Welfare Guarantees in Schelling Segregation

Joint work with Warut Suksompong and Alexandros Voudouris

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Segregation on a checkerboard

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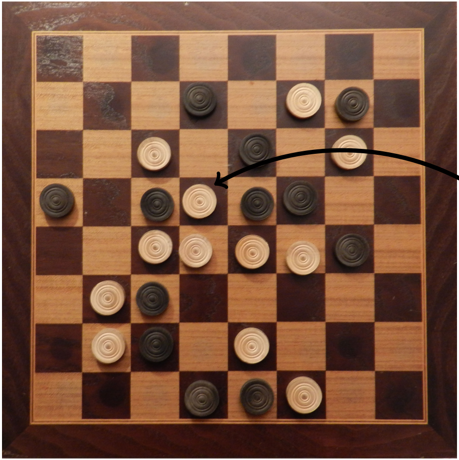

neighbors define happiness

Segregation on a checkerboard

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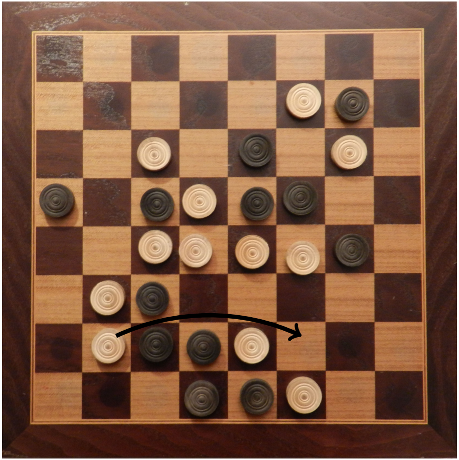
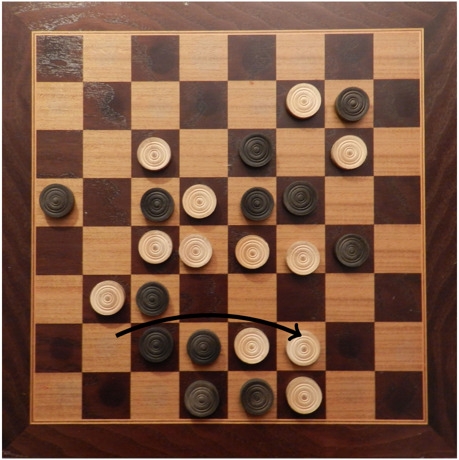


Segregation on a checkerboard



Segregation on a checkerboard

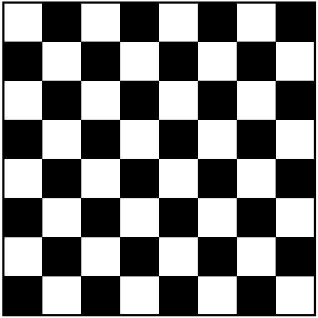
Segregation on a checkerboard

Schelling segregation

- Seeks to explain segregation in metropolitan areas
- First applied to grids (checkers) and lines (Schelling 1969, 1971)
- Surprising convergence under low threshold for movement
- Recently: Game-theoretic approach (Chauhan et al. 2018, Echzell et al. 2019, Elkind et al. 2019)

Formal model

- Set of n agents
- Partitioning into two classes
- Topology graph with at least n vertices



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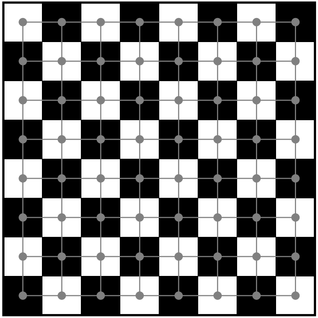

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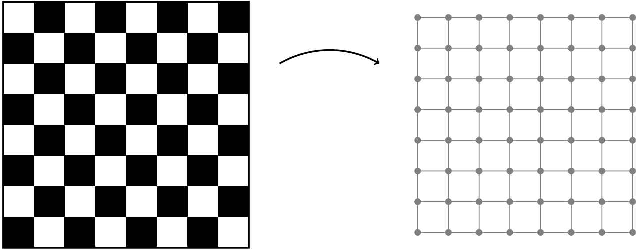
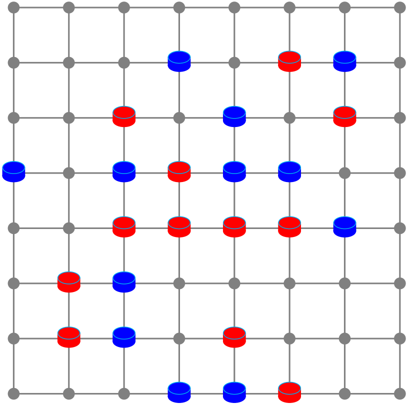


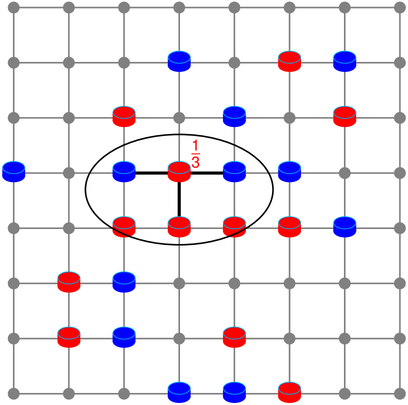
Utilities and Social welfare

- Output: assignment of agents to nodes



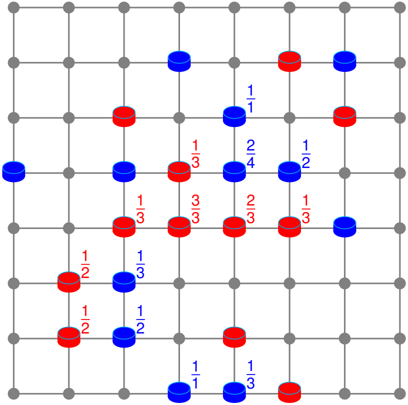
Utilities and Social welfare

- Output: assignment of agents to nodes
- Utilities: fractions friends/neighbors (0 if no neighbors)



Utilities and Social welfare

- Output: assignment of agents to nodes
- Utilities: fractions friends/neighbors (0 if no neighbors)
- Social welfare: sum of utilities



Social Welfare$= 7\frac{5}{6}$

Maximum welfare

Theorem

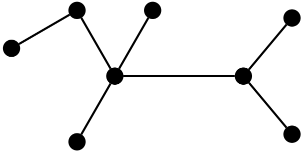
It is NP-complete to maximize social welfare in Schelling instances, even for the class of instances where the number of agents is equal to the number of nodes.

- Very restrictive class of instances
- Previous reductions require auxiliary agent type
- Approximation of social welfare?

Let's play a game!

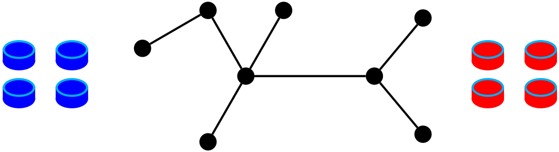
Schelling Welfare Game

- Game board: topology graph



Schelling Welfare Game

- Game board: topology graph
- Red and blue pieces placed cooperatively



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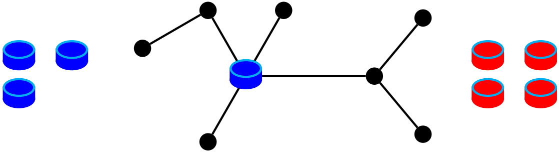

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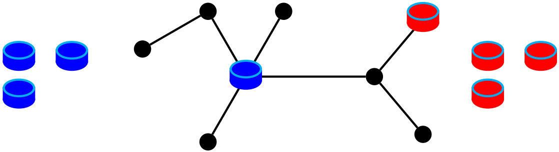
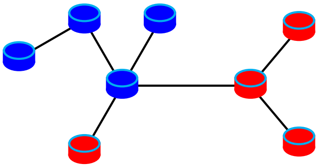


Schelling Welfare Game

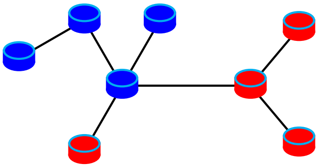
- Game board: topology graph
- Red and blue pieces placed cooperatively
- Award: social welfare of assignment



$$SW = 3.5 + 2.67 = 6.17$$

Schelling Welfare Game

- Game board: topology graph
- Red and blue pieces placed cooperatively
- Award: social welfare of assignment
- What is a good move?



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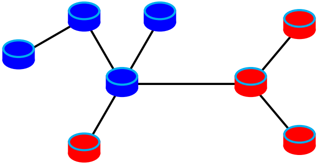


Position with **high expected welfare under uniform distribution**

Approximation of welfare

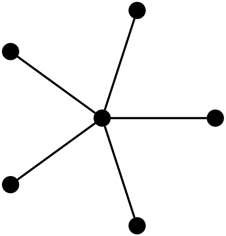
Theorem

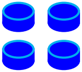
For any Schelling instance with n agents, there exists an assignment with social welfare at least $\frac{n}{2} - 1$, which can be computed in polynomial time.

- Consider assignment chosen uniformly at random
- Deramdomize this selection
- Tight bound slightly larger

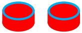
Two striking examples

Star topology



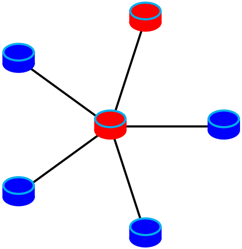


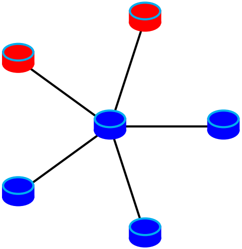
$n - 2$ blue agents



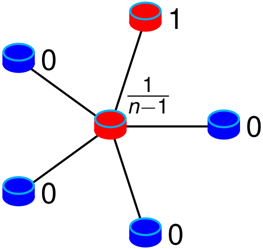
2 red agents

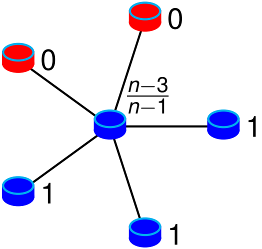
Star topology





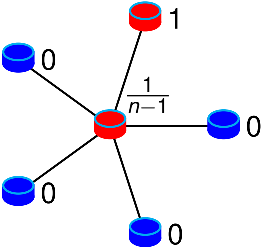
Star topology

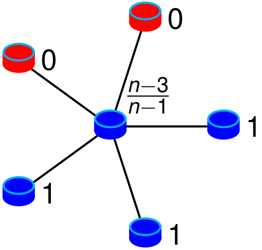




- All assignments are Pareto optimal
- Social welfare linear factor apart

Star topology

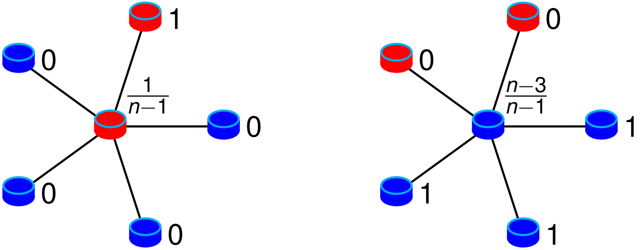




$OUV = (1, \frac{1}{n-1}, 0, \dots, 0)$ $\qquad$ $OUV = (1, \dots, 1, \frac{n-3}{n-1}, 0, 0)$

- All assignments are Pareto optimal
- Social welfare linear factor apart
- *Ordered utility vector* of right assignment dominates left one

Star topology



$$\left(1, 1, 1, \tfrac{n-3}{n-1}, 0, 0\right)$$
$$\left(1, \tfrac{1}{n-1}, 0, 0, 0, 0\right)$$

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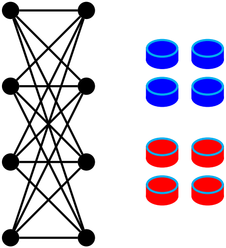
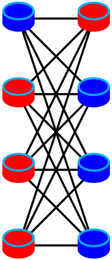
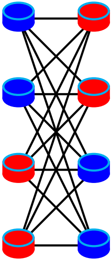
Complete bipartite topology



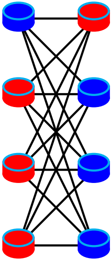
Complete bipartite topology

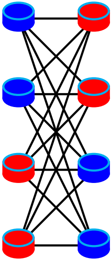




Complete bipartite topology

$SW = 4 - \frac{8}{n}$



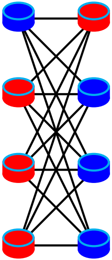


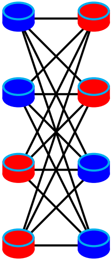
$SW = \frac{n}{2}$

- Both assignments are Pareto optimal
- Social welfare linear factor apart

Complete bipartite topology

$SW = 4 - \frac{8}{n}$





$SW = \frac{n}{2}$

$OUV = (\frac{n-2}{n}, \frac{n-2}{n}, \frac{2}{n}, \ldots, \frac{2}{n})$

$OUV = (\frac{1}{2}, \ldots, \frac{1}{2})$

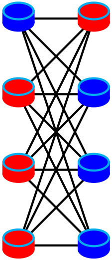
- Both assignments are Pareto optimal
- Social welfare linear factor apart
- Ordered utility vectors undominated

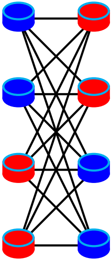
Complete bipartite topology

$SW = 4 - \frac{8}{n}$

$SW_B = 2 - \frac{4}{n}$

$SW_R = 2 - \frac{4}{n}$





$SW = \frac{n}{2}$

$SW_B = \frac{n}{4}$

$SW_R = \frac{n}{4}$

$OUV = (\frac{n-2}{n}, \frac{n-2}{n}, \frac{2}{n}, \ldots, \frac{2}{n})$ $\qquad$ $OUV = (\frac{1}{2}, \ldots, \frac{1}{2})$

- Both assignments are Pareto optimal
- Social welfare linear factor apart
- Ordered utility vectors undominated
- Domination in *group welfare*

Welfare notions

$$
\begin{tikzcd}
 & \text{Group-welfare optimality} \arrow[dr] & \\
\text{Maximum welfare} \arrow[ur] \arrow[dr] & & \text{Pareto optimality} \\
 & \text{Utility-vector optimality} \arrow[ur] &
\end{tikzcd}
$$

Welfare guarantees

- Group-welfare optimality guarantees welfare $n/(n-1)$
- Utility-vector optimality guarantees welfare 1
- Pareto optimality guarantees welfare $1/\sqrt{n}$

$$
\begin{tikzcd}
 & \text{Group-welfare optimality: } n/(n-1) \arrow[dr] & \\
\text{Maximum welfare: } n/2-1 \arrow[ur] \arrow[dr] & & \text{Pareto optimality: } 1/\sqrt{n} \\
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\end{tikzcd}
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Welfare guarantees

- Group-welfare optimality guarantees welfare $n/(n-1)$
- Utility-vector optimality guarantees welfare 1
- Pareto optimality guarantees welfare $1/\sqrt{n}$
- Pareto optimality guarantees welfare $n/(n-1)$ for tree topologies

$$\begin{tikzcd}
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\text{Maximum welfare: } n/2-1 \arrow[ur] \arrow[dr] & & \text{Pareto optimality: } 1/\sqrt{n} \\
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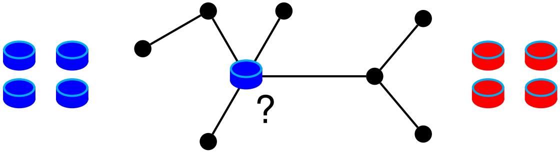
\end{tikzcd}$$

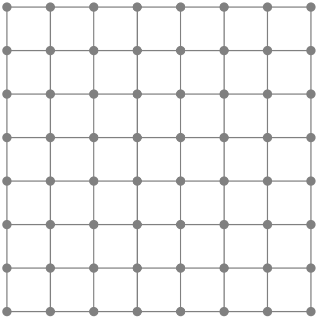
Agents of positive utility

- Not all agents may obtain positive utility



Agents of positive utility

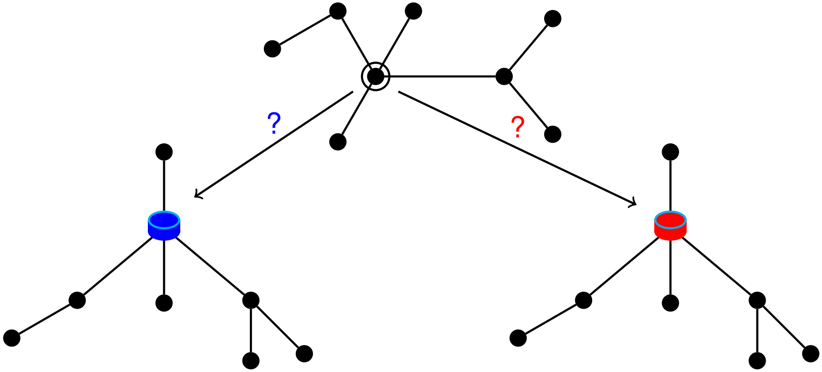
- Not all agents may obtain positive utility
- Minimum degree of 2 allows to give everyone positive utility*



* if number of agents equals number of nodes

Agents of positive utility

- Not all agents may obtain positive utility
- Minimum degree of 2 allows to give everyone positive utility*
- Efficient decidability on tree topologies



Conclusion

- Maximum welfare can be approximated well
- New welfare notions differentiate Pareto-optimal assignments
- Basic happiness of all agents can often be achieved