is SD-efficient since agents have opposed preferences.

$$1\colon a, b, c, d, e$$
$$2\colon e, d, c, b, a$$

However, $p = [0, \frac{1}{2}, 0, \frac{1}{2}, 0]$ is neither DL nor UL-efficient. The lottery $[\frac{1}{2}, 0, 0, 0, \frac{1}{2}]$ DL-dominates p, whereas $[0, 0, 1, 0, 0]$ UL-dominates it. □

We showed earlier that the set of BD-efficient lotteries is convex. Table 2 reveals that this is not true for the other efficiency notions we discussed. Both the lotteries which yield c or d for sure respectively are e-efficient for $e \in \{SD, PC, UL, DL\}$. However, no proper convex combination of these two is even SD-efficient. This shows that none of the sets of e-efficient lotteries is convex. Note that for strict preferences, the set of DL-efficient lotteries is convex, while the set of SD, PC, and UL-efficient lotteries is not. In fact even stronger statements hold for weak preferences, e.g., the convex hull of the set of PC-efficient lotteries is not contained in the set of SD-efficient lotteries. However, the convex hull of SD-efficient lotteries is contained in the set of Pareto-optimal lotteries. In fact, even equality holds, i.e., every Pareto-optimal lottery can be written as a convex combination of SD-efficient lottery. Clearly, every Pareto-optimal lottery can be written as a convex combination of degenerate Pareto-optimal lotteries. These are SD-efficient. Figure 2 provides an overview of inclusion relationship among the efficiency notions discussed in this section. Naturally this reverses the ordering of the respective dominance relations on lotteries. We found that efficiency depends on the support of a lottery only for BD, SD, DL, and UL. It would be interesting to find sufficient conditions for lottery extension which imply support dependence of efficiency.

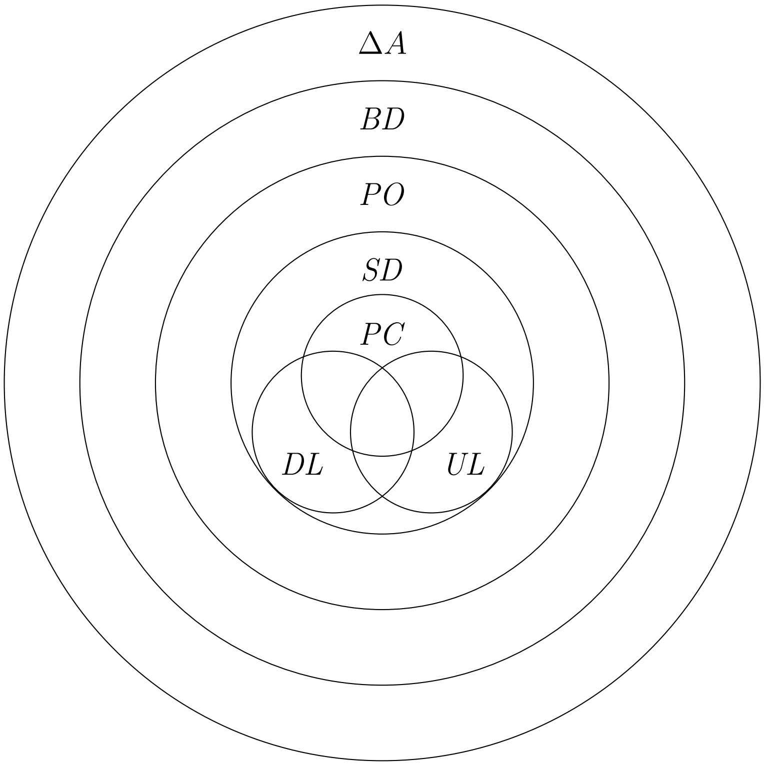

Figure 2: Relations between efficiency notions

5.2 Social Welfare

Besides ensuring efficiency, another objective may be to maximize social welfare. To measure overall welfare, every agent is equipped with a utility function. Social welfare is maximized, if the sum of all utilities is maximal over all lotteries. It can be seen easily that if a lottery maximizes social welfare for some profile of vNM-utility functions it is SD-efficient. Bogomolnaia and Moulin (2001) conjecture that for random assignments the converse is also true, i.e., if a lottery is SD-efficient there is some profile of vNM-utility functions consistent with the ordinal preferences for which it maximizes social welfare. This statement known as the *ordinal efficiency welfare theorem* was proven by McLennan (2002) using a variant of the separating polyhedron hyperplane theorem. Later, Manea (2008) and Athanassoglou (2010) provide constructive proofs for the same theorem. A generalization of this statement was shown by Carroll (2010) with a proof technique similar to McLennan's, for general social choice. He allows for agents to have incompletely known preferences, i.e., sets of vNM-utility functions, not necessarily the same for all agents, which satisfy certain geometric properties. If for some lottery there is no lottery which yields higher utility for all agents and all their possible utility functions, the former is efficient. Carroll's theorem shows that for every efficient lottery there exists a utility profile with utility functions from each agent's set of plausible utility functions, such that this lottery maximizes social welfare for some weighting of agents utilities. In this section we proceed as follows. First, we provide a generalization of the result by Carroll to SSB-preferences, which we will refer to as the generalized efficiency welfare theorem. In the second part, we prove a special case of this theorem where the set of utility functions contains all SSB-functions for every agent. This proof makes use of the same technique as Athanassoglou, providing a constructive proof. However, our statement applies to general social choice.

To make the notion of efficiency mentioned above formal, we state a definition here. We assume every agent has a set of possible SSB-utility functions Φ_i. Then a lottery p *dominates* q *with respect to* $(\Phi_1, \ldots, \Phi_n)$ if $\phi_i(p, q) \geq 0$ for all $\phi_i \in \Phi_i$ and all $i \in N$ and $\phi_i(p, q) > 0$ for all $\phi_i \in \Phi_i$ for some $i \in N$. If p is not dominated by any lottery, it is efficient with respect to $(\Phi_1, \ldots, \Phi_n)$. To keep things short, we will just write efficient and omit the respective sets of utility functions. If for example we let Φ_i be the set of all SSB-functions for all i, the corresponding efficiency notion is *BD*-efficiency.

We start by stating a geometrical lemma shown by Carroll. This lemma is the core for the proof of Theorem 12.

Lemma 3. *Let $U, V \subset \mathbb{R}^m$ be nonempty, convex sets such that U is relatively open and V is a polyhedron. Let $v_0 \in V$. Suppose that for every $v \in V$ there exists $u \in U$ such that $u \cdot (v - v_0) \leq 0$. Then there exists $u \in U$ such that $u \cdot (v - v_0) \leq 0$ for all $v \in V$.*

The basic idea is that SSB-functions can be identified with vNM-utility-functions once the second argument is fixed. In our case, the latter will be the lottery which is assumed to be efficient and V is the set of lotteries. U is the set of functions which map a lottery to its social welfare level, i.e., the (possibly weighted) sum of all agents utilities. If then v_0 is some efficient lottery, we get a utility profile u for which this lottery maximizes social welfare. For this proof we identify every SSB-function with a matrix in $\mathbb{R}^{m \times m}$ and vNM-utility functions with vectors in $\mathbb{R}^m$.

Theorem 12. *Let $\Phi_1, \ldots, \Phi_n$ be nonempty, convex, and relatively open sets of SSB-functions. Suppose p is a lottery which is efficient (w.r.t. $\Phi_1, \ldots, \Phi_n$). Then there exist SSB-functions $\phi_i \in \Phi_i$ and positive weights λ_i such that for all $q \in \Delta A$:*

$$\sum_{i=1}^{n} \lambda_i \phi_i(q, p) \leq 0.$$

Proof. The set of lotteries is a polytope in $\mathbb{R}^m$, as such a polyhedron. Let p be an efficient lottery. For $i = 1, \dots, n$, define $U_i = \Phi_i p = \{\phi p \colon \phi \in \Phi_i\}$. Since the Φ_i are assumed nonempty, convex, and relatively open, the same holds for the U_i by linearity and continuity of matrix multiplication. Let $U \subset \mathbb{R}^m$ be the set of weighted sums of plausible utility functions, i.e.,

$$U = \{\lambda_1 u_1 + \dots + \lambda_n u_n \colon \lambda_i > 0, u_i \in U_i\}.$$

By Lemma 4 of Carroll (2010), U is nonempty, convex, and relatively open.

First note that $u_i \cdot p = p^T \phi_i p = 0$ for all $u_i \in U_i$ and $i = 1, \dots, n$ by skew symmetry of the ϕ_i. Hence $u \cdot p = 0$ for all $u \in U$. Since p is efficient, for every $q \in \Delta A$ there are $u_i \in U_i$ such that either $u_i \cdot q < 0$ for some i or $u_i \cdot q = 0$ for all i. In the former case, choose $u_j \in U_j$ arbitrary for $j \neq i$ and λ_i sufficiently large compared to the λ_j. Then for $u = \sum_{j=1}^{n} \lambda_j u_j$, we have $u \cdot q \leq 0$. In the latter case, let $u = \sum_{j=1}^{n} u_j$, which implies $u \cdot q = 0$. In either case, we have $u \in U$ such that $u \cdot q \leq 0 = u \cdot p$.

Since all the requirements are met we can apply Lemma 3 and obtain $u \in U$ such that $u \cdot q \leq 0$ for all $q \in \Delta A$. Translating things back to SSB-utility, we get $u = \sum_i \lambda_i u_i$ such that

$$\begin{aligned} &\sum_i \lambda_i u_i \cdot q \leq 0 \text{ for all } q \in \Delta A \\ \Leftrightarrow &\sum_i \lambda_i q^T \phi_i p = \sum_i \lambda_i \phi_i(q,p) \leq 0 \text{ for all } q \in \Delta A \end{aligned}$$

for $\phi_i \in \Phi_i$ such that $u_i = \phi_i p$. Thus, p maximizes social welfare if agents have SSB-preferences according to the ϕ_i as obtained before. $\square$

However, it might be undesirable to weight agents utilities differently. If more restrictions on the set of possible utility functions are made, Theorem 12 can be phrased without using weights. For an alternative version of the efficiency welfare theorem we assume each agents set of plausible SSB-functions to be conic, i.e., $\phi_i^1, \dots, \phi_i^k \in \Phi_i$ implies $\sum \lambda_k \phi_i^k \in \Phi_i$ for all $\lambda_1, \dots, \lambda_k > 0$. This condition can be thought of as knowing an agents intensity of pairwise comparison, but not the scale of his utility. Then Theorem 12 can be stated as follows:

Theorem 13. *Let $\Phi_1, \dots, \Phi_n$ be nonempty, conic, and relatively open sets of SSB-functions. Suppose p is a lottery which is efficient (w.r.t. $\Phi_1, \dots, \Phi_n$). Then there exist*

SSB-functions $\phi_i \in \Phi_i$ *such that for all*

$$q \in \Delta A\colon \quad \sum_{i=1}^{n} \phi_i(q,p) \leq 0.$$

Note that Φ_i being conic implies convexity. To show the power of Theorem 12, we state some corollaries. One special case is the ordinal efficiency welfare theorem where each agent has all vNM-utility functions consistent with his ordinal preferences at his disposal. Also Theorem 14 below, which we prove separately, is a consequence. An interesting corollary is derived if all agents are assumed to have preferences according to pairwise comparison, i.e., $\Phi_i = \{\phi_i^{PC}\}$. Clearly, any one element set satisfies all the requirements needed to apply Theorem 12.

Corollary 3. *A lottery* p *is* PC*-efficient if and only if there are* $\lambda_1, \dots, \lambda_n > 0$ *such that*

$$\sum_{i=1}^{n} \lambda_i \phi_i^{PC}(q,p) \leq 0$$

for all lotteries q*.*

A subset of PC-efficient lotteries are maximal lotteries introduced by Kreweras (1965) and Fishburn (1984a). Maximal lotteries maximize social welfare according to ϕ^{PC} if $\lambda_i = 1$ for all $i \in N$. Corollary 3 implies for example that every maximal lottery is PC-efficient. The minimax theorem by von Neumann (1928) implies that the set of maximal and therefore PC-efficient lotteries is non-empty even for intransitive preferences over alternatives.

A special case of the above theorem can be shown by applying a method used by Athanassoglou to prove the ordinal efficiency welfare theorem. He formulates a linear program for which feasible points are lotteries which dominate some efficient lottery p. To obtain the constraints, the characterization of stochastic dominance is used. By efficiency of p, we can fix the optimal value of this LP. Then he states the corresponding dual LP, which has the same optimal target value by strong duality. An optimal solution of the dual LP can be used to construct utility functions for which p maximizes social welfare. We will show using the same technique that every BD-efficient lottery maximizes social welfare for some profile of SSB-functions.

Theorem 14. *In social choice, a lottery* p *is* BD*-efficient with respect to a preference profile* R *if and only if there are SSB-functions* $\phi_1, \dots, \phi_n$ *compatible with* R *for which* *p maximizes social welfare.*

Proof. For the direction from right to left, assume p is not BD-efficient at R, i.e., there exists a lottery q such that

$$\begin{aligned} q\ R_i^{BD}\ p \quad &\text{for all } i \in N, \\ q\ P_i^{BD}\ p \quad &\text{for some } i \in N. \end{aligned}$$

This implies $\phi_i(q,p) \geq 0$ for all $\phi_i \in \Phi$ compatible with R_i and all $i \in N$ and strict for one. Thus, p is not welfare maximizing at R for any profile of SSB-functions.
The direction from left to right is shown by considering the following linear programs:

Primal LP:

$$\begin{aligned} \min_{q,r} \quad & \sum_{i=1}^{n} \sum_{j P_i k} -r_{ijk} \\ \text{subject to:} \quad & q_j p_k - q_k p_j - r_{ijk} = 0 \quad \text{for all } i \in N, j\ P_i\ k && (19) \\ & \sum_{j=1}^{m} q_j = 1 && (20) \\ & q \geq 0, r \geq 0. \end{aligned}$$

By (19), every lottery feasible for the primal LP BD-dominates p. So if p is BD-efficient the optimal target value of the primal LP is 0. The corresponding dual LP writes as follows:

Dual LP:

$$\begin{aligned} \max_{x,y} \quad & y \\ \text{subject to:} \quad & \sum_{i=1}^{n} \Big(\sum_{j R_i k} x_{ijk} p_k - \sum_{k P_i j} x_{ijk} p_k\Big) + y \leq 0 \quad j = 1, \ldots, m && (21) \\ & x \geq 1 \\ & y \text{ free variable.} && (22) \end{aligned}$$

By strong duality, the optimal target value for the dual LP is also 0. Let $(\hat{x}, \hat{y})$ be an

optimal solution of the dual LP, then $\hat{y} = 0$.
We define SSB-functions $\phi_1, \ldots, \phi_n$ for which p will be shown to maximize social welfare as follows:

$$\phi_i(j,k) = \begin{cases} \hat{x}_{ijk} & if\ j\ P_i\ k, \\ -\hat{x}_{ijk} & if\ k\ P_i\ j, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

Now let q be a feasible lottery. We want to show that q does not yield higher welfare than p for the profile $\phi_1, \ldots, \phi_n$. That is

$$\sum_i \phi_i(q,p) = \sum_i \sum_{j,k} \phi_i(j,k) q_j p_k = \sum_j q_j \sum_i \sum_k \phi_i(j,k) p_k$$
$$\overset{(23)}{=} \sum_j q_j \sum_i (\sum_{j P_i k} \hat{x}_{ijk} p_k - \sum_{k P_i j} \hat{x}_{ijk} p_k) \overset{(21)}{\leq} \sum_j q_j(-\hat{y}) = 0.$$

The first and second equality is just applying definitions and reordering of the sum respectively. For the third equality we use the definition of ϕ_i. The inequality holds since $(\hat{x}, \hat{y})$ is feasible for the dual LP. $\hat{y} = 0$ finishes the proof. □

SD-efficiency is stronger than *BD*-efficiency, while welfare maximizing for some profile of vNM-utility functions is also stronger than for a profile of SSB-functions. Therefore, there is no relation between Theorem 14 and the ordinal efficiency welfare theorem, so neither one implies the other. Even though this theorem if weaker than the generalized efficiency welfare theorem, we stated the proof here to illustrate how the proof technique is adapted to social choice and SSB-utility functions.

5.3 Social Decision Schemes

A *social decision scheme* (SDS) is a function $f\colon \mathcal{R}^n \to \Delta A$, mapping a preference profile to a lottery. The notion of efficiency carries over to SDSs, i.e., an SDS is e-efficient if it maps any profile to an e-efficient lottery. Besides efficiency, another desirable property is strategyproofness. We say an SDS is *manipulable* if in some preference profile some agent can get a preferred outcome by lying about his preferences. Formally, f is e-manipulable if there exist a profile R, a agent $i \in N$ and a preference relation R_i' such that $f(R_i', R_{-i})\ R_i^e\ f(R_i, R_{-i})$, where R_{-i} denotes the preferences of all agents except i. If a SDS is not manipulable, it is *strategyproof.* For a stronger notion of strategyproofness, we say an SDS is strong strategyproof if whenever a single agent submits a lie, she receives a lottery which is weakly dominated by the lottery obtained from truthful voting, i.e., $f(R_i, R_{-i})\ R_i^e\ f(R_i', R_{-i})$. Clearly, the strength of these two notions of strategyproofness only differs for incomplete relations. The notion of strategyproofness can be extended to groups of agents. A SDS is *group-strategyproof* if no group of agents can get an outcome preferred by all of them by misrepresenting their preferences. Additionally, we demand basic fairness and symmetry properties. A SDS is *anonymous*, if permuting agents does not change the outcome, i.e., for any permutation π of N we have $f(R_1, \dots, R_n) = f(R_{\pi(1)}, \dots, R_{\pi(n)})$. Similarly, an SDS is *neutral* if permuting alternatives permutes probabilities in the outcome the same way. For a permutation π of A and a preference relation R_i, $\pi(x)\ R_i^\pi\ \pi(y)$ if and only if $x\ R_i\ y$. Then anonymity is formally defined as $f(R)_x = f(R^\pi)_{\pi(x)}$ for all x in A.

A well-studied SDS is *random serial dictatorship (RSD)*, which is a randomization over serial dictatorships. A serial dictatorship determines the outcome as follows. For some permutation of agents, the first agent gets to decide for her most preferred alternatives and we restrict attention to those. Then the second agent chooses her most preferred alternatives among the remaining ones and so on until all agents have been invoked once. For simplicity we assume there can be no two alternatives such that all agents are indifferent between them. Thus, serial dictatorship always determines one alternative uniquely. Let Π denote the set of all possible permutations of n agents and π some permutation. We define

$$\begin{aligned}
f^1(R, \pi) &= \max_{R_{\pi(1)}} A, \\
f^k(R, \pi) &= \max_{R_{\pi(k)}} f^{k-1}(R, \pi) \quad \text{for } k = 2, \dots, n.
\end{aligned}$$

The outcome of serial dictatorship for π is then $f^n(R, \pi)$. We identify $f^n(R, \pi)$ with the lottery which yields this alternative for sure. The lottery obtained from RSD is the average outcome over all serial dictatorships.

$$RSD(R) = \frac{1}{n!} \sum_{\pi \in \Pi} f^n(R, \pi).$$

To ensure anonymity of RSD, we assume that every permutation is equally likely. As a convex combination of dictatorships, RSD does very well on the strategyproofness front. For every agent it is a (stochastically) dominant strategy to vote truthful once she is asked for her most preferred alternatives in some set. Therefore, RSD is strong *SD*-strategyproof. It can be easily seen that RSD is Pareto-optimal and as such *BD*-efficient.

However, the example in Figure 2 shows that it is not *SD*-efficient. The resulting lottery is $p = [a\colon \frac{11}{30}, b\colon \frac{11}{30}, c\colon \frac{4}{30}, d\colon \frac{4}{30}]$, which is *SD*-dominated by $q = [a\colon \frac{1}{2}, b\colon \frac{1}{2}, c\colon 0, d\colon 0]$.
In addition two facts about strategyproofness can be derived from Figure 2. The first is that RSD violates *SD*-group-strategyproofness. If agents 1 to 4 rank a of b uniquely first, depending on which one they prefer more, the RSD outcome is q instead of p. In fact, there is a strong connection between group-strategyproofness and efficiency, which is worth further examination. Furthermore, RSD is not strongly *BD*-strategyproof. This can be seen by breaking one of the indifferences for first rank for some agent in Table 2. However, this notion is extremely strong and we will not devote further attention to it. Despite the efficiency loss, RSD has nice incentive properties as discussed above. Another benefit is its easy implementability, even though it can be hard to calculate the resulting lottery (cf. Aziz et al., 2013a).
One special class of mechanisms are majoritarian SDSs. A SDS is majoritarian if it only depends on the pairwise majority comparisons, so the unweighted tournament associated with the majority relation. This definition already implies anonymity and neutrality. Even though majoritarianism is a widely used for social choice functions, it appears to be very restrictive for SDSs. The following theorem shows that even mild requirements on efficiency and strategyproofness cannot be achieved by a majoritarian SDS.

Theorem 15. *There is no Pareto-optimal, BD-strategyproof, majoritarian SDS.*

Proof. Consider a preference profile with two agents and preferences

$$\begin{aligned} 1&\colon a, c, b, d \\ 2&\colon b, d, a, c \end{aligned} \tag{24}$$

By Pareto-dominance and majoritarianism, the resulting lottery is $p = [\frac{1}{2}, \frac{1}{2}, 0, 0]$. Now we consider a second profile:

$$\begin{aligned} 1&\colon a, c, \{b, d\} \\ 2&\colon \{b, d\}, a, c \end{aligned} \tag{25}$$

Again c is Pareto dominated and both agents are indifferent between b and d. Hence any majoritarian SDS yields a lottery of the form $q = [1 - 2\lambda, \lambda, 0, \lambda]$. We aim to show

$\lambda < \frac{1}{3}$. First assume for a contradiction $\lambda = \frac{1}{3}$. The profile

$$\begin{aligned} &1\colon a, c, b, d \\ &2\colon \{b, d\}, \{a, c\} \\ &3\colon a, c, \{b, d\} \\ &4\colon \{b, d\}, c, a \end{aligned} \tag{26}$$

has the same majority graph as the profile in (24), thus yields lottery p. If agent 4 reports $dbca$ instead, the majority graph is the same as in (25) and results in lottery q. It can be checked that $\phi_4(q, p) = \frac{1}{3} \cdot \frac{1}{2} \cdot \phi_4(a, b) + \frac{1}{3} \cdot \frac{1}{2} \cdot \phi_4(b, a) + \frac{1}{3} \cdot \frac{1}{2} \cdot \phi_4(d, b) = \frac{1}{3} \cdot \frac{1}{2} \cdot \phi_4(d, b) > 0$, which contradicts BD-strategyproofness. An easy proof shows that $\lambda > \frac{1}{3}$ yields a contradiction as well.

To finish, we consider 4 agents with preferences

$$\begin{aligned} &1\colon a, c, \{b, d\} \\ &2\colon \{b, d\}, a, c \\ &3\colon \{b, c, d\}, a \\ &4\colon a, \{b, c, d\} \end{aligned} \tag{27}$$

Profile (27) has the same majority graph as profile (25) and therefore yields q. If agent 3 was to report $\{bd\}ca$ instead, she receives the lottery $r = [\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}]$ and $\phi_3(r, q) = \frac{1}{3} \cdot \lambda \cdot (\phi_3(a, b) + \phi_3(a, d)) + \frac{1}{3} \cdot (1 - 2\lambda) \cdot (\phi_3(b, a) + \phi_3(d, a)) > 0$ if $\lambda < \frac{1}{3}$. Thus, agent 3 can benefit from misrepresenting her preferences. $\square$

It seems especially hard to satisfies any reasonable notion of strategyproofness if restricted to majoritarian SDSs. Aziz et al. (2013b) discuss this issue in detail and consider strict maximal lotteries (SML). Even the majoritarian variant of SML is ST-strategyproof and SD-efficient (Fishburn, 1984a). However, SML is not BD-strategyproof and hence not SD-strategyproof. It is an interesting an challenging question, whether there is any anonymous SDS which is SD-efficient and SD-strategyproof. We hope address to this issue in future work and show an impossibility for a weaker set of conditions for the time being.

Theorem 16. *There is no anonymous and neutral SDS which is UL-efficient and UL-strategyproof.*

Proof. We consider three preference profiles:

R^1	R^2	R^3
$1\colon a, b, c$	$1\colon a, c, b$	$1\colon c, a, b$
$2\colon b, a, c$	$2\colon b, a, c$	$2\colon b, a, c$

Let f be an SDS with properties as stated. *UL*-efficiency, neutrality, and anonymity imply that $f(R^1) = [\frac{1}{2}, \frac{1}{2}, 0]$. By *UL*-efficiency $f(R^2)_c = 0$. Hence by *UL*-strategyproofness, $f(R^2) = f(R^1)$. Again by *UL*-strategyproofness $f(R^3)_b = f(R^2)_b = \frac{1}{2}$, otherwise agent 1 could benefit from misrepresenting her preferences according to *UL*-preferences, either by reporting *cab* instead of *acb* or the other way round. By neutrality and anonymity, $f(R^3)_b = f(R^3)_c = \frac{1}{2}$. But the lottery $[0, \frac{1}{2}, \frac{1}{2}]$ is not *UL*-efficient for R^3, which is as a contradiction. □

6 Random Assignment

The domain of random assignment is a special case of the voting problem discussed in Section 5. We consider this as a matching problem with one-sided preferences. Objects from a set O are matched to agents in N, where agents have preferences over objects, but objects do not have preferences. The preference profile will be denoted by R as before. We will restrict to strict preferences for assignments and assume every agent receives exactly one object, so in particular $|O| = |N|$. Every deterministic or pure assignment can be identified with a permutation matrix in $\mathbb{R}^{n\times n}$. If pure assignments are identified with alternatives, any assignments problem can be thought of as a social choice problem. Agents preferences over alternatives extend naturally to preferences over assignments with every agent being indifferent between all assignments in which she receives the same object. Particularly for assignments, it is clearly desirable not to restrict attention to deterministic outcomes, since this produces unfairness. If two agents have the same preferences over objects, there is no fair way to assign one object for sure to both. Therefore, we consider random assignments, which are probability distributions over deterministic assignments. Assignments will be denoted by $Q \in \mathbb{R}^{n\times n}$ and Q_i is the assignment of agent i. By definition, Q is a bistochastic matrix, i.e., each row and each column of Q sums up to one. By a well-known Theorem of Birkhoff (1946), each bistochastic matrix can be expressed as a convex combination of permutation matrices or in our case deterministic assignments. Efficiency and strategyproofness with respect to some extension e translates from social choice in the obvious way.

In this section we proceed by shortly addressing efficiency in random assignments. Then we discuss random serial dictatorship and the probabilistic serial rule, two solution concepts for assignment problems.

6.1 Efficiency

In an influential paper, Bogomolnaia and Moulin (2001) characterize *SD*-efficiency for assignments by a binary relation on alternatives, which shows in particular that *SD*-efficiency depends on the support of an assignment only. As mentioned earlier, *DL* and *UL*-efficiency are equivalent to *SD*-efficiency for assignments as shown by Cho and hence depend on the support only. These charazterizations are special cases of Theorems 8 and 11 respectively. However, the characterization of *BD*-efficiency does not carry over from social choice. Translating an assignment problem to voting results in one alternative per deterministic assignment. But in social choice, if an agent is indifferent between alternatives Q and Q' and S and S' respectively, this does not imply $\phi(Q,S) = \phi(Q',S')$ for all SSB-functions. On the other hand, this equality does hold for the corresponding pure assignments since the agent receives the same object in Q and Q' and S and S'. This shows that in the context of SSB-utility, assignment problems cannot be regarded as voting in the way described above.

Remark 3. *In the assignment domain BD-efficiency does not depend on the support of an assignment only.*

Proof. Let us consider the following 3 agent profile.

$$1\colon a, b, c$$
$$2\colon b, c, a$$
$$3\colon b, c, a$$

We compare two assignments with full support.

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \end{pmatrix} \end{array} \qquad S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix} \end{array}$$

It is easy to see that Q is BD-dominated by Q' defined as follows:

$$Q' = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

Now we aim to show that S is BD-efficient. Assume there is an assignment S' which BD-dominates S. By Theorem 4 and S' being bistochastic, we get the following equations among others:

$$S'_{1b} \geq \frac{1}{2}S'_{1c} \qquad S'_{2b} \geq \frac{3}{2}S'_{2c} \qquad S'_{3b} \geq \frac{3}{2}S'_{3c} \tag{28}$$

$$\sum_{i=1}^{3} S'_{ib} = 1 \qquad \sum_{i=1}^{3} S'_{ic} = 1 \tag{29}$$

Summing up the the inequalities in (28) and using the equalities in (29), we get

$$1 \geq \frac{1}{2}S'_{1c} + \frac{3}{2}(S'_{2c} + S'_{3c}) = \frac{1}{2}S'_{1c} + \frac{3}{2}(1 - S'_{1c}) = \frac{3}{2} - S'_{1c} \tag{30}$$

Clearly $S'_{1c} \leq S_{1c}$, which together with (30) implies $S'_{1c} = S_{1c} = \frac{1}{2}$. By the first inequality in (28) and S' being bistochastic, we have $S'_{1b} = S_{1b}$ and $S'_{1a} = S_{1a}$. From this $S' = S$ follows immediately. Therefore, the only assignment which BD-dominates S is S itself. □

The same statements holds for PC-efficiency. This implies the corresponding remark about PC-efficiency for social choice.

Remark 4. *In the assignment domain PC-efficiency does not only depend on the support of an assignment.*

Proof. Consider a preference profile with 3 agents and preferences abc for each agent. Then the assignment

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \end{array} \tag{31}$$

is PC-dominated by

$$Q' = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} \frac{5}{8} & 0 & \frac{3}{8} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \end{array}$$

Agent 1 prefers his lottery in Q to his lottery in Q'. Agents 2 and 3 are indifferent between both assignments. Whereas

$$S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{array} \tag{32}$$

has the same support as Q, but is PC-efficient. We show this by identifying S as the only assignment which dominates S.
Suppose there is an assignment S' which PC-dominates S. Then the following equations hold:

$$\sum_{i=1}^{3} S'_{ij} = 1 \tag{33}$$

$$\sum_{j=1}^{3} S'_{ij} = 1 \tag{34}$$

$$\frac{2}{3}(S'_{i1} - S'_{i3}) \geq 0 \text{ for } i = 1, 2, 3. \tag{35}$$

But (33) - (35) imply that $S'_{ij} = \frac{1}{3}$ for $i, j = 1, 2, 3$. Thus, $S' = S$ which completes the proof. □

All efficiency notions we considered here can be checked efficiently using linear programming. The proofs in the remarks on BD and PC-efficiency illustrate how to construct appropriate LP's.

6.2 Assignment Mechanisms

An assignment mechanism f maps a preference profile to a random assignment. The definitions for efficiency and strategyproofness carry over from SDSs. For assignments we introduce additional notions of fairness. A mechanism satisfies *equal treatment of equals* if identical preference relations for two agents imply that they receive the same assignment, i.e., $R_i = R_j$ implies $f(R)_i = f(R)_j$. Another desirable property is envy-freeness, meaning every agent should prefer his own assignment to the assignment of any other agent. Since preferences over lotteries depend on the extension considered, envy-freeness is defined with respect to some extension. We say an assignment is *e-envy-free* if $Q_i \; R_i \; Q_j$ for all $j \neq i$ and all $i \in N$. If an assignment mechanism returns an envy-free assignment for any profile, it is envy-free. For discussion of assignment mechanisms and proofs of the facts we state in this section, we refer to Bogomolnaia and Moulin (2001). A standard mechanism is *random serial dictatorship* (random priority) for assignments, which works similarly as for social choice. First a random order of agents is drawn in which they act. If it is an agents turn, she receives her most preferred object among the remaining objects. This mechanism is equivalent to translating the assignment problem to social choice and then applying RSD. The efficiency and incentive properties of RSD as an assignment mechanism are not stronger than in social choice. Strong *SD*-strategyproofness naturally carries over from social choice. Any assignment obtained from RSD is a convex combination of Pareto-optimal assignments, commonly referred to as ex-post-efficiency in this context. However, RSD is not *SD*-efficient and not *SD*-envy-free. Especially the lack of efficiency gives rise to consider alternative mechanisms. A promising solution to this problem, the *probabilistic serial rule (PS)*, was introduced by Bogomolnaia and Moulin. The intuition is that agents eat fractions of their most preferred object until it is eaten up, then they start eating from their second ranked object until eaten up and so on. Therefore, this mechanism is sometimes referred to as the simultaneous eating algorithm. PS features a number of attractive properties such as *SD*-efficiency, *SD*-strategyproofness and *SD*-envy-freeness. Cho (2012) shows that PS is even *DL*-efficient and *DL*-strategyproof. It is however not strongly *SD*-strategyproof. A number of generalizations of PS have been made for different settings, e.g., for weak preferences (Katta and Sethuraman, 2006) or more than one object per agent (Kojima, 2009). In an interesting paper, Che and Kojima (2010) show that RSD and PS are asymptotically equivalent, meaning the RSD and the PS assignment do not differ too much for many copies of each object and many agents. This has two immediate implications. For one, the inefficiency of RSD becomes small in large economies. On the other hand, PS asymptotically inherits the strategyproofness properties from RSD. Since PS has to our knowledge not been considered before in connection with pairwise comparison, we will check it for PC-efficiency and PC-strategyproofness.

Theorem 17. *The probabilistic serial rule is neither PC-efficient nor PC-strategyproof.*

Proof. First we show PC-inefficiency. Consider the following preference profile with 4 agents:

$$\begin{aligned} 1,2,3\colon & a,b,c,d \\ 4\colon & b,a,c,d \end{aligned}$$

The PS assignment

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \end{array}$$

is PC-dominated by

$$Q' = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{15} & \frac{1}{2} & \frac{1}{10} \\ 0 & \frac{3}{5} & 0 & \frac{2}{5} \end{pmatrix} \end{array}$$

since agents $1,2$, and 4 are indifferent between Q' and Q according to PC and agent 3 strictly prefers Q' to Q.

To prove PC-manipulability, consider the profile R defined as follows:

$$\begin{aligned} 1,2\colon & a_1,a_2,a_3,\dots,a_{25} \\ 3\colon & a_2,a_1,a_3,\dots,a_{25} \\ 4,\dots,25\colon & a_1,a_5,a_6,\dots,a_{25},\dots \end{aligned}$$

PS applied to R yields the assignment below for agent 3.

$$Q_3 = [0, \frac{27}{96}, \frac{1}{4}, \frac{1}{4}, 0, \dots, 0, \frac{3}{160}, \frac{1}{25}, \dots, \frac{1}{25}].$$

If instead agent 3 reports the same preferences as 1 and 2, her assignment is

$$Q'_3 = [\frac{1}{25}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \dots, 0, \frac{1}{100}, \frac{1}{25}, \dots, \frac{1}{25}].$$

It can be checked that $Q'_3 \; P_3^{PC} \; Q_3$. □

These negative results raise the question whether there exists any assignments mechanism which is PC-efficient, PC-strategyproof, and satisfies either PC-no-envy or equal treatment of equals. Since PC is asymmetric, the latter two conditions are independent. In the following, we deduce an impossibility for an unrelated set of conditions (since SD-no-envy is stronger than PC-no-envy and equal treatment of equals).

Theorem 18. *For $n \geq 3$ there is no mechanism which satisfies SD-efficiency, SD-no-envy, and UL-strategy-proofness.*

Proof. Bogomolnaia and Moulin show that on three alternatives PS is characterized by SD-efficiency, SD-strategyproofness, and SD-envy-freeness. Furthermore they show that on three alternatives PS is not UL-strategyproof. These two observations imply the claim. □

This theorem highlights the fundamental and possibly unexpected difference between DL and UL, despite their duality. It would be desirable to strengthen Theorem 18 by replacing SD-no-envy with UL-no-envy or equal treatment of equals.
A wide variety of practical applications suppose to look at assignment problems independently from social choice. Results like the equivalence of SD and DL/UL-efficiency show that the assignment domain is a proper restriction of the general social choice setting.

7 Conclusions

We started our study with a comparison between the traditional linear vNM-utility theory and SSB-utilities. Since the axioms imposed to obtain the linear utility representation are often violated in practical applications, the more general SSB-theory is

certainly worth considering and interesting for future research. In this light, the *BD*-extension fits nicely in the inventory of lottery extensions between established notions like the *SD* and *DD*-extension. Our work on efficiency in randomized social choice shows that BD, SD, DL, and *UL*-efficiency are support dependent, which generalizes the respective statements by Bogomolnaia and Moulin and Cho for assignments.
The efficiency welfare theorem shows an alternative way to characterize efficiency. It relates efficiency and social welfare maximization. Finding a utility profile for which some efficient lottery maximizes social welfare can be regarded as solving a minimax problem. The objective is to minimize over all feasible utility profiles the additional utility any lottery can yield. The duality proof based on a method by Athanassoglou for a special case of this theorem illustrates how to construct such a utility profile. However, this proof technique makes crucial use of the characterization of *BD*-efficiency or *SD*-efficiency in case of the ordinal efficiency welfare theorem. Thus, this method does not seem to work for a broader class of lottery extensions.
The framework of extensions is the basis for analysis of social decision schemes. A standard mechanism in randomized social choice is random serial dictatorship. However, RSD fails to satisfy *SD*-efficiency. Most other mechanisms do not even meet very weak strategyproofness requirements. For the class of majoritarian SDSs we prove an impossibility demanding weak notions of efficiency and strategyproofness. We also show that *UL*-efficiency and *UL*-strategyproofness are incompatible provided basic symmetry properties. It would be clearly desirable to strengthen these results to a broader class of SDSs or weaker notions of efficiency and strategyproofness respectively. In the last part we proceed with a similar study for the special case of random assignment.
The focus of our work lay on efficiency. However, for designing new mechanisms and proving impossibilities it would help to get a better grasp on strategyproofness as well.

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