

A tournament of order 24 with two disjoint TEQ-retentive sets

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Brandt et al. (2013) have recently disproved a conjecture by Schwartz (1990) by non-constructively showing the existence of a counterexample with about 10^{136} alternatives. We provide a concrete counterexample for Schwartz’s conjecture with only 24 alternatives.

1 Introduction

A *tournament* T is a pair (A, \succ) , where A is a set of alternatives and \succ is an asymmetric and complete (and thus irreflexive) binary relation on A , usually referred to as the *dominance relation*. The dominance relation can be extended to sets of alternatives by writing $X \succ Y$ when $x \succ y$ for all $x \in X$ and $y \in Y$. For a tournament (A, \succ) , an alternative $x \in A$, and a subset $X \subseteq A$ of alternatives, we denote by $\overline{D}_X(x) = \{y \in X \mid y \succ x\}$ the *dominators* of x . A *tournament solution* is a function that maps a tournament to a nonempty subset of its alternatives (see, e.g., Laslier, 1997, for further information). Given a tournament $T = (A, \succ)$ and a tournament solution S , a nonempty subset of alternatives $X \subseteq A$ is called *S-retentive* if $S(\overline{D}_A(x)) \subset X$ for all $x \in X$ such that $\overline{D}_A(x) \neq \emptyset$. Schwartz (1990) defined the tournament equilibrium set (*TEQ*) of a given tournament $T = (A, \succ)$ recursively as the union of all inclusion-minimal *TEQ*-retentive sets in T .

Schwartz conjectured that every tournament contains a *unique* inclusion-minimal *TEQ*-retentive set, which was later shown to be equivalent to *TEQ* satisfying any one of a number of desirable properties for tournament solutions (Laffond et al., 1993; Houy, 2009a,b; Brandt et al., 2010a; Brandt, 2011b; Brandt and Harrenstein, 2011; Brandt, 2011a). This conjecture was recently disproved by Brandt et al. (2013) who have shown the existence of a counterexample with about 10^{136} alternatives using the probabilistic method. Since it was shown that *TEQ* satisfies the above mentioned desirable properties for all tournaments

that are smaller than the smallest counterexample to Schwartz’s conjecture, the search for smaller counterexamples remains of interest. In fact, the counterexample found by Brandt et al. (2013) is so large that it has no practical consequences whatsoever for TEQ .

In this note, we provide a tournament of order 24 with two disjoint TEQ -retentive sets. In contrast to the previous counterexample, both TEQ -retentive sets are of the same order and even isomorphic. On the other hand, the tournament does not constitute a counterexample to a weakening of Schwartz’s conjecture by Brandt (2011b).

Let n_{TEQ} be the greatest natural number n such that all tournaments of order n or less admit a unique minimal TEQ -retentive set. Together with earlier results by Brandt et al. (2010b), we now have that $12 \leq n_{TEQ} \leq 23$.

The counterexample is based on new structural insights about tournament solutions that we will explore further in future work.

2 The Counterexample

We define a tournament $T = (A \succ)$ with 24 alternatives that has two disjoint TEQ -retentive sets $X = \{x_1, \dots, x_{12}\}$ and $Y = \{y_1, \dots, y_{12}\}$ with $A = X \cup Y$. The two induced subtournaments $T|_X$ and $T|_Y$ are isomorphic.

Let $X_1 = \{x_1, \dots, x_6\}$, $X_2 = \{x_7, \dots, x_{12}\}$, $Y_1 = \{y_1, \dots, y_6\}$, and $Y_2 = \{y_7, \dots, y_{12}\}$. Then, the dominance relation between alternatives in X and Y is defined as illustrated in Figure 1.

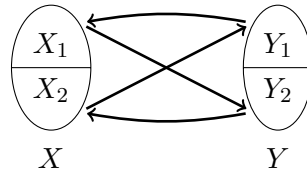


Figure 1: The structure of the counterexample. The two subtournaments $T|_X$ and $T|_Y$ are isomorphic and of order 12. Furthermore, $X_1 \succ Y_2$, $X_2 \succ Y_1$, $Y_1 \succ X_1$, and $Y_2 \succ X_2$.

The dominator sets in $T|_X$ (and equivalently in $T|_Y$) are defined as follows:

$$\begin{aligned}
 \overline{D}_X(x_1) &= \{x_4, x_5, x_6, x_8, x_9, x_{12}\}, & \overline{D}_X(x_2) &= \{x_1, x_6, x_7, x_{10}, x_{12}\}, \\
 \overline{D}_X(x_3) &= \{x_1, x_2, x_6, x_7, x_9, x_{10}\}, & \overline{D}_X(x_4) &= \{x_2, x_3, x_7, x_8, x_{11}\}, \\
 \overline{D}_X(x_5) &= \{x_2, x_3, x_4, x_8, x_{10}, x_{11}\}, & \overline{D}_X(x_6) &= \{x_4, x_5, x_9, x_{11}, x_{12}\}, \\
 \overline{D}_X(x_7) &= \{x_1, x_5, x_6, x_{11}, x_{12}\}, & \overline{D}_X(x_8) &= \{x_2, x_3, x_6, x_7, x_{12}\}, \\
 \overline{D}_X(x_9) &= \{x_2, x_4, x_5, x_7, x_8\}, & \overline{D}_X(x_{10}) &= \{x_1, x_4, x_6, x_7, x_8, x_9\}, \\
 \overline{D}_X(x_{11}) &= \{x_1, x_2, x_3, x_8, x_9, x_{10}\}, & \text{and } \overline{D}_X(x_{12}) &= \{x_3, x_4, x_5, x_9, x_{10}, x_{11}\}.
 \end{aligned}$$

A rather tedious check reveals that

$$\begin{aligned}
 TEQ(\overline{D}_A(x_1)) &= \{x_4, x_8, x_{12}\} \subset X, & TEQ(\overline{D}_A(x_2)) &= \{x_6, x_{10}, x_{12}\} \subset X, \\
 TEQ(\overline{D}_A(x_3)) &= \{x_6, x_7, x_9\} \subset X, & TEQ(\overline{D}_A(x_4)) &= \{x_2, x_7, x_{11}\} \subset X, \\
 TEQ(\overline{D}_A(x_5)) &= \{x_2, x_8, x_{10}\} \subset X, & TEQ(\overline{D}_A(x_6)) &= \{x_4, x_9, x_{11}\} \subset X, \\
 TEQ(\overline{D}_A(x_7)) &= \{x_1, x_5, x_{11}\} \subset X, & TEQ(\overline{D}_A(x_8)) &= \{x_3, x_6, x_{12}\} \subset X, \\
 TEQ(\overline{D}_A(x_9)) &= \{x_2, x_5, x_7\} \subset X, & TEQ(\overline{D}_A(x_{10})) &= \{x_4, x_6, x_7\} \subset X, \\
 TEQ(\overline{D}_A(x_{11})) &= \{x_1, x_2, x_8\} \subset X, \text{ and} & TEQ(\overline{D}_A(x_{12})) &= \{x_3, x_4, x_9\} \subset X.
 \end{aligned}$$

Hence, X is TEQ -retentive in T . Moreover, it can be checked that $TEQ(\overline{D}_A(y_i)) \subset Y$ for all $i \in \{1, \dots, 12\}$, which implies that Y is TEQ -retentive as well. In fact, we even have that, for all $i, j \in \{1, \dots, 12\}$, $y_j \in TEQ(\overline{D}_A(y_i))$ if and only if $x_j \in TEQ(\overline{D}_A(x_i))$.

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