# Pareto-Optimality in Cardinal Hedonic Games

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## ABSTRACT

Pareto-optimality and individual rationality are among the most natural requirements in coalition formation. We study classes of hedonic games with cardinal utilities that can be succinctly represented by means of complete weighted graphs, namely additively separable (ASHG), fractional (FHG), and modified fractional (MFHG) hedonic games. Each of these can model different aspects of dividing a society into groups. For all classes of games, we give algorithms that find Pareto-optimal partitions under some natural restrictions. While the output is also individually rational for modified fractional hedonic games, combining both notions is NP-hard for symmetric ASHGs and FHGs. In addition, we prove that welfare-optimal and Pareto-optimal partitions coincide for simple, symmetric MFHGs, solving an open problem from Elkind et al. [9]. For general MFHGs, our algorithm returns a 2-approximation to welfare. Interestingly, welfare-optimal partitions in MFHGs only require coalitions of at most three agents.

## KEYWORDS

Coalition Formation; Hedonic Games; Pareto-optimality; Welfare

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## 1 INTRODUCTION

Coalition formation is a central problem in multi-agent systems and has been extensively studied, ever since the publication of von Neumann and Morgenstern's *Theory of Games and Economic Behavior* in 1944. In coalition formation, every agent of a group seeks to be in a desirable coalition. As an important special case therein, in clustering problems, a society is observed, for which a structuring into like-minded groups, or communities, is to be identified [16]. Coalition scenarios can be modeled by letting the agents submit preferences, subject to which the happiness of an individual agent with her coalition, or the like-mindedness of a coalition, can be measured. The goal of every individual agent is to maximize the value of her coalition.

In many settings, it is natural to assume that an agent is only concerned about her own coalition, i.e., externalities are ignored. As a consequence, much of the research on coalition formation now concentrates on these so-called *hedonic games* [7]. Still, the number of coalitions an agent can be part of is exponential in the number of agents, and therefore it is desirable to consider expressive, but succinctly representable classes of hedonic games. This can be established by encoding preferences by means of a complete directed weighted graph where an edge weight $v_x(y)$ is a cardinal value or utility that agent $x$ assigns to agent $y$. Still, this underlying structure leaves significant freedom, how to obtain (cardinal) preferences over coalitions. An especially appealing type of preferences is to have the sum of values of the other individuals as the value of a coalition. This constitutes the important class of additively separable hedonic games (ASHGs) [6].

In ASHGs, an agent is willing to accept an additional agent into her coalition as long as her valuation of this agent is non-negative and in this sense, ASHGs are not sensitive to the intensities of single-agent preferences. In particular, if all valuations are non-negative, forming the grand coalition consisting of all agents is a best choice for every agent. In contrast, fractional hedonic games (FHGs) define preferences over coalitions by dividing the sum of values by the size of the coalition [1]. This incentivizes agents to form dense cliques, and therefore appropriately models like-mindedness in the sense of clustering problems.

On the other hand, agents in FHGs may improve with a new agent whose valuation is below the average of their current coalition partners. To avoid this, it is also natural to define the value of a coalition as the average value of other agents, i.e., the denominator in the definition of FHGs is replaced by the size of the coalition minus 1. This defines the class of modified fractional hedonic games (MFHGs) [17]. For all measures of stability and optimality that have been investigated, these games do not guarantee the formation of large cliques and are therefore less suitable for clustering problems. In fact, partitions of agents into coalitions satisfying many desirable properties can be computed in polynomial time, often simply by computing maximum (weight) matchings. However, these games ensure a certain degree of homogeneity of the agents, as agents tend to contribute uniformly to each other's utility in stable coalitions.

Having selected a representation for modeling the coalition formation process, one needs a measure for evaluating the quality of a partition. Various such measures, also called *solution concepts*, have been proposed in the literature. Most of them aim to guarantee a certain degree of stability—preventing single agents or groups from agents to break apart from their coalitions—and optimality—guaranteeing a globally measured outcome that is good for the society as entity. A good overview of solution concepts is by Aziz and Savani [5]. The most undisputed measure of optimality is *Pareto-optimality*, i.e., there should be no other partition, such that every agent is weakly, and some agent strictly, better off. Apart from its optimality guarantee, Pareto-optimality can also be seen as a measure of stability, because a Pareto-optimal partition disallows an agent to propose a partition she prefers without having another agent vetoing this proposal. A stronger notion of optimality is that

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of *(utilitarian) welfare-optimality*, which aims to maximize the sum of utilities of all agents.

Pareto-optimal outcomes still might be extremely disadvantageous for single agents that receive large negative utility in order to give small positive utility to another agent. Therefore, it is also desirable that agents receive at least the utility they would receive in a coalition of their own (in all our models this means non-negative utility). This condition is called *individual rationality*. Clearly, a Pareto-optimal and individually rational outcome can always be found by the simple local search algorithm that starts with the partition of the agents into singleton coalitions and moves to Pareto-improvements as long as these exist. In general, this basic algorithm need not run in polynomial time and the output need not be welfare-optimal. It can even occur that no welfare-optimal partition is individually rational. As we will see, it is even often NP-hard to compute a Pareto-optimal and individually rational partition (Theorem 5.2, Theorem 6.4).

We study Pareto-optimality in all three classes of games, and give polynomial-time algorithms for computing Pareto-optimal partitions in important subclasses, including symmetric ASHGs and MFHGs. In addition, we prove that welfare-optimal and Pareto-optimal partitions coincide for simple symmetric MFHGs, closing the bounds on the price of Pareto-optimality for this class of games left by Elkind et al. [9]. In the weighted case, our algorithm for Pareto-optimality in MFHGs gives a 2-approximation of welfare and its output is always individually rational. While we can prove that even in the weighted case, welfare-optimality is attained by a partition consisting only of coalitions of size two and three, the complexity of computing a welfare-optimal partition remains open. On the other hand, we prove that computing a Pareto-optimal and individually rational partition is NP-hard for symmetric ASHGs and FHGs, thus extending a result by Aziz et al. [3] for general ASHGs. Note that symmetry is a significant restriction for hardness reductions, because non-symmetric games allow for the phenomenon of non-mutual interest.

## 2 RELATED WORK

Hedonic games were first introduced by Drèze and Greenberg [7]. Since then, a great amount of research has been devoted to the study of algorithmic and mathematical properties of axiomatic concepts regarding stability and optimality, representability of preferences, and the discovery of well-behaved, yet expressive classes of hedonic games. The survey by Hajduková [12] gives a critical overview of preference representations and conditions that allow for the existence and efficient computability of central stability notions, such as Nash and core stability.

Pareto-optimality can be studied in many classes of hedonic games by exploiting a strong relationship between Pareto-optimality and perfection, i.e., partitions that put every agent in one of her most preferred partitions [2]. This gives rise to the preference refinement algorithm (PRA) which finds Pareto-optimal partitions under certain conditions by means of a perfection-oracle. The resulting Pareto-optimal partitions are even individually rational. The assumptions required for the algorithm include the efficient computation of preference refinements, which is not possible for ASHGs and FHGs. Indeed, during the PRA, one has to search a set of hedonic games that interpolates between two preference profiles and can contain games not implementable as an ASHG or FHG, respectively. In fact, for ASHGs and FHGs, perfect partitions can be computed in polynomial time (Theorem 5.4, Theorem 6.6), while computing Pareto-optimal and individually rational partitions is NP-hard (Theorem 5.2, Theorem 6.4).

For cardinal hedonic games, stability and optimality have been studied to some extent. Gairing and Savani [10, 11] settled the complexity of the individual-player stability notions of Drèze and Greenberg for symmetric ASHGs by treating them as local search problems. For FHGs, hardness and approximation results for welfare are given by Aziz et al. [4], while Aziz et al. [1] study stability. Monaco et al. [15] show the tractability of some stability notions and welfare-optimality for simple symmetric MFHGs, and the computability of partitions in the core for weighted MFHGs. The approximation results for FHGs and most of the positive results for MFHGs rely on computing specific matchings.

Pareto-optimality for cardinal hedonic games was mainly studied by Elkind et al. [9] in terms of the price of Pareto-optimality (PPO), a worst-case ratio of Pareto-optimal and welfare-optimal partitions. The focus lies on simple symmetric graphs and their main result is to bound the PPO between 1 and 2 for simple, symmetric MFHGs. Since we show that, for these games, every Pareto-optimal partition is welfare-optimal, we will close this gap. Pareto-optimality for ASHGs was considered by Aziz et al. [3]. However, they only dealt with a restricted class of preferences that guarantees unique top-ranked coalitions and therefore one can apply a simple serial dictatorship algorithm.

## 3 PRELIMINARIES

The primary ingredient of our model is a set of agents $N$ that assign hedonic preferences over *partitions* of $N$ (also called *coalition structures*), where the only information of a partition an agent is interested in, is her own coalition. Preferences of agent $i$ are therefore given over $\mathcal{N}_i = \{C \subseteq N : i \in C\}$, i.e., the subsets of agents including herself, by valuation functions $v_i : \mathcal{N}_i \to \mathbb{R}$. We investigate a partition $\pi$ of the agents for notions of optimality and stability, most importantly *Pareto-optimality*. Given a partition $\pi$, we denote by $\pi(i)$ the partition of agent $i$ and the utility she received from this partition by $v_i(\pi) = v_i(\pi(i))$. A partition $\pi'$ is a *Pareto-improvement* over $\pi$ if, for all agents $i \in N$, $v_i(\pi') \geq v_i(\pi)$ and there exists an agent $j \in N$ with $v_j(\pi') > v_j(\pi)$. In this case, we also say that $\pi'$ *Pareto-dominates* $\pi$. A partition $\pi$ is *Pareto-optimal* if it is not Pareto-dominated.

A stronger requirement is that of *(utilitarian) welfare-optimality*. For a subset $M \subseteq N$ of agents, we denote $v_M(\pi) = \sum_{i \in M} v_i(\pi)$. The *social welfare* of a coalition $C$ is $\mathcal{SW}(C) = v_C(\pi)$. The social welfare of a partition $\pi$ is $\mathcal{SW}(\pi) = \sum_{C \in \pi} \mathcal{SW}(C) = \sum_{i \in N} v_i(\pi)$. A partition $\pi$ is called *welfare-optimal* if it maximizes the function $\mathcal{SW}$ amongst all partitions of agents. Welfare-optimal partitions are Pareto-optimal.

A partition $\pi$ is *individually rational* for agent $i$ if $v_i(\pi) \geq v_i(\{i\})$, i.e., agent $i$ does not prefer to stay alone. In addition, $\pi$ is *individually rational* if it is individually rational for every agent. Partitions that are welfare-optimal or individually rational and Pareto-optimal always exist.

Preferences are succinctly represented by a family of cardinal utility functions $(v_i)_{i\in N}$ where $v_i : N \to \mathbb{R}$ with $v_i(i) = 0$ that can be aggregated to preferences over coalitions. A natural representation is by means of a complete, directed, and weighted graph $G = (N, E, v)$ where the weights are defined by the utility functions. A game is called *symmetric* if, for all pairs of agents $i, j \in N$, $v_i(j) = v_j(i)$. In this case, the underlying graph is symmetric and we denote $v(e) = v(i, j) = v_i(j) = v_j(i)$ for a 2-elementary set of agents $e = \{i, j\}$. A game is called *simple* if $v_i(j) \in \{0, 1\}$ for all agents $i, j \in N$. A hedonic game with simple and symmetric preferences can therefore be represented by an unweighted and undirected (but incomplete) graph.

We define the aggregated utilities for ASHGs, FHGs, and MFHGs for partition $\pi$ and agent $i$ by

$$v_i^{ASHG}(\pi) = \sum_{j\in\pi(i)} v_i(j)$$

$$v_i^{FHG}(\pi) = \frac{v_i^{ASHG}(\pi)}{|\pi(i)|}, \text{ and}$$

$$v_i^{MFHG}(\pi) = \begin{cases} \frac{v_i^{ASHG}(\pi)}{|\pi(i)|-1} & \text{for } \pi(i) \neq \{i\} \\ 0 & \text{for } \pi(i) = \{i\}. \end{cases}$$

If it is clear from the context which game is considered, we omit the superscripts of the utility functions.

We use the following notation from graph theory. For an arbitrary graph $G = (V, E)$, a vertex set $W \subseteq V$ and an edge set $F \subseteq E$, denote by $G[W]$ and $G[F]$ the subgraph of $G$ induced by $W$ and $F$, respectively, and denote by $E(G)$ its edge set.

# 4 MODIFIED FRACTIONAL HEDONIC GAMES

In this section we focus on symmetric MFHGs. The analysis of Pareto-optimality on this class of games relies on an extension of maximum matchings to cliques. Given a graph, a set $C$ of vertex-disjoint cliques, each of size at least 2, is called a *clique matching*. A vertex that is part of any clique in $C$ is called *covered* or *matched*. We are interested in clique matchings that cover a maximum number of vertices. We call the corresponding search problem `MaxCliqueMatching`. Interestingly, a clique matching is maximum if and only if it is *inclusion-maximal w.r.t. vertices*, i.e., there exists no other clique matching covering a strict superset of vertices (Theorem 4.5), and can be computed in polynomial time. We can further simplify the analysis to the special case that only triangles and edges are allowed.

Given a graph, a set of vertex disjoint cliques, each of size 2 or 3, is called a *3-clique matching*. By splitting larger cliques, `MaxCliqueMatching` is equivalent to `Max3CliqueMatching`, i.e., finding a maximum 3-clique matching. Given a 3-clique matching $C$, we denote by $M(C)$ and $T(C)$ its cliques of size 2 (edges) and size 3 (triangles), respectively.

Theorem 4.1 (Hell and Kirkpatrick [13]). *The problem* `Max3CliqueMatching` *can be solved in polynomial time.*

We prove that `MaxCliqueMatching` is equivalent to finding a Pareto-optimal partition on an MFHG. Note that a relationship between clique matchings and *simple* symmetric MFHGs was already exploited by Monaco et al. [15] for computing welfare-optimal partitions.

Theorem 4.2. `MaxCliqueMatching` *is equivalent to finding a Pareto-optimal partition on a symmetric MFHG (under Turing reductions). Moreover, if we can solve* `MaxCliqueMatching` *in polynomial time, we can even compute a Pareto-optimal and individually rational partition for a symmetric MFHG in polynomial time.*

Proof. Assume first that we are given an algorithm to find a Pareto-optimal partition on a symmetric MFHG and let $G = (V, E)$ be an instance of `MaxCliqueMatching`. We transform $G$ into an MFHG with the underlying weighted symmetric graph $G' = (V, E', v)$ where $E' = \{e \subseteq V : |e| = 2\}$ and

$$v(e) = \begin{cases} 1 & \text{if } e \in E \\ -\Delta - 1 & \text{else,} \end{cases}$$

where $\Delta$ is the maximum degree of a vertex in $G$.

Let $\pi$ be a Pareto-optimal partition of vertices into coalitions for the symmetric MFHG with underlying graph $G'$. Define $C = \{P \in \pi : |P| \geq 2\}$. Then, $C$ consists of cliques in $G$. Otherwise, by construction of the utilities, there is one agent who receives negative utility. Assume for contradiction that there exists a coalition $P \in C$ such that some agents in $P$ receive negative utility. Let $S = \{p \in P : v(p, p') = 1 \text{ for all } p' \in P\backslash\{p\}\}$ be the set of agents that receive non-negative utility from all other agents in $P$. Since some agents receive negative utility, there exists an agent $q \in P\backslash S$. But then, the coalition $S_q = S \cup \{q\}$ forms a clique in $G$ and the partition $\pi' = (\pi\backslash\{P\}) \cup \{S_q\} \cup \{\{p'\} : p' \in P\backslash S_q\}$ is a Pareto improvement over $\pi$. Hence, $\pi$ consists only of cliques and, by design of the MFHG utilities, assigns utility 1 to agents in a clique of size at least 2, and 0 to agents in singleton coalitions. Consequently, $C$ is inclusion-maximal w.r.t. vertices, because every clique matching that covers strictly more agents gives rise to a Pareto-improvement that assigns utility 1 to a strict superset of agents that already receive utility 1. Hence, we can solve `MaxCliqueMatching` by computing a Pareto-optimal partition of the MFHG induced by $G'$.

Conversely, assume that we can solve `MaxCliqueMatching`. Consider a symmetric MFHG induced by a complete weighted graph $G = (N, E, v)$. Algorithm 1 computes a Pareto-optimal partition in polynomial time given an algorithm `MaxCliqueMatching` that computes a vertex-maximal clique matching in polynomial time. The idea is to restrict attention to the unweighted subgraph induced by edges with the largest positive weight still available.

The running time of the algorithm is clearly polynomial, including polynomially many calls of `MaxCliqueMatching`. We prove its correctness. Let $\pi$ be the output of the algorithm. First, note that all non-singleton coalitions are cliques in $G$ with identical positive utility within each clique. Hence, the output is individually rational.

For the proof of Pareto optimality, we assume that the while loop took $m$ iterations and we subdivide $\pi = \mathcal{S} \cup \bigcup_{k=1}^{m} C_k$, where $C_k$ is the clique matching in iteration $k$, and $\mathcal{S}$ consists of the singleton coalitions that are added to $\pi$ after the while loop. We will show by induction over $m$ that if the algorithm uses $m$ iterations of the while loop, then the output is Pareto-optimal. Let $\pi'$ be any coalition such that, for all agents $i \in N$, $v_i(\pi) \leq v_i(\pi')$. We will prove that this implies, for all agents $i \in N$, $v_i(\pi) = v_i(\pi')$.

**Input:** Symmetric MFHG induced by graph $G = (N, E, v)$
**Output:** Pareto-optimal and individually rational partition $\pi$

$\pi \leftarrow \emptyset, A \leftarrow N, G_r \leftarrow G[\{e \in E \colon v(e) > 0\}]$
**while** $E(G_r) \neq \emptyset$ **do**
  $v_{\max} \leftarrow \max\{v(e) \colon e \in E(G_r)\}$
  $E_H \leftarrow \{e \in E(G_r) \colon v(e) = v_{\max}\}$
  $H \leftarrow G_r[E_H]$
  $C \leftarrow \texttt{MaxCliqueMatching}(H)$
  $\pi \leftarrow \pi \cup C$
  $A \leftarrow \{a \in A \colon a$ not covered by $C\}$
  $G_r \leftarrow G_r[A]$
**return** $\pi \cup \{\{a\} \colon a \in A\}$

**Algorithm 1: Pareto-optimal partition of a sym. MFHG**

If $m = 0$, $G$ contains no edges of positive weight and therefore, for all agents $i \in N$, $v_i(\pi) = 0 \geq v_i(\pi')$. For the induction step, let $m \geq 1$. Let $H$ be the auxiliary graph of the first while loop. Note that within $\pi$, agents in $C_1$ can only be matched with agents in $H$ since they receive the highest possible utility of any agent in any coalition in $G$ and every other agent diminishes their MFHG utility. In particular, they cannot be better off. Since $C_1$ is a vertex-maximal clique matching on $H$, no agent in $H$ not covered by $C_1$ can be in a coalition with an agent in $C_1$ in $\pi'$. Define $W = \{i \in N \colon i$ not covered by $C_1\}$ and consider $\hat{G} = G[W]$. Then $\hat{\pi} = \mathcal{S} \cup \bigcup_{k=1}^{m-1} C_{k+1}$ is a possible outcome of the algorithm for $\hat{G}$. Furthermore, $\pi'$ restricted to agents in $W$ is weakly better for any agent in $W$ than $\hat{\pi}$. Hence, by induction, also no agent outside $C_1$ can be better off. □

Even though Pareto-optimal outcomes may be worse than welfare-optimal outcomes by an arbitrarily large factor, the output of the above algorithm guarantees significant social welfare.

Theorem 4.3. *Let a symmetric MFHG be given. Let $\pi$ be the partition computed by Algorithm 1 for this game. Then, for any welfare-optimal partition $\pi^*$, it holds that $2\mathcal{SW}(\pi) \geq \mathcal{SW}(\pi^*)$.*

Proof. Consider a symmetric MFHG induced by graph $G = (N, E, v)$. Let $\pi$ be a partition computed by Algorithm 1 for this game and let $\pi^*$ be welfare-optimal. We will show that, for any coalition $C \in \pi^*$, it holds that $2v_C(\pi) \geq v_C(\pi^*)$, which implies the assertion.

Before we prove this, we make the observation that, for each edge $\{x, y\} = e \in E$, it holds that $v_x(\pi) \geq v(x, y)$ or $v_y(\pi) \geq v(x, y)$. Indeed, if $v(x, y) \leq 0$, then this follows from the individual rationality of $\pi$. If we reach a maximum weight $v_{\max} \leq v(x, y)$ in the while loop and $x \notin G_r$ or $y \notin G_r$ then $v_x(\pi) > v(x, y)$ or $v_y(\pi) > v(x, y)$. Otherwise, we reach an iteration where $v_{\max} = v(x, y)$ and any maximum clique matching matches at least one of them.

Now let any coalition $C \in \pi^*$ be given where $|C| = k$. The next step is to sort the agents in $C$ by means of Algorithm 2. The resulting order places the agents essentially in decreasing value $w_i$ of an incident edge of high utility. Let $(c_1, \dots, c_k)$, $(w_1, \dots, w_k)$ be an outcome of this algorithm. We claim that for every $i \in \{1, \dots, k\}$, $v_{c_i}(\pi^*) \leq v_{c_i}(\pi) + \sum_{1 \leq j < i} \frac{w_j}{k-j}$. Note that for all $1 \leq j < i$,

**Input:** Coalition $C \in \pi^*$
**Output:** Order $(c_1, \dots, c_k)$ of $C$ and weights $(w_1, \dots, w_k)$

$H \leftarrow C, F \leftarrow \{\{x, y\} \in E \colon x, y \in C, v(x, y) > 0\}, j \leftarrow 1$
**while** $F \neq \emptyset$ **do**
  $v_{\max} \leftarrow \max\{v(e) \colon e \in F\}$
  Choose $\{x, y\} \in \operatorname{argmax}\{v(x, y) \colon \{x, y\} \in F\}$ with $v_x(\pi) \geq v(x, y)$
  $c_j \leftarrow x, w_j \leftarrow v_{\max}$
  $H \leftarrow H \backslash \{x\}, F \leftarrow \{e \in F \colon x \notin e\}, j \leftarrow j + 1$
Order $H = (c_j, \dots, c_k)$ arbitrarily, $w_i \leftarrow 0$ for $i = j, \dots, k$
**return** $(c_1, \dots, c_k)$, $(w_1, \dots, w_k)$

**Algorithm 2: Special ordering for weight distribution**

$v(c_i, c_j) \leq w_j$ and for all $j > i$, $v(c_i, c_j) \leq w_i$. Hence,

$$\begin{aligned} v_{c_i}(\pi^*) &= \sum_{j \neq i} \frac{v(c_i, c_j)}{k-1} \leq \sum_{j > i} \frac{w_i}{k-1} + \sum_{j < i} \frac{w_j}{k-1} \\ &\leq \sum_{j > i} \frac{w_i}{k-i} + \sum_{j < i} \frac{w_j}{k-j} = w_i + \sum_{j < i} \frac{w_j}{k-j} \leq v_{c_i}(\pi) + \sum_{j < i} \frac{w_j}{k-j}. \end{aligned}$$

We infer that

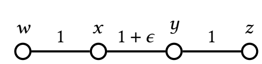
$$\begin{aligned} v_C(\pi^*) &= \sum_{i=1}^{k} v_{c_i}(\pi^*) \leq \sum_{i=1}^{k} \left( v_{c_i}(\pi) + \sum_{j < i} \frac{w_j}{k-j} \right) \\ &= v_C(\pi) + \sum_{i=1}^{k} \sum_{j < i} \frac{w_j}{k-j} = v_C(\pi) + \sum_{j=1}^{k} (k-j) \frac{w_j}{k-j} \leq 2v_C(\pi). \end{aligned}$$

□

Note that the approximation guarantee of the theorem extends to the case of non-symmetric weights, because the symmetrization $v'(x, y) = \frac{1}{2}(v_x(y) + v_y(x))$ preserves the welfare.

Moreover, the factor of 2 is the best possible approximation guarantee of Algorithm 1. Let $\epsilon > 0$ and the complete graph on vertex set $V = \{w, x, y, z\}$ be given with weights as

$$v(e) = \begin{cases} 1 + \epsilon & \text{if } e = \{x, y\} \\ 1 & \text{if } e \in \{\{w, x\}, \{y, z\}\} \\ 0 & \text{else.} \end{cases}$$




Then, the output of Algorithm 1 is $\pi = \{\{x, y\}, \{w\}, \{z\}\}$ with $\mathcal{SW}(\pi) = 2 + 2\epsilon$ while $\pi^* = \{\{w, x\}, \{y, z\}\}$ is welfare-optimal with $\mathcal{SW}(\pi^*) = 4$.

In the special case of simple symmetric games, the output is welfare-optimal. The proof uses the characterization of Pareto-optimal partitions by Elkind et al. [9, Lemma 15] that have observed that all coalitions in such partitions are stars or cliques. The proof shows that some welfare-optimal partition is a clique matching and therefore, all maximum clique matchings are welfare-optimal. We omit the proof due to space restrictions.

Theorem 4.4. *Let a simple symmetric MFHG be given. Let $\pi$ be a clique matching of the underlying unweighted graph. Then, $\pi$ is* *welfare-optimal if and only if it is a maximum clique matching. In particular, the output of Algorithm 1 is welfare-optimal.*

As we have seen in the above example, Pareto-optimal partitions need not be welfare-optimal. However, this was shown by Elkind et al. [9] for simple, symmetric MFHGs induced by a bipartite graph, and we will extend their result to simple symmetric MFHGs. While their results rely on estimating the welfare of partitions in terms of minimum vertex covers, we will exploit a combinatorial description of `Max3CliqueMatching`. We will develop it using terminology closely related to the famous blossom algorithm by Edmonds [8] to show the close relationship of computing maximum cardinality matchings and maximum size 3-clique matchings.

The blossom algorithm deals with odd cycles by finding subgraphs called *flowers*. Let a graph $G = (V, E)$ together with a matching $M$ be given. A path is called *$M$-alternating* if it alternately uses edges of $M$ and outside $M$. An *$M$-augmenting* path is an $M$-alternating path starting and ending at vertices not covered by $M$. An *$M$-stem* is an $M$-alternating path of even length. The uncovered endpoint is its *root*, the covered endpoint is its *tip*. An *$M$-blossom* is an odd cycle $C = (B, E_B)$ of $G$ such that all vertices except one are covered by $M \cap E_B$. Let $C = (B, E_B)$ be an $M$-blossom such that $b \in B$ is uncovered and let $b_1, b_2 \in B$ be the neighbors of $b$ on $C$. $B$ is called *$M$-chordal* if $\{b_1, b_2\} \in E$. An *$M$-(chordal) flower* is the union of a stem and a (chordal) blossom that intersect exactly in the tip of the stem. An example of a chordal flower together with an augmentation is given in Figure 1. If the matching is clear from the context, we will not specify it for the previous notation.

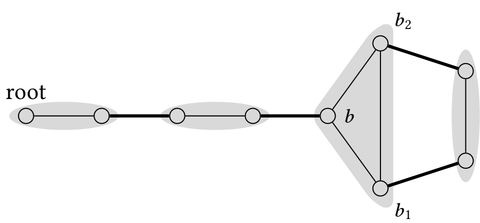



**Figure 1: The bold matching indicates a chordal flower that can be augmented via the gray clique cover.**

Theorem 4.5 (Hell and Kirkpatrick [13]). *Let $G = (V, E)$ be a graph. Then $\mathcal{C}$ is a maximum size 3-clique matching if and only if*

1. *There exists no $M(\mathcal{C})$-augmenting path.*
2. *There exists no $M(\mathcal{C})$-alternating path starting at an uncovered vertex and ending at a vertex covered by a triangle.*
3. *There exists no $M(\mathcal{C})$-chordal flower.*

*In particular, a clique matching is maximum if and only if is vertex-maximal.*

Note that compared to the classical characterization of maximum cardinality matchings, the second condition allows for improvement by deleting a triangle, and the third one by creating one in order to improve a 3-clique matching.

Theorem 4.6. *Let a simple, symmetric MFHG be given. Let $\pi$ be a partition of the agents. Then, $\pi$ is welfare-optimal if and only if it is Pareto-optimal.*

Proof sketch. Let an MFHG induced by a unweighted graph $G = (V, E)$ be given. Clearly, welfare-optimal partitions are Pareto-optimal.

For the reverse implication, let $\pi$ be a Pareto-optimal partition which consists of cliques and stars only [9, Lemma 15]. By splitting larger cliques, we may assume that all cliques are of size 2 and 3 (splitting the cliques yields a partition with identical utilities). Let $\mathcal{S} \subseteq \pi$ be its star-coalitions. For any star $S$ with $k_S \geq 2$ leaves, let $c^S$ be its center and $l_1^S, \dots, l_{k_S}^S$ be its leaves. We define a new partition $\pi' = \{C \in \pi : C \notin \mathcal{S}\} \cup \{\{c^S, l_1^S\}, \{l_2^S\}, \dots, \{l_{k_S}^S\} : S \in \mathcal{S}\}$. Then, $\mathcal{SW}(\pi') = \mathcal{SW}(\pi)$. The core of the proof is that $\pi'$ is a maximum clique matching and therefore by Theorem 4.4 welfare-optimal.

Define the set $M_{\mathcal{S}} = \{\{c^S, l_1^S\} : S \in \mathcal{S}\}$, i.e., the set of edges that are added to the partition $\pi'$. Then, $M(\pi') = M(\pi) \cup M_{\mathcal{S}}$ is the set of 2-cliques of the 3-clique matching $\pi'$.

We will prove that the conditions of the combinatorial characterization of Theorem 4.5 are satisfied. First, assume that there exists an $M(\pi')$-augmenting path $P$. An illustration of this step is given after the proof with the aid of Figure 2.

For $e \in M_{\mathcal{S}}$, we denote by $S_e$ the star which the edge $e$ originates from. An edge $e \in M_{\mathcal{S}} \cap P$ is called *exterior* if $c^{S_e}$, the center vertex of $S_e$, is the second or second-last vertex on the path. Otherwise, we call the edge *interior*.

For an exterior edge $e$, we denote by $t(e)$ the endpoint of $P$ that is a neighbor of $c^{S_e}$ on $P$. The first step is to modify $P$. In a second step, the modified path will yield a Pareto-improvement over $\pi$. An exterior edge $e$ is called *saturated* if $t(e)$ is a leaf of $S_e$. An interior edge $e$ is called *saturated* if all leaves of $S_e$ are covered by $P$.

First, we may assume that every exterior edge is saturated or there exists only one exterior edge $e^*$ which corresponds to a star $S$ with two leaves and $l_2^S$ is the endpoint of $P$ that is not $t(e^*)$.

To this end, assume first that there exist two exterior edges $e$ and $f$ originating from stars $S$ and $T$, respectively. Replacing $t(e)$ and $t(f)$ by $l_2^S$ and $l_2^T$ leaves both edges saturated. Otherwise, if $e^*$ is the only exterior edge originating from star $S$, and is not saturated, then $S$ has only two leaves, or we can replace $t(e^*)$ by a leaf of $S$ uncovered by $P$. In addition, if $l_2^S$ is not the other endpoint of $P$, we can replace $t(e^*)$ by $l_2^S$. This establishes the claim.

Second, we show that we can additionally assume that all interior edges are saturated. Indeed, if $e$ is an interior edge and $l$ is a leaf of $S_e$ not covered by $P$, then we can replace $P$ by the augmenting path that starts with $l$ and proceeds on $P$ with $e$. Assume that the path ends in an exterior edge $f$. If $f$ is not saturated, we replace $t(f)$ by $l_2^{S_f}$. Otherwise, we follow the path to the end. In any case, this procedure yields a path $P'$ such that all exterior edges are as after the first step and all interior edges are saturated.

We will show how to obtain a Pareto-improvement over $\pi$ from $P$. Label the vertices of the path $p_0, \dots, p_m$ for some (odd) integer $m$. If the first matching-edge of the path is exterior and saturated, we delete $p_0$ and $p_1$ from the path. If the last matching-edge is exterior and saturated, we delete $p_{m-1}$ and $p_m$ from the path. This leaves a path $P'$ on vertices $p'_0, \dots, p'_{m'}$ such that all leaves corresponding to stars of edges in $M_{\mathcal{S}} \cap P'$ are covered by $P'$. Let $\mathcal{T}$ be the set of star coalitions $T$ such that $c^T \notin \{p'_0, \dots, p'_{m'}\}$, but some leaf of $T$ is a endpoint of $P'$.

Consider $\pi' = \{C \in \pi \colon C \cap \{p_0, \dots, p_m\} = \emptyset\} \cup \{T \setminus \{p'_0, p'_{m'}\} \colon T \in \mathcal{T}\} \cup (P' \setminus M(\pi'))$, which is a Pareto-improvement over $\pi$. Hence, $\pi$ is not Pareto-optimal, which is a contradiction. Therefore, the first condition of the combinatorial characterization is satisfied.

An example for this case is given in Figure 2. The path $P$ is formed by the straight lines and the partition $\pi'$ by the bold edges. Dashed edges indicate leaves of the stars $S$ and $T$. The edge $\{c^S, l_1^S\}$ is exterior and saturated, and the edge $\{c^T, l_1^T\}$ is interior and saturated. The gray partition yields a Pareto-improvement over $\pi$.

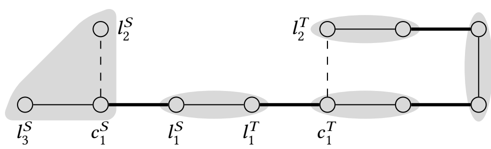



**Figure 2: Example of a Pareto-improvement that can be constructed from an $M(\pi')$-augmenting path.**

The proofs that the second and third condition hold are similar, and use that the first condition is already proved.

Hence, $\pi'$ is a maximum clique matching and is therefore welfare-optimal. Since $\mathcal{SW}(\pi') = \mathcal{SW}(\pi)$, it follows that $\pi$ is welfare-optimal. □

As a corollary, we obtain efficient verification of Pareto-optimality for simple symmetric MFHGs.

Theorem 4.7. *The problem of verifying Pareto-optimality can be done in polynomial time for simple symmetric MFHGs.*

Proof. Let an MFHG be given and a partition $\pi$ that is to be checked for Pareto-optimality. Simply compute a partition $\pi^*$ via Algorithm 1 and compare their social welfare. By Theorem 4.2 and Theorem 4.1, this runs in polynomial time. By Theorem 4.4, $\pi^*$ is welfare-optimal. Finally, by Theorem 4.6, comparing social welfare checks for Pareto-optimality. □

For general, weighted MFHGs, it is still of interest as to whether one can also find a welfare-optimal partition in polynomial time. While this question remains open, we can at least narrow down the search to coalitions of small size. Then, a weighted version of `Max3CliqueMatching` might give rise to an efficient algorithm.

The proof of the proposition relies on the fact that we can split a coalition $C$ of size at least 4 into an edge $e$ and the remainder $C \setminus e$ such that $\mathcal{SW}(e) + \mathcal{SW}(C \setminus e) \geq \mathcal{SW}(C)$. The edge $e$ maximizes a cleverly chosen objective function that relies on the weight of $e$, the weight of the cut between $e$ and $C \setminus e$, and the welfare of $C \setminus e$.

Proposition 4.8. *Let a partition $\pi$ of the agents of a general MFHG be given. Then, there exists a partition $\pi'$ with $|C| \leq 3$ for all $C \in \pi'$ with $\mathcal{SW}(\pi') \geq \mathcal{SW}(\pi)$. In particular, there exists a welfare-optimal partition consisting of coalitions of size at most* 3.

# 5 ADDITIVELY SEPARABLE HEDONIC GAMES

In this section, we will survey Pareto-optimality on ASHGs. Positive results exist so far only for a very restrictive class that does not allow for 0-weights, and unique top-ranked coalitions are therefore guaranteed [3, Theorem 11]. We extend this result to a very general class of ASHGs that includes symmetric ASHGs.

An ASHG is called *mutually indifferent* if $v_i(j) = 0$ implies $v_j(i) = 0$ for every pair of agents $i, j$. Note that every symmetric ASHG is mutually indifferent.

Theorem 5.1. *A Pareto-optimal outcome for mutually indifferent ASHGs can be computed in polynomial time. In particular, a Pareto-optimal outcome for symmetric ASHGs can be computed in polynomial time.*

Proof. Consider Algorithm 3. The algorithm can be seen as a variant of serial dictatorship where in every coalition formed through a dictator $d_i$, the dictator asks the agents in her coalition to improve using a strict rank order such that none of higher rank becomes worse off.

**Input:** Mutually indifferent ASHG induced by $G = (N, E, v)$
**Output:** Pareto-optimal partition $\pi$

$\pi \leftarrow \emptyset, D \leftarrow N, i \leftarrow 1$
**while** $D \neq \emptyset$ **do**
  Pick $d_i \in D$
  $C_i \leftarrow \{d_i\} \cup \{j \in D \colon v_{d_i}(j) > 0\}, I_i \leftarrow \{j \in D \colon v_{d_i}(j) = 0\}$
  $H_i \leftarrow C_i \setminus \{d_i\}$
  **while** $H_i \neq \emptyset \wedge I_i \neq \emptyset$ **do**
    Pick $h \in H_i$
    $C_i \leftarrow C_i \cup \{j \in I_i \colon v_h(j) > 0\}$
    $H_i \leftarrow (H_i \cup \{j \in I_i \colon v_h(j) > 0\}) \setminus \{h\}$
    $I_i \leftarrow \{j \in I_i \colon v_h(j) = 0\}$
  $\pi \leftarrow \pi \cup \{C_i\}, D \leftarrow D \setminus C_i, i \leftarrow i + 1$
**return** $\pi$

**Algorithm 3: Pareto-optimality for mutual indifference**

The running time is polynomial since every edge in the graph underlying the ASHG is checked at most once.

For correctness, denote by $C_1, \dots, C_k$ the coalitions that form in the order of the algorithm, and $I_i$ the respective indifference sets (some may be empty) at the end of the inner while-loop. Note that for all $i \in 1, \dots, k, a \in C_i, b \in I_i$ holds that $v_a(b) = 0$. For correctness, assume that $\pi'$ is a partition such that for every agent $i \in N$, $v_i(\pi') \geq v_i(\pi)$. If $|C_i| > 1$, then $C_i \subseteq \pi'(d_i) \subseteq C_i \cup I_i$. Hence, no agent in such a coalition will be better off. In addition, no agent in a singleton coalition $C_i = \{d_i\} \in \pi$ will be better off, since they can only form coalitions with other singleton coalitions or with coalitions such that they are in the set $I_j$, both of which give them 0 utility. Therefore, no agent's utility has improved. □

Slight modifications of the above algorithm give computational tractability of Pareto-optimality even for more general classes of ASHGs. The same algorithm works for the class of ASHGs such that $v_i(j) = 0$ implies $v_j(i) \leq 0$ for every pair of agents $i, j$. Hence, the only edges remaining for the full domain of ASHGs are *critical* edges of the form $\{i, j\}$ such that $v_i(j) > 0$ while $v_j(i) = 0$. One idea towards obtaining an algorithm for a more general class of ASHGs is to use a pivoting rule that selects dictators. This allows, for example, for a positive result for the class of ASHGs such that the critical edges form a directed acyclic graph (using a topological order on the agents for a pivoting rule).

The outcome of the algorithm can, however, have an arbitrarily large gap to the maximum social welfare that is obtained in a welfare-optimal outcome. In addition, all but one agent may obtain a worst coalition. On the other hand, a Pareto-optimal and individually rational outcome does always exist, but computing such a partition is intractable. The following is a strengthening of a result by Aziz et al. [3] who dealt with the whole class of ASHGs and established weak NP-hardness.

The reduction is from the NP-complete problem `Exact3Cover` [14]. An instance $(R, S)$ of `Exact3Cover` (X3C) consists of a ground set $R$ together with a set $S$ of 3-element subsets of $R$. A 'yes' instance is an instance so that there exists a subset $S' \subseteq S$ that partitions $R$.

Theorem 5.2. *Finding a Pareto-optimal and individually rational partition for symmetric ASHGs is (strongly) NP-hard, even if all weights are integers bounded from above by* 3.

Proof sketch. We provide a Turing reduction, illustrated in Figure 3, from X3C. Given an instance $(R, S)$ of X3C, we construct the symmetric ASHG with agent set $N = R \cup V$ where $V = \{s_i : i = 1, \dots, 5, s \in S\}$ consists of 5 copies of agents for the sets in $S$. Preferences are given by weights $v$ as

- $v(i, j) = 0, i, j \in R, i \neq j$
- $v(i, s_1) = 2, s \in S, i \in s$
- $v(s_1, s_2) = v(s_1, s_3) = v(s_2, s_4) = v(s_3, s_5) = v(s_4, s_5) = 3$, $v(s_2, s_3) = 0, s \in S$, and
- all other weights are set to $-13$.

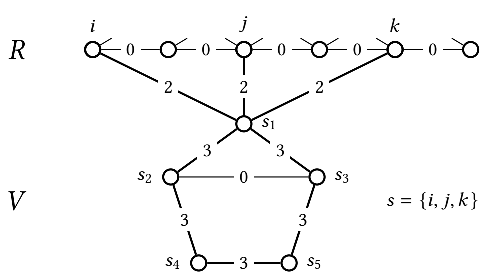



**Figure 3: ASHG for the reduction. Indicated edges between $R$-agents have weight 0, other omitted edges have weight $-13$.**

We will argue that if we can compute a Pareto-optimal and individually rational partition, we can decide X3C. Amongst all individually rational partitions, the highest utility that the agents can obtain is 2, 6, and 3 for agents in $R$, $\{s_1 : s \in S\}$, and $\{s_i : i = 2, \dots, 4, s \in S\}$, respectively. It can be shown that there exists an individually rational partition that attains these bounds for every agent if and only if there exists a 3-partition of $R$ through sets in $S$.

Hence, we can solve X3C in polynomial time by computing a Pareto-optimal and individually rational partition for the corresponding ASHG, and check whether every agent receives the utility of a best partition amongst individually rational partitions. □

By applying a local search algorithm that starts with the singleton partition, we obtain the following corollary.

Corollary 5.3. *Finding a Pareto-improvement is NP-hard for symmetric ASHGs.*

Finally, it is interesting to see why the strong relation between Pareto-optimality and perfection exploited by Aziz et al. [2] does not hold for ASHGs. The preference refinement algorithm computes an individually rational, Pareto-optimal partition given an oracle that decides whether there exists a perfect partition and, in the case there exists one, can compute one. While the former problem is NP-hard, the latter can be solved in polynomial time by forming a top-ranked coalition, adding requisite agents, and adding outside agents that need an inside agent. This theorem even holds for the more general class of separable hedonic games [3, Theorem 9].

Theorem 5.4. *The problem of, given a general ASHG, computing a perfect partition or deciding that no such partition exists, can be done in polynomial time.*

## 6 FRACTIONAL HEDONIC GAMES

The serial dictatorship version used for ASHGs in Algorithm 3 implicitly exploits the fact that ASHGs have the property that top-ranked coalitions in subgames are the restrictions of top-ranked coalitions in the original game. This is not the case anymore for FHGs. However, the set of top-ranked coalitions can be described using the following observation.

Proposition 6.1. *Let an FHG be given based on a graph $G = (N, E, v)$ and let $d \in N$. Let $C$ be a top-ranked coalition of $d$ and set $\mu = v_d(C)$. Then, $C = \{a \in N \setminus \{d\} : v_d(a) > \mu\} \cup W$ for some $W \subseteq \{a \in N \setminus \{d\} : v_d(a) = \mu\}$.*

Hence, to obtain the top-ranked coalitions of an agent, one can order the other agents in decreasing value and add them until another agent is not beneficial. If there are agents that give exactly the utility of a top-ranked coalition, they may or may not be added.

We will now show efficient computability of Pareto-optimal partitions for certain classes of FHGs. An FHG satisfies the *equal affection condition* if $v_x(y), v_x(z) > 0$ implies $v_x(y) = v_x(z)$ for every triple of agents $x$, $y$, and $z$. An FHG is called *generic* if $v_x(y) \neq v_x(z)$ for every triple of agents $x$, $y$, and $z$, i.e., the utilities over the remaining set of agents are pairwise distinct for every agent.

Since the equal affection condition guarantees unique top-ranked coalitions for every agent, we obtain the following theorem, which applies in particular to simple FHGs.

Theorem 6.2. *Finding a Pareto-optimal partition for FHGs satisfying the equal affection condition can be done in polynomial time.*

Another variant of serial dictatorship finds Pareto-optimal partitions on generic FHGs.

Theorem 6.3. *Finding a Pareto-optimal partition for generic FHGs can be done in polynomial time.*

Proof sketch. Let an FHG be based on the graph $G = (N, E, v)$. We give an algorithm based on serial dictatorship that exploits a dynamically created hierarchy for the dictatorship. The next dictator is chosen based on the top choices of the previous dictator.

By Proposition 6.1, we know the structure of the top-ranked coalitions of an agent. Let an agent set $M \subseteq N$ be given, that induces the FHG on the subgraph $G[M]$, and let $d \in M$. There exists a unique smallest top-ranked coalition, which we denote by $T_d(M)$. Furthermore, for a generic FHG, there exist at most two top-ranked coalitions. Denote in this case by $t_d(M)$ the number of such coalitions for agent $d$ in the subgame and if $t_d(M) = 2$, let $\alpha_d(M) \in M$ be the unique agent such that $T_d(M) \cup \{\alpha_d(M)\}$ is the other top-ranked coalition.

We are ready to formulate the recursive algorithm $\mathcal{A}$ that computes a Pareto-optimal partition by adding iteratively coalitions to a partial partition. The actual Pareto-optimal partition is obtained by choosing an arbitrary first dictator $d \in N$ and calling $\mathcal{A}(N, \emptyset, d)$.

**Input:** Non-empty agent set $M$, partial partition $\pi$, pivot agent $d$
**Output:** Partition $\pi$

```
if t_d(M) = 1 then
    if T_d(M) = M then
        return π ∪ {T_d(M)}
    else
        Pick d_new ∈ M\T_d(M)
        Execute A(M\T_d(M), π ∪ {T_d(M)}, d_new)
else
    if v_{α_d(M)}(T_d(M) ∪ {α_d(M)}) > v_{α_d(M)}(M\T_d(M)) then
        if T_d(M) ∪ {α_d(M)} = M then
            return π ∪ {T_d(M) ∪ {α_d(M)}}
        else
            Pick d_new ∈ M\(T_d(M) ∪ {α_d(M)})
            Execute A(M\(T_d(M) ∪ {α_d(M)}), π ∪ {T_d(M) ∪
            {α_d(M)}}, d_new)
    else if v_{α_d(M)}(T_d(M) ∪ {α_d(M)}) < v_{α_d(M)}(M\T_d(M)) or
    v_{α_d(M)}(T_d(M) ∪ {α_d(M)}) = v_{α_d(M)}(M\T_d(M)) = 0 then
        Execute A(M\T_d(M), π ∪ {T_d(M)}, α_d(M))
    else
        Pick d_new ∈ argmax{v_{α_d(M)}(x): x ∈ M\T_d(M)}
        if α_d(M) ∈ T_{d_new}(M\T_d(M)) then
            Execute A(M\T_d(M), π ∪ {T_d(M)}, d_new)
        else
            Execute A(M\(T_d(M) ∪ {α_d(M)}), π ∪ {T_d(M) ∪
            {α_d(M)}}, d_new)
```

**Algorithm 4: Pareto-optimal partition for generic FHG by the recursive algorithm $\mathcal{A}$**

By the top-ranked coalition structure of agents in generic FHGs, every step of Algorithm 4 can be executed and as argued after Proposition 6.1, top-ranked coalitions can be efficiently computed. Hence, the algorithm runs in polynomial time.

It can be checked that $\pi = \mathcal{A}(N, \emptyset, d)$ returns a Pareto-optimal partition, provided that the input evolves from a generic FHG. □

Finally, similar statements as for ASHGs hold for FHGs. The proofs are similar.

THEOREM 6.4. *Finding a Pareto-optimal and individually rational partition for symmetric FHGs is NP-hard.*

THEOREM 6.5. *Finding a Pareto-improvement is NP-hard for symmetric FHGs.*

THEOREM 6.6. *The problem of, given an FHG, computing a perfect partition or deciding that no such partition exists, can be done in polynomial time.*

## 7 CONCLUSION

We have investigated Pareto-optimality in three types of cardinal hedonic games. The main findings and important related results are collected in Table 1. We can efficiently find Pareto-optimal partitions in symmetric MFHGs and AHSGs, and reasonable classes of FHGs including simple FHGs. The key insight for MFHGs is the equivalence with an extension of matchings to cliques. The combinatorial view of the problem allowed us to completely understand Pareto-optimal outcomes on simple, symmetric MFHGs, where they coincide with welfare-optimal outcomes. This motivates the study of the weighted case, where Pareto-optimal outcomes have no guarantee on the welfare, and yet our algorithm returns a 2-approximation to social welfare. The complexity of welfare-optimization in the weighted case is an interesting open problem. We are at least able to prove that coalitions of size 2 and 3 suffice, which is in line with the research on any other solution concept for MFHGs.

| | *Pareto optimality* | | *Welfare optimality* | |
|---|---|---|---|---|
| | PO | PO∧IR | Deterministic | Approximation |
| MFHG | P (sym, Thm. 4.2) | P (sym, Thm. 4.2) | P (0/1 sym, [15]) | 2 (sym, Thm. 4.3) |
| ASHG | P (sym, Thm. 5.1) | NP-h (sym, Thm. 5.2) | NP-h (sym, [3]) | open |
| FHG | P (0/1 sym, Thm. 6.2) | NP-h (sym, Thm. 6.4) | NP-h (0/1 sym, [4]) | 4 (sym, [4]) |

**Table 1: Complexity of Pareto- and welfare-optimality for cardinal hedonic games. Preference restrictions are given in parenthesis, where (0/1) sym denotes (simple) symmetric preferences. For welfare-optimality, the best known efficiently attainable approximation ratio is given.**

The key technique for positive results on ASHGs and FHGs are refinements of serial dictatorship algorithms. Further enhancements, e.g., with respect to the order of selection of the dictators, might yield even better results. On the other hand, computing Pareto-optimal outcomes that satisfy further properties will often be intractable. Computational hardness is obtained if we require individual rationality in addition. Since it is even hard to compute Pareto-improvements, local search heuristics based on Pareto-optimality cannot be exploited.

Partitions on simple MFHGs can simultaneously satisfy high demands in terms of stability and optimality. Interesting further directions for research therefore concern weighted MFHGs as well as Pareto-optimality on the general domains of cardinal hedonic games.

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# A APPENDIX: PROOFS

In the appendix, we provide the proofs omitted in the main part as well as completions for the proof sketches.

## A.1 Modified Fractional Hedonic Games

Proof of Theorem 4.4. Let an MFHG induced by a unweighted graph $G = (V, E)$ be given and let $\pi$ be a partition of the agents into cliques. By Theorem 4.5, if it is not a maximum clique matching, it is not vertex-maximal, and we find a Pareto-improvement.[1] Hence, $\pi$ was not welfare-optimal.

For the reverse implication, assume that $\pi$ is a maximum clique matching. In the case of a simple, symmetric MFHG, Elkind et al. [9, Lemma 15] have observed that all coalitions in Pareto-optimal partitions are stars or cliques. Therefore, it suffices to prove that there exists a welfare-optimal maximum clique matching.

If we can prove that, then for any clique matching $\pi'$, $\mathcal{SW}(\pi) = |W|$, where $W$ is the set of vertices covered by $\pi$. Hence, $\pi'$ is welfare-optimal amongst clique matchings if it forms a maximum clique matching.

Let $\pi^*$ be any welfare-optimal partition (that is in particular Pareto-optimal) and assume it contains at least one star. For any star $S$ with $k_S \geq 2$ leaves, let $c^S$ be its center and $l_1^S, \dots, l_{k_S}^S$ be its leaves. We obtain a new partition $\pi' = \{C \in \pi^* : C \text{ clique}\} \cup \{\{c^S, l_1^S\}, \{l_2^S\}, \dots, \{l_{k_S}^S\} : S \in \pi^* \text{ star}\}$. The partition $\pi'$ arises from $\pi$ by substituting any star by a 2-clique and singleton coalitions. If $S \in \pi$ is a star, then $v_S(\pi) = k_S \cdot \frac{1}{k_S} + 1 = 2 = v_S(\pi')$. Hence, $\mathcal{SW}(\pi') = \mathcal{SW}(\pi)$. In addition, $\pi'$ is Pareto-optimal. Otherwise, consider a Pareto-optimal Pareto-improvement $\pi''$ over $\pi'$. Then, $\mathcal{SW}(\pi'') > \mathcal{SW}(\pi') = \mathcal{SW}(\pi^*)$, contradicting the welfare-optimality of $\pi^*$. Hence, there exists a welfare-optimal partition such that every coalition is a clique.

Since the output of Algorithm 1 on unweighted graphs is a maximum clique matching, it is welfare-optimal. □

Proof details for Theorem 4.6. We show why the second and third condition of the combinatorial characterization hold.

If the second condition of Theorem 4.5 is violated, we can treat the endpoint of the triangle as one of the uncovered end-vertices in the proof of the first condition, and obtain a Pareto-improvement.

Finally, assume that the third condition of Theorem 4.5 is violated, i.e., there exists an $M(\pi')$-chordal flower. Denote its stem vertices by $s_0, \dots, s_m$ where $s_0$ is the root and $s_m$ is the tip of the stem, and its blossom vertices in circular order by $b_0, \dots, b_r$, where $b_0 = s_m$. If no edge of $M_\mathcal{S}$ is contained in the flower, we can augment the flower and obtain a Pareto-improvement (improving the root of the flower that was a leaf or an uncovered vertex before). Assume therefore that the flower contains an edge in $e \in M_\mathcal{S}$ corresponding to star $S$. We may assume that either $s_1 = c^S$ or that the flower contains (all vertices of) $S$. Indeed, if $e$ is part of the blossom, all vertices of $S$ have to be covered by the flower or we find an $M(\pi')$-augmenting path from the root of the flower to an uncovered vertex, which is already excluded. Assume therefore $e = \{s_t, s_{t+1}\}$. If $s_{t+1} = c^S$, we find again an augmenting path unless every leaf of $S$ is covered by the flower. Finally, if $s_t = c^S$, we can assume that the root of the flower is $l_2^S$ and the stem is on the vertices $l_2^S, s_t, \dots, s_m$.

In particular, there can be at most one matching-edge of $M_\mathcal{S}$ part of the flower. If all vertices of $S$ are covered by the flower, we augment the flower to improve the utility of the leaves of $S$. Otherwise, we delete $s_0$ and $s_1$ from the flower and augment the resulting flower, improving the leaves of $S$.

□

Proof of Proposition 4.8. Using the symmetrization described after Theorem 4.3, it suffices to prove the proposition for symmetric MFHGs.

Let a symmetric MFHG be given, induced by a weighted graph $G = (N, E, v)$. For any set of agents $M \subseteq N$, define $v(M) = \sum_{\{x,y\} \subseteq M, x \neq y} v(x, y)$ as the utility of all pairs of agents. This means, $\mathcal{SW}(M) = \frac{2v(M)}{|M|-1}$. In addition, for $F \subseteq E$, define $v(F) = \sum_{f \in F} v(f)$.

Given a set of agents $M \subseteq N$, we define $\delta(M) = \{\{x, y\} : x \in M, y \in N \setminus M\}$, i.e., the cut induced by $M$.

It suffices to show that for every coalition $C$ of size $|C| \geq 4$, there exists an edge $e \subseteq C$ with $|e| = 2$ and $\mathcal{SW}(e) + \mathcal{SW}(C \setminus e) \geq \mathcal{SW}(C)$. Indeed, if we can prove that, the claim follows by iteratively splitting coalitions of size at least 4.

Therefore, let a coalition $C$ of agents be given with $|C| \geq 4$.

Given a 2-element set of vertices $e$, define $\omega_e = \frac{1}{2}v(e) + v(\delta(e)) + \left(1 - \frac{1}{2(|C|-3)}\right) v(C \setminus e)$. We define $\omega = \min\{\omega_e : e \subseteq V, |e| = 2\}$ and let $e^* \in \operatorname{argmin}\{\omega_e : e \subseteq V, |e| = 2\}$. The claim is that $\mathcal{SW}(C \setminus e^*) + \mathcal{SW}(e^*) \geq \mathcal{SW}(C)$.

Note that

$$
\begin{aligned}
2(|C|-2)\omega \leq & \sum_{f \in \delta(e^*)} \omega_f \\
= & v(e^*)(|C|-2)2 \\
& + v(\delta(e^*)) \left( \frac{1}{2} + (|C|-2) + \left(1 - \frac{1}{2(|C|-3)}\right)(|C|-3) \right) \\
& + v(C \setminus e^*) \left( 4 + \left(1 - \frac{1}{2(|C|-3)}\right)(|C|-4)2 \right).
\end{aligned}
$$

The equality follows from counting how often the edge $e^*$, the edges in the cut $\delta(e^*)$, and the edges between vertices in $C \setminus e^*$ play each of the roles of the sums of all $\omega_f$. Inserting $\omega = \frac{1}{2}v(e^*) + v(\delta(e^*)) + \left(1 - \frac{1}{2(|C|-3)}\right) v(C \setminus e^*)$ and rearranging terms yields

$$
v(\delta(e^*)) \leq (|C|-2)v(e^*) + \frac{2}{|C|-3} v(C \setminus e^*).
$$

On the other hand,

$$
\begin{aligned}
& 2v(e) + \frac{2}{|C|-3} v(C \setminus e) = \mathcal{SW}(e) + \mathcal{SW}(C \setminus e) \\
\geq & \mathcal{SW}(C) = \frac{2}{|C|-1} \left( v(e) + v(\delta(e)) + v(C \setminus e) \right) \\
\Leftrightarrow & v(\delta(e)) \leq (|C|-2)v(e) + \left( \frac{|C|-1}{|C|-3} - 1 \right) v(C \setminus e) \\
= & (|C|-2)v(e) + \frac{2}{|C|-3} v(C \setminus e).
\end{aligned}
$$

Hence, the edge $e^*$ has the desired property. □

[1] Note that the equivalence of maximality and vertex-maximality follows from the combinatorial description of Theorem 4.5 and is independently achieved.

## A.2 Additively Separable Hedonic Games

Proof details for Theorem 5.2. If $S' \subseteq S$ is a 3-partition of $R$, we find a desired partition via $\pi = \{\{i, j, k, s_1\}, \{s_2, s_4\}, \{s_3, s_5\} : s \in S'\} \cup \{\{s_1, s_2, s_3\}, \{s_4, s_5\} : s \in S \setminus S'\}$.

Conversely, assume that no 3-partition of $R$ through sets in $S$ exists, and let $\pi$ be an individually rational partition such that every agent in $R$ has utility 2. Then, for some $s \in S$, an agent $s_1$ forms a coalition with some, but less than 3 agents in $R$, and due to individual rationality with no other agent. Therefore, $u(s_1) < 6$. □

For the sake of self-containment, we provide a proof for Theorem 5.4, a special case of [3, Theorem 9]. As we will see, a similar algorithm works for FHGs.

Proof of Theorem 5.4. Let an ASHG be given which is induced by the graph $G = (V, E, v)$.

We find a partition $\pi$ applying Algorithm 5. Basically, the algorithm finds coalitions iteratively by adding agents to a current coalition $C$ if they need an agent inside $C$ to be in a top-ranked coalition, or if an agent inside $C$ needs them to be in a top-ranked coalition.

**Input:** ASHG induced by graph $(N, E, v)$
**Output:** Partition $\pi$

$\pi \leftarrow \emptyset$
$U \leftarrow N$
**while** $U \neq \emptyset$ **do**
  Pick $a \in U$
  $C \leftarrow \{a\}$
  **while** $\exists i \in C, j \in U \setminus C : v_i(j) > 0 \vee v_j(i) > 0$ **do**
    $C \leftarrow C \cup \{j\}$
  $\pi \leftarrow \pi \cup \{C\}$
  $U \leftarrow U \setminus C$
**return** $\pi$

**Algorithm 5: Candidate for perfect partition on ASHG**

Note that for an agent $i \in N$, the set of top-ranked coalitions is given by $\{\{j \in N : v_i(j) > 0\} \cup W : W \subseteq \{j \in N : v_i(j) = 0\}\}$. As a consequence, it is quickly checked that there exists a perfect partition if and only if $\pi$ is a perfect partition, which is the case if and only if, for all $i \in N$, $v_i(\pi) = \sum_{j \in N : v_i(j) > 0} v_i(j)$. □

## A.3 Fractional Hedonic Games

Proof of Proposition 6.1. Let an FHG be given based on a graph $G = (N, E, v)$ and let $d \in N$. Let $C$ be a top-ranked coalition of $d$ and set $\mu = v_d(C)$. We may assume that there exists an agent $a \in N \setminus \{d\}$ with $v_d(a) > 0$, because otherwise the assertion is immediate.

For $a \in N \setminus C$, it holds that $v_d(C \cup \{a\}) = \frac{1}{|C|+1} v_d(a) + \frac{|C|}{|C|+1} \mu > \mu$ if and only if $v_d(a) > \mu$. Consequently, $\{a \in N \setminus \{d\} : v_d(a) > \mu\} \subseteq C$. In addition, removing an agent $a \in C \setminus \{d\}$ with $v_d(a) < \mu$ would increase the value of the coalition $C$, contradicting its maximality. Since the above equivalence also holds with equality replacing inequality, removing and adding agents with $v_d(a) = \mu$ do not change the utility of the coalition. □

Proof of Theorem 6.2. Let an FHG be given that satisfies the equal affection condition. Let $M$ be the set of agents that assigns positive utility to at least one neighbor. By Proposition 6.1, every agent in $M$ has a unique top-ranked coalition. Hence, a serial dictatorship, i.e., letting agents iteratively pick a top-ranked coalition, that priorizes agents in $M$ finds a Pareto-optimal partition. □

Proof details for Theorem 6.3. We prove that $\pi = \mathcal{A}(N, \emptyset, d)$ returns a Pareto-optimal partition provided that the input evolves from a generic FHG.

Assume that $\pi'$ is a partition such that for every agent $a \in N$, $v_a(\pi') \geq v_a(\pi)$. We will prove, by induction over $k = |\pi|$ that then for every agent $a \in N$, $v_a(\pi') = v_a(\pi)$.

Assume first that $k = 1$ and let $d$ be the first dictator. Then either $\pi = \{T_d(N)\}$ and $\pi' = \pi$ (since otherwise $d$ is worse off), or $\pi = \{T_d(N) \cup \{\alpha_d(N)\}\}$ and $\pi' = \pi$ (since otherwise $\alpha_d(N)$ or $d$ is worse off).

For the induction step, assume $k \geq 2$ and let $C$ be the first coalition formed through the initial dictator $d$. If $t_d(N) = 1$, then $T_d(N) \in \pi'$ (otherwise $d$ is worse off) and the claim follows by induction on the subgame induced by $N \setminus T_d(N)$ for the coalitions $\pi \setminus \{T_d(N)\}$ and $\pi' \setminus \{T_d(N)\}$.

Assume now $t_d(N) = 2$ and denote $a = \alpha_d(N)$. Note that then $T_d(N) \in \pi'$ or $T_d(N) \cup \{a\} \in \pi'$, since otherwise $d$ is worse off. If $v_a(T_d(N) \cup \{a\}) > v_a(N \setminus T_d(N))$, then $T_d(N) \cup \{a\} \in \pi' \cap \pi$ (otherwise $a$ is worse off) and we can apply induction.

If $v_a(T_d(N) \cup \{a\}) < v_a(N \setminus T_d(N))$, then $a$ is the next dictator, hence $v_a(\pi) > v_a(T_d(N) \cup \{a\})$. Therefore, $T_d(N) \in \pi'$ and we can apply induction.

Assume next $v_a(T_d(N) \cup \{a\}) = v_a(N \setminus T_d(N)) = 0$. If $|\pi(a)| = 1$ then we can remove the two coalitions $T_d(N)$ and $\{a\}$ and apply induction. Otherwise, $\pi(a) = \{a, \alpha_a(N \setminus T_d(N))\}$. Unless $v_{\alpha_a(N \setminus T_d(N))}(\pi(a)) = v_{\alpha_a(N \setminus T_d(N))}(N \setminus (T_d(N) \cup \pi(a))) > 0$, $\pi'(d) = \pi(d)$ and we can apply induction. Otherwise, $\alpha_a(N \setminus T_d(N))$ can only leave if she joins the coalition of the third dictator $e$. In this case, $\pi(d) \cup \pi(a) \cup \pi(e) = \pi'(d) \cup \pi'(a) \cup \pi'(e)$ and all of these agents obtain the same utility. Hence we can apply induction by removing all three coalitions.

Finally, if $v_a(T_d(N) \cup \{a\}) = v_a(N \setminus T_d(N)) > 0$, let $e$ be the second dictator. The only possibility that $a$ leaves her coalition is if $T_d(N) \cup \{a\} \in \pi$ and $a \cup \pi(e)$ is the second top-ranked coalition of $e$ and is simultaneously a top-ranked coalition of $a$ in $N \setminus T_d(N)$. In this case we can remove $\pi(d) \cup \pi(e)$ and apply induction.

□

Proof of Theorem 6.4. The same reduction as in Theorem 5.2 works with the following adaptations of edge weights. Replace edge weights by $v(i, j) = \frac{1}{2}$ for $i, j \in R, i \neq j$, $v(s_1, s_2) = v(s_1, s_3) = v(s_2, s_4) = v(s_3, s_5) = v(s_4, s_5) = \frac{9}{8}$, $s \in S$, and $v(s_2, s_3) = \frac{9}{16}$, $s \in S$. □

Proof of Theorem 6.5. Consider the reduction of Theorem 6.4. If we start with the partition $\pi = \{\{r\} : r \in R\} \cup \{\{s_1, s_2, s_3\}, \{s_4, s_5\} : s \in S\}$, every subsequent Pareto-improvement must increase the number coalitions of the type $\{i, j, k, s_1\}$ where

$s = \{i, j, k\} \in S$. Hence, there can be at most $|R|/3$ Pareto-improvements. Hence, the local search algorithm finds a Pareto-optimal partition in polynomial time which can be used to solve X3C. □

Proof of Theorem 6.6. In the beginning, compute for every agent $i$ the value $\tau(i)$ of her utility in a top-ranked coalition. Then, Algorithm 5 still works if we replace the condition in the while-loop by $\exists i \in C, j \in U \setminus C : v_i(j) > \tau(i) \vee v_j(i) > \tau(j)$. □