Incentives in Social Decision Schemes with Pairwise Comparison Preferences

Felix Brandt[1] , **Patrick Lederer**[1] , **Warut Suksompong**[2]
[1]Technische Universität München
[2]National University of Singapore
{brandtf, ledererp}@in.tum.de, warut@comp.nus.edu.sg

Abstract

Social decision schemes (SDSs) map the preferences of individual voters over multiple alternatives to a probability distribution over the alternatives. In order to study properties such as efficiency, strategyproofness, and participation for SDSs, preferences over alternatives are typically lifted to preferences over lotteries using the notion of stochastic dominance (SD). However, requiring strategyproofness or strict participation with respect to this preference extension only leaves room for rather undesirable SDSs such as random dictatorships. Hence, we focus on the natural but little understood pairwise comparison (PC) preference extension, which postulates that one lottery is preferred to another if the former is more likely to return a preferred outcome. In particular, we settle three open questions raised by Brandt (2017): *(i)* there is no Condorcet-consistent SDS that satisfies PC-strategyproofness; *(ii)* there is no anonymous and neutral SDS that satisfies PC-efficiency and PC-strategyproofness; and *(iii)* there is no anonymous and neutral SDS that satisfies PC-efficiency and strict PC-participation. All three impossibilities require $m \geq 4$ alternatives and turn into possibilities when $m \leq 3$.

1 Introduction

Incentives constitute a central aspect when designing mechanisms for multiple agents: mechanisms should incentivize agents to participate and to act truthfully (see, e.g., Nisan et al., 2007; Shoham and Leyton-Brown, 2009; Brandt et al., 2016). However, for many applications, guaranteeing these properties—usually called *participation* and *strategyproofness*—is a notoriously difficult task. This is particularly true for collective decision making, which studies the aggregation of preferences of multiple voters into a group decision, because strong impossibility theorems show that these axioms are in variance with other elementary properties (see, e.g., Gibbard, 1973; Satterthwaite, 1975; Moulin, 1988). For instance, the Gibbard-Satterthwaite theorem shows that every strategyproof voting rule is either dictatorial or imposing, and Moulin's No-Show paradox demonstrates that all Condorcet-consistent voting rules violate participation. A natural escape route in light of these negative results is to allow for randomization in the output of the voting rule. Rather than returning a single winner, a *social decision scheme (SDS)* selects a lottery over the alternatives and the winner is eventually drawn at random according to the given probabilities.

In order to study properties such as efficiency, strategyproofness, and participation for SDSs, preferences over alternatives are typically lifted to preferences over lotteries using the notion of *stochastic dominance (SD)*, i.e., one lottery is preferred to another lottery if the expected utility of the former exceeds that of the latter for *every* utility representation consistent with the voter's preferences over alternatives (see, e.g., Gibbard, 1977; Bogomolnaia and Moulin, 2001; Brandl et al., 2018). When demanding SD-efficiency, SD-strategyproofness, and anonymity, the only SDS that does the job is *uniform random dictatorship (RD)*, which selects a voter uniformly at random and then returns his favorite alternative (Gibbard, 1977). Moreover, RD satisfies strict SD-participation, which means that voters are *strictly* better off participating (unless their top choice already receives probability 1). Unsurprisingly, RD has some severe shortcomings. It is often criticized for its inability to compromise: if all voters agree on a second best alternative but disagree on the best one, the uniform random dictatorship will not choose the common second best option. Furthermore, it cannot be extended to weak preferences without giving up SD-strategyproofness or SD-efficiency (Brandl et al., 2016b). On top of these criticisms, the representation of preferences over lotteries via expected utility functions has come under scrutiny (e.g., Allais, 1953; Kahneman and Tversky, 1979; Machina, 1989; Anand, 2009).

As an alternative to traditional expected utility representations, some authors have proposed to postulate that one lottery is preferred to another if the former is more likely to return a preferred outcome (Blyth, 1972; Packard, 1982; Blavatskyy, 2006). The resulting preference extension is known as *pairwise comparison (PC)* and represents a special case of Fishburn's skew-symmetric bilinear utility functions (Fishburn, 1982). Brandl et al. (2019) have shown that the No-Show paradox can be circumvented using PC preferences. Moreover, Brandl and Brandt (2020) proved that PC preferences constitute the *only* domain of preferences within a rather broad class of preferences over lotteries that allow for preference aggregation that satisfies independence of irrelevant alternatives and efficiency, thus avoiding Arrow's impossibility. In both cases, the resulting SDS is the set of *maximal lotteries (ML)*, which was proposed by Fishburn (1984a) and has recently attracted significant attention (Brandl et al., 2016a, 2022; Hoang, 2017).

It is known that PC-efficiency is stronger than SD-efficiency and violated by RD. PC-strategyproofness and strict PC-participation, on the other hand, are weaker than their SD counterparts. ML satisfies Condorcet-consistency, PC-efficiency, and PC-participation (Aziz et al., 2018; Brandl et al., 2019). Furthermore, it is PC-strategyproof in preference profiles that admit a Condorcet winner (Hoang, 2017; Brandl et al., 2022). These encouraging results lead to the natural question of whether there are attractive SDSs that satisfy PC-strategyproofness or strict PC-participation. We address this question by proving the following theorems, all of which settle open problems raised by Brandt (2017, p. 18):

- There is no Condorcet-consistent SDS that satisfies PC-strategyproofness.
- There is no anonymous and neutral SDS that satisfies PC-efficiency and PC-strategyproofness.
- There is no anonymous and neutral SDS that satisfies PC-efficiency and strict PC-participation.

All three theorems hold for strict preferences and require $m \geq 4$ alternatives; we show that they turn into possibilities when $m \leq 3$ by constructing two new SDSs. The second theorem strengthens Theorem 5 by Aziz et al. (2018), which requires weak preferences. Our theorems demonstrate that efficiency, strategyproofness, and strict participation—which are satisfied by RD if we extend preferences using SD—are not compatible for PC preferences. Hence, there is no equivalent of random dictatorships for PC preferences. This also means that—unlike with Arrow's impossibility and the No-Show paradox—PC preferences do not help to circumvent the Gibbard-Satterthwaite theorem. As a consequence, we face a tradeoff between efficiency and incentive-compatibility, which implies that *no* SDS can combine the advantages of ML and RD when using PC to compare lotteries. Hence, among the known SDSs, RD is the most attractive one when aiming for incentive-compatibility and ML when aiming for efficiency (and other properties such as Condorcet-consistency).

2 The Model

Let $A = \{a_1, \dots, a_m\}$ be a finite set of m alternatives and $\mathbb{N} = \{1, 2, 3, \dots\}$ an infinite set of voters. We denote by $\mathcal{F}(\mathbb{N})$ the set of all finite and non-empty subsets of $\mathbb{N}$. Intuitively, $\mathbb{N}$ is the set of all potential voters, whereas $N \in \mathcal{F}(\mathbb{N})$ is a concrete electorate. Given an electorate $N \in \mathcal{F}(\mathbb{N})$, every voter $i \in N$ has a *preference relation* $\succ_i$, which is a complete, transitive, and anti-symmetric binary relation on A. In particular, we do not allow for ties (which only makes our results stronger). We write preference relations as comma-separated lists and denote the set of all preference relations by $\mathcal{R}$. A *preference profile* R on an electorate $N \in \mathcal{F}(\mathbb{N})$ contains a preference relation $\succ_i$ for every voter $i \in N$, i.e., $R \in \mathcal{R}^N$. When writing preference profiles, we use sets before preference relations to indicate the voters who report the same preference relation. To this end, we define $[j \dots k] = \{i \in N : j \leq i \leq k\}$ and note that $[j \dots k] = \emptyset$ if $j > k$. For instance, $[1 \dots 3]$: a, b, c means that voters 1, 2, and 3 prefer a to b to c. We omit the brackets for singleton sets. Given a preference profile $R \in \mathcal{R}^N$, the *majority margin* between two alternatives $x, y \in A$ is $g_R(x, y) = |\{i \in N : x \succ_i y\}| - |\{i \in N : y \succ_i x\}|$, i.e., the majority margin indicates how many more voters prefer x to y than *vice versa*. Furthermore, we define $n_R(x)$ as the number of voters who prefer alternative x the most in the profile R. Next, we denote by $R_{-i} = (\succ_1, \dots, \succ_{i-1}, \succ_{i+1}, \dots, \succ_n)$ the profile derived from $R \in \mathcal{R}^N$ by removing voter $i \in N$. Finally, $\mathcal{R}^{\mathcal{F}(\mathbb{N})}$ is the set of all possible preference profiles.

In this paper, we study *social decision schemes (SDSs)*, which map preference profiles to lotteries over the alternatives. A *lottery* p is a probability distribution over the alternatives, i.e., a function $p : A \to [0, 1]$ such that $p(x) \geq 0$ for all $x \in A$ and $\sum_{x \in A} p(x) = 1$. The set of all lotteries on A is denoted by $\Delta(A)$. Then, an SDS f formally is a function of type $f : \mathcal{R}^{\mathcal{F}(\mathbb{N})} \to \Delta(A)$. We define $f(R, x)$ as the probability assigned to x by $f(R)$ and extend this notion to sets $X \subseteq A$ by $f(R, X) = \sum_{x \in X} f(R, x)$.

2.1 Fairness and Decisiveness

Next, we formalize desirable properties of SDSs. Two basic fairness notions are anonymity and neutrality, which require that voters and alternatives are treated equally, respectively. Formally, an SDS f is *anonymous* if $f(\pi(R)) = f(R)$ for all electorates $N \in \mathcal{F}(\mathbb{N})$, preference profiles $R \in \mathcal{R}^N$, and permutations $\pi : N \to N$. Here, $R' = \pi(R)$ is defined by $\succ'_i = \succ_{\pi(i)}$ for all $i \in N$. Analogously, *neutrality* requires of an SDS f that $f(\pi(R)) = \pi(f(R))$ for all electorates $N \in \mathcal{F}(\mathbb{N})$, preference profiles $R \in \mathcal{R}^N$, and permutations $\pi : A \to A$. This time, $R' = \pi(R)$ is the profile such that for all $i \in N$ and $x, y \in A$, $\pi(x) \succ'_i \pi(y)$ if and only if $x \succ_i y$. Another fairness condition is cancellation, which demands that the outcome does not change if two voters with inverse preferences join the electorate. Hence, an SDS f satisfies *cancellation* if $f(R) = f(R')$ for all preference profiles $R, R' \in \mathcal{R}^{\mathcal{F}(\mathbb{N})}$ such that R' is derived from R by adding two voters with inverse preferences.

A natural desideratum in randomized social choice is decisiveness: randomization should be avoided whenever possible. For instance, Condorcet-consistency formalizes this idea. We say an alternative x is a *Condorcet winner* in a profile R if $g_R(x, y) > 0$ for all $y \in A \setminus \{x\}$. Then, *Condorcet-consistency* requires of an SDS f that the Condorcet winner is chosen with probability 1 whenever it exists, i.e., $f(R, x) = 1$ for all preference profiles $R \in \mathcal{R}^{\mathcal{F}(\mathbb{N})}$ with Condorcet winner x. A weaker decisiveness condition is the absolute winner property. An *absolute winner* is an alternative x that is top-ranked by more than half of the voters in $R \in \mathcal{R}^N$, i.e., $n_R(x) > \frac{|N|}{2}$. Then, the *absolute winner property* requires that $f(R, x) = 1$ for all profiles $R \in \mathcal{R}^{\mathcal{F}(\mathbb{N})}$ with absolute winner x. Since absolute winners are also Condorcet winners, Condorcet-consistency implies the absolute winner property.

2.2 PC and SD Preferences

We assume that the voters' preferences over alternatives are lifted to preferences over lotteries via the pairwise comparison (PC) extension (see, e.g., Aziz et al., 2015, 2018; Brandt, 2017; Brandl and Brandt, 2020). According to this notion, a voter prefers lottery p to lottery q if the probability that p returns a better outcome than q is at least as large as the probability that q returns a better outcome than p, i.e.,

$$p \succsim_i^{PC} q \iff \sum_{x,y\in A: x\succ_i y} p(x)q(y) \geq \sum_{x,y\in A: x\succ_i y} q(x)p(y).$$

The relation $\succsim_i^{PC}$ is known to be complete but intransitive. An appealing interpretation of PC preferences is *ex ante* regret minimization, i.e., given two lotteries, a voter prefers the one which is less likely to result in *ex post* regret. Despite the simple definition, PC preferences are quite difficult to work with and even simple notions such as PC-efficiency are little understood (Aziz et al., 2015).

Another well-known way to compare lotteries is *stochastic dominance (SD)* (e.g., Gibbard, 1977; Brandl et al., 2018):

$$p \succsim_i^{SD} q \iff \forall x \in A: \sum_{y\in A: y\succ_i x} p(y) \geq \sum_{y\in A: y\succ_i x} q(y).$$

It follows from a result by Fishburn (1984b) that $p \succsim_i^{SD} q$ implies $p \succsim_i^{PC} q$ for all preference relations $\succ_i$ and all lotteries p and q (see also Aziz et al., 2015). In other words, the SD relation is a subrelation of the PC relation. For both $\mathcal{X} \in \{PC, SD\}$, we say a voter *strictly $\mathcal{X}$-prefers* p to q, denoted by $p \succ_i^{\mathcal{X}} q$, if $p \succsim_i^{\mathcal{X}} q$ and not $q \succsim_i^{\mathcal{X}} p$. Note that $p \succ_i^{SD} q$ implies $p \succ_i^{PC} q$.

2.3 Efficiency and Incentives

We can now define efficiency, strategyproofness, and participation. All of these axioms can be defined for both SD and PC; we thus define the concepts for $\mathcal{X} \in \{PC, SD\}$. First, we discuss efficiency, which requires that no lottery is unanimously preferred to the lottery chosen by the SDS. To formalize this, we say a lottery p *$\mathcal{X}$-dominates* another lottery q in a profile $R \in \mathcal{R}^N$ if $p \succsim_i^{\mathcal{X}} q$ for all voters $i \in N$ and $p \succ_{i^*}^{\mathcal{X}} q$ for some voter $i^* \in N$. Conversely, a lottery p is *$\mathcal{X}$-efficient* in R if it is not $\mathcal{X}$-dominated, and an SDS f is *$\mathcal{X}$-efficient* if $f(R)$ is $\mathcal{X}$-efficient for all preference profiles $R \in \mathcal{R}^{\mathcal{F}(\mathbb{N})}$. Both PC-efficiency and SD-efficiency imply *ex post* efficiency. For introducing this concept, we say an alternative x *Pareto-dominates* another alternative y in a profile $R \in \mathcal{R}^N$ if $x \succ_i y$ for all voters $i \in N$.[1] Then, *ex post efficiency* requires that $f(R, x) = 0$ for all profiles $R \in \mathcal{R}^{\mathcal{F}(\mathbb{N})}$ and alternatives $x \in A$ that are Pareto-dominated in R.

Next, we introduce strategyproofness, which demands that no voter can benefit by lying about his true preferences. Formally, an SDS f is *$\mathcal{X}$-strategyproof* if $f(R) \succsim_i^{\mathcal{X}} f(R')$ for all electorates $N \in \mathcal{F}(\mathbb{N})$, voters $i \in N$, and preference profiles $R, R' \in \mathcal{R}^N$ with $R_{-i} = R'_{-i}$.[2] Conversely, an SDS is *$\mathcal{X}$-manipulable* if it is not $\mathcal{X}$-strategyproof. Since strategyproofness does not require a variable electorate, we usually specify the electorates for which an SDS is strategyproof or manipulable. Similarly to strategyproofness, participation requires that voters should not be able to benefit by abstaining from the election. Hence, an SDS f satisfies *$\mathcal{X}$-participation* if $f(R) \succsim_i^{\mathcal{X}} f(R_{-i})$ for all electorates $N \in \mathcal{F}(\mathbb{N})$, voters $i \in N$, and preference profiles $R \in \mathcal{R}^N$. In this paper, we are interested in *strict $\mathcal{X}$-participation* introduced by Brandl et al. (2015), which demands of an SDS f that, for all $N \in \mathcal{F}(\mathbb{N})$, $i \in N$, and $R \in \mathcal{R}^N$, it holds that $f(R) \succsim_i^{\mathcal{X}} f(R_{-i})$ and, moreover, $f(R) \succ_i^{\mathcal{X}} f(R_{-i})$ if there is a lottery p with $p \succ_i^{\mathcal{X}} f(R_{-i})$. That is, if possible, a voter strictly benefits from voting compared to abstaining.

Since $p \succ_i^{SD} q$ implies $p \succ_i^{PC} q$ and $p \succsim_i^{SD} q$ implies $p \succsim_i^{PC} q$, the concepts of SD-efficiency, SD-strategyproofness, and SD-participation are related to the analogous concepts for PC: PC-efficiency entails SD-efficiency, whereas SD-strategyproofness and strict SD-participation are stronger than the corresponding notions for PC (cf. Brandt, 2017). See Figure 1 for an overview of these axioms.

2.4 Random Dictatorship and Maximal Lotteries

The following two important SDSs help to put our results into perspective: the uniform random dictatorship (RD) and maximal lotteries (ML). These SDSs are well-known and all subsequent claims are taken from the survey by Brandt (2017). The *uniform random dictatorship (RD)* assigns probabilities proportional to $n_R(x)$, i.e., $RD(R,x) = \frac{n_R(x)}{\sum_{y\in A} n_R(y)}$ for every alternative $x \in A$ and preference profile $R \in \mathcal{R}^{\mathcal{F}(\mathbb{N})}$. RD is known to satisfy SD-strategyproofness, strict SD-participation, and SD-efficiency. Even more, when additionally imposing anonymity, it is the only SDS that satisfies these axioms. Since SD-strategyproofness and strict SD-participation imply the corresponding concepts for PC, RD satisfies our incentive axioms also if we extend preferences using PC. However, RD fails PC-efficiency and Condorcet-consistency, as can be seen in the following profile.

R: 1: a, b, c 2: b, a, c 3: c, a, b

For this profile, $RD(R, x) = \frac{1}{3}$ for all $x \in A$, but a is the Condorcet winner and the lottery that puts probability 1 on a PC-dominates $RD(R)$.

In order to define ML, let $ML(R) = \{p \in \Delta(A) : \sum_{x,y\in A} p(x)q(y)g_R(x,y) \geq 0 \text{ for all } q \in \Delta(A)\}$ be the set of maximal lotteries for profile R. $ML(R)$ is non-empty by the minimax theorem and almost always a singleton. For all our claims about ML, it does not matter how ties are broken and any maximal lottery can be returned for a profile that admits multiple maximal lotteries. ML satisfies PC-efficiency, PC-participation, and Condorcet-consistency. However, ML fails PC-strategyproofness and strict PC-participation. The former can be seen by considering the following profiles.

R: $\{1,2\}$: a, b, c $\{3,4\}$: b, c, a 5: c, a, b
R': $\{1,2\}$: a, b, c 3: b, c, a $\{4,5\}$: c, a, b

The unique maximal lotteries in R and R', respectively, are p and q with $p(a) = q(c) = \frac{3}{5}$ and $p(b) = p(c) = q(a) = q(b) = \frac{1}{5}$. Since $\succ_i = \succ'_i$ for all $i \in \{1, 2, 3, 5\}$ and $q \succ_4^{PC} p$, voter 4 can PC-manipulate by deviating from R to R'. This raises the question of whether there is an SDS that unifies the advantages of ML and RD. As we show, this is not the case.

[1] Recall that ties in $\succ_i$ are not allowed.

[2] Another version is to require that $f(R') \not\succ_i^{\mathcal{X}} f(R)$; these two versions coincide for PC because the PC extension is complete.

3 Results

We are now ready to present our results. The results for PC-strategyproofness are given in Section 3.1 while those for strict PC-participation are given in Section 3.2. Due to space restrictions, we defer most proofs to the appendix and discuss proof sketches instead.

3.1 PC-strategyproofness

In this section, we show that every Condorcet-consistent and every anonymous, neutral, and PC-efficient SDS is PC-manipulable when there are $m \geq 4$ alternatives. These results show that no SDS simultaneously satisfies PC-strategyproofness and the desirable properties of maximal lotteries. Moreover, since PC-strategyproofness is weaker than SD-strategyproofness, the incompatibility of PC-strategyproofness and Condorcet-consistency is a strengthening of the well-known impossibility of Condorcet-consistent and SD-strategyproof SDSs. Perhaps more surprising is the impossibility involving PC-efficiency: while anonymity, neutrality, SD-strategyproofness, and SD-efficiency characterize the uniform random dictatorship, the axioms become incompatible when moving from SD to PC. Since both impossibilities require $m \geq 4$ alternatives, we also show that they turn into possibilities if $m \leq 3$.

We start by discussing the impossibility of Condorcet-consistent and strategyproof SDSs.

Theorem 1. *Every Condorcet-consistent SDS is PC-manipulable if $|N| \geq 5$ is odd and $m \geq 4$.*

Proof. Assume for contradiction that there is a Condorcet-consistent and PC-strategyproof SDS f for $m \geq 4$ alternatives. Subsequently, we focus on the electorate $N = \{1, \dots, 5\}$ because we can generalize the result to any larger electorate with an odd number of voters by adding pairs of voters with inverse preferences. These voters do not change the Condorcet winner and hence will not affect our analysis.

As the first step, consider the profiles R^1 to R^4. The $*$ symbol is a placeholder for all missing alternatives.

R^1: 1: $a, b, d, c, *$ 2: $d, b, a, c, *$ 3: $a, d, c, b, *$
4: $*, c, d, b, a$ 5: $c, b, a, d, *$

R^2: 1: $a, b, d, c, *$ 2: $d, b, a, c, *$ 3: $a, b, c, d, *$
4: $*, c, d, b, a$ 5: $c, b, a, d, *$

R^3: 1: $a, b, d, c, *$ 2: $d, b, a, c, *$ 3: $a, d, c, b, *$
4: $*, c, d, a, b$ 5: $c, b, a, d, *$

R^4: 1: $a, b, d, c, *$ 2: $d, b, a, c, *$ 3: $d, a, c, b, *$
4: $*, c, d, b, a$ 5: $c, b, a, d, *$

Note that b is the Condorcet winner in R^2, a in R^3, and d in R^4. Thus, Condorcet-consistency entails that $f(R^2, b) = f(R^3, a) = f(R^4, d) = 1$. In contrast, there is no Condorcet winner in R^1 and we use PC-strategyproofness to derive $f(R^1)$. For instance, this axiom postulates that $\sum_{x,y\in A: x \succ_3^2 y} f(R^2, x) f(R^1, y) \geq \sum_{x,y\in A: x \succ_3^2 y} f(R^1, x) f(R^2, y)$ as voter 3 can PC-manipulate by deviating from R^2 to R^1 otherwise. By substituting $f(R^2, b) = 1$ and $f(R^2, x) = 0$ for $x \in A \setminus \{b\}$, we thus derive that

$$f(R^1, a) \leq f(R^1, A \setminus \{a, b\}). \qquad (1)$$

Analogously, PC-strategyproofness between R^1 and R^3 and between R^1 and R^4 entails the following inequalities because voter 4 needs to PC-prefer $f(R^3)$ to $f(R^1)$ and voter 3 needs to PC-prefer $f(R^1)$ to $f(R^4)$.

$$f(R^1, A \setminus \{a, b\}) \leq f(R^1, b) \qquad (2)$$

$$f(R^1, A \setminus \{a, d\}) \leq f(R^1, a) \qquad (3)$$

Chaining the inequalities together, we get $f(R^1, A \setminus \{a, d\}) \leq f(R^1, a) \leq f(R^1, A \setminus \{a, b\}) \leq f(R^1, b)$, so $f(R^1, A \setminus \{a, b, d\}) = 0$. Simplifying (1), (2), and (3) then results in $f(R^1, a) \leq f(R^1, d) \leq f(R^1, b) \leq f(R^1, a)$, so $f(R^1, a) = f(R^1, b) = f(R^1, d) = \frac{1}{3}$.

Next, we analyze the profiles R^5 to R^8.

R^5: 1: $a, b, d, c, *$ 2: $b, d, a, c, *$ 3: $a, d, c, b, *$
4: $*, c, d, b, a$ 5: $c, b, a, d, *$

R^6: 1: $a, b, d, c, *$ 2: $b, d, a, c, *$ 3: $a, d, c, b, *$
4: $*, c, d, b, a$ 5: $b, c, a, d, *$

R^7: 1: $a, b, d, c, *$ 2: $b, d, a, c, *$ 3: $a, d, c, b, *$
4: $*, c, d, a, b$ 5: $c, b, a, d, *$

R^8: 1: $a, b, d, c, *$ 2: $b, c, d, a, *$ 3: $a, d, c, b, *$
4: $*, c, d, b, a$ 5: $c, b, a, d, *$

Just as for the profiles R^1 to R^4, there is no Condorcet winner in R^5, whereas b is the Condorcet winner in R^6, a in R^7, and c in R^8. Consequently, Condorcet-consistency requires that $f(R^6, b) = f(R^7, a) = f(R^8, c) = 1$. Next, we use PC-strategyproofness to derive $f(R^5)$. In particular, we infer the following inequalities as voter 5 needs to PC-prefer $f(R^5)$ to $f(R^6)$, voter 4 needs to PC-prefer $f(R^7)$ to $f(R^5)$, and voter 2 needs to PC-prefer $f(R^8)$ to $f(R^5)$.

$$f(R^5, A \setminus \{b, c\}) \leq f(R^5, c) \qquad (4)$$

$$f(R^5, A \setminus \{a, b\}) \leq f(R^5, b) \qquad (5)$$

$$f(R^5, b) \leq f(R^5, A \setminus \{b, c\}) \qquad (6)$$

Analogous computations as for R^1 now show that $f(R^5, a) = f(R^5, b) = f(R^5, c) = \frac{1}{3}$. Finally, note that R^1 and R^5 only differ in the preferences of voter 2. This means that voter 2 can PC-manipulate by deviating from R^5 to R^1 since he PC-prefers $f(R^1)$ to $f(R^5)$. Hence, f fails PC-strategyproofness, which contradicts our assumptions. □

It is open whether Theorem 1 also holds if $|N|$ is even.

Next, we turn the focus to our second impossibility result: every anonymous and neutral SDS that satisfies PC-efficiency is PC-manipulable. For proving this theorem, we first show that every SDS that satisfies the absolute winner property and PC-efficiency is PC-manipulable.

Lemma 1. *Every PC-efficient SDS that satisfies the absolute winner property is PC-manipulable if $|N| \geq 3$, $|N| \notin \{4, 6\}$, and $m \geq 4$.*

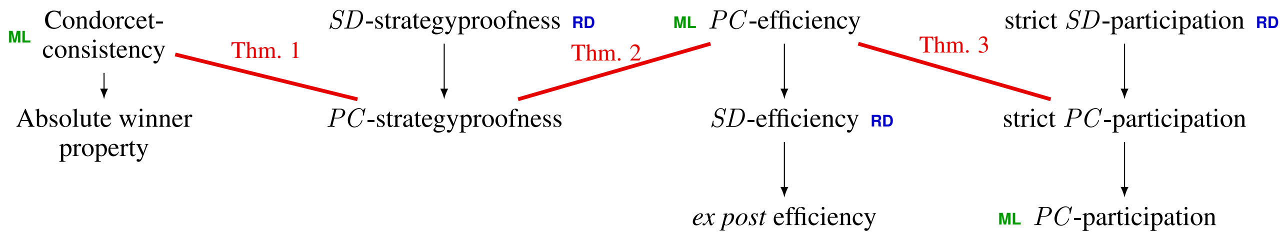

Figure 1: Overview of results. An arrow from an axiom X to another axiom Y indicates that X implies Y. The red lines between axioms represent impossibility theorems. Note that Theorems 2 and 3 additionally require anonymity and neutrality. Axioms labeled with ML are satisfied by maximal lotteries, and axioms labelled with RD are satisfied by the uniform random dictatorship.

Proof sketch. Assume for contradiction that there are an SDS f and an electorate $N \in \mathcal{F}(\mathbb{N})$ that satisfy the requirements of the lemma. In this sketch, we focus on $m = 4$ alternatives as we can extend the construction to more alternatives by adding Pareto-dominated ones. Moreover, we suppose that $n = |N|$ is odd; the argument for even n is similar but more involved. Now, consider the profiles R and R' shown below.

R: $[1 \dots \frac{n-1}{2}]$: a, d, b, c $\quad$ $\frac{n+1}{2}$: b, c, d, a
$\quad$ $[\frac{n+3}{2} \dots n]$: c, a, d, b

R': $[1 \dots \frac{n-1}{2}]$: a, d, b, c $\quad$ $\frac{n+1}{2}$: b, d, c, a
$\quad$ $[\frac{n+3}{2} \dots n]$: c, a, d, b

The goal is to show that $f(R, a) = f(R, b) = f(R, c) = \frac{1}{3}$ and $f(R', a) = f(R', c) = f(R', d) = \frac{1}{3}$. Then, voter $\frac{n+1}{2}$ can PC-manipulate by deviating from R' to R because he PC-prefers $f(R)$ to $f(R')$. We proceed in three steps to derive the lottery $f(R)$: first, we use the absolute winner property and PC-strategyproofness to show that $f(R, c) > 0$. Next, we infer from PC-efficiency that $f(R, d) = 0$. Finally, we repeatedly apply PC-strategyproofness and the absolute winner property to prove that $f(R, a) \geq f(R, c) \geq f(R, b) \geq f(R, a)$, which implies that $f(R, x) = \frac{1}{3}$ for $x \in \{a, b, c\}$. An analogous argument can be used to derive $f(R')$. ☐

Note that Lemma 1 is a rather strong impossibility itself. Next, we use it to prove that every anonymous, neutral, and PC-efficient SDS is PC-manipulable.

Theorem 2. *Every anonymous and neutral SDS that satisfies PC-efficiency is PC-manipulable if $|N| \geq 3$, $|N| \notin \{4, 6\}$, and $m \geq 4$.*

Proof sketch. We prove this theorem by showing that the given axioms imply the absolute winner property; then, Lemma 1 implies the impossibility. Consider an arbitrary SDS f that satisfies all given axioms and an electorate $N \in \mathcal{F}(\mathbb{N})$ with $n = |N| \geq 3$. Moreover, we focus on three alternatives because, as in Lemma 1, we can extend the argument to more alternatives by assuming that these are Pareto-dominated. We proceed with a case distinction with respect to the parity of n and, in this sketch, restrict attention to even n. Consider the following profile R.

R: $\quad$ 1: a, b, c $\quad$ 2: a, c, b
$\quad$ $[3 \dots \frac{n}{2}{+}1]$: b, a, c $\quad$ $[\frac{n}{2}{+}2 \dots n]$: c, a, b

Anonymity and neutrality require that $f(R, b) = f(R, c)$. Next, PC-efficiency implies that $f(R, b) = f(R, c) = 0$ and hence $f(R, a) = 1$. Based on this insight, one can prove that a is chosen with probability 1 whenever the voters $i \in [1 \dots \frac{n}{2}{+}1]$ report it as their best alternative. Due to anonymity and neutrality, this statement is equivalent to the absolute winner property. ☐

Since both Theorems 1 and 2 require $m \geq 4$ alternatives, we can still hope for a possibility if $m \leq 3$. Indeed, for $m = 2$, ML satisfies Condorcet-consistency, PC-efficiency, PC-strategyproofness, anonymity, and neutrality. However, as shown in Section 2.4, ML fails PC-strategyproofness if $m = 3$. Thus, we construct another SDS that satisfies all given axioms. To this end, let $CW(R)$ be the set of Condorcet winners in R, and let $WCW(R) = \{x \in A \colon g_R(x, y) \geq 0 \text{ for all } y \in A \setminus \{x\}\}$ be the set of weak Condorcet winners if $CW(R) = \emptyset$, and $WCW(R) = \emptyset$ otherwise. Then, define the SDS f^1 as follows.

$$f^1(R) = \begin{cases} [x : 1] & \text{if } CW(R) = \{x\} \\ [x : \frac{1}{2}; y : \frac{1}{2}] & \text{if } WCW(R) = \{x, y\} \\ [x : \frac{3}{5}; y : \frac{1}{5}; z : \frac{1}{5}] & \text{if } WCW(R) = \{x\} \\ [x : \frac{1}{3}; y : \frac{1}{3}; z : \frac{1}{3}] & \text{otherwise} \end{cases}$$

It is easy to see that f^1 is Condorcet-consistent. The next proposition characterizes f^1 as the *only* SDS that satisfies cancellation and the axioms of Theorem 2.

Proposition 1. *For $m = 3$, f^1 is the only SDS that satisfies PC-efficiency, PC-strategyproofness, neutrality, anonymity, and cancellation.*

Proof sketch. The definition of f^1 immediately implies that this SDS satisfies anonymity, neutrality, and cancellation. Moreover, tedious case distinctions establish that f^1 satisfies PC-strategyproofness and PC-efficiency. For the reverse direction, we show first that every SDS that satisfies all given axioms is Condorcet-consistent. Building on this insight, we use the given axioms to infer the outcomes for all profiles. ☐

Remark 1. Of all the axioms in this section besides cancellation, ML only fails PC-strategyproofness, dictatorships only fail anonymity and Condorcet-consistency, and the uniform random dictatorship only fails PC-efficiency and Condorcet-consistency. This shows that all axioms of Theorem 2 but neutrality are required for the result, and that both axioms are required for Theorem 1. We conjecture that Theorem 2 holds even without neutrality. Proposition 1 shows that $m \geq 4$ alternatives are required for both Theorems 1 and 2.

3.2 Strict PC-participation

In this section, we investigate strict PC-participation and prove that this axiom is incompatible with PC-efficiency. This result is rather surprising given that multiple SDSs are known to satisfy SD-efficiency and strict SD-participation (see Brandl et al., 2015). Moreover, our impossibility can be seen as a complement to the work of Brandl et al. (2019) which shows that ML satisfies both PC-participation and PC-efficiency. In particular, our result demonstrates that maximal lotteries satisfy a maximal degree of participation subject to PC-efficiency. Finally, since Theorem 3 requires $m \geq 4$ alternatives, we construct an SDS that satisfies all our requirements when $m \leq 3$.

We first discuss the impossibility theorem.

Theorem 3. *No neutral and anonymous SDS satisfies both PC-efficiency and strict PC-participation if $m \geq 4$.*

Proof. Assume for contradiction that there is a neutral and anonymous SDS f that satisfies both PC-efficiency and strict PC-participation. In what follows, we focus on the case $m = 4$ because we can generalize our construction to $m > 4$ by adding $m - 4$ alternatives at the bottom of all voters' preference rankings. Since these dummy alternatives are Pareto-dominated by the original alternatives, PC-efficiency requires these alternatives to be assigned probability 0, and thus they will not affect our subsequent analysis.

First, consider the following profile with ten voters.

R^1:	1: a, b, c, d	2: a, b, d, c	3: a, c, b, d
	4: a, c, d, b	5: a, d, b, c	6: a, d, c, b
	7: b, a, c, d	8: b, a, d, c	
	9: c, a, b, d	10: c, a, d, b	

Observe that d is Pareto-dominated by a, so by PC-efficiency, $f(R^1, d) = 0$. Moreover, since b and c are symmetric in this profile, neutrality and anonymity imply that $f(R^1, b) = f(R^1, c)$. If $f(R^1, b) = f(R^1, c) > 0$, then f is not PC-efficient because all voters weakly prefer the degenerate lottery that puts probability 1 on a, with voters 1–6 strictly preferring this lottery. Hence, $f(R^1, b) = f(R^1, c) = 0$ which means that $f(R^1, a) = 1$.

Next, consider profile R^2, which is obtained by adding voter 11 with the preference d, a, b, c to R^1. We infer from strict PC-participation that $f(R^2, d) > f(R^2, b) + f(R^2, c)$.

Finally, consider profile R^3, which is obtained by adding voter 12 with the preference d, a, c, b to R^2. Observe that b, c, and d are symmetric in R^3, so by neutrality and anonymity, $f(R^3, b) = f(R^3, c) = f(R^3, d)$. If $f(R^3, b) = f(R^3, c) = f(R^3, d) > 0$, then f is not PC-efficient because all voters strictly prefer the degenerate lottery that puts probability 1 on a. Hence, $f(R^3, b) = f(R^3, c) = f(R^3, d) = 0$, which means that $f(R^3, a) = 1$. Since $f(R^2, d) > f(R^2, b) + f(R^2, c)$, voter 12 has a disincentive to participate in R^3, thereby contradicting the strict PC-participation of f. □

Since Theorem 3 requires $m \geq 4$, a natural question is whether the impossibility also holds for $m \leq 3$. As we demonstrate, the impossibility ceases to hold. If $m = 2$, it is easy to see that the uniform random dictatorship satisfies all axioms of Theorem 3. For $m = 3$, however, the uniform random dictatorship fails PC-efficiency (see Section 2.4). In light of this, we construct a new SDS that satisfies all axioms used in Theorem 3. To this end, let B denote the set of alternatives that are never bottom-ranked. Then, the SDS f^2 is defined as follows: return the uniform random dictatorship if $|B| \in \{0, 2\}$; otherwise, we delete the alternatives $x \in A \setminus B$ that minimize $n_R(x)$ (if there is a tie, delete both alternatives) and return the outcome of the uniform random dictatorship for the reduced profile. As the following proposition demonstrates, f^2 indeed satisfies all axioms of Theorem 3 if $m = 3$.

Proposition 2. *For $m = 3$, f^2 satisfies anonymity, neutrality, PC-efficiency, and strict PC-participation.*

Proof sketch. The definition of f^2 immediately implies that this SDS is anonymous and neutral. Next, for proving that f^2 is PC-efficient, we consider the case distinction used in the definition of this SDS: if $|B| \in \{0, 2\}$, randomizing over the top-ranked alternatives is PC-efficient. On the other hand, if $|B| = 1$, f^2 is PC-efficient as it ignores one of the top-ranked alternatives. Finally, the strict PC-participation of f^2 follows from a tedious case distinction with respect to B. □

Remark 2. Each of PC-efficiency and strict PC-participation is by itself compatible with anonymity and neutrality, as witnessed by maximal lotteries and the uniform random dictatorship, respectively. Hence, these two axioms are required for Theorem 3. In contrast, we do not know whether anonymity and neutrality are needed for this result. Proposition 2 shows that $m \geq 4$ is required.

Remark 3. Strict PC-participation is also incompatible with Condorcet-consistency, i.e., a statement analogous to Theorem 1 holds. This follows from the fact that a single voter cannot always change the Condorcet winner by joining the electorate, even if it is his least preferred outcome. In such cases, Condorcet-consistency implies that the outcome does not change while strict PC-participation requires the opposite.

4 Conclusion

We have studied incentive properties of social decision schemes (SDSs) based on the pairwise comparison (PC) lottery extension, and answered open questions raised by Brandt (2017) by proving three strong impossibilities. In particular, we showed that PC-strategyproofness and strict PC-participation are incompatible with PC-efficiency and Condorcet-consistency (see also Figure 1). We highlight three important aspects and consequences of our results. Firstly, when moving from the standard approach of stochastic dominance (SD) to PC, previously compatible axioms become incompatible. Secondly, our results show that—unlike with other classical impossibilities—PC does not help to circumvent the Gibbard-Satterthwaite theorem. Finally, our impossibilities identify a tradeoff between incentive-compatibility and efficiency. In light of this tradeoff, two SDSs seem particularly appealing: the uniform random dictatorship because it satisfies PC-strategyproofness and strict PC-participation, and maximal lotteries because it satisfies PC-strategyproofness in all profiles that admit a Condorcet winner, Condorcet-consistency, PC-efficiency, and PC-participation.

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A Omitted Proofs

In this section, we present the proofs omitted from the main body. In particular, we discuss in Appendix A.1 the proofs of Theorem 2 and Lemma 1. To increase the readability of these proofs, we defer all computations for proving PC-(in)efficiency to Appendix A.2. Finally, we discuss in Appendix A.3 the proofs of our two propositions.

A.1 Proof of Theorem 2

We start by proving Theorem 2 and present first two auxiliary lemmas about the consequences of PC-strategyproofness for lotteries with small support. The *support* of a lottery is defined as $supp(p) = \{x \in A \colon p(x) > 0\}$. Our first lemma focuses on the case where $|supp(f(R))| = 2$ and states that, if the support does not change after a manipulation and the manipulator does not reorder the alternatives in the support, the outcome is not allowed to change.

Lemma 2. *Consider two preference profiles $R, R' \in \mathcal{R}^N$ for some electorate $N \in \mathcal{F}(\mathbb{N})$, a voter $i \in N$, and two alternatives $a, b \in A$ such that $R_{-i} = R'_{-i}$ and $a \succ_i b$ iff*

$a \succ_i' b$. *Every PC-strategyproof SDS f satisfies $f(R) = f(R')$ if $supp(f(R)) \subseteq \{a, b\}$ and $supp(f(R')) \subseteq \{a, b\}$.*

Proof. Let $R, R' \in \mathcal{R}^N$ denote two preference profiles for some electorate $N \in \mathcal{F}(\mathbb{N})$, let $i \in N$ denote a voter, and let $a, b \in A$ denote two alternatives such that $R_{-i} = R'_{-i}$ and $a \succ_i b$ iff $a \succ_i' b$. Without loss of generality, we assume in this proof that $a \succ_i b$ and $a \succ_i' b$; the case where voter i prefers b to a is symmetric. Moreover, let f denote a PC-strategyproof SDS and assume that $supp(f(R)) \subseteq \{a, b\}$ and $supp(f(R')) \subseteq \{a, b\}$. Finally, assume for contradiction that $f(R) \neq f(R')$. Since both lotteries only put positive probability on a and b, this means either that $f(R, a) < f(R', a)$ and $f(R, b) > f(R', b)$, or that $f(R, a) > f(R', a)$ and $f(R, b) < f(R', b)$. First, suppose that $f(R, a) < f(R', a)$. Then, $f(R') \succ_i^{PC} f(R)$, and voter i can thus PC-manipulate by deviating from R to R'. On the other hand, if $f(R, a) > f(R', a)$, voter i can PC-manipulate by deviating from R' to R as he PC-prefers $f(R)$ to $f(R')$ with respect to $\succ_i'$. Hence, both cases result in a PC-manipulation, contradicting the PC-strategyproofness of f. This proves that $f(R) = f(R')$. ☐

Next, we focus on the case $|supp(f(R))| \leq 3$, where only a significantly weaker implication holds. In more detail, we show that if $supp(f(R)) \subseteq \{a, b, c\}$, $f(R, a) < f(R, c)$, and $a \succ_i b \succ_i c$ for some voter i, then this voter cannot change the fact that a gets less probability than c in the outcome.

Lemma 3. *Consider two preference profiles $R, R' \in \mathcal{R}^N$ for some electorate $N \in \mathcal{F}(\mathbb{N})$, a voter $i \in N$, and three alternatives a, b, c such that $R_{-i} = R'_{-i}$ and $a \succ_i b \succ_i c$. Every PC-strategyproof SDS f satisfies $f(R', a) < f(R', c)$ if $f(R, a) < f(R, c)$, $supp(f(R)) \subseteq \{a, b, c\}$, and $supp(f(R')) \subseteq \{a, b, c\}$.*

Proof. Let $R, R' \in \mathcal{R}^N$ be two preference profiles for some electorate $N \in \mathcal{F}(\mathbb{N})$, $i \in N$ be a voter, and $a, b, c \in A$ be three distinct alternatives such that $R_{-i} = R'_{-i}$ and $a \succ_i b \succ_i c$. Furthermore, consider a PC-strategyproof SDS f and suppose that $supp(f(R)) \subseteq \{a, b, c\}$ and $supp(f(R')) \subseteq \{a, b, c\}$. For simplicity, we define $p = f(R)$ and $q = f(R')$ and assume for contradiction that $p(a) < p(c)$ and $q(a) \geq q(c)$. Next, we use PC-strategyproofness to relate p and q. In particular, we infer the following equation from the PC-strategyproofness between R and R'. Note that the alternatives $x \in A \setminus \{a, b, c\}$ can be omitted as $p(x) = q(x) = 0$.

$$\begin{aligned} &p(a)q(b) + p(a)q(c) + p(b)q(c) \\ &\qquad\geq q(a)p(b) + q(a)p(c) + q(b)p(c) \end{aligned}$$

Using the fact that $1 = q(a) + q(b) + q(c)$ and $1 = p(a) + p(b) + p(c)$, we have two possibilities of rewriting this inequality.

$$\begin{aligned} &p(a)(1 - q(a)) + p(b)q(c) \geq q(a)(1 - p(a)) + q(b)p(c) \\ &\iff p(a) + p(b)q(c) \geq q(a) + q(b)p(c) \end{aligned}$$

$$\begin{aligned} &p(a)q(b) + (1 - p(c))q(c) \geq q(a)p(b) + (1 - q(c))p(c) \\ &\iff p(a)q(b) + q(c) \geq q(a)p(b) + p(c) \end{aligned}$$

Summing up these two inequalities results in the following inequality.

$$\begin{aligned} &p(a)(1 + q(b)) + q(c)(1 + p(b)) \\ &\qquad\geq q(a)(1 + p(b)) + p(c)(1 + q(b)) \\ &\iff (p(a) - p(c))(1 + q(b)) \geq (1 + p(b))(q(a) - q(c)) \end{aligned}$$

Our assumption that $p(a) < p(c)$ implies that the left-hand side of the inequality is smaller than 0. On the other hand, we have $q(a) \geq q(c)$, so the right-hand side is non-negative. This is a contradiction, which proves that our assumptions on p and q are in conflict with PC-strategyproofness. Hence, if $f(R, a) < f(R, c)$, then $f(R', a) < f(R', c)$. ☐

As the next step, we prove Lemma 1. Note that, since the proof of this lemma is rather involved and relies on a case distinction, we split its proof across two separate lemmas: one for electorates with an odd number of voters and one for electorates with an even number of voters.

Lemma 1a). *Every PC-efficient SDS that satisfies the absolute winner property is PC-manipulable if $|N| \geq 3$ is odd and $m \geq 4$.*

Proof. Consider an arbitrary electorate $N \in \mathcal{F}(\mathbb{N})$ with an odd number of voters $n = |N| \geq 3$ and suppose there are $m \geq 4$ alternatives. We assume for contradiction that there is an PC-efficient SDS f that satisfies the absolute winner property and PC-strategyproofness for N. In the sequel, we will focus on profiles on the alternatives $\{a, b, c, d\}$; all other alternatives are always ranked below these alternatives and therefore Pareto-dominated. Hence, PC-efficiency entails for all subsequent profiles that $f(R, x) = 0$ for all $x \in A \setminus \{a, b, c, d\}$, which means that these alternatives do not affect our further analysis. In slight abuse of notation, we therefore assume that $A = \{a, b, c, d\}$.

We derive a contradiction by focusing on the profiles R and R' shown below. Specifically, our goal is to show that $f(R, a) = f(R, b) = f(R, c) = \frac{1}{3}$ and $f(R', a) = f(R', c) = f(R', d) = \frac{1}{3}$. This implies that voter $\frac{n+1}{2}$ can PC-manipulate by switching from R' to R as he prefers b to d, i.e., these claims result in a contradiction to PC-strategyproofness.

R:	$[1 \dots \frac{n-1}{2}]$: a, d, b, c	$\frac{n+1}{2}$: b, c, d, a
	$[\frac{n+3}{2} \dots n]$: c, a, d, b	
R':	$[1 \dots \frac{n-1}{2}]$: a, d, b, c	$\frac{n+1}{2}$: b, d, c, a
	$[\frac{n+3}{2} \dots n]$: c, a, d, b	

Claim 1: $f(R, a) = f(R, b) = f(R, c) = \frac{1}{3}$

For proving this claim, our first goal is to establish that $f(R, c) > 0$. Hence, assume for contradiction that this is not the case, i.e., that $f(R, c) = 0$. Then, we consider the profile R^1 derived from R by swapping a and c in the preference relation of voter n. Since a is top-ranked by more than half of the voters in R^1, it follows from the absolute winner property that $f(R^1, a) = 1$. Hence, PC-strategyproofness from R to R^1 implies that $f(R, c) \geq f(R, b) + f(R, d)$. Because we assume that $f(R, c) = 0$, this means that $f(R, b) = f(R, d) = 0$ and that $f(R, a) = 1$. On the other hand, if voter $\frac{n+1}{2}$ swaps b and c in R, we derive a profile R^2 with $f(R^2, c) = 1$ because of the absolute winner property. Hence, PC-strategyproofness requires that $f(R,b) \geq f(R,a) + f(R,d)$, which conflicts with $f(R,a) = 1$. Thus, the initial assumption that $f(R,c) = 0$ is incorrect, i.e., it holds that $f(R,c) > 0$. Departing from this insight, PC-efficiency entails that $f(R,d) = 0$. In more detail, Observation 2 (in Appendix A.2) proves that every lottery q with $q(d) > 0$ and $q(c) > 0$ fails PC-efficiency for R. Since we already know that $f(R,c) > 0$, it follows therefore that $f(R,d) = 0$.

Next, note that the inequalities derived from PC-strategyproofness on R^1 and R^2 remain valid, even if $f(R,c) > 0$. Combined with the fact that $f(R,d) = 0$, this means that $f(R,c) \geq f(R,b) \geq f(R,a)$. Hence, we prove Claim 1 by showing that $f(R,a) \geq f(R,c)$. Consider for this the profiles $\bar{R}^i$ for $i \in \{0, \dots, \frac{n-1}{2}\}$, which are defined as follows.

$\bar{R}^i$: $[1 \dots i]$: a, d, b, c $\quad$ $[i{+}1 \dots \frac{n-1}{2}]$: b, a, d, c
$\quad\frac{n+1}{2}$: b, c, d, a $\quad$ $[\frac{n+3}{2} \dots n]$: c, a, d, b

First, note that $\bar{R}^{\frac{n-1}{2}} = R$ and that $f(\bar{R}^0, b) = 1$ because $\frac{n+1}{2}$ voters report b as their favorite alternative in this profile. Furthermore, Observation 2 shows that $f(\bar{R}^i, d) = 0$ for every profile $\bar{R}^i$ with $i < \frac{n-1}{2}$ because all lotteries q with $q(d) > 0$ fail PC-efficiency for $\bar{R}^i$.

Finally, by a repeated application of Lemma 3, we derive that $f(R,a) \geq f(R,c)$. To this end, consider a fixed index $i \in \{1, \dots, \frac{n-1}{2}\}$. If $f(\bar{R}^i, a) < f(\bar{R}^i, c)$, this lemma requires that $f(\bar{R}^{i-1}, a) < f(\bar{R}^{i-1}, c)$. Hence, if $f(R,a) < f(R,c)$, we can repeatedly apply this argument to derive that $f(\bar{R}^0, a) < f(\bar{R}^0, c)$. However, this contradicts the absolute winner property, and thus we must have that $f(R,a) \geq f(R,c)$. This proves Claim 1.

Claim 2: $f(R',a) = f(R',c) = f(R',d) = \frac{1}{3}$

As second claim, we prove that f assigns a probability of $\frac{1}{3}$ to a, c, and d in R'. For this, we proceed analogously to Claim 1 and first show that $f(R',c) > 0$. Assume for contradiction that $f(R',c) = 0$, and consider the profile R^1 derived from R' by letting voter n swap a and c. As a consequence of the absolute winner property, $f(R^1, a) = 1$ because a is top-ranked by $\frac{n+1}{2}$ voters. Therefore, PC-strategyproofness requires that $f(R',c) \geq f(R',b) + f(R',d)$. Since $f(R',c) = 0$ by assumption, it follows that $f(R',b) = f(R',d) = 0$ and therefore $f(R',a) = 1$. Next, consider the profile R^2 derived from R' by letting voter $\frac{n+1}{2}$ make c into his best alternative. It follows again from the absolute winner property that $f(R^2, c) = 1$ because c is now top-ranked by all voters $i \in [\frac{n+1}{2} \dots n]$. Hence, we infer from PC-strategyproofness that $f(R',b) + f(R',d) \geq f(R',a)$, which contradicts that $f(R',a) = 1$. This shows that the initial assumption $f(R',c) = 0$ is wrong, i.e., it must be that $f(R',c) > 0$. As a consequence, we have that $f(R',b) = 0$ because of PC-efficiency: Observation 3 proves that if $f(R',b) > 0$, then $f(R',a) = 0$, and Observation 4 that if $f(R',b) > 0$ and $f(R',a) = 0$, then $f(R',c) = 0$. (Both observations and their proofs can be found in Appendix A.2.) Since we have already proven that $f(R',c) > 0$, it must therefore follow that $f(R',b) = 0$.

Just as for R, we can use the fact that $f(R',b) = 0$ to simplify the inequalities derived from PC-strategyproofness on R^1 and R^2. In particular, we infer that $f(R',c) \geq f(R',d) \geq f(R',a)$ from these observations. Hence, Claim 2 will follow by showing that $f(R',a) \geq f(R',c)$. Consider for this the profiles $\bar{R}^i$ for $i \in \{0, \dots, \frac{n-1}{2}\}$, which are defined as follows.

$\bar{R}^i$: $[1 \dots i]$: a, d, b, c $\quad$ $[i{+}1 \dots \frac{n-1}{2}]$: d, a, b, c
$\quad\frac{n+1}{2}$: b, d, c, a $\quad$ $[\frac{n+3}{2} \dots n]$: c, a, d, b

First, observe that $f(\bar{R}^i, b) = 0$ for all $i \in \{0, \dots, \frac{n-3}{2}\}$ because of PC-efficiency and PC-strategyproofness. Indeed, assume for contradiction that this is not true, i.e., $f(\bar{R}^i, b) > 0$ for some $i \in \{0, \dots, \frac{n-3}{2}\}$. Observations 3 and 4 and PC-efficiency require then that $f(\bar{R}^i, a) = 0$ and $f(\bar{R}^i, c) = 0$. Equivalently, $supp(f(\bar{R}^i)) \subseteq \{b, d\}$. However, this entails that one of the voters $i \in [\frac{n+3}{2} \dots n]$ can PC-manipulate. This follows by considering the subsequent preference profiles $\bar{R}^{i,j}$ for $j \in \{\frac{n+1}{2}, \dots, n\}$.

$\bar{R}^{i,j}$: $[1 \dots i]$: a, d, b, c $\quad$ $[i{+}1 \dots \frac{n-1}{2}]$: d, a, b, c
$\quad\frac{n+1}{2}$: b, d, c, a $\quad$ $[\frac{n+3}{2} \dots j]$: c, a, d, b
$\quad[j{+}1 \dots n]$: d, a, b, c

Note that $\bar{R}^{i,n} = \bar{R}^i$, and that $f(\bar{R}^{i,\frac{n+1}{2}}, d) = 1$ because more than half of the voters report d as their favorite choice. On the other hand, we claim that if $f(\bar{R}^{i,j}, b) > 0$, then $f(\bar{R}^{i,j-1}, b) > 0$. Observe for this that the voter types in $\bar{R}^i$ and $\bar{R}^{i,j}$ coincide, and thus PC-efficiency also requires that $f(\bar{R}^{i,j}, a) = f(\bar{R}^{i,j}, c) = 0$ if $f(\bar{R}^{i,j}, b) > 0$. Moreover, PC-strategyproofness requires that the deviating voter j PC-prefers $f(\bar{R}^{i,j})$ to $f(\bar{R}^{i,j-1})$. Now, if $f(\bar{R}^{i,j-1}, b) = 0$, deviating to $\bar{R}^{i,j-1}$ is a PC-manipulation for voter j because $supp(f(\bar{R}^{i,j}))$ consists of his worst two alternatives. Hence, $f(\bar{R}^{i,j}, b) > 0$ implies that $f(\bar{R}^{i,j-1}, b) > 0$ and, by repeatedly applying this argument, we infer that if $f(\bar{R}^i, b) > 0$, then $f(\bar{R}^{i,\frac{n+1}{2}}, b) > 0$. However, this contradicts $f(\bar{R}^{i,\frac{n+1}{2}}, d) = 1$, so we have that $f(\bar{R}^i, b) = 0$ for all $i \in \{0, \dots, \frac{n-3}{2}\}$.

In particular, this argument proves also for the profile $\bar{R}^0$ that $f(\bar{R}^0, b) = 0$. We derive next that $f(\bar{R}^0, d) = 1$. Consider for this the profile $\hat{R}$ derived from $\bar{R}^0$ by letting voter $\frac{n+1}{2}$ swap b and d. We have that $f(\hat{R}, d) = 1$ because of the absolute winner property. Hence, PC-strategyproofness requires that $f(\bar{R}^0, b) \geq f(\bar{R}^0, c) + f(\bar{R}^0, a)$. Now, since $f(\bar{R}^0, b) = 0$, this means that $f(\bar{R}^0, c) = f(\bar{R}^0, a) = 0$ and therefore $f(\bar{R}^0, d) = 1$. Based on this observation, we can now use Lemma 3 to derive that $f(R',a) \geq f(R',c)$. Consider for this an index $i \in \{1, \dots, \frac{n-1}{2}\}$ and suppose that $f(\bar{R}^{i-1}, a) \geq f(\bar{R}^{i-1}, c)$. The contraposition of Lemma 3 shows that $f(\bar{R}^i, a) \geq f(\bar{R}^i, c)$ because the deviating voter i prefers a to d to c. Finally, since $f(\bar{R}^0, c) = f(\bar{R}^0, a) = 0$, repeatedly applying the previous argument and noting that $R' = \bar{R}^{\frac{n-1}{2}}$, we obtain $f(R',a) \geq f(R',c)$. This establishes Claim 2. □

Lemma 1b). *Every PC-efficient SDS that satisfies the absolute winner property is PC-manipulable if $|N| \geq 8$ is even and $m \geq 4$.*

Proof. Consider an arbitrary electorate $N \in \mathcal{F}(\mathbb{N})$ with $n = |N| \geq 8$ even and assume for contradiction that there is an SDS f for $m \geq 4$ alternatives that satisfies PC-efficiency, PC-strategyproofness, and the absolute winner property on N. We focus on the case $m = 4$ because we can generalize the constructions to larger values of m by simply ranking the additional alternatives at the bottom. Then, PC-efficiency requires that these alternatives obtain probability 0 and they therefore do not affect our analysis. We derive a contradiction by analyzing the following two profiles.

R: $[1 \dots \frac{n}{2}-1]$: a, d, b, c $\quad \{\frac{n}{2}, \frac{n}{2}+1\}$: b, c, d, a
$\quad [\frac{n}{2}+2 \dots n]$: c, a, d, b

R': $[1 \dots \frac{n}{2}-1]$: a, d, b, c $\quad \{\frac{n}{2}, \frac{n}{2}+1\}$: b, d, c, a
$\quad [\frac{n}{2}+2 \dots n]$: c, a, d, b

In more detail, we show in Claims 1 and 2 that $f(R, a) = f(R, b) = f(R, c) = \frac{1}{3}$ and $f(R', a) = f(R', c) = f(R', d) = \frac{1}{3}$. These two claims are in conflict with PC-strategyproofness, as the following analysis shows. Let R'' denote the profile "between" R and R' in which voter $\frac{n}{2}$ reports b, d, c, a and voter $\frac{n}{2} + 1$ reports b, c, d, a. Moreover, let $p = f(R)$, $q = f(R')$, and $r = f(R'')$ denote the outcome of f in these profiles. PC-strategyproofness from R' to R'' results in the following inequality because $q(a) = q(c) = q(d) = \frac{1}{3}$.

$$q(b)\Big(r(d) + r(c) + r(a)\Big) + q(d)\Big(r(c) + r(a)\Big) + q(c)r(a)$$
$$\geq r(b)\Big(q(d) + q(c) + q(a)\Big) + r(d)\Big(q(c) + q(a)\Big) + r(c)q(a)$$
$$\iff \frac{1}{3}r(c) + \frac{2}{3}r(a) \geq r(b) + \frac{2}{3}r(d) + \frac{1}{3}r(c)$$
$$\iff r(a) \geq \frac{3}{2}r(b) + r(d)$$

Moreover, we can also use PC-strategyproofness from R'' to R and the fact that $p(a) = p(b) = p(c) = \frac{1}{3}$ to infer the following inequality.

$$r(b)\Big(p(d) + p(c) + p(a)\Big) + r(d)\Big(p(c) + p(a)\Big) + r(c)p(a)$$
$$\geq p(b)\Big(r(d) + r(c) + r(a)\Big) + p(d)\Big(r(c) + r(a)\Big) + p(c)r(a)$$
$$\iff \frac{2}{3}r(b) + \frac{2}{3}r(d) + \frac{1}{3}r(c) \geq \frac{1}{3}r(d) + \frac{1}{3}r(c) + \frac{2}{3}r(a)$$
$$\iff r(b) + \frac{1}{2}r(d) \geq r(a)$$

Combining these two inequalities entails that $r(b) + \frac{1}{2}r(d) \geq \frac{3}{2}r(b) + r(d)$, which is true only if $r(b) = r(d) = 0$. Moreover, the second inequality upper bounds $r(a)$, and thus $r(a) = 0$. This means that $f(R'', c) = r(c) = 1$. However, c is the worst alternative of the voters $i \in [1 \dots \frac{n}{2}-1]$ and PC-strategyproofness hence requires that these voters cannot affect the outcome by misreporting their preferences. On the other hand, if we let these voters one after another change their preference relation to b, d, a, c, we arrive at a profile $\hat{R}$, where b is top-ranked by more than half of the voters. Hence, the absolute winner property requires that $f(\hat{R}, b) = 1$, which is in conflict with the observation that these voters are not able to affect the outcome. This is the desired contradiction. Hence, to complete the proof of Lemma 1b), it remains to show the claims on $f(R)$ and $f(R')$.

Claim 1: $f(R, a) = f(R, b) = f(R, c) = \frac{1}{3}$

Just as in the case of odd n, our first goal is to prove that $f(R, d) = 0$. As the first step in proving this statement, we show that $f(R, a) < 1$. Hence, assume for contradiction that $f(R, a) = 1$, which means that the least preferred lottery of voters $\frac{n}{2}$ and $\frac{n}{2} + 1$ is chosen. Moreover, if both of these voters swap b and c, c is top-ranked by more than half of the voters, so the absolute winner property requires c to receive a probability of 1. This is, however, in conflict with PC-strategyproofness, which requires that these voters cannot affect the outcome. Hence, the assumption that $f(R, a) = 1$ must have been wrong, i.e., $f(R, a) < 1$.

Based on this insight, we show by contradiction that $f(R, c) > 0$, i.e., suppose that $f(R, c) = 0$. Next, consider the profiles R^1 and R^2 which are derived from R by letting voters $n - 1$ and n one after another swap a and c. Hence, a is top-ranked by $\frac{n}{2} + 1$ voters in R^2, which means that $f(R^2, a) = 1$ because of the absolute winner property. Now, using PC-strategyproofness from R^1 to R^2, we derive the following inequality since voter n's preference relation in R^1 is c, a, d, b.

$$f(R^1, c) \geq f(R^1, d) + f(R^1, b)$$

In particular, observe that this inequality requires that $f(R^1, a) = 1$ if $f(R^1, c) = 0$. However, in that case, voter $n - 1$ can PC-manipulate by deviating from R to R^1: since $f(R, c) = 0$ and $f(R, a) < 1$, it follows that $f(R^1) \succ^{SD}_{n-1} f(R)$ and therefore also $f(R^1) \succ^{PC}_{n-1} f(R)$. Hence, it must hold that $f(R^1, c) > 0$. Next, we use PC-strategyproofness from R to R^1 and vice versa to derive the following two inequalities, where $p = f(R)$ and $q = f(R^1)$.

$$p(c)\Big(q(a) + q(d) + q(b)\Big) + p(a)\Big(q(d) + q(b)\Big) + p(d)q(b)$$
$$\geq q(c)\Big(p(a) + p(d) + p(b)\Big) + q(a)\Big(p(d) + p(b)\Big) + q(d)p(b)$$

$$q(a)\Big(p(c) + p(d) + p(b)\Big) + q(c)\Big(p(d) + p(b)\Big) + q(d)p(b)$$
$$\geq p(a)\Big(q(c) + q(d) + q(b)\Big) + p(c)\Big(q(d) + q(b)\Big) + p(d)q(b)$$

Adding these two inequalities and cancelling common terms yields $p(c)q(a) \geq q(c)p(a)$. Since $p(c) = 0$ by assumption and $q(c) > 0$ because of our previous analysis, this inequality can only be true if $p(a) = 0$. Using the facts that $p(c) =$

$p(a) = 0$ and $q(c) \geq q(b) + q(d)$, we can therefore vastly simplify the first inequality.

$$\begin{aligned} p(d)q(b) &\geq q(c) + q(a) + q(d)p(b) \\ &\geq q(d) + q(b) + q(a) + q(d)p(b) \end{aligned}$$

It is easy to see that this inequality can only be true if $f(R, d) = p(d) = 1$. We now derive a contradiction to this insight. First, from R, let voters $\frac{n}{2}$ and $\frac{n}{2} + 1$ make d into their favorite alternative. This leads to the profile R^3 and PC-strategyproofness (one step at a time) requires that $f(R^3, d) = 1$. Moreover, note that d Pareto-dominates b in R^3. We let voters $n-1$ and n swap a and c to obtain the profile R^4. PC-efficiency still requires that $f(R^4, b) = 0$ as this alternative is still Pareto-dominated, and PC-strategyproofness requires in turn that $f(R^4, d) = 1$ as any other lottery with support $\{a, c, d\}$ is a PC-manipulation for voters $n-1$ and n. However, this contradicts the absolute winner property as $\frac{n}{2}+1$ voters report a as their favorite alternative in R^4, so it cannot be the case that $f(R, d) = 1$. Thus, no feasible outcome for $f(R)$ remains, which demonstrates that the assumption that $f(R, c) = 0$ is wrong.

As the next step, we can use PC-efficiency to derive that $f(R, d) = 0$. In more detail, Observation 2 shows that $f(R, d) = 0$ or $f(R, c) = 0$, and we excluded the latter case already.

Based on this insight, we show now that $f(R, c) \geq f(R, b) \geq f(R, a) \geq f(R, c)$, which implies that $f(R, a) = f(R, b) = f(R, c) = \frac{1}{3}$. For the first inequality, consider the profiles R^5 and R^6 which are derived from R by letting voters $n-1$ and n replace their preference relation with a, c, b, d one after another. Observation 2 and PC-efficiency imply that $f(R^5, d) = 0$ and the absolute winner property that $f(R^6, a) = 1$. In particular, $f(R, d) = f(R^5, d) = f(R^6, d) = 0$ and we thus can use Lemma 3 to derive that $f(R, c) \geq f(R, b)$ since $f(R^6, c) = f(R^6, b) = 0$.

Next, we show that $f(R, b) \geq f(R, a)$. Consider for this the profiles R^7 and R^8 derived from R by replacing the preference relations of voters $\frac{n}{2}$ and $\frac{n}{2}+1$ sequentially with c, a, b, d. First, observe that $f(R^8, c) = 1$ as all voters $i \in [\frac{n}{2} \dots n]$ report c as their best choice. Furthermore, Observation 2 shows that every lottery q with $q(d) > 0$ is PC-inefficient in R^7, which means that $f(R, d) = f(R^7, d) = f(R^8, d) = 0$. Applying Lemma 3 twice then shows that $f(R, b) \geq f(R, a)$ because $f(R^8, b) = f(R^8, a) = 0$.

Finally, we prove that $f(R, a) \geq f(R, c)$. Consider for this the profiles $\bar{R}^k$ for $k \in \{0, \dots, \frac{n}{2} - 1\}$ defined as follows.

$\bar{R}^k$: $[1 \dots k]$: a, d, b, c $\quad$ $[k{+}1 \dots \frac{n}{2}{-}1]$: b, a, d, c
$\{\frac{n}{2}, \frac{n}{2}{+}1\}$: b, c, d, a $\quad$ $[\frac{n}{2}{+}2 \dots n]$: c, a, d, b

Note that $R = \bar{R}^{\frac{n}{2}-1}$ and that $f(\bar{R}^0, b) = 1$ because of the absolute winner property. Moreover, $f(\bar{R}^k, d) = 0$ for all $k \in \{0, \dots, \frac{n}{2} - 2\}$ because of PC-efficiency: once again, Observation 2 shows that any lottery q with $q(d) > 0$ is PC-dominated by the lottery p with $p(a) = q(a) + \frac{\epsilon}{q(d)+q(a)}$, $p(b) = q(b) + \frac{\epsilon}{q(d)+q(b)}$, $p(c) = q(c)$, and $p(d) = q(d) - \frac{\epsilon}{q(d)+q(a)} - \frac{\epsilon}{q(d)+q(b)}$ (where $\epsilon > 0$ is so small that $p(d) \geq 0$). Hence, d has probability 0 for all of these profiles. We also know that $f(R, d) = 0$, so $f(\bar{R}^k, d) = 0$ for all $k \in \{0, \dots, \frac{n}{2} - 1\}$. By inductively applying Lemma 3, we derive that $f(R, a) \geq f(R, c)$ because $f(\bar{R}^0, a) = f(\bar{R}^0, c) = 0$.

This completes the proof of Claim 1.

Claim 2: $f(R', a) = f(R', c) = f(R', d) = \frac{1}{3}$

For proving this claim, we show as the first step that $f(R', b) = 0$. Note for this that an analogous argument as in Claim 1 proves that $f(R', c) > 0$. Indeed, we can let voters $n-1$ and n deviate to derive this result and, as these voters have the same preferences in R and R', the argument does not change. Based on this insight, PC-efficiency entails that $f(R', b) = 0$ because Observations 3 and 4 imply in combination that every lottery q with $q(b) > 0$ and $q(c) > 0$ fails PC-efficiency for R'. Hence, we have $f(R', b) = 0$.

Using this observation, we show next that $f(R', c) \geq f(R', d) \geq f(R', a) \geq f(R', c)$, which implies that all three alternatives receive a probability of $\frac{1}{3}$. First, we prove that $f(R', c) \geq f(R', d)$ by considering the profiles R^1 and R^2 derived from R' by replacing the preference relations of voters $n-1$ and n one after another with a, d, b, c. Note that $f(R^2, a) = 1$ because more than half of the voters rank a top in R^2. Consequently, PC-strategyproofness entails that $f(R^1, c) \geq f(R^1, d) + f(R^1, b)$. Hence, if $f(R^1, c) = 0$, then $f(R^1, a) = 1$ and another application of PC-strategyproofness shows that $f(R, c) \geq f(R, d)$ because $f(R, b) = 0$. On the other hand, if $f(R^1, c) > 0$, PC-efficiency requires that $f(R^1, b) = 0$: Observation 3 implies that $f(R^1, a) = 0$ if $f(R^1, b) > 0$ and, in turn, Observation 4 shows then that $f(R^1, c) = 0$. However, this contradicts $f(R^1, c) > 0$, so it must be that $f(R^1, b) = 0$. That is, we have $f(R', b) = f(R^1, b) = f(R^2, b) = 0$. Hence, we derive from a repeated application of Lemma 3 that $f(R', c) \geq f(R', d)$ because $f(R^2, c) = f(R^2, d) = 0$.

Next, we derive that $f(R', d) \geq f(R', a)$. To this end, consider the profiles R^3 and R^4 derived from R' by replacing the preference relations of voters $\frac{n}{2}$ and $\frac{n}{2} + 1$ sequentially with c, a, d, b. It follows from the absolute winner property that $f(R^4, c) = 1$ as more than half of the voters report c as their best alternative. Moreover, we can use the same construction as for R (in Claim 1) to derive that $f(R^3, c) > 0$. Indeed, voters n and $n+1$ have the same preference relations in R and R^3 and they also can make a into the absolute winner by swapping a and c in R^3. Analogously to R', it now follows from PC-efficiency and Observations 3 and 4 that $f(R^3, b) = 0$. Finally, a repeated application of Lemma 3 shows that $f(R', d) \geq f(R', a)$ because $f(R', b) = f(R^3, b) = f(R^4, b) = 0$ and $f(R^4, d) = f(R^4, a) = 0$.

It remains to show that $f(R', a) \geq f(R', c)$. Consider the sequence of profiles $\hat{R}^k$ for $k \in \{0, \dots, \frac{n}{2} - 1\}$.

$\hat{R}^k$: $[1 \dots k]$: a, d, b, c $\quad$ $[k{+}1 \dots \frac{n}{2}{-}1]$: d, a, b, c
$\{\frac{n}{2}, \frac{n}{2}{+}1\}$: b, d, c, a $\quad$ $[\frac{n}{2}{+}2 \dots n]$: c, a, d, b

First, note that $R' = \hat{R}^{\frac{n}{2}-1}$, which means that $f(\hat{R}^{\frac{n}{2}-1}, b) = 0$. Next, we show that the same is true for all other profiles $\hat{R}^k$ with $k \in \{0, \dots, \frac{n}{2} - 2\}$. To this end, note that Observation 3 shows that either $f(\hat{R}^k, b) = 0$ or $f(\hat{R}^k, a) = 0$. Moreover, if $f(\hat{R}^k, a) = 0$, then Observation 4 shows that $f(\hat{R}^k, b) = 0$ or $f(\hat{R}^k, c) = 0$. Now, assume for contradiction that $f(\hat{R}^k, b) > 0$ for a fixed k and hence $f(\hat{R}^k, a) = f(\hat{R}^k, c) = 0$. We proceed with a case distinction on k. First, assume that $\frac{n}{2} - 2 \geq k \geq 2$, which means that at least two voters in $[1 \dots k]$ top-rank a. For this case, we consider the profiles $\hat{R}^{k,j}$ for $j \in \{\frac{n}{2}+1, \dots, n\}$.

$\hat{R}^{k,j}$:	$[1 \dots k]$: a, d, b, c	$[k{+}1 \dots \frac{n}{2}{-}1]$: d, a, b, c
	$\{\frac{n}{2}, \frac{n}{2}{+}1\}$: b, d, c, a	$[\frac{n}{2}{+}2 \dots j]$: a, d, b, c
	$[j{+}1 \dots n]$: c, a, d, b	

It holds by definition that $\hat{R}^k = \hat{R}^{k,\frac{n}{2}+1}$. Moreover, PC-efficiency requires that either $f(\hat{R}^{k,j}, a) + f(\hat{R}^{k,j}, c) = 0$ or $f(\hat{R}^{k,j}, b) = 0$ for every $j \in \{\frac{n}{2}+1, \dots, n\}$ (see Observations 3 and 4). This implies for every j that if $f(\hat{R}^{k,j}, b) > 0$, then $f(\hat{R}^{k,j+1}, b) > 0$. In more detail, $f(\hat{R}^{k,j}, b) > 0$ requires that $f(\hat{R}^{k,j}, b) + f(\hat{R}^{k,j}, d) = 1$ because of PC-efficiency. Hence, if $f(\hat{R}^{k,j+1}, b) = 0$, voter $j+1$ can PC-manipulate by deviating from $\hat{R}^{k,j}$ to $\hat{R}^{k,j+1}$ since b and d are his worst two alternatives in $\hat{R}^{k,j}$. By repeatedly applying this argument, it follows that if $f(\hat{R}^k, b) > 0$, then $f(\hat{R}^{k,n}, b) > 0$. However, this is in conflict with the absolute winner property as $\frac{n}{2} - 1 + k$ voters top-rank a in $\hat{R}^{k,n}$. This proves that $f(\hat{R}^k, b) = 0$ for all $k \in \{2, \dots, \frac{n}{2} - 2\}$.

Note that the argument above fails if $k \leq 1$ as no more than $\frac{n}{2}$ voters top-rank a in $\hat{R}^{1,n}$. Hence, we investigate the case $k \leq 1$ separately and consider for this the profiles $\tilde{R}^{k,j}$ for $j \in \{\frac{n}{2}+1, \dots, n\}$.

$\tilde{R}^{k,j}$:	$[1 \dots k]$: a, d, b, c	$[k{+}1 \dots \frac{n}{2}{-}1]$: d, a, b, c
	$\{\frac{n}{2}, \frac{n}{2}{+}1\}$: b, d, c, a	$[\frac{n}{2}{+}2 \dots j]$: d, a, b, c
	$[j{+}1 \dots n]$: c, a, d, b	

It holds by definition that $\hat{R}^k = \tilde{R}^{k,\frac{n}{2}+1}$. Note that the profiles $\tilde{R}^{k,j}$ consist of the same preference relations as $\hat{R}^k$ and hence, Observations 3 and 4 show that $f(\tilde{R}^{k,j}, b) = 0$ or $f(\tilde{R}^{k,j}, a) = f(\tilde{R}^{k,j}, c) = 0$. (When $j = n$, even though the preference relation c, a, d, b is not present, a similar argument still holds.) Moreover, if $f(\tilde{R}^{k,j}, b) > 0$, then $f(\tilde{R}^{k,j+1}, b) > 0$. The reason for this is that if $f(\tilde{R}^{k,j}, b) > 0$, then $f(\tilde{R}^{k,j}, b) + f(\tilde{R}^{k,j}, d) = 1$. Hence, if $f(\tilde{R}^{k,j+1}, b) = 0$, voter $j+1$ can PC-manipulate by deviating from $\tilde{R}^{k,j}$ to $\tilde{R}^{k,j+1}$ as b and d are his least preferred alternatives in $\tilde{R}^{k,j}$. By repeatedly applying this argument, we derive that if $f(\hat{R}^k, b) > 0$, then $f(\tilde{R}^{k,n}, b) > 0$. However, this contradicts the absolute winner property as at least $(\frac{n}{2} - 1 - k) + (\frac{n}{2} - 1) \geq n - 3 > \frac{n}{2}$ voters top-rank d in $\tilde{R}^{k,n}$. (Here we use the assumption that $n \geq 8$.) Hence, we also have that $f(\hat{R}^k, b) = 0$ if $k \leq 1$.

As the last point, we prove that $f(\hat{R}^0, d) = 1$. Then, it follows from a repeated application of Lemma 3 that $f(R', a) \geq f(R', c)$ because $f(\hat{R}^0, a) = f(\hat{R}^0, c) = 0$. Thus, consider the profiles R^5 and R^6 derived from $\hat{R}^0$ by sequentially replacing the preference of voter $\frac{n}{2}$ and $\frac{n}{2} + 1$ with d, a, b, c. Note that an absolute majority top-ranks d in R^6, which means that $f(R^6, d) = 1$. Hence, PC-strategyproofness entails for R^5 that $f(R^5, b) \geq f(R^5, c) + f(R^5, a)$. If $f(R^5, b) = 0$, we derive then that $f(R^5, d) = 1$, and an application of PC-strategyproofness between $\hat{R}^0$ and R^5 shows that $f(\hat{R}^0, b) \geq f(\hat{R}^0, c) + f(\hat{R}^0, a)$. Since we already established that $f(\hat{R}^0, b) = 0$, this proves that $f(\hat{R}^0, d) = 1$. On the other hand, if $f(R^5, b) > 0$, PC-efficiency requires that $f(R^5, a) = f(R^5, c) = 0$ (see Observations 3 and 4). However, then voter n can PC-manipulate in R^5 by reporting d as his favorite alternative. Then, d must be chosen with probability 1 because it is top-ranked by $\frac{n}{2} + 1$ voters. However, voter n PC-prefers this lottery to $f(R^5)$ if $f(R^5, b) > 0$, which contradicts the PC-strategyproofness of f. Hence, it must indeed hold that $f(R^5, b) = 0$ and therefore also $f(\hat{R}^0, d) = 1$. Finally, as mentioned earlier in this paragraph, a repeated application of Lemma 3 shows now that $f(R', a) \geq f(R', c)$. Therefore, we have that $f(R', c) \geq f(R', d) \geq f(R', a) \geq f(R', c)$ and $f(R', b) = 0$, which implies that $f(R', a) = f(R', c) = f(R', d) = \frac{1}{3}$. ☐

Since Lemmas 1a) and 1b) combined are clearly equivalent to Lemma 1, this concludes the first step for the proof of Theorem 2. Next, we show that every anonymous, neutral, PC-efficient and PC-strategyproof SDS also satisfies the absolute winner property if there are $m \geq 3$ alternatives. This insight together with Lemma 1 immediately implies Theorem 2.

Lemma 4. *Assume $m \geq 3$. Every SDS that satisfies PC-efficiency, PC-strategyproofness, neutrality, and anonymity also satisfies the absolute winner property.*

Proof. Let f denote an SDS that satisfies anonymity, neutrality, PC-efficiency, and PC-strategyproofness for $m \geq 3$ alternatives. First, note that for electorates N with $n = |N| \leq 2$, the absolute winner property requires that an alternative is chosen with probability 1 if it is top-ranked by all voters. Since PC-efficiency also requires that an alternative is chosen with probability 1 if it is unanimously top-ranked, f satisfies the absolute winner property for such electorates. Hence, we suppose in the sequel that $n \geq 3$. To establish the lemma, we first prove an auxiliary claim that f satisfies the absolute winner property for an electorate $N \in \mathcal{F}(\mathbb{N})$ if there is a profile R and an alternative $a \in A$ such that $f(R, a) = 1$ even though $k = \lfloor \frac{n-1}{2} \rfloor$ voters top-rank another alternative b. Based on a case distinction with respect to the parity of n, we then show that such a profile exists for every electorate, which establishes the lemma.

Claim 1: f satisfies the absolute winner property for an electorate N if there are $R \in \mathcal{R}^N$ and $a \in A$ such that $f(R, a) = 1$ even though $k = \lfloor \frac{n-1}{2} \rfloor$ voters top-rank another alternative $b \in A \setminus \{a\}$.

Consider an arbitrary electorate $N \in \mathcal{F}(\mathbb{N})$ with $n = |N| \geq 3$ and assume that such a profile R exists. Due to anonymity, we may assume that the voters $i \in [1 \dots k]$ top-rank alternative b in R. Our goal is to show that $f(R', a) = 1$ for all preference profiles $R' \in \mathcal{R}^N$ in which the voters $i \in [k{+}1 \dots n]$ report a as their best alternative. Since anonymity allows us to reorder the voters and neutrality to exchange the roles of alternatives, this claim is equivalent to showing that f satisfies the absolute winner property.

As the first step, we consider the sequence of preference profiles $\bar{R}^i$ for $i \in \{k+1, \dots, n+1\}$ such that $\bar{R}^{k+1} = R$ and $\bar{R}^{i+1}$ is derived from $\bar{R}^i$ by replacing the preference relation of voter i with $a, b, *$. Here, the $*$ symbol indicates that all missing alternatives can be ordered arbitrarily below a and b. It is easy to see that for all $i \in \{k+1, \dots, n\}$, if $f(\bar{R}^i, a) = 1$, then $f(\bar{R}^{i+1}, a) = 1$ because of PC-strategyproofness; otherwise voter i can manipulate by reverting this step. Hence, this process results in a profile $R^1 = \bar{R}^{n+1}$ such that $f(R^1, a) = 1$ and b Pareto-dominates all alternatives $x \in A \setminus \{a, b\}$; the latter is true since all voters $i \in [1 \dots k]$ top-rank b and the voters $i \in [k+1 \dots n]$ prefer b to all alternatives but a. As the next step, we focus on the profiles $\hat{R}^i$ for $i \in \{1, \dots, k+1\}$ which are constructed as follows: $\hat{R}^1 = R^1$ and $\hat{R}^{i+1}$ evolves out of $\hat{R}^i$ by replacing the preference relation of voter i with $b, *, a$. First, note that b Pareto-dominates all alternatives but a in every profile $\hat{R}^i$ and thus $f(\hat{R}^i, x) = 0$ for all $i \in \{1, \dots, k+1\}$ and $x \in A \setminus \{a, b\}$. This implies that $supp(f(\hat{R}^i)) \subseteq \{a, b\}$ and hence, Lemma 2 proves that $f(\hat{R}^i) = f(\hat{R}^{i+1})$ for all $i \in \{1, \dots, k\}$ as the deviating voter i never reorders a and b. Consequently, this process results in a profile $R^2 = \hat{R}^{k+1}$ such that $f(R^2, a) = 1$, all voters $i \in [1 \dots k]$ bottom-rank a, and all voters $i \in [k+1 \dots n]$ top-rank a.

Finally, this insight implies that $f(R', a) = 1$ for all preference profiles R' in which the voters $i \in [k+1 \dots n]$ report a as their best alternative. The reason for this is that, because of PC-strategyproofness, the voters $i \in [1 \dots k]$ can change their preference relation arbitrarily without affecting the outcome as any other lottery would give rise to a manipulation, and similarly the voters $i \in [k+1 \dots n]$ can reorder the alternatives $A \setminus \{a\}$ arbitrarily without affecting the resulting lottery. This proves that, if there is a profile R such that $f(R, a) = 1$ even though $k = \lfloor \frac{n-1}{2} \rfloor$ voters top-rank b, f satisfies the absolute winner property.

Claim 2: For every electorate N with $n = |N| \geq 4$ even, there are $R \in \mathcal{R}^N$ and $a \in A$ such that $f(R, a) = 1$ even though k voters prefer b the most.

Assume that N consists of an even number of voters $n \geq 4$. We will focus on the case of three alternatives, as we can extend the argument to more alternatives by appending the additional alternatives below $\{a, b, c\}$ in the preference relation of every voter. Then, PC-efficiency guarantees that these alternatives receive probability 0 and they thus do not affect our analysis. We consider the profile R.

R: $\quad 1$: $a, b, c \qquad 2$: a, c, b
$\quad [3 \dots \frac{n}{2}+1]$: $b, a, c \qquad [\frac{n}{2}+2 \dots n]$: c, a, b

First, note that anonymity and neutrality require that $f(R, b) = f(R, c)$ because these alternatives are symmetric in R. Furthermore, PC-efficiency requires that $f(R, b) = f(R, c) = 0$ because every lottery q with $q(b) = q(c) > 0$ is PC-inefficient for R (see Observation 1). Hence, it follows from PC-efficiency that $f(R, a) = 1$. Since the k voters in $i \in [3 \dots \frac{n}{2}+1]$ prefer b the most, all conditions of Claim 1 are met and f satisfies the absolute winner property for N.

Claim 3: For every electorate N with $n = |N| \geq 3$ odd, there are $R \in \mathcal{R}^N$ and $a \in A$ such that $f(R, a) = 1$ even though k voters prefer b the most.

Assume now that N consists of an odd number of voters $n \geq 3$. Analogously to the previous case, we focus again only on three alternatives a, b, c as we can extend the argument to more alternatives by bottom-ranking the additional alternatives. We start our analysis by considering the profiles R^1 and R^2 described below.

R^1: $\quad [1 \dots \frac{n-1}{2}]$: $b, a, c \qquad \frac{n+1}{2}$: a, b, c
$\quad [\frac{n+3}{2} \dots n]$: c, a, b

R^2: $\quad [1 \dots \frac{n-1}{2}]$: $b, a, c \qquad \frac{n+1}{2}$: a, c, b
$\quad [\frac{n+3}{2} \dots n]$: c, a, b

First, note that anonymity and neutrality imply that $f(R^1, a) = f(R^2, a)$, $f(R^1, b) = f(R^2, c)$, and $f(R^1, c) = f(R^2, b)$. Furthermore, PC-efficiency shows that $f(R^1, b) = f(R^2, c) = 0$ or $f(R^1, c) = f(R^2, b) = 0$ (see Observation 1 for the details). Subsequently, we show that $f(R^1, b) = f(R^1, c) = 0$ must be true, which means that $f(R^1, a) = 1$.

Assume for contradiction that $f(R^1, c) = f(R^2, b) > 0$. Then, our previous observation implies that $f(R^2, c) = f(R^1, b) = 0$. However, this means that voter $\frac{n+1}{2}$ can manipulate by deviating from R^1 to R^2 because he PC-prefers $f(R^2)$ to $f(R^1)$ (he even SD-prefers $f(R^2)$ to $f(R^1)$). Hence, f is PC-manipulable if $f(R^1, c) > 0$, contradicting our assumptions.

As the second case, assume for contradiction that $f(R^1, b) = f(R^2, c) > 0$ (note that this is *not* symmetric to the case studied in the previous paragraph) and consider the following sequence of preference profiles $\bar{R}^i$ for $i \in \{0, \dots, \frac{n-1}{2}\}$.

$\bar{R}^i$: $\quad [1 \dots \frac{n-1}{2}]$: $b, a, c \qquad \frac{n+1}{2}$: a, b, c
$\quad [\frac{n+3}{2} \dots n-i]$: $c, a, b \qquad [n-i+1 \dots n]$ a, c, b

First, note that $R^1 = \bar{R}^0$ and that Observation 1 shows for all profiles $\bar{R}^i$ that $f(\bar{R}^i, b) = 0$ or $f(\bar{R}^i, c) = 0$ because of PC-efficiency. Moreover, PC-strategyproofness and PC-efficiency imply that if $f(\bar{R}^i, b) > 0$, then $f(\bar{R}^{i+1}) = f(\bar{R}^i)$. The reason for this is that if $f(\bar{R}^i, b) > 0$, then $f(\bar{R}^i, c) = 0$ because of PC-efficiency. This means that every lottery with $f(\bar{R}^{i+1}, b) = 0$ is a PC-manipulation for the deviating voter $n - i$ as he even SD-prefers $f(\bar{R}^{i+1})$ to $f(\bar{R}^i)$. Hence, $f(\bar{R}^{i+1}, b) > 0$, and we can now use PC-efficiency to derive that $f(\bar{R}^{i+1}, c) = f(\bar{R}^i, c) = 0$. Finally, Lemma 2 implies that $f(\bar{R}^{i+1}) = f(\bar{R}^i)$. As a consequence, this sequence ends at a profile $R^3 = \bar{R}^{\frac{n-1}{2}}$ with $f(R^3) = f(R^1)$.

Next, consider the profile R^4 which is derived from R^3 by swapping b and c in the preference relation of voter $\frac{n+1}{2}$.

R^4: $\quad [1 \dots \frac{n-1}{2}]$: $b, a, c \qquad \frac{n+1}{2}$: a, c, b
$\quad [\frac{n+3}{2} \dots n]$: a, c, b

Since a Pareto-dominates c in R^4, it follows that $f(R^4, c) = 0$. Hence, we can use again Lemma 2 to conclude that $f(R^4) = f(R^3) = f(R^1)$.

As the last step, consider the sequence of profiles $\hat{R}^i$ for $i \in \{0, \dots, \frac{n-1}{2}\}$, which leads from R^4 to R^2.

$\hat{R}^i$: $[1 \dots \frac{n-1}{2}]$: b, a, c $\quad$ $\frac{n+1}{2}$: a, c, b $\quad$ $[\frac{n+3}{2} \dots n-i]$: a, c, b $\quad$ $[n-i+1 \dots n]$ c, a, b

First, observe that $\hat{R}^0 = R^4$ and $\hat{R}^{\frac{n-1}{2}} = R^2$. Moreover, PC-efficiency requires again for every profile $\hat{R}^i$ that either $f(\hat{R}^i, b) = 0$ or $f(\hat{R}^i, c) = 0$. Even more, since $f(\hat{R}^0, c) = 0$ and $f(\hat{R}^{\frac{n-1}{2}}, b) = 0$, there is at least one index i such that $f(\hat{R}^i, c) = f(\hat{R}^{i+1}, b) = 0$. Let $i^* \in \{0, \dots, \frac{n-3}{2}\}$ denote the smallest such index, which means that $f(\hat{R}^i, c) = 0$ for all $i \in \{0, \dots, i^*\}$. Therefore, we can again use Lemma 2 to conclude that $f(\hat{R}^{i^*}) = f(\hat{R}^0) = f(R^1)$, which means in particular that $f(\hat{R}^{i^*}, b) = f(R^1, b) > 0$. Now, if $f(\hat{R}^{i^*+1}, a) \geq f(R^{i^*}, a)$, voter $n - i^*$ can PC-manipulate by deviating from $\hat{R}^{i^*}$ to $\hat{R}^{i^*+1}$. This follows as voter $n - i^*$, whose preference is a, c, b in R^{i^*}, SD-prefers (and therefore also PC-prefers) $f(\hat{R}^{i^*+1})$ to $f(\hat{R}^{i^*})$ in this case. Hence, PC-strategyproofness requires that $f(\hat{R}^{i^*+1}, a) < f(\hat{R}^{i^*}, a)$. Since $f(\hat{R}^{i^*+1}, b) = f(\hat{R}^{i^*}, c) = 0$, this implies that $f(\hat{R}^{i^*+1}, c) > f(\hat{R}^{i^*}, b) = f(R^1, b)$.

Finally, we prove that $f(\hat{R}^{i+1}, c) \geq f(\hat{R}^i, c)$ for all $i > i^*$. Assume for contradiction that there is an index j where this is not the case. Then, there is also a minimal index $j^* > i^*$ such that $f(\hat{R}^{j^*+1}, c) < f(\hat{R}^{j^*}, c)$. In particular, it follows from the minimality of j^* that $f(\hat{R}^{j^*}, c) \geq f(\hat{R}^{i^*+1}, c) > 0$ and PC-efficiency then shows that $f(\hat{R}^{j^*}, b) = 0$. Now, note that voter $n - j^*$'s preference relation in $\hat{R}^{j^*+1}$ is c, a, b. Hence, if $f(R^{j^*+1}, c) = 0$, he clearly PC-prefers $f(\hat{R}^{j^*})$ to $f(\hat{R}^{j^*+1})$. This means that $f(\hat{R}^{j^*+1}, c) > 0$ and consequently $f(\hat{R}^{j^*+1}, b) = 0$ because of PC-efficiency. However, then voter j^* still PC-prefers $f(\hat{R}^{j^*})$ to $f(\hat{R}^{j^*+1})$ because $f(\hat{R}^{j^*+1}, c) < f(\hat{R}^{j^*}, c)$. Hence, voter j^* can either way PC-manipulate by deviating from $\hat{R}^{j^*+1}$ to $\hat{R}^{j^*}$. This contradicts the PC-strategyproofness of f, which proves $f(\hat{R}^{i+1}, c) \geq f(\hat{R}^i, c)$ for all $i > i^*$. In particular, this implies that $f(R^2, c) \geq f(\hat{R}^{i^*+1}, c) > f(\hat{R}^{i^*}, b) = f(R^1, b)$ because $R^2 = \hat{R}^{\frac{n-1}{2}}$. However, this observation contradicts anonymity and neutrality between R^2 and R^1, and thus, the assumption that $f(R^1, b) > 0$ must be wrong. It follows that $f(R^1, b) = f(R^1, c) = 0$, and so $f(R^1, a) = 1$. We infer now from Claim 1 that f satisfies the absolute winner property for N as R^1 satisfies all requirements. □

Finally, we use Lemmas 1 and 4 to prove Theorem 2.

Theorem 2. *Every anonymous and neutral SDS that satisfies PC-efficiency is PC-manipulable if $|N| \geq 3$, $|N| \notin \{4, 6\}$, and $m \geq 4$.*

Proof. Assume for contradiction that there is a PC-strategyproof, PC-efficient, anonymous, and neutral SDS f for $m \geq 4$ alternatives. First, note that Lemma 4 implies that f satisfies the absolute winner property for all electorates. However, we show in Lemma 1a) and Lemma 1b) that every PC-efficient SDS that satisfies the absolute winner property fails PC-strategyproofness for all electorates N with $|N| \notin \{4, 6\}$ and $|N| \geq 3$. This contradicts our initial assumption and thus establishes the theorem. □

A.2 Computations for PC-efficiency

In this subsection, we prove the claims on PC-(in)efficiency used in Appendix A.1. In more detail, we analyze when voters PC-prefer a lottery p to q. It is straightforward to verify the claims on PC-efficiency in Appendix A.1 based on our observations because they imply PC-dominance for large classes of lotteries. Recall for this section that $p \succ_i^{SD} q$ implies $p \succ_i^{PC} q$ for all lotteries $p, q \in \Delta(A)$ and all preference relations $\succ_i$. We start with the claims on PC-efficiency in the proof of Lemma 4.

Observation 1. *Let $\succ^1 = a, b, c$, $\succ^2 = a, c, b$, $\succ^3 = b, a, c$, and $\succ^4 = c, a, b$. Moreover, consider lotteries p and q on $\{a, b, c\}$ with $q(b) > 0$, $q(c) > 0$, $p(a) = q(a) + \frac{\epsilon}{q(a)+q(b)} + \frac{\epsilon}{q(a)+q(c)}$, $p(b) = q(b) - \frac{\epsilon}{q(a)+q(b)}$, and $p(c) = q(c) - \frac{\epsilon}{q(a)+q(c)}$ (where $\epsilon > 0$ is sufficiently small so that $p(b) \geq 0$ and $p(c) \geq 0$).*

All voters i with $\succ_i \in \{\succ^1, \succ^2\}$ strictly PC-prefer p to q and all voters i with $\succ_i \in \{\succ^3, \succ^4\}$ weakly PC-prefer p to q.

Proof. Consider a lottery q with $q(b) > 0$ and $q(c) > 0$ and let p be defined as in the statement. First, note that voters whose preference relation is $\succ^1$ or $\succ^2$ strictly SD-prefer p to q because $p(a) > q(a)$, $p(a) + p(b) > q(a) + q(b)$, and $p(a) + p(c) > q(a) + q(c)$. Since $\succ_i^{SD} \subseteq \succ_i^{PC}$, it follows that these voters also strictly PC-prefer p to q.

Next, consider a voter with preference relation $\succ^3 = b, a, c$. As the following calculation shows, such a voter is indifferent between p and q.

$$\begin{aligned}
&p(b)\Big(q(a) + q(c)\Big) + p(a)q(c)\\
&= \Big(q(b) - \frac{\epsilon}{q(a)+q(b)}\Big)\Big(q(a) + q(c)\Big)\\
&\quad + \Big(q(a) + \frac{\epsilon}{q(a)+q(b)} + \frac{\epsilon}{q(a)+q(c)}\Big)q(c)\\
&= q(b)\Big(q(a) + q(c)\Big) + q(a)q(c)\\
&\quad - \frac{\epsilon q(a)}{q(a)+q(b)} + \frac{\epsilon q(c)}{q(a)+q(c)}\\
&= q(b)\Big(q(a) + q(c)\Big) + q(a)q(c)\\
&\quad - \Big(\epsilon - \frac{\epsilon q(b)}{q(a)+q(b)}\Big) + \Big(\epsilon - \frac{\epsilon q(a)}{q(a)+q(c)}\Big)\\
&= q(b)\Big(q(a) + \frac{\epsilon}{q(a)+q(b)} + q(c)\Big)\\
&\quad + q(a)\Big(q(c) - \frac{\epsilon}{q(a)+q(c)}\Big)\\
&= q(b)\Big(p(a) + p(c)\Big) + q(a)p(c)
\end{aligned}$$

A symmetric calculation shows that voters with preference relation $\succ^4$ are indifferent between p and q. □

Next, we discuss the PC-efficiency applications used in Claim 1 of Lemmas 1a) and 1b).

Observation 2. *Let* $\succ^1 = a, d, b, c$, $\succ^2 = b, c, d, a$, $\succ^3 = c, a, d, b$, $\succ^4 = b, a, d, c$, $\succ^5 = a, c, b, d$, *and* $\succ^6 = c, a, b, d$. *Moreover, consider lotteries* p *and* q *on* $\{a, b, c, d\}$ *such that* $q(d) > 0$, $p(a) = q(a) + \frac{\epsilon}{q(d)+q(a)}$, $p(b) = q(b) + \frac{\epsilon}{q(d)+q(b)}$, $p(c) = q(c)$, *and* $p(d) = q(d) - \frac{\epsilon}{q(d)+q(a)} - \frac{\epsilon}{q(d)+q(b)}$ *(where* $\epsilon > 0$ *is so small that* $p(d) \geq 0$*).*

All voters i *with* $\succ_i \in \{\succ^4, \succ^5, \succ^6\}$ *strictly* PC*-prefer* p *to* q*, and voters* i *with* $\succ_i \in \{\succ^1, \succ^2, \succ^3\}$ *weakly* PC*-prefer* p *to* q*. Furthermore, if* $q(c) > 0$*, then voters* i *with* $\succ_i = \succ^2$ *strictly* PC*-prefer* p *to* q*.*

Proof. Consider an arbitrary lottery q with $q(d) > 0$ and let p be defined as in the statement. First, consider voters i with $\succ_i = \succ^4$. These voters strictly SD-prefer p to q because $p(b) > q(b)$, $p(b) + p(a) > q(b) + q(a)$, and $p(b) + p(a) + p(d) = q(b) + q(a) + q(d)$. Analogous computations also show that voters i with $\succ_i \in \{\succ^5, \succ^6\}$ strictly SD-prefer p to q. Hence, it follows that voters with one of these three preference relations strictly PC-prefer p to q.

Next, consider a voter i with preference relation $\succ_i = \succ^1$. The following chain of equations shows that he is indifferent between p and q.

$$\begin{aligned}
&p(a)\Big(q(d) + q(b) + q(c)\Big) + p(d)\Big(q(b) + q(c)\Big) + p(b)q(c) \\
&= \Big(q(a) + \frac{\epsilon}{q(d) + q(a)}\Big)\Big(q(d) + q(b) + q(c)\Big) \\
&\quad + \Big(q(d) - \frac{\epsilon}{q(d) + q(a)} - \frac{\epsilon}{q(d) + q(b)}\Big)\Big(q(b) + q(c)\Big) \\
&\quad + \Big(q(b) + \frac{\epsilon}{q(d) + q(b)}\Big)q(c) \\
&= q(a)\Big(q(d) + q(b) + q(c)\Big) + q(d)\Big(q(b) + q(c)\Big) \\
&\quad + q(b)q(c) + \frac{\epsilon q(d)}{q(d) + q(a)} - \frac{\epsilon q(b)}{q(d) + q(b)} \\
&= q(a)\Big(q(d) + q(b) + q(c)\Big) + q(d)\Big(q(b) + q(c)\Big) \\
&\quad + q(b)q(c) + \Big(\epsilon - \frac{\epsilon q(a)}{q(d) + q(a)}\Big) - \Big(\epsilon - \frac{\epsilon q(d)}{q(d) + q(b)}\Big) \\
&= q(a)\Big(q(d) - \frac{\epsilon}{q(d) + q(a)} + q(b) + q(c)\Big) \\
&\quad + q(d)\Big(q(b) + \frac{\epsilon}{q(d) + q(b)} + q(c)\Big) + q(b)q(c) \\
&= q(a)\Big(p(d) + p(b) + p(c)\Big) + q(d)\Big(p(b) + p(c)\Big) \\
&\quad + q(b)p(c)
\end{aligned}$$

Based on an analogous chain, we also derive that the voters i with $\succ_i = \succ^3$ are indifferent between p and q.

$$\begin{aligned}
&p(c)\Big(q(a) + q(d) + q(b)\Big) + p(a)\Big(q(d) + q(b)\Big) + p(d)q(b) \\
&= q(c)\Big(q(a) + q(d) + q(b)\Big) \\
&\quad + \Big(q(a) + \frac{\epsilon}{q(d) + q(a)}\Big)\Big(q(d) + q(b)\Big) \\
&\quad + \Big(q(d) - \frac{\epsilon}{q(d) + q(a)} - \frac{\epsilon}{q(d) + q(b)}\Big)q(b) \\
&= q(c)\Big(q(a) + q(d) + q(b)\Big) + q(a)\Big(q(d) + q(b)\Big) \\
&\quad + q(d)q(b) + \frac{\epsilon q(d)}{q(d) + q(a)} - \frac{\epsilon q(b)}{q(d) + q(b)} \\
&= q(c)\Big(q(a) + q(d) + q(b)\Big) + q(a)\Big(q(d) + q(b)\Big) \\
&\quad + q(d)q(b) + \Big(\epsilon - \frac{\epsilon q(a)}{q(d) + q(a)}\Big) - \Big(\epsilon - \frac{\epsilon q(d)}{q(d) + q(b)}\Big) \\
&= q(c)\Big(q(a) + q(d) + q(b)\Big) \\
&\quad + q(a)\Big(q(d) - \frac{\epsilon}{q(d) + q(a)} + q(b)\Big) \\
&\quad + q(d)\Big(q(b) + \frac{\epsilon}{q(d) + q(b)}\Big) \\
&= q(c)\Big(p(a) + p(d) + p(b)\Big) + q(a)\Big(p(d) + p(b)\Big) \\
&\quad + q(d)p(b)
\end{aligned}$$

Finally, we derive that voters i with $\succ_i = \succ^2$ weakly PC-prefer p to q. Note that the inequality is strict if $q(c) > 0$ because we replace $\frac{\epsilon q(c)}{q(d)+q(b)}$ with $\frac{-\epsilon q(c)}{q(d)+q(b)}$.

$$\begin{aligned}
&p(b)\Big(q(c) + q(d) + q(a)\Big) + p(c)\Big(q(d) + q(a)\Big) + p(d)q(a) \\
&= \Big(q(b) + \frac{\epsilon}{q(d) + q(b)}\Big)\Big(q(c) + q(d) + q(a)\Big) \\
&\quad + q(c)\Big(q(d) + q(a)\Big) \\
&\quad + \Big(q(d) - \frac{\epsilon}{q(d) + q(a)} - \frac{\epsilon}{q(d) + q(b)}\Big)q(a) \\
&= q(b)\Big(q(c) + q(d) + q(a)\Big) + q(c)\Big(q(d) + q(a)\Big) \\
&\quad + q(d)q(a) + \frac{\epsilon q(c)}{q(d) + q(b)} + \frac{\epsilon q(d)}{q(d) + q(b)} - \frac{\epsilon q(a)}{q(d) + q(a)} \\
&\geq q(b)\Big(q(c) + q(d) + q(a)\Big) + q(c)\Big(q(d) + q(a)\Big) \\
&\quad + q(d)q(a) - \frac{\epsilon q(c)}{q(d) + q(b)} + \Big(\epsilon - \frac{\epsilon q(b)}{q(d) + q(b)}\Big) \\
&\quad - \Big(\epsilon - \frac{\epsilon q(d)}{q(d) + q(a)}\Big) \\
&= q(b)\Big(q(c) + q(d) - \frac{\epsilon}{q(d) + q(b)} + q(a)\Big) \\
&\quad + q(c)\Big(q(d) - \frac{\epsilon}{q(d) + q(b)} + q(a)\Big) \\
&\quad + q(d)\Big(q(a) + \frac{\epsilon}{q(d) + q(a)}\Big) \\
&= q(b)\Big(p(c) + p(d) + p(a)\Big) + q(c)\Big(p(d) + p(a)\Big) \\
&\quad + q(d)p(a)
\end{aligned}$$

This completes the proof. ◻

We now discuss two more applications of PC-efficiency used in Claim 2 of Lemmas 1a) and 1b).

Observation 3. *Let* $\succ^1 = a, d, b, c$, $\succ^2 = b, d, c, a$, $\succ^3 = c, a, d, b$, *and* $\succ^4 = d, a, b, c$. *Moreover, consider lotteries* p

and q on $\{a,b,c,d\}$ such that $q(b)>0$, $q(a)>0$, $p(a)=q(a)-\frac{\epsilon}{q(d)+q(a)}$, $p(b)=q(b)-\frac{\epsilon}{q(d)+q(b)}$, $p(c)=q(c)$, and $p(d)=q(d)+\frac{\epsilon}{q(d)+q(a)}+\frac{\epsilon}{q(d)+q(b)}$ (where $\epsilon>0$ is so small that $p(a)\geq 0$, $p(b)\geq 0$).

All voters i with $\succ_i=\succ^4$ strictly PC-prefer p to q, and all voters i with $\succ_i\in\{\succ^1,\succ^2,\succ^3\}$ weakly PC-prefer p to q. Furthermore, if $q(c)>0$, then voters i with $\succ_i=\succ^2$ strictly PC-prefer p to q.

Proof. Consider a lottery q with $q(b)>0$ and $q(a)>0$ and let p be defined as in the statement. First, note that voters i with $\succ_i=\succ^4$ strictly SD-prefer p to q because $p(d)>q(d)$, $p(d)+p(a)>q(d)+q(a)$, and $p(d)+p(a)+p(b)=q(d)+q(a)+q(b)$. Since $\succ_i^{SD}\subseteq\succ_i^{PC}$, these voters also strictly PC-prefer p to q.

Next, we prove that every voter i with $\succ_i=\succ^1$ is indifferent between p and q.

$$\begin{aligned}
&p(a)\Big(q(d)+q(b)+q(c)\Big)+p(d)\Big(q(b)+q(c)\Big)+p(b)q(c)\\
&=\Big(q(a)-\frac{\epsilon}{q(d)+q(a)}\Big)\Big(q(d)+q(b)+q(c)\Big)\\
&\quad+\Big(q(d)+\frac{\epsilon}{q(d)+q(a)}+\frac{\epsilon}{q(d)+q(b)}\Big)\Big(q(b)+q(c)\Big)\\
&\quad+\Big(q(b)-\frac{\epsilon}{q(d)+q(b)}\Big)q(c)\\
&=q(a)\Big(q(d)+q(b)+q(c)\Big)+q(d)\Big(q(b)+q(c)\Big)\\
&\quad+q(b)q(c)-\frac{\epsilon q(d)}{q(d)+q(a)}+\frac{\epsilon q(b)}{q(d)+q(b)}\\
&=q(a)\Big(q(d)+q(b)+q(c)\Big)+q(d)\Big(q(b)+q(c)\Big)\\
&\quad+q(b)q(c)-\Big(\epsilon-\frac{\epsilon q(a)}{q(d)+q(a)}\Big)+\Big(\epsilon-\frac{\epsilon q(d)}{q(d)+q(b)}\Big)\\
&=q(a)\Big(q(d)+\frac{\epsilon}{q(d)+q(a)}+q(b)+q(c)\Big)\\
&\quad+q(d)\Big(q(b)-\frac{\epsilon}{q(d)+q(b)}+q(c)\Big)+q(b)q(c)\\
&=q(a)\Big(p(d)+p(b)+p(c)\Big)+q(d)\Big(p(b)+p(c)\Big)\\
&\quad+q(b)p(c)
\end{aligned}$$

Similarly, the following chain of equations proves that voters i with $\succ_i=\succ^3$ are indifferent between p and q.

$$\begin{aligned}
&p(c)\Big(q(a)+q(d)+q(b)\Big)+p(a)\Big(q(d)+q(b)\Big)+p(d)q(b)\\
&=q(c)\Big(q(a)+q(d)+q(b)\Big)\\
&\quad+\Big(q(a)-\frac{\epsilon}{q(d)+q(a)}\Big)\Big(q(d)+q(b)\Big)\\
&\quad+\Big(q(d)+\frac{\epsilon}{q(d)+q(a)}+\frac{\epsilon}{q(d)+q(b)}\Big)q(b)\\
&=q(c)\Big(q(a)+q(d)+q(b)\Big)+q(a)\Big(q(d)+q(b)\Big)\\
&\quad+q(d)q(b)-\frac{\epsilon q(d)}{q(d)+q(a)}+\frac{\epsilon q(b)}{q(d)+q(b)}\\
&=q(c)\Big(q(a)+q(d)+q(b)\Big)+q(a)\Big(q(d)+q(b)\Big)\\
&\quad+q(d)q(b)-\Big(\epsilon-\frac{\epsilon q(a)}{q(d)+q(a)}\Big)+\Big(\epsilon-\frac{\epsilon q(d)}{q(d)+q(b)}\Big)\\
&=q(c)\Big(q(a)+q(d)+q(b)\Big)\\
&\quad+q(a)\Big(q(d)+\frac{\epsilon}{q(d)+q(a)}+q(b)\Big)\\
&\quad+q(d)\Big(q(b)-\frac{\epsilon}{q(d)+q(b)}\Big)\\
&=q(c)\Big(p(a)+p(d)+p(b)\Big)+q(a)\Big(p(d)+p(b)\Big)\\
&\quad+q(d)p(b)
\end{aligned}$$

Finally, we show that voters i with $\succ_i=\succ^2$ weakly PC-prefer p to q. Note that the preference is strict if $q(c)>0$ because we replace $\frac{\epsilon q(c)}{q(d)+q(a)}$ with $\frac{-\epsilon q(c)}{q(d)+q(a)}$ in the third step.

$$\begin{aligned}
&p(b)\Big(q(d)+q(c)+q(a)\Big)+p(d)\Big(q(c)+q(a)\Big)+p(c)q(a)\\
&=\Big(q(b)-\frac{\epsilon}{q(d)+q(b)}\Big)\Big(q(d)+q(c)+q(a)\Big)\\
&\quad+\Big(q(d)+\frac{\epsilon}{q(d)+q(a)}+\frac{\epsilon}{q(d)+q(b)}\Big)\Big(q(c)+q(a)\Big)\\
&\quad+q(c)q(a)\\
&=q(b)\Big(q(d)+q(c)+q(a)\Big)+q(d)\Big(q(c)+q(a)\Big)\\
&\quad+q(c)q(a)-\frac{\epsilon q(d)}{q(d)+q(b)}+\frac{\epsilon q(a)}{q(d)+q(a)}+\frac{\epsilon q(c)}{q(d)+q(a)}\\
&\geq q(b)\Big(q(d)+q(c)+q(a)\Big)+q(d)\Big(q(c)+q(a)\Big)\\
&\quad+q(c)q(a)-\Big(\epsilon-\frac{\epsilon q(b)}{q(d)+q(b)}\Big)+\Big(\epsilon-\frac{\epsilon q(d)}{q(d)+q(a)}\Big)\\
&\quad-\frac{\epsilon q(c)}{q(d)+q(a)}\\
&=q(b)\Big(q(d)+\frac{\epsilon}{q(d)+q(b)}+q(c)+q(a)\Big)\\
&\quad+q(d)\Big(q(c)+q(a)-\frac{\epsilon}{q(d)+q(a)}\Big)\\
&\quad+q(c)\Big(q(a)-\frac{\epsilon}{q(d)+q(a)}\Big)\\
&=q(b)\Big(p(d)+p(c)+p(a)\Big)+q(d)\Big(p(c)+p(a)\Big)\\
&\quad+q(c)p(a)
\end{aligned}$$

This completes the proof. □

Observation 4. *Let $\succ^1=a,d,b,c$, $\succ^2=b,d,c,a$, $\succ^3=c,a,d,b$, and $\succ^4=d,a,b,c$. Moreover, consider lotteries p and q on $\{a,b,c,d\}$ such that $q(a)=0$, $q(b)>0$, $q(c)>0$, $p(a)=0$, $p(b)=q(b)-\frac{\epsilon}{q(d)+q(b)}$, $p(c)=q(c)-\frac{\epsilon}{q(d)+q(c)}$, and $p(d)=q(d)+\frac{\epsilon}{q(d)+q(b)}+\frac{\epsilon}{q(d)+q(c)}$ (where $\epsilon>0$ is so small that $p(b)\geq 0$ and $p(c)\geq 0$).*

All voters i with $\succ_i\in\{\succ^1,\succ^4\}$ strictly PC-prefer p to q, and all voters i with $\succ_i\in\{\succ^2,\succ^3\}$ weakly PC-prefer p to q.

Proof. Let q and p be defined as in the statement. Since $p(a) = q(a) = 0$, we ignore this alternative in all subsequent calculations. First, observe that voters i with $\succ_i = \succ^1$ strictly SD-prefer p to q: $p(a) = q(a)$, $p(a) + p(d) > q(a) + q(d)$, and $p(a) + p(d) + p(b) > q(a) + q(d) + q(b)$. Similarly, voters i with $\succ_i = \succ^4$ also strictly SD-prefer p to q since $p(d) > q(d)$, $p(d) + p(a) > q(d) + p(a)$, and $p(d) + p(a) + p(b) > q(d) + q(a) + q(b)$. Hence, such voters also strictly PC-prefer p to q.

Next, we prove that voters i with $\succ_i = \succ^2$ are indifferent between p and q.

$$\begin{aligned}
&p(b)\Big(q(d) + q(c)\Big) + p(d)q(c)\\
&= \Big(q(b) - \frac{\epsilon}{q(d)+q(b)}\Big)\Big(q(d) + q(c)\Big)\\
&\quad + \Big(q(d) + \frac{\epsilon}{q(d)+q(b)} + \frac{\epsilon}{q(d)+q(c)}\Big)q(c)\\
&= q(b)\Big(q(d) + q(c)\Big) + q(d)q(c) - \frac{\epsilon q(d)}{q(d)+q(b)}\\
&\quad + \frac{\epsilon q(c)}{q(d)+q(c)}\\
&= q(b)\Big(q(d) + q(c)\Big) + q(d)q(c) - \Big(\epsilon - \frac{\epsilon q(b)}{q(d)+q(b)}\Big)\\
&\quad + \Big(\epsilon - \frac{\epsilon q(d)}{q(d)+q(c)}\Big)\\
&= q(b)\Big(q(d) + \frac{\epsilon}{q(d)+q(b)} + q(c)\Big)\\
&\quad + q(d)\Big(q(c) - \frac{\epsilon}{q(d)+q(c)}\Big)\\
&= q(b)\Big(p(d) + p(c)\Big) + q(d)p(c)
\end{aligned}$$

Finally, a symmetric argument shows that voters i with $\succ_i = \succ^3$ are also indifferent between p and q.

$$\begin{aligned}
&p(c)\Big(q(d) + q(b)\Big) + p(d)q(b)\\
&= \Big(q(c) - \frac{\epsilon}{q(d)+q(c)}\Big)\Big(q(d) + q(b)\Big)\\
&\quad + \Big(q(d) + \frac{\epsilon}{q(d)+q(b)} + \frac{\epsilon}{q(d)+q(c)}\Big)q(b)\\
&= q(c)\Big(q(d) + q(b)\Big) + q(d)q(b) - \frac{\epsilon q(d)}{q(d)+q(c)}\\
&\quad + \frac{\epsilon q(b)}{q(d)+q(b)}\\
&= q(c)\Big(q(d) + q(b)\Big) + q(d)q(b) - \Big(\epsilon - \frac{\epsilon q(c)}{q(d)+q(c)}\Big)\\
&\quad + \Big(\epsilon - \frac{\epsilon q(d)}{q(d)+q(b)}\Big)\\
&= q(c)\Big(q(d) + \frac{\epsilon}{q(d)+q(c)} + q(b)\Big)\\
&\quad + q(d)\Big(q(b) - \frac{\epsilon}{q(d)+q(b)}\Big)\\
&= q(c)\Big(p(d) + p(b)\Big) + q(d)p(b)
\end{aligned}$$

This completes the proof. □

A.3 Proofs of Propositions 1 and 2

In this subsection, we prove Propositions 1 and 2, which show that the impossibilities for $m \geq 4$ in Theorems 1 to 3 turn into possibilities when $m = 3$. First, we investigate PC-efficiency in more detail because both propositions involve this axiom.

Lemma 5. *Consider a profile $R \in \mathcal{R}^{\mathcal{F}(\mathbb{N})}$ on three alternatives $A = \{a, b, c\}$. A lottery p is PC-efficient for R if it satisfies the following conditions.*

- *$p(x) = 0$ if x is Pareto-dominated in R.*
- *For an alternative $x \in A$ that is never bottom-ranked and at least once top-ranked in R, there is $y \in A \setminus \{x\}$ with $p(y) = 0$.*
- *For an alternative $x \in A$ that is never top-ranked and at least once bottom-ranked in R, $p(x) = 0$.*

Proof. Consider an arbitrary electorate $N \in \mathcal{F}(\mathbb{N})$ and let $R \in \mathcal{R}^N$ denote a profile. We proceed with a case distinction with respect to R.

Case 1: There is an alternative $x \in A$ that is top-ranked by all voters in R.
As the first case, we suppose that an alternative, say a, is top-ranked by all voters. In this case, both b and c are Pareto-dominated and thus, the first condition only allows for the lottery p with $p(a) = 1$. Furthermore, it is straightforward to see that this lottery is PC-efficient as this is the most preferred lottery of every voter. Because of this observation, we suppose for all subsequent cases that at least two alternatives are top-ranked.

Case 2: There is an alternative $x \in A$ that is never bottom-ranked and at least once top-ranked in R.
Assume without loss of generality that a is an alternative that is never bottom-ranked and at least once top-ranked in R. Moreover, we suppose that a second alternative b is top-ranked as otherwise Case 1 applies. We prove that every lottery p with $supp(p) \subseteq \{a, b\}$ is PC-efficient and show later why this suffices for this case.

Let p denote a lottery with $supp(p) \subseteq \{a, b\}$ and assume for contradiction that there is a lottery q that PC-dominates p. First, note that p is not PC-dominated by a lottery q with $q(b) < p(b)$. This follows as there is a voter $i \in N$ with $\succ_i = b, a, c$; such a voter exists because b is at least once top-ranked and a never bottom-ranked. Hence, p can only be PC-dominated by a lottery q with $q(b) \geq p(b)$. Next, recall that there is a voter $j \in N$ who prefers a the most in R. This voter strictly SD-prefers p to all lotteries q with $q(a) < p(a)$ and $q(b) \geq p(b)$. In more detail, this follows by a case distinction: if b is the second best alternative of voter j, we have that $p(a) > q(a)$ and $p(a) + p(b) = 1 \geq q(a) + q(b)$. On the other hand, if b is the least preferred alternative of voter j, we have that $p(a) > q(a)$ and $p(a) + p(c) = 1 - p(b) \geq 1 - q(b) = q(a) + q(c)$. Hence, we must also have that $q(a) \geq p(a)$. Since $p(c) = 0$, $q(a) \geq p(a)$ and $q(b) \geq p(b)$ imply that $q = p$, which contradicts that p is PC-dominated by q. Hence, a lottery p with $supp(p) \subseteq \{a, b\}$ is PC-efficient.

If c is never top-ranked, then it is Pareto-dominated by a, and the first condition implies that $p(c) = 0$; hence, by the previous paragraph, p is PC-efficient. On the other hand, suppose that c is top-ranked by some voter. It follows from a symmetric argument as above that any lottery p with $supp(p) \subseteq \{a, c\}$ is PC-efficient. Since a is never bottom-ranked and at least once top-ranked, by the second condition, $supp(p) \subseteq \{a, b\}$ or $supp(p) \subseteq \{a, c\}$. Hence, p is PC-efficient.

Case 3: There is an alternative $x \in A$ that is never top-ranked and at least once bottom-ranked in R.

Assume without loss of generality that a is an alternative that is never top-ranked but at least once bottom-ranked. Now, if there is an alternative x that is top-ranked by all voters, Case 1 applies and we hence suppose that both b and c are top-ranked at least once. It suffices to show that every lottery p with $supp(p) \subseteq \{b, c\}$ is PC-efficient for R, because our third condition requires that $p(a) = 0$.

Suppose for contradiction that a lottery p with $supp(p) \subseteq \{b, c\}$ is PC-dominated by another lottery q in R. Since a is bottom-ranked by at least one voter, there is a voter with preference b, c, a or c, b, a. We focus on the first case as the other one is symmetric. First, note that p is not PC-dominated by a lottery q with $q(b) < p(b)$ because the voter i with $\succ_i = b, c, a$ strictly SD-prefers p to q. This is true since $p(b) > q(b)$ and $p(b) + p(c) = 1 \geq q(b) + q(c)$. Moreover, no lottery q with $q(b) \geq p(b)$ and $q(c) < p(c)$ PC-dominates p because there is a voter who top-ranks c. If this voter second-ranks b, he strictly SD-prefers p to q because $p(c) > q(c)$ and $p(c)+p(b) = 1 \geq q(c)+q(b)$. On the other hand, if this voter bottom-ranks b, then we have that $p(c) + p(a) = 1 - p(b) \geq 1 - q(b) = q(c) + q(a)$. Hence, it must hold that $q(b) \geq p(b)$ and $q(c) \geq p(c)$. However, since $p(a) = 0$, this means that $q = p$, a contradiction. Thus, p is PC-efficient.

Case 4: There is an alternative $x \in A$ that is second-ranked by all voters in R.

Without loss of generality, suppose that a is second-ranked by all voters $i \in N$. Once again, if all voters top-rank the same alternative, Case 1 applies and we thus assume that both b and c are top-ranked at least once. This means that we have two types of voters: some report b, a, c and the others c, a, b. In this case, we will show that all lotteries are PC-efficient. Consider an arbitrary lottery p and suppose for contradiction that p is PC-dominated by another lottery q. This means that the following inequalities must be true as both types of voters PC-prefer q to p.

$$q(b)p(a) + q(b)p(c) + q(a)p(c) \geq p(b)q(a) + p(b)q(c) + p(a)q(c)$$

$$q(c)p(a) + q(c)p(b) + q(a)p(b) \geq p(c)q(a) + p(c)q(b) + p(a)q(b)$$

It is easy to see that both inequalities can be true at the same time only if

$$q(b)p(a) + q(b)p(c) + q(a)p(c) = p(b)q(a) + p(b)q(c) + p(a)q(c)$$

However, in this case there is no strict PC-preference, contradicting that p PC-dominates q. Hence, every lottery is PC-efficient if a is second-ranked by all voters and both b and c are top-ranked at least once.

Case 5: Every alternative is bottom-ranked and top-ranked at least once each in R.

As the last case, suppose that every alternative is both top-ranked and bottom-ranked at least once. In this case, we claim that every lottery p is PC-efficient for R. Assume for contradiction that this is not the case, which means that there is a lottery p that is PC-dominated by another lottery q in R. Without loss of generality, suppose that $p(a) > q(a)$ since the argument is the same for each alternative. By assumption, there is a voter i who top-ranks a and we assume that c is his worst alternative because the argument for b is symmetric. Now if $q(c) \geq p(c)$, it is straightforward that voter i SD-prefers p to q, contradicting that q PC-dominates p. Hence, it must be the case that $p(c) > q(c)$. This means also that $q(b) > p(b)$. Finally, there is a voter who bottom-ranks b. Now, it is again easy to verify that this voter strictly SD-prefers p to q, contradicting that q PC-dominates p. Hence, the assumption that p is not PC-efficient is wrong, i.e., all lotteries are PC-efficient in this case. □

Next, we use Lemma 5 to prove Proposition 1. Recall the definition of f^1 (where $CW(R)$ is the set of Condorcet winners in R and $WCW(R)$ the set of weak Condorcet winners).

$$f^1(R) = \begin{cases} [x:1] & \text{if } CW(R) = \{x\} \\ [x:\frac{1}{2}; y:\frac{1}{2}] & \text{if } WCW(R) = \{x, y\} \\ [x:\frac{3}{5}; y:\frac{1}{5}; z:\frac{1}{5}] & \text{if } WCW(R) = \{x\} \\ [x:\frac{1}{3}; y:\frac{1}{3}; z:\frac{1}{3}] & \text{otherwise} \end{cases}$$

Proposition 1. *For $m = 3$, f^1 is the only SDS that satisfies PC-efficiency, PC-strategyproofness, neutrality, anonymity, and cancellation.*

Proof. The proposition consists of two claims: on the one hand, we need to show that f^1 satisfies all axioms of the proposition, and on the other hand, that f^1 is the only SDS satisfying these axioms. We consider both claims separately and start by showing that f^1 satisfies all axioms of the proposition.

Claim 1: f^1 satisfies anonymity, neutrality, cancellation, PC-efficiency and PC-strategyproofness.

First, note that f^1 satisfies cancellation because adding two voters with inverse preferences does not affect whether an alternative is a (weak) Condorcet winner. Furthermore, the definition of f^1 immediately shows that it is anonymous and neutral.

For proving that f^1 is PC-efficient, we consider an arbitrary preference profile $R \in \mathcal{R}^{\mathcal{F}(\mathbb{N})}$. We use a case distinction with respect to R to prove that f^1 is PC-efficient:

- First, assume that an alternative x is unanimously top-ranked in R. Then, $f^1(R, x) = 1$ because $CW(R) = \{x\}$ and Lemma 5 shows that this is PC-efficient.
- Next, suppose that there is an alternative x that is never top-ranked and at least once bottom-ranked in R. It follows for the other two alternatives y and z that either one

of them is top-ranked by more than half of the voters, or both y and z are top-ranked by exactly half of the voters. In the former case there is a Condorcet winner while in the latter case y and z are weak Condorcet winners, but x is not. Hence, $f(R^1, x) = 0$ and Lemma 5 implies that f^1 is PC-efficient in this case.

- As the third case, we assume that there is an alternative x that is never bottom-ranked and at least once top-ranked in R. Now, if x is the Condorcet winner in R, then $f^1(R, x) = 1$, which is PC-efficient by Lemma 5. If $CW(R) \neq \{x\}$, there is another alternative y that is top-ranked by at least half of the voters. In particular, this means that z is top-ranked by less than half of the voters and thus $g_R(x, z) > 0$. Hence, either $CW(R) = \{y\}$ if strictly more than half of the voters top-rank it, or $WCW(R) = \{x, y\}$ if exactly half of the voters top-rank y. In all cases, $f^1(R, z) = 0$ and Lemma 5 implies that this outcome is PC-efficient.
- In all remaining cases, Lemma 5 implies that all lotteries are PC-efficient. In particular, note here that if an alternative x is Pareto-dominated, there is either an alternative y that is unanimously top-ranked, or never bottom-ranked and at least once top-ranked. Both cases have already been discussed.

Finally, we need to show that f^1 is PC-strategyproof. Assume for contradiction that this is not the case. Then, there are an electorate N, two preference profiles $R, R' \in \mathcal{R}^N$, and a voter $i \in N$ such that $f^1(R') \succ_i^{PC} f^1(R)$ and $R_{-i} = R'_{-i}$. Subsequently, we discuss a case distinction with respect to the definition to f^1. In more detail, we have for both R and R' five different options: there is a Condorcet winner (CW), or there is no Condorcet winner but $k \in \{0, 1, 2, 3\}$ weak Condorcet winners ($kWCW$). We label the cases with a shorthand notation: for instance, $CW \to 1WCW$ is the case where there is a Condorcet winner in R and a single weak Condorcet winner in R'.

Before discussing the case distinction, we present a symmetry argument to reduce the number of cases: if f^1 is PC-strategyproof in a case $X \to Y$, it is also PC-strategyproof for the case $Y \to X$. Assume this is not the case, i.e., that voter i can PC-manipulate in the case $Y \to X$. Then, we can add two new voters i^* and j^* to R' such that voter i^* has the same preference relation as voter i in R and voter j^* has exactly the inverse preference relation. We call this new profile R^1 and note that $f^1(R^1) = f^1(R')$ because adding two voters with inverse preferences does not affect the majority margins. Next, consider the profile R^2 derived from R^1 by assigning voter j^* the inverse preference relation of voter i in R'. It is not difficult to see that $f^1(R^2) = f^1(R)$ since $g_R = g_{R^2}$. In particular, this follows as the preference relations of voters j^* and i cancel each other out, while voter i^* reports the same preference relation as voter i in R. Finally, since voter i PC-prefers $f^1(R')$ to $f^1(R)$ according to $\succ_i$ and voter j^* has exactly the inverse preference relation in R^1, it follows that voter j^* PC-prefers $f^1(R^2) = f^1(R)$ to $f^1(R^1) = f^1(R')$ in R^1. Hence, if f^1 is PC-manipulable in the case $Y \to X$, it is also PC-manipulable in the case $X \to Y$. Or conversely, if it is PC-strategyproof for $X \to Y$, it is also PC-strategyproof for $Y \to X$.

Next, we start the case distinction and first suppose that R and R' are defined by an odd number of voters n, which means that there are no weak Condorcet winners. Hence, there are only four possible types of manipulations, one of which is covered by our symmetry argument. We leave it to the reader to verify the claims on when a voter PC-prefers $f^1(R)$ to $f^1(R')$.

- $CW \to CW$: Suppose that a is the Condorcet winner in R. If a is also the Condorcet winner in R', then $f^1(R) = f^1(R')$ and deviating from R to R' is no PC-manipulation. On the other hand, if another alternative b is the Condorcet winner in R', we must have $a \succ_i b$ in R. Since $f^1(R, a) = f^1(R', b) = 1$, this is again no PC-manipulation.
- $CW \to 0WCW$: Suppose that a is the Condorcet winner in R, and there is no Condorcet winner in R'. This means that voter i reinforces an alternative b against a, i.e., a is at least his second best alternative. Since $f^1(R, a) = 1$ and $f^1(R', x) = \frac{1}{3}$ for all $x \in A$, this proves that deviating from R to R' is no PC-manipulation.
- $0WCW \to 0WCW$: We have $f^1(R) = f^1(R')$ in this case, which contradicts that voter i can PC-manipulate.

Since these are all the possible manipulations, f^1 is PC-strategyproof if there are an odd number of voters n. Next, we focus on the case where n is even.

- First, suppose that a is the Condorcet winner in R, which means that $f^1(R, a) = 1$. The assumption that n is even implies that $g_R(a, b) \geq 2$ and $g_R(a, c) \geq 2$ and thus a is at least a weak Condorcet winner in R'.
  - $CW \to CW$: It is not possible to change the Condorcet winner from a to another alternative x since $g_R(a, x) \geq 2$. Hence, f^1 is PC-strategyproof in this case.
  - $CW \to 0WCW$: As discussed above, this case is not possible since a is at least a weak Condorcet winner in R'.
  - $CW \to 1WCW$: Suppose that there is a single weak Condorcet winner in R'. Our previous observation implies that a is this weak Condorcet winner. Hence, voter i has to weaken a against another alternative b to derive R' because a is not a strict Condorcet winner in R'. This means that a is not his worst alternative in R. Since $f^1(R', a) = \frac{3}{5}$ and $f^1(R', b) = f(R^1, c) = \frac{1}{5}$, this insight shows that f^1 is PC-strategyproof in this case.
  - $CW \to 2WCW$: Suppose that there are two weak Condorcet winners in R', one of which is a. Voter i needs to reinforce the second weak Condorcet winner b against a to ensure that $g_{R'}(a, b) = 0$, which means that $a \succ_i b$ in R. Since $f^1(R', a) = f^1(R', b) = \frac{1}{2}$, this is no PC-manipulation for voter i.
  - $CW \to 3WCW$: Suppose that every alternative is a weak Condorcet winner in R'. In particular, this

means that voter i reinforces both b and c against a. Hence, a is his favorite alternative in R and thus this is no PC-manipulation.

- As the second case, suppose that there is no weak Condorcet winner in R, which means that $f^1(R,x) = \frac{1}{3}$ for all $x \in A$. Since every alternative is involved with a negative majority margin, we assume without loss of generality that $g_R(a,b) \geq 2$, $g_R(b,c) \geq 2$, and $g_R(c,a) \geq 2$.
  - $0WCW \to 0WCW$: This case is no manipulation as $f^1(R) = f^1(R')$.
  - $0WCW \to 1WCW$: Suppose without loss of generality that a is the only weak Condorcet winner in R'. This means that voter i reinforces a against c, i.e., $c \succ_i a$. Since $f^1(R',a) = \frac{3}{5}$ and $f^1(R',b) = f^1(R',c) = \frac{1}{5}$, f^1 is PC-strategyproof in this case.
  - $0WCW \to 2WCW$: Suppose without loss of generality that a and b are the weak Condorcet winners in R'. Hence, voter i needs to reinforce a against c and b against a, which means that $c \succ_i a \succ_i b$ in R. Since $f^1(R',a) = f^1(R',b) = \frac{1}{2}$, this is no PC-manipulation for voter i.
  - $0WCW \to 3WCW$: This case is no PC-manipulation because $f^1(R) = f^1(R')$.
- As the third case, suppose that a is the only weak Condorcet winner in R, which means that $f^1(R,a) = \frac{3}{5}$ and $f^1(R,b) = f^1(R,c) = \frac{1}{5}$. Since b and c are no weak Condorcet winners, they are affected with a negative majority margin. In particular, we assume without loss of generality that $g_R(a,b) \geq 2$, $g_R(b,c) \geq 2$, and $g_R(a,c) = 0$.
  - $1WCW \to 1WCW$: Suppose there is a single weak Condorcet winner in R'. If this is a, then $f^1(R) = f^1(R')$ and deviating from R to R' is no PC-manipulation. If $WCW(R') = \{b\}$, voter i needs to reinforce b and c against a to ensure that all majority margins of b are non-negative and that one majority margin of a is negative. This means that a is voter i's favorite alternative in R and, since $f^1(R',b) = \frac{3}{5}$ and $f^1(R',a) = f^1(R',c) = \frac{1}{5}$, this is no PC-manipulation for voter i. Finally, suppose that c is the weak Condorcet winner in R'. Then, voter i needs to reinforce c against a and b, i.e., c is voter i's least preferred alternative in R. Since $f^1(R',c) = \frac{3}{5}$ and $f^1(R',a) = f^1(R',b) = \frac{1}{5}$, this is again no PC-manipulation for voter i.
  - $1WCW \to 2WCW$: Suppose there are two weak Condorcet winners in R'. Assume first that $CWC(R') = \{a,b\}$. Hence, voter i reinforces b against a, which means that $a \succ_i b$. Since $f^1(R',a) = f^1(R',b) = \frac{1}{2}$, deviating from R to R' is no PC-manipulation for voter i. Next, suppose $CWC(R') = \{a,c\}$. This means that voter i reinforces c against b, i.e., $b \succ_i c$ and $c \succ'_i b$. This is sufficient to prove that voter i cannot PC-manipulate because $f^1(R',a) = f^1(R',c) = \frac{1}{2}$. Finally, suppose that $CWC(R') = \{b,c\}$. Then, voter i reinforces b against a and c against b, i.e., $a \succ_i b \succ_i c$. Since $f^1(R',b) = f^1(R',c) = \frac{1}{2}$, f^1 is also PC-strategyproof in this case.
  - $1WCW \to 3WCW$: Finally, suppose that every alternative is a weak Condorcet winner in R'. This means that voter i reinforces b against a and c against b, i.e., he deviates from $a \succ_i b \succ_i c$ to $c \succ'_i b \succ'_i a$. However, then $g_{R'}(c,a) > 0$, contradicting that a is a weak Condorcet winner. Hence, this case is not possible.
- As the fourth case, we assume that $CWC(R) = \{a,b\}$, which means that $f^1(R,a) = f^1(R,b) = \frac{1}{2}$. Moreover, we have that $g_R(a,b) = 0$ and that at least one majority margin of c is negative. Without loss of generality, we suppose that $g_R(a,c) \geq 2$ and $g_R(b,c) \geq 0$.
  - $2WCW \to 2WCW$: Suppose that there are two weak Condorcet winners in R. If these are a and b, then $f^1(R) = f^1(R')$ and voter i cannot PC-manipulate. Hence, c is a weak Condorcet winner, which means that voter i reinforces c against a, i.e., $a \succ_i c$. If b and c are the weak Condorcet winners in R', the fact that $a \succ_i c$ is enough to prove that $f^1(R) \succ_i^{PC} f^1(R')$ because $f^1(R',b) = f^1(R',c) = \frac{1}{2}$. Hence, suppose that a and c are the weak Condorcet winners in R'. This means that voter i also needs to reinforce a or c against b to ensure that one of the majority margins of b is negative. In particular, we infer that $b \succ_i c$ since we already know that $a \succ_i c$, and hence, this is again no PC-manipulation for voter i.
  - $2WCW \to 3WCW$: Suppose that all three alternatives are weak Condorcet winners in R'. This means that voter i reinforced c against a, which implies that $a \succ_i c$. Since $f^1(R',x) = \frac{1}{3}$ for all $x \in A$, it can be checked that this is no PC-manipulation.
- As the last case, suppose that all alternatives are weak Condorcet winners in R. This means that $f^1(R,x) = \frac{1}{3}$ for all $x \in A$. We only need to check the case $3WCW \to 3WCW$ because all other cases are covered by our symmetry argument. For this case, we have $f^1(R) = f^1(R')$, and f^1 is thus PC-strategyproof.

Since we have enumerated all possible deviations and none leads to a manipulation, f^1 is indeed PC-strategyproof.

Claim 2: f^1 is the only SDS that satisfies anonymity, neutrality, cancellation, PC-efficiency, and PC-strategyproofness.

Consider an arbitrary SDS f for $m = 3$ alternatives that satisfies all given axioms. We show that $f(R) = f^1(R)$ for all profiles $R \in \mathcal{R}^{\mathcal{F}(\mathbb{N})}$, which proves this claim. For this, we name the six possible preference relations $\succ_1 = a,b,c$, $\succ_2 = c,b,a$, $\succ_3 = b,c,a$, $\succ_4 = a,c,b$, $\succ_5 = c,a,b$, and $\succ_6 = b,a,c$. Moreover, given a profile R, let n_i denote the number of voters who report preference relation $\succ_i$ in R. Using this notation, we can describe the majority margins of R as follows.

$$g_R(a,b) = (n_1 - n_2) - (n_3 - n_4) + (n_5 - n_6)$$
$$g_R(b,c) = (n_1 - n_2) + (n_3 - n_4) - (n_5 - n_6)$$

$$g_R(c,a) = -(n_1 - n_2) + (n_3 - n_4) + (n_5 - n_6)$$

It is not difficult to derive from these equations that

$$n_1 = \frac{g_R(a,b) + g_R(b,c)}{2} + n_2$$
$$n_3 = \frac{g_R(b,c) + g_R(c,a)}{2} + n_4$$
$$n_5 = \frac{g_R(c,a) + g_R(a,b)}{2} + n_6.$$

Next, consider an arbitrary preference profile R. Based on cancellation, we can use the above equations to remove pairs of voters with inverse preferences from R until $n_{2k} = 0$ or $n_{2k-1} = 0$ for all $k \in \{1, 2, 3\}$. Unless all majority margins are 0, this leads to a minimal profile R', which we consider in the subsequent case distinction. Note that the removal of voters with inverse preferences does not affect the majority margins and therefore also not the (weak) Condorcet winners. In particular, this means that $f^1(R) = f^1(R')$. Analogously, cancellation proves for f that $f(R) = f(R')$. Hence, we will consider multiple cases depending on the structure of R' and prove that $f(R) = f(R') = f^1(R') = f^1(R)$ in every case. On the other hand, if all majority margins are 0, we need a separate argument, which we discuss in our first case below. Taken together, our cases imply that $f(R) = f^1(R)$ for every profile R.

Case 2.1: $g_R(a,b) = g_R(b,c) = g_R(c,a) = 0$.

First, suppose that $g_R(a,b) = g_R(b,c) = g_R(c,a) = 0$, which means that all three alternatives are weak Condorcet winners in R. Moreover, our equations show that $n_1 = n_2$, $n_3 = n_4$, and $n_5 = n_6$. Let n^* denote the maximum among all n_i. Using cancellation, we can add pairs of voters with inverse preferences such that $n_k = n^*$ for every $k \in \{1, \dots, 6\}$. Moreover, cancellation implies that $f(R) = f(R'')$ for the new profile R''. Finally, all alternatives are symmetric to each other in R'' since all preference relations appear equally often. Hence, anonymity and neutrality require that $f(R'', x) = \frac{1}{3}$ for all $x \in A$, which means that $f(R) = f(R'') = f^1(R)$ for all profiles R with three weak Condorcet winners.

Case 2.2: An alternative x is top-ranked by more than half of the voters in R'.

As the second case, suppose that R' is well-defined and that an alternative x is top-ranked by more than half of the voters in this profile. Then, it holds that $f(R', x) = 1$ because Lemma 4 implies that f satisfies the absolute winner property. Since x is the Condorcet winner in R', it holds that $f(R') = f^1(R')$.

Case 2.3: Two alternatives are top-ranked by exactly half of the voters in R'.

Next, suppose that not all majority margins in R are 0 and that two alternatives are top-ranked by exactly half of the voters in R'. Without loss of generality, suppose that these alternatives are a and b. Then, a and b are weak Condorcet winners. Moreover, c is not a weak Condorcet winner in R' since we supposed that not all majority margins are 0, which implies that there is a voter who ranks c last. Due to symmetry, we assume that this voter's preference relation is a, b, c. Now, if there is a voter with preference relation a, c, b in R', then the last possible preference relation is b, a, c; otherwise, R' is not minimal. Hence, a Pareto-dominates c in R'. Similarly, if there is no voter with a, c, b, all voters prefers b to c and c is again Pareto-dominated. Therefore, it follows in both cases that $f(R', c) = 0$ because of PC-efficiency. Moreover, we can let the voters with a, c, b and b, c, a (if any) push down c. Then, c stays Pareto-dominated and therefore still receives probability 0 from f. Hence, Lemma 2 shows that the probability of a and b does not change during these steps. Finally, this process results in a profile R'' in which half of the voters report a, b, c and the other half b, a, c. Anonymity, neutrality, and PC-efficiency imply for this profile that $f(R'', a) = f(R'', b) = \frac{1}{2}$. Hence, we have that $f(R') = f(R'') = f^1(R')$ because a and b are the only weak Condorcet winners in R'.

Case 2.4: Each alternative is top-ranked at least once and one alternative is top-ranked by exactly half of the voters in R'.

Next, suppose that an alternative is top-ranked by exactly half of the voters and the other two alternatives are top-ranked at least once. Without loss of generality, assume that there is a voter with preference relation a, b, c in R'. Since c is top-ranked by a voter, there is also a voter with preference relation c, a, b; note for this that no voter can report c, b, a in R' because of the minimality of R'. By an analogous argument, we also derive that there is a voter with preference relation b, c, a. In summary, we have that $n_1 > 0$, $n_3 > 0$, $n_5 > 0$, and $n_2 = n_4 = n_6 = 0$. Moreover, one alternative is top-ranked by half of the voters; suppose without loss of generality that this alternative is a. Hence, $n_1 = n_3 + n_5$. We prove that $f(R', a) = \frac{3}{5}$ and $f(R', b) = f(R', c) = \frac{1}{5}$ by considering the following preference profiles, where $l = n_1 + n_3$.

R^{1,n_3,n_5}:	$[1 \dots n_1]$: a, b, c	$[n_1{+}1 \dots l]$: b, c, a
	$[l{+}1 \dots n{-}1]$: c, a, b	n: c, b, a
R^{2,n_3,n_5}:	$[1 \dots n_1]$: a, b, c	$[n_1{+}1 \dots l]$: b, c, a
	$[l{+}1 \dots n]$: c, a, b	
R^{3,n_3,n_5}:	$[1 \dots n_1]$: a, b, c	$[n_1{+}1 \dots l{-}1]$: b, c, a
	l: c, b, a	$[l{+}1 \dots n]$: c, a, b

Anonymity implies that that $f(R') = f(R^{2,n_3,n_5})$. Hence, our goal is to show that $f(R^{2,n_3,n_5}, a) = \frac{3}{5}$ and $f(R^{2,n_3,n_5}, b) = f(R^{2,n_3,n_5}, c) = \frac{1}{5}$ for all $n_3 > 0$ and $n_5 > 0$. Note for this that, in R^{1,n_3,n_5} and R^{3,n_3,n_5}, we can use cancellation to remove voters 1 and n or voters 1 and l, respectively. This step leads to the profile R^{2,n_3,n_5-1} or R^{2,n_3-1,n_5}, which proves that $f(R^{1,n_3,n_5}) = f(R^{2,n_3,n_5-1})$ and $f(R^{3,n_3,n_5}) = f(R^{2,n_3-1,n_5})$. Moreover, note that if $n_3 = 0$, then a and c are top-ranked by half of the voters in R^{2,n_3,n_5}. Hence, we have that $f(R^{2,0,n_5}, a) = f(R^{2,0,n_5}, c) = f(R^{3,1,n_5}, a) = f(R^{3,1,n_5}, c) = \frac{1}{2}$ by Case 2.3. An analogous argument also shows that $f(R^{2,n_3,0}, a) = f(R^{2,n_3,0}, b) = f(R^{1,n_3,1}, a) = f(R^{1,n_3,1}, b) = \frac{1}{2}$. Based on these insights, we now prove our claim on $f(R^{2,n_3,n_5})$ with an induction on $n_3 + n_5$.

First, we consider the induction basis that $n_3 = n_5 = 1$. The previous paragraph implies that $f(R^{1,n_3,n_5}, a) = f(R^{1,n_3,n_5}, b) = f(R^{3,n_3,n_5}, a) = f(R^{3,n_3,n_5}, c) = \frac{1}{2}$. Hence, PC-strategyproofness from R^{1,n_3,n_5} to R^{2,n_3,n_5} and from R^{2,n_3,n_5} to R^{3,n_3,n_5} entails the following inequalities, where $p = f(R^{2,n_3,n_5})$.

$$\frac{1}{2}p(a) \geq p(c) + \frac{1}{2}p(b)$$
$$p(b) + \frac{1}{2}p(c) \geq \frac{1}{2}p(a)$$

Moreover, note that voter n can ensure in R^{2,n_3,n_5} that a is chosen with probability 1 by reporting it as his favorite alternative because of the absolute winner property. Hence, we also get that $p(c) \geq p(b)$ from PC-strategyproofness. Finally, it is easy to see these three inequalities are true at the same time only if $p(a) = 3p(b) = 3p(c)$. Using the fact that $p(a) + p(b) + p(c) = 1$, we hence derive that $p(a) = \frac{3}{5}$ and $p(b) = p(c) = \frac{1}{5}$.

Next, we prove the induction step and thus consider some fixed $n_3 > 0$ and $n_5 > 0$ such that $n_3 + n_5 > 2$. The induction hypothesis is that $f(R^{2,n_3',n_5'}, a) = \frac{3}{5}$ and $f(R^{2,n_3',n_5'}, b) = f(R^{2,n_3',n_5'}, c) = \frac{1}{5}$ for all $n_3' > 0$ and $n_5' > 0$ with $n_3' + n_5' = n_3 + n_5 - 1$. In particular, if $n_5 > 1$, then $f(R^{1,n_3,n_5}) = f(R^{2,n_3,n_5-1})$ because of our previous insights. Hence, $f(R^{1,n_3,n_5}, a) = \frac{3}{5}$ and $f(R^{1,n_3,n_5}, b) = f(R^{1,n_3,n_5}, c) = \frac{1}{5}$ because of the induction hypothesis. PC-strategyproofness from R^{1,n_3,n_5} to R^{2,n_3,n_5} implies therefore the following inequality, where $p = f(R^{2,n_3,n_5})$.

$$\frac{2}{5}p(a) + \frac{1}{5}p(b) \geq \frac{4}{5}p(c) + \frac{3}{5}p(b)$$
$$\iff \frac{1}{2}p(a) \geq p(c) + \frac{1}{2}p(b)$$

On the other hand, if $n_5 = 1$, then $f(R^{1,n_3,n_5}, a) = f(R^{1,n_3,n_5}, b) = \frac{1}{2}$, and PC-strategyproofness results in the same inequality.

Similarly, if $n_3 > 1$, then $f(R^{3,n_3,n_5}, a) = \frac{3}{5}$ and $f(R^{3,n_3,n_5}, b) = f(R^{3,n_3,n_5}, c) = \frac{1}{5}$ because of the induction hypothesis and cancellation. Hence, we derive the following inequality from PC-strategyproofness between R^{2,n_3,n_5} and R^{3,n_3,n_5}.

$$\frac{4}{5}p(b) + \frac{3}{5}p(c) \geq \frac{1}{5}p(c) + \frac{2}{5}p(a)$$
$$\iff p(b) + \frac{1}{2}p(c) \geq \frac{1}{2}p(a)$$

On the other hand, if $n_3 = 1$, then $f(R^{3,n_3,n_5}, a) = f(R^{3,n_3,n_5}, c) = \frac{1}{2}$. Applying PC-strategyproofness in this case results in the same inequality as above.

Finally, it must hold that $p(c) \geq p(b)$. Indeed, otherwise voter n could PC-manipulate in R^{2,n_3,n_5} by reporting a his favorite option—a would then chosen with probability 1 because of the absolute winner property. Since $p(a) + p(b) + p(c) = 1$, it can be verified that the only possible solution to the three inequalities that we have derived is $p(a) = \frac{3}{5}$ and $p(b) = p(c) = \frac{1}{5}$. This proves the induction step and therefore that $f(R') = f(R^{2,n_3,n_5}) = f^1(R')$.

Case 2.5: Every alternative is top-ranked by less than half of the voters in R'.

As the last case, suppose that every alternative is top-ranked by less than half of the voters in R'. In particular, this means that every alternative is top-ranked at least once. We suppose again without loss of generality that a voter reports a, b, c in R' and hence, the same analysis as in the last case shows that the only possible preference relations in R' are a, b, c ($\succ_1$), b, c, a ($\succ_3$), and c, a, b ($\succ_5$). In particular, we have that $n_1 > 0$, $n_3 > 0$, $n_5 > 0$, and $n_2 = n_4 = n_6 = 0$. Moreover, since no alternative is top-ranked by at least half of the voters, we have that $n_1 < n_3 + n_5$, $n_3 < n_1 + n_5$, and $n_5 < n_1 + n_3$. This shows that there is not even a weak Condorcet winner in R', and hence our goal is to show that $f(R', x) = \frac{1}{3}$ for all $x \in A$. Suppose that this is not the case, which means that that either $f(R', a) < f(R', c)$, $f(R', b) < f(R', a)$, or $f(R', c) < f(R', b)$; otherwise, $f(R', a) \geq f(R', c) \geq f(R', b) \geq f(R', a)$, which implies that all alternatives get a probability of $\frac{1}{3}$. We assume in the sequel that $f(R', a) < f(R', c)$ as all cases are symmetric. Now, in this case, we let the voters i with preference relation a, b, c one after another swap a and b. For each step, Lemma 3 implies that the probability of a remains smaller than that of c. However, this process results in a profile R'' in which $n_1 + n_3$ voters report b as their favorite alternative. Since $n_1 + n_3 > n_5$, b is the Condorcet winner and our previous observation proves that $f(R'', b) = 1$. However, this contradicts that $f(R'', a) < f(R'', c)$ and hence, the claim that $f(R', a) < f(R', c)$ must be wrong. This shows that $f(R', x) = \frac{1}{3} = f^1(R', x)$ for all $x \in A$. □

Finally, we prove Proposition 2. Recall for this that $n_R(x)$ denotes the number of voters who top-rank alternative x in R and that $B(R)$ is the set of alternatives that are never bottom-ranked in R. Moreover, the uniform random dictatorship RD is defined by $RD(R, x) = \frac{n_R(x)}{\sum_{y \in A} n_R(y)}$ for all $x \in A$ and $R \in \mathcal{R}^{\mathcal{F}(\mathbb{N})}$. As discussed in Section 2.4, RD is known to satisfy strict SD-participation and therefore satisfies also strict PC-participation, but fails PC-efficiency. We consider the following variant of RD called f^2: if $|B(R)| \in \{0, 2\}$, then $f^2(R, x) = RD(R, x)$. On the other hand, if $|B(R)| = 1$, let x denote the single alternative in $B(R)$ and let C denote the set of alternatives $y \in A \setminus \{x\}$ with minimal $n_y(R)$. Then, $f^2(R, x) = \frac{n_R(x) + \sum_{y \in C} n_R(y)}{\sum_{y \in A} n_R(y)}$, $f(R^2, y) = 0$ for $y \in C$, and $f^2(R, z) = RD(R, z)$ for $z \notin C \cup \{x\}$. Intuitively, if $|B(R)| = 1$, f^2 removes the alternatives in $A \setminus B(R)$ with minimal $n_R(x)$ and then computes RD.

Proposition 2. *For $m = 3$, f^2 satisfies anonymity, neutrality, PC-efficiency, and strict PC-participation.*

Proof. First note that f^2 is anonymous and neutral since its definition does not depend on the identities of voters or alternatives.

Next, we discuss why f^2 satisfies strict PC-participation—in fact, we prove the even stronger claim that it satisfies strict SD-participation. Consider an arbitrary electorate $N \in \mathcal{F}(\mathbb{N})$, a voter $i \in N$, and two preference profiles $R \in \mathcal{R}^N$ and $R' \in \mathcal{R}^{N \setminus \{i\}}$ such that $R' = R_{-i}$. We need to show that if i's top alternative is not already chosen with probability 1 in $f^2(R')$, then $f^2(R) \succ_i^{SD} f^2(R')$. First, note that this is obvious if $f^2(R) = RD(R)$ and $f^2(R') = RD(R')$ because RD satisfies strict SD-participation. Moreover, $|B(R')| -$

$1 \leq |B(R)| \leq |B(R')|$ because voter i can only bottom-rank a single alternative. These two observations leave us with three interesting cases: $|B(R')| = 2$ and $|B(R)| = 1$, $|B(R')| = |B(R)| = 1$, and $|B(R')| = 1$ and $|B(R)| = 0$.

First, consider the case where $|B(R')| = 1$ and $|B(R)| = 0$. Without loss of generality, we assume that $B(R') = \{a\}$, which means that a is voter i's least preferred alternative. Moreover, we call voter i's best alternative $z \in \{b, c\}$. The following case distinction proves that f^2 satisfies strict SD-participation under the given assumptions.

- If $n_{R'}(b) = n_{R'}(c)$, then $f^2(R', a) = 1$ and it is obvious that $f^2(R) \succ_i^{SD} f^2(R')$ because a is voter i's least preferred outcome and $f^2(R, z) = RD(R, z) > 0$.
- If $n_{R'}(b) > n_{R'}(c)$, we have that $f^2(R', a) = \frac{n_{R'}(a)+n_{R'}(c)}{\sum_{x\in A} n_{R'}(x)} > \frac{n_{R'}(a)}{1+\sum_{x\in A} n_{R'}(x)} = f^2(R, a)$ and $f^2(R', z) \leq \frac{n_{R'}(z)}{\sum_{x\in A} n_{R'}(x)} < \frac{1+n_{R'}(z)}{1+\sum_{x\in A} n_{R'}(x)} = f^2(R, z)$. It is now easy to see that $f^2(R) \succ_i^{SD} f^2(R')$.
- The case $n_{R'}(b) < n_{R'}(c)$ is symmetric to the last one.

Next, consider the case where $|B(R')| = 2$ and $|B(R)| = 1$. Without loss of generality, we suppose that $B(R') = \{a, b\}$ and $B(R) = \{a\}$, which means that voter i bottom-ranks b. Moreover, note that all voters in $N \setminus \{i\}$ bottom-rank c as otherwise $B(R') = \{a, b\}$ is not possible. This means that $f^2(R', c) = 0$, $n_{R'}(c) = 0$, and $n_R(c) \leq 1$. We consider again several subcases.

- If $n_R(b) > n_R(c)$, then $f^2(R, c) = 0 = f^2(R', c)$, $f^2(R, a) \geq \frac{1+n_{R'}(a)}{1+\sum_{x\in A} n_{R'}(x)} > \frac{n_{R'}(a)}{\sum_{x\in A} n_{R'}(x)} = f^2(R', a)$, and thus $f^2(R, b) < f^2(R', b)$. Hence, $f^2(R) \succ_i^{SD} f^2(R')$ as b is voter i's worst alternative.
- If $n_R(c) > n_R(b)$, then $f^2(R, b) = 0 \leq f^2(R', b)$ and $f^2(R, c) > 0 = f^2(R', c)$. If i top-ranks c, we have $f^2(R) \succ_i^{SD} f^2(R')$. Else, i top-ranks a, and we have $f^2(R, a) \geq \frac{1+n_{R'}(a)}{1+\sum_{x\in A} n_{R'}(x)} > \frac{n_{R'}(a)}{\sum_{x\in A} n_{R'}(x)} = f^2(R', a)$, so again $f^2(R) \succ_i^{SD} f^2(R')$.
- If $n_R(c) = n_R(b) = 0$, all voters (including i) report a as their best option and thus $f^2(R', a) = f^2(R, a) = 1$, which satisfies strict SD-participation because $f^2(R)$ is voter i's favorite lottery.
- If $n_R(c) = n_R(b) = 1$, then voter i's preference relation is c, a, b. Moreover, $f^2(R', c) = 0 \leq f^2(R, c)$ and $f^2(R', b) > 0 = f^2(R, b)$. This proves again that $f^2(R) \succ_i^{SD} f^2(R')$

As the last case, suppose that $|B(R')| = |B(R)| = 1$ and let a denote the alternative in $B(R) = B(R')$. Since $a \in B(R)$, voter i does not bottom-rank a. We consider again a case distinction.

- First, suppose that voter i top-ranks a, which means that $n_R(b) = n_{R'}(b)$ and $n_R(c) = n_{R'}(c)$.
  - If $n_{R'}(b) = n_{R'}(c)$, we have that $f^2(R, a) = f^2(R', a) = 1$ and strict PC-participation holds as this is voter i's favorite lottery.
  - If $n_{R'}(b) > n_{R'}(c)$. Then, $f^2(R, a) = \frac{n_{R'}(a)+n_{R'}(c)+1}{1+\sum_{x\in A} n_{R'}(x)} > \frac{n_{R'}(a)+n_{R'}(c)}{\sum_{x\in A} n_{R'}(x)} = f^2(R', a)$, $f^2(R, c) = f^2(R', c) = 0$, and hence $f^2(R, b) < f^2(R', b)$. It is now easy to verify that $f(R) \succ_i^{SD} f(R')$.
  - The case $n_{R'}(b) < n_{R'}(c)$ is symmetric to the last one.
- Next, suppose that voter i places a second. We assume without loss of generality that $\succ_i = b, a, c$ because the case $\succ_i = c, a, b$ is symmetric. This assumption means that $n_{R'}(b) + 1 = n_R(b)$ and $n_{R'}(x) = n_R(x)$ for $x \in \{a, c\}$.
  - If $n_{R'}(b) \geq n_{R'}(c)$, then $f^2(R', b) \leq \frac{n_{R'}(b)}{\sum_{x\in A} n_{R'}(x)} < \frac{1+n_{R'}(b)}{1+\sum_{x\in A} n_{R'}(x)} = f^2(R, b)$, $f^2(R', c) = f^2(R, c) = 0$, and hence $f^2(R', a) > f^2(R, a)$, which proves that $f(R) \succ_i^{SD} f(R')$.
  - If $n_{R'}(b) + 1 = n_{R'}(c)$, then $f^2(R', b) = 0 = f^2(R, b)$, $f^2(R', c) > 0 = f^2(R, c)$, and thus $f^2(R', a) < 1 = f^2(R, a)$. It can be again verified that $f(R) \succ_i^{SD} f(R')$.
  - If $n_{R'}(b) + 1 < n_{R'}(c)$, then $f^2(R', b) = 0 = f^2(R, b)$, $f^2(R', c) = \frac{n_{R'}(c)}{\sum_{x\in A} n_{R'}(x)} > \frac{n_{R'}(c)}{1+\sum_{x\in A} n_{R'}(x)} = f^2(R, c)$, and hence $f^2(R', a) < f^2(R, a)$. Once again, it holds that $f^2(R) \succ_i^{SD} f^2(R')$.

Lastly, we show that f^2 satisfies PC-efficiency by proving that $f^2(R)$ satisfies for all profiles R the three conditions of Lemma 5. To this end, note first that f^2 is *ex post* efficient: it only puts positive probability on an alternative that is never top-ranked if it is second-ranked by all voters and both other alternatives are top-ranked at least once. In this case, all three alternatives are Pareto-optimal, and thus f^2 is *ex post* efficient. This argument also shows that an alternative that is never top-ranked and at least once bottom-ranked is always assigned probability 0. Finally, if an alternative is never bottom-ranked and at least once top-ranked, only two alternatives can have positive probability. In more detail, either $|B(R)| = 2$, which means that one alternative is bottom-ranked by all voters and receives probability 0, or $|B(R)| = 1$ and an alternative in $A \setminus B(R)$ gets probability 0 by definition of f^2. Hence, all conditions of Lemma 5 hold, which implies that f^2 is PC-efficient. □