Online Coalition Formation under Random Arrival or Coalition Dissolution

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Abstract

Coalition formation considers the question of how to partition a set of n agents into disjoint coalitions according to their preferences. We consider a cardinal utility model with additively separable aggregation of preferences and study the online variant of coalition formation, where the agents arrive in sequence and whenever an agent arrives, they have to be assigned to a coalition immediately. The goal is to maximize social welfare. In a purely deterministic model, the greedy algorithm, where an agent is assigned to the coalition with the largest gain, is known to achieve an optimal competitive ratio, which heavily relies on the range of utilities.

We complement this result by considering two related models. First, we study a model where agents arrive in a random order. We find that the competitive ratio of the greedy algorithm is $\Theta\left(\frac{1}{n^2}\right)$, whereas an alternative algorithm, which is based on alternating between waiting and greedy phases, can achieve a competitive ratio of $\Theta\left(\frac{1}{n}\right)$. Second, we relax the irrevocability of decisions by allowing to dissolve coalitions into singleton coalitions, presenting a matching-based algorithm that once again achieves a competitive ratio of $\Theta\left(\frac{1}{n}\right)$. Hence, compared to the base model, we present two ways to achieve a competitive ratio that precisely gets rid of utility dependencies. Our results also give novel insights in weighted online matching.

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1 Introduction

Coalition formation is a vibrant topic in multi-agent systems for at least two decades. The goal is to partition a set of n agents into disjoint coalitions. We consider the framework of hedonic games, where the agents have preferences for the coalitions they are part of by disregarding externalities [11]. More specifically, we assume that each agent has cardinal utilities for each other agent and that utilities for coalitions are aggregated in an additively separable way by taking sums [4].

Most of the hedonic games literature considers an offline setting, where a fully specified instance is given. By contrast, in many real-life situations, such as the formation of teams in a company, the agents are not all present in the beginning but rather join an ongoing coalition formation process over time. With this motivation in mind, Flammini et al. [15] proposed an online version of hedonic games, where agents arrive one by one. Upon arrival, an agent reveals their preferences for coalitions containing the agents present thus far, and has to be added immediately and irrevocably to an existing coalition (or to a new singleton coalition). Flammini et al. [15] seek to find algorithms that have high social welfare in a worst-case analysis against an adversary that is allowed to select an arbitrary instance and an arbitrary arrival order of the agents. The performance of an algorithm is measured with respect to its competitive ratio, that is, the worst-case ratio of the social welfare of the computed solution compared to the social welfare of the optimal offline solution. Their main result is that the greedy algorithm, which adds every agent to the best possible coalition upon arrival, has the optimal competitive ratio of $\Theta\left(n\frac{U_{max}}{U_{min}}\right)$, where U_{max} and U_{min} are the maximum and minimum absolute value of a non-zero single-agent utility in the adversarial instance, respectively. In other words, for a fixed number of agents the competitive ratio is infinite if the maximum or minimum utility is not bounded.

Arguably, unbounded utilities give a lot of power to an adversary. The following example by Flammini et al. [15] exploits this. Consider the situation where the first two arriving agents have a mutual utility of 1. If an algorithm adds the second agent to the coalition of the first agent, then the adaptive adversary can set the utility of 1 as the minimum utility and add agents with large positive and negative utilities that bring no gain to this coalition. Hence, the welfare of the large positive utilities is lost, and the algorithm performs badly. On the other hand, if an algorithm puts the second agent into a singleton coalition, then the adversary can set the utility of other arriving agents to 0, and the loss of the welfare of the single positive edge leads to an unbounded competitive ratio.

As a consequence, we want to see whether we can perform better if the adversary has less power in two related models. First, we consider a model where the adversary is still capable of fixing a bad instance. However, agents arrive in a random order according to a permutation of the agents selected uniformly at random. We show that the greedy algorithm achieves a competitive ratio of $\Theta\left(\frac{1}{n^2}\right)$. Moreover, we present an alternative algorithm with a competitive ratio of $\Theta\left(\frac{1}{n}\right)$, which is based on alternating between waiting and greedy phases.

Second, we allow an algorithm to dissolve a coalition into singleton coalitions to revert bad previous decisions. We present a $\frac{1}{6}$-competitive online matching algorithm, which can be used to achieve a competitive ratio of $\Theta\left(\frac{1}{n}\right)$ for online coalition formation. Hence, compared to the deterministic model, we present a novel algorithm, whose competitive ratio gets precisely rid of utility dependencies.

We conclude by showing boundaries for optimal algorithms in both settings. Moreover, we discuss why the worst-case analysis for our presented algorithm does not generalize for any algorithm.

2 Related Work

Hedonic games originated from economic theory [11] more than four decades ago. However, their broad consideration only started about two decades later [3, 4, 9]. In particular, Bogomolnaia and Jackson [4] introduced additively separable hedonic games (ASHGs), which are since then an ongoing subject of study. A large part of the research on ASHGs focuses on the computational complexity of stability measures [2, 5, 8, 10, 15, 16, 22, 23], but some more recent studies also consider economic efficiency in the sense of Pareto optimality [7, 12] or strategyproofness [14]. In general, maximizing social welfare is NP-hard, even if utilities are symmetric and only attain the values -1 and 1 [2]. Moreover, even though Bullinger [7] presents a polynomial-time algorithm to compute Pareto-optimal partitions for symmetric ASHG, the partitions computed by this algorithm are no approximation of social welfare. All of the literature discussed thus far considers ASHG in an offline setting.

The online variant only came under scrutiny very recently and has been researched far less. Flammini et al. [15] introduce the problem and focus on deterministic algorithms with guarantees to social welfare. Strictly speaking, they consider a variant of games equivalent to ASHGs, where utilities are scaled by a factor of 2. In particular, they consider games restricted by a maximum coalition size or maximum number of coalitions.

From the online algorithms literature, most related to our work is the online matching problem, which was first considered in the seminal paper by Karp et al. [20] considering online bipartite matching. In this work, an unweighted, bipartite graph is given, and the agents of one side appear online. The goal is to find a matching of maximum cardinality. Karp et al. [20] introduce the famous ranking algorithm, which achieves a competitive ratio of $1 - \frac{1}{e}$. An overview of this research is presented in the very recent book chapter by Huang and Tröbst [18].

Our model of online coalition formation can be viewed as a generalization of the setting by Karp et al. [20], with the following modifications: (i) there are edge weights, (ii) all vertices arrive online, (iii) the underlying graph is not necessarily bipartite, and (iv) coalitions can be arbitrary subsets of agents. While condition (iv) is specific to coalition formation, conditions (i)-(iii) have been studied in the literature, albeit, to the best of our knowledge, not in the combination of all three. Feldman et al. [13] consider condition (i), i.e., a bipartite setting with edge weights, where one side of the vertices arrives online. They show that the natural greedy algorithm is $\frac{1}{2}$-competitive, and they provide an algorithm matching the competitive ratio of $1 - \frac{1}{e}$ from Karp et al. [20]. An important condition in this setting is *free disposal*, which essentially requires that a previous matching can be dissolved upon the arrival of a better option. Wang and Wong [24] consider condition (ii), i.e., online bipartite matching where both sides arrive online. They can beat the greedy algorithm by a primal-dual algorithm achieving a competitive ratio of $0.532 < 1 - \frac{1}{e}$. Finally, Huang et al. [17] consider the conjunction of conditions (ii) and (iii), i.e., fully online (non-bipartite) matching. They extend the ranking algorithm to this setting and show that it is 0.5211-competitive. Both Wang and Wong [24] and Huang et al. [17] show that a competitive ratio of $1-\frac{1}{e}$ is impossible to achieve in their respective settings. Interestingly, the competitive ratio of $1 - \frac{1}{e}$ can be beaten in the random arrival model [19, 21].

3 Preliminaries

In this section, we present our model. For an integer $i \in \mathbb{N}$, we define $[i] := \{1, \dots, i\}$. Additionally, for any set N, define $\binom{N}{2} := \{e \subseteq N : |e| = 2\}$.

3.1 Additively Separable Hedonic Games

Let N be a finite set of n *agents*. Any subset of N is called a *coalition*. We denote the set of all possible coalitions containing agent $i \in N$ by $\mathcal{N}_i = \{C \subseteq N : i \in C\}$. A *coalition structure* (or *partition*) is a partition of the agents. Given an agent $i \in N$ and a partition π, let $\pi(i)$ denote the coalition of i, i.e., the unique coalition $C \in \pi$ with $i \in C$.

A *hedonic game* is a pair $(N, \succsim)$ consisting of a set N of agents and a preference profile $\succsim = (\succsim_i)_{i\in N}$, where $\succsim_i$ is a weak order over $\mathcal{N}_i$ that represents the preferences of agent i. A hedonic game is called an *additively separable hedonic game* (ASHG) if there exists a complete, undirected, and weighted graph $G = (N, E, w)$ with edge set $E = \binom{N}{2}$ and weight function $w : E \to \mathbb{Q}$, such that, for every agent $i \in N$ and every pair of coalitions $C, C' \in \mathcal{N}_i$, it holds that $C \succsim_i C'$ if and only if $\sum_{j\in C} w(\{i, j\}) \geq \sum_{j\in C'} w(\{i, j\})$ [4].[1] We then speak of the ASHG *given by* G. We abbreviate $w(i, j) = w(\{i, j\})$. Moreover, since we only consider complete graphs, we shorten notation and write $G = (N, w)$, where $w : \binom{N}{2} \to \mathbb{Q}$, instead of $G = (N, E, w)$ to fully specify an underlying graph. For an agent $i \in N$ and a coalition $C \in \mathcal{N}_i$ or a partition π, we define the *utility* of i for C or π by $u_i(C) := \sum_{j\in C} w(i, j)$ and $u_i(\pi) := u_i(\pi(i))$, respectively.

Additionally, we extend the weight function to sets of edges $F \subseteq \binom{N}{2}$ by $w(F) := \sum_{e\in F} w(e)$. A *matching* is a coalition structure π such that, for all $C \in \pi$, it holds that $|C| \leq 2$. A matching π is represented by its edge set $M(\pi) := \{C \in \pi : |C| = 2\} \subseteq \binom{N}{2}$. We then write $w(\pi) := w(M(\pi))$ for the weight of matching π.

3.2 Online Coalition Formation

In this section, we introduce our model of online coalition formation and appropriate objectives. Consider an ASHG given by $G = (N, w)$. Given a subset of agents $N' \subseteq N$, let $G[N']$ be the graph induced by agent set N'. Moreover, given a partition π of a set N and a subset of agents $N' \subseteq N$, we define $\pi[N']$ as the *partition restricted to* N' as $\pi[N'] := \{C \cap N' : C \in \pi, C \cap N' \neq \emptyset\}$. Specifically, if $N' = N \setminus \{i\}$ for some agent $i \in N$, we write $\pi - i$ instead of $\pi[N']$.

In an online setting, previous decisions influence the capabilities of an algorithm to form a partition in the next step. Given a partition π and an agent i not covered by i, let $\mathcal{A}(\pi, i)$ denote the set of available partitions, when the tentative partition is π and the newly arriving agent is i. As a default, we assume the standard setting where $\mathcal{A}(\pi, i) = \mathcal{A}^S(\pi, i) := \{\pi' : \pi' - i = \pi\}$. We also consider algorithms that have the capability to dissolve a coalition completely. We say that an algorithm acts under *free dissolution* if $\mathcal{A}(\pi, i) = \mathcal{A}^D(\pi, i) = \mathcal{A}^S(\pi, i) \cup \bigcup_{C\in\pi, j\in C}\{(\pi \setminus \{C\}) \cup \{\{i, j\}\} \cup \{\{k\} : k \in C \setminus \{j\}\}\}$. Free dissolution is the natural extension of free disposal by Feldman et al. [13] in the domain of matching adapted to coalitions of size larger than 2. In addition, we define $\Sigma(N) := \{\sigma : [|N|] \to N \text{ bijective}\}$ as the set of all *orders* of the agent set N.

An instance of an *online coalition formation* problem is a tuple (G, σ), where $G = (N, w)$ defines an ASHG and $\sigma \in \Sigma(N)$. An *online coalition formation algorithm* for instance (G, σ) gets as input the sequence $G_1, \ldots, G_n$, where, for every $i \in [n]$, $G_i = G[\{\sigma(1), \ldots, \sigma(i)\}]$. Then, for every $i \in [n]$, the algorithm produces a partition π_i of $\{\sigma(1), \ldots, \sigma(i)\}$ such that the algorithm has only access to G_i and for $i \geq 2$, it holds that $\pi_i \in \mathcal{A}(\pi_{i-1}, \sigma(i))$.

[1] Since our focus will be on social welfare, the consideration of *undirected* graphs is without loss of generality because the social welfare of a partition is invariant under the symmetrization $w^S(\{x, y\}) = \frac{1}{2}(w_x(y) + w_y(x))$ given directed edges with weights $w_i(j)$.

The output of the algorithm is the partition π_n. Given an online coalition formation algorithm ALG, let $ALG(G,\sigma)$ be its output for instance (G,σ).

In other words, the algorithm iteratively builds a partition such that, whenever an agent arrives, the only knowledge is the game restricted to the present agents, and the algorithm has to irrevocably assign the new agent to an existing coalition or start a new coalition (or, under free dissolution, dissolve any coalition before making its decision). If an online coalition formation algorithm creates a matching in any step, we speak of an *online matching algorithm*.

Our benchmark algorithm is the greedy algorithm as introduced by Flammini et al. [15].

▶ **Definition 1** (Greedy algorithm). *On input (G,σ), in the ith step, $i \geq 2$, the* greedy algorithm *(GDY) forms $\pi_i = \arg\max_{\pi \in \mathcal{A}(\pi_{i-1},\sigma(i))} \mathcal{SW}(\pi)$ if there exists $\pi \in \mathcal{A}(\pi_{i-1},\sigma(i))$ with $\mathcal{SW}(\pi) > \mathcal{SW}(\pi_{i-1})$, and $\pi_i = \pi_{i-1} \cup \{\{\sigma(i)\}\}$, otherwise.*

Hence, GDY assigns each arriving agent to the available coalition such that the increase in social welfare is maximized, or creates a new singleton coalition if no increase is possible.

3.3 Competitive Analysis

The competitive analysis needs a quantifiable objective and we follow Flammini et al. [15] by considering social welfare. The *social welfare* of a partition π is defined as $\mathcal{SW}(\pi) = \sum_{i \in N} u_i(\pi)$. A partition π is said to be *welfare-optimal* if, for every partition π', it holds that $\mathcal{SW}(\pi) \geq \mathcal{SW}(\pi')$. Given a hedonic game G, let $\pi^*(G)$ be a welfare-optimal partition. We say that an online coalition formation algorithm ALG is *c-competitive*[2] if

$$\inf_{G} \min_{\sigma \in \Sigma(N)} \frac{\mathcal{SW}(ALG(G,\sigma))}{\mathcal{SW}(\pi^*(G))} \geq c.$$

Equivalently, this means that, for all instances (G,σ), it holds that $\mathcal{SW}(ALG(G,\sigma)) \geq c\,\mathcal{SW}(\pi^*(G))$. The *competitive ratio* of an algorithm ALG, denoted by c_{ALG}, is the maximum c such that ALG is c-competitive. Note that the competitive ratio is always at most 1.

In addition, we consider online coalition formation with a random arrival order, where we assume that the arrival order is selected uniformly at random. In the random arrival model, the competitive ratio of an algorithm ALG is defined as

$$\inf_{G} \frac{\mathbb{E}_{\sigma}\left[\mathcal{SW}(ALG(G,\sigma))\right]}{\mathcal{SW}(\pi^*(G))} \geq c.$$

There, the expectation is over the uniform selection of an arrival order σ from $\Sigma(N)$. Note that if π is a matching, then $w(\pi) = \frac{1}{2}\mathcal{SW}(M(\pi))$. Hence, in the competitive analysis of online matching algorithms, we can as well consider the weight of matchings instead of their social welfare.

4 Random Arrival Model

In this section, we analyze algorithms aiming at achieving a high social welfare in the random arrival model. Interestingly, while in the deterministic arrival model, the specific utility values affect the competitive ratio of the greedy algorithm (and any other algorithm) [15], in the random arrival model, the dependency is solely on the number of agents. For our

[2] Here, we use the convention that $\frac{0}{0} = 1$ and $\frac{x}{0} = 0$ for any $x \in \mathbb{Q}_{<0}$.

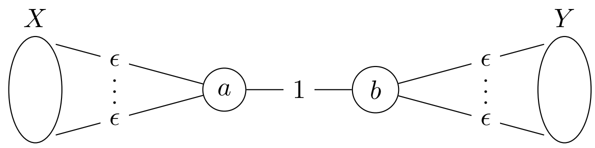

Figure 1 The example contains $n = 2m + 2$ agents. There are m agents in each of the sets X and Y. The utility between a and any agent in X and b and any agent in Y is ϵ. All omitted edges represent utilities of -1.

analysis of algorithms in this section we use the notation $x \prec^\sigma y$ to say that $\sigma^{-1}(x) < \sigma^{-1}(y)$ for $x, y \in N$ and an arrival order σ.

▶ **Theorem 2.** *The competitive ratio of GDY for ASHGs under a random arrival order satisfies* $\Theta\left(\frac{1}{n^2}\right)$.

Proof. First, we show that the competitive ratio of GDY satisfies $O\left(\frac{1}{n^2}\right)$ by providing an example where it performs "bad". Let $\epsilon > 0$ and consider the ASHG given by $G^\epsilon = (N, w^\epsilon)$ depicted in Figure 1. The agent set is $N = X \cup Y \cup \{a, b\}$, where $|X| = |Y| = m$. Utilities are given by $w^\epsilon(a, b) = 1$, $w^\epsilon(a, x) = w^\epsilon(b, y) = \epsilon$ for $x \in X$ and $y \in Y$, and all other weights are -1. Clearly, for small ϵ, the optimal solution has a value of 1 (with $\{a, b\}$ the only non-singleton coalition). By inspecting the limit case for ϵ tending to 0, the value of the greedy algorithm (and therefore also its competitive ratio) is at most the probability of matching a with b, i.e.,

$$\begin{aligned} c_{GDY} &= \inf_G \frac{\mathbb{E}_\sigma\left[\mathcal{SW}(GDY(G, \sigma))\right]}{\mathcal{SW}(\pi^*(G))} \\ &\leq \inf_{\epsilon > 0} \frac{\mathbb{E}_\sigma\left[\mathcal{SW}(GDY(G^\epsilon, \sigma))\right]}{1} \\ &\leq \mathbb{P}_\sigma(\{a, b\} \in GDY(G^\epsilon, \sigma)). \end{aligned}$$

We compute

$$\begin{aligned} \mathbb{P}_\sigma(\{a, b\} \in GDY(G^\epsilon, \sigma)) &= \mathbb{P}_\sigma(\{a, b\} \in GDY(G^\epsilon, \sigma) \mid a \prec^\sigma b)\mathbb{P}_\sigma(a \prec^\sigma b) \\ &\quad + \mathbb{P}_\sigma(\{a, b\} \in GDY(G^\epsilon, \sigma) \mid b \prec^\sigma a)\mathbb{P}_\sigma(b \prec^\sigma a) \\ &= 2\mathbb{P}_\sigma(\{a, b\} \in GDY(G^\epsilon, \sigma) \mid a \prec^\sigma b)\mathbb{P}_\sigma(a \prec^\sigma b). \end{aligned}$$

The second equality follows by symmetry. Next, we sum over all possible arrival positions of b by summing over the number of alternatives that arrive before b in addition to a. Note that if more than m agents arrive before b, excluding a, then, by the pigeonhole principle, some $x \in X$ arrives before b and forms a coalition with a. This prevents the coalition $\{a, b\}$ from forming so all terms of the sum for $i > m$ are 0. Conditioned on a arriving before b, the coalition $\{a, b\}$ forms if and only if all i agents arriving before b are from Y since those agents will not form a coalition with a. We derive[3]

[3] For the first inequality, note that it holds that $\mathbb{P}(A \mid B)\mathbb{P}(B) = \mathbb{P}(A, B) = \sum_C \mathbb{P}(A, B, C) = \sum_C \mathbb{P}(A \mid B, C)\mathbb{P}(B, C) = \sum_C \mathbb{P}(A \mid B, C)\mathbb{P}(B \mid C)\mathbb{P}(C)$ for arbitrary events A, B, and C and probability measures $\mathbb{P}$.

$$\begin{aligned}c_{GDY} &\leq 2\sum_{i=0}^{m}\mathbb{P}_\sigma(\{a,b\}\in GDY(G^\epsilon,\sigma)\mid a\prec^\sigma b,\sigma^{-1}(b)=i+2)\\&\qquad\cdot\mathbb{P}_\sigma(a\prec^\sigma b\mid\sigma^{-1}(b)=i+2)\mathbb{P}_\sigma(\sigma^{-1}(b)=i+2)\\&=2\sum_{i=0}^{m}\mathbb{P}_\sigma(\{d\colon d\prec^\sigma b\}\setminus\{a\}\subseteq Y\mid a\prec^\sigma b,\sigma^{-1}(b)=i+2)\\&\qquad\cdot\mathbb{P}_\sigma(a\prec^\sigma b\mid\sigma^{-1}(b)=i+2)\mathbb{P}_\sigma(\sigma^{-1}(b)=i+2).\end{aligned}$$

We compute all individual terms in the previous sum. First, the probability that the i agents arriving before b are all from Y is $\mathbb{P}_\sigma(\{d\colon d\prec^\sigma b\}\setminus\{a\}\subseteq Y\mid a\prec^\sigma b,\sigma^{-1}(b)=i+2)=\frac{\binom{m}{i}}{\binom{2m}{i}}$, i.e., the number of possibilities to draw i agents from Y divided by the number of possibilities to draw i agents from $Y\cup X$.

Second, the probability that a arrives before b when b arrives in position $i+2$ is $\mathbb{P}_\sigma(a\prec^\sigma b\mid\sigma^{-1}(b)=i+2)=\frac{i+1}{2m+1}$ because we have $i+1$ chances to draw a among the remaining $2m+1$ alternatives. Finally, due to the random arrival order, the probability that agent b arrives in a certain fixed position is $\mathbb{P}_\sigma(\sigma^{-1}(b)=i+2)=\frac{1}{2m+2}$. Together, we obtain

$$c_{GDY}\leq 2\sum_{i=0}^{m}\frac{\binom{m}{i}}{\binom{2m}{i}}\frac{i+1}{2m+1}\frac{1}{2m+2}=\frac{2}{m^2+3m+2}\in O\left(\frac{1}{m^2}\right)=O\left(\frac{1}{n^2}\right).$$

Next, we show that the competitive ratio of GDY satisfies $\Omega\left(\frac{1}{n^2}\right)$. Consider an arbitrary ASHG given by $G=(N,w)$ and let $E^+(G)=\{e\in\binom{n}{2}\colon w(e)>0\}$ be the set of agent pairs with positive weights. Then, the welfare-optimal partition $\pi^*(G)$ satisfies $\mathcal{SW}(\pi^*(G))\leq 2\sum_{e\in E^+(G)}w(e)$. Furthermore, the social welfare of GDY is at least twice the utility between the first arriving pair of agents from $E^+(G)$. Indeed, until the arrival of a pair of agents with positive utility, every agent is assigned to a singleton coalition. Thus, since every pair in $E^+(G)$ has equal probability to be the first such pair, the expected social welfare of GDY is at least the average, i.e., $\mathbb{E}_\sigma[\mathcal{SW}(GDY(G,\sigma))]\geq\frac{\sum_{e\in E^+(G)}2w(e)}{|E^+(G)|}$ and the competitive ratio is then at least

$$c_{GDY}=\inf_G\frac{\mathbb{E}_\sigma[\mathcal{SW}(GDY(G,\sigma))]}{\mathcal{SW}(\pi^*(G))}\geq\inf_G\frac{\frac{\sum_{e\in E^+(G)}2w(e)}{|E^+(G)|}}{2\sum_{e\in E^+(G)}w(e)}=\inf_G\frac{1}{|E^+(G)|}\in\Omega\left(\frac{1}{n^2}\right).$$

Altogether, we have shown that c_{GDY} is of order $\Theta\left(\frac{1}{n^2}\right)$. ◀

The natural question is whether we can obtain better algorithms than the greedy algorithm. A simple attempt to achieve a better algorithm is to make use of randomization. The performance of the greedy algorithm was bounded because in the worst case, the value of greedy is equal to the average weight of a positive edge. However, we can easily achieve the average weight of a random matching, improving the performance to $1/n$. For simplicity, we assume first that n is even and known to the algorithm in advance. We first analyze a simple online matching algorithm.

▶ **Definition 3** (Random matching algorithm). *The* random matching algorithm *(RMA) leave the first $\frac{n}{2}$ agents unmatched. Then, we select a bijection $\phi\colon\left[\frac{n}{2}\right]\to\left[\frac{n}{2}\right]$ uniformly at random. For $1\leq i\leq\frac{n}{2}$, if the weight between the $\left(\frac{n}{2}+i\right)$th agent and the $\phi(i)$th agent is positive, then these are matched. Otherwise, the $\left(\frac{n}{2}+i\right)$th remains unmatched.*

We defer the proof of the next proposition to the appendix. Also, all other missing proofs can be found there.

▶ **Proposition 4.** *RMA has a competitive ratio of* $\Theta\left(\frac{1}{n}\right)$ *for matching instances under a random arrival order when* n *is known.*

The natural question is whether we can derandomize this algorithm while maintaining the same competitive ratio. Indeed, a natural way to do this is to have the first half of the agents form singleton coalitions, and the second half of the agents joining the best available coalition. This yields a deterministic algorithm achieving a competitive ratio of $\frac{1}{n}$. First, we maintain the assumption that n is even and known to the algorithm beforehand. However, after we have analyzed this algorithm, we will show that we can drop this additional assumption and that a variation of the algorithm still achieves the same competitive ratio asymptotically.

▶ **Definition 5** (Waiting greedy algorithm)**.** *The* waiting greedy algorithm *(WGDY) places the first* $\frac{n}{2}$ *agents in singleton coalitions. Then, for the remaining* $\frac{n}{2}$ *agents, it assigns coalitions greedily.*

The upper bound of the performance of *WGDY* is again attained by the game depicted in Figure 1. The analysis is more involved and relies on investigating the distribution of the agents in the second phase. We defer the details to the appendix and restrict attention to the lower bound.

▶ **Theorem 6.** *The competitive ratio of WGDY for ASHGs under a random arrival and known and even* n *is* $\Theta\left(\frac{1}{n}\right)$*.*

Proof of lower bound. We show that $WGDY \in \Omega\left(\frac{1}{n}\right)$. Consider an arbitrary ASHG given by $G = (N, w)$. Let $\Pi(n) = \left\{(A, B) \colon A \cup B = N, |A| = |B| = \frac{n}{2}\right\}$ be the set of partitions of the agent set N into two equally-sized subsets.

Let $(A, B) \in \Pi(n)$. Define $E^+(A, B) = \left\{e = \{a, b\} \in \binom{N}{2} \colon w(e) > 0, a \in A, b \in B\right\}$. We claim that if the agents in A and B arrive in the first and second stage, respectively, then the weight of the obtained partition is at least $\frac{1}{n}w(E^+(A, B))$. Let $P_{A,B}$ denote the event that the partition (A, B) realizes.

Consider an arbitrary agent $b \in B$. Let $a_1(b), \dots, a_{r(b)}(b) \in A$ be the agents in A such that $\{b, a_i(b)\} \in E^+(A, B)$, where $r(b) \in \mathbb{N}$ is their number. Assume that $w(a_1(b), b) \geq w(a_2(b), b) \geq \dots \geq w(a_{r(b)}(b), b)$. Let $i \in [r(b)]$. In the event $P_{A,B}$, an agent $b \in B$ arrives as the $(\frac{n}{2} + i)$th agent (i.e., the ith agent within B) with a probability of $\frac{2}{n}$. Moreover, when b arrives as the $(\frac{n}{2} + i)$th agent, then at most $i - 1$ coalitions have formed so far, and therefore some agent in $\{a_1(b), \dots, a_i(b)\}$ is still in a singleton coalition. Hence, the increase in weight caused by b is at least the weight of forming a singleton coalition with the worst partner in $\{a_1(b), \dots, a_i(b)\}$, that is, $w(a_i(b), b)$. Let I_b be the random variable denoting the gain in social welfare caused by the arrival of b.

Hence, in the event $P_{A,B}$, the attained expected social welfare satisfies

$$\begin{aligned}
&\mathbb{E}_\sigma[\mathcal{SW}(\mathit{WGDY}(G,\sigma)) \mid P_{A,B}] \geq \sum_{b\in B} \mathbb{E}_\sigma[I_b \mid P_{A,B}] \\
&= \sum_{b\in B}\sum_{i=1}^{r(b)} \mathbb{E}_\sigma[I_b \mid P_{A,B}, \sigma^{-1}(b) = \frac{n}{2} + i]\mathbb{P}_\sigma(\sigma^{-1}(b) = \frac{n}{2} + i \mid P_{A,B}) \\
&= \sum_{b\in B}\sum_{i=1}^{r(b)} \mathbb{E}_\sigma[I_b \mid P_{A,B}, \sigma^{-1}(b) = \frac{n}{2} + i]\frac{2}{n} \\
&\geq \sum_{b\in B}\sum_{i=1}^{r(b)} w(a_i(b), b)\frac{2}{n} = \frac{2}{n}\sum_{b\in B}\sum_{e\in E^+(A,B)\colon b\in e} w(e) = \frac{2}{n}w(E^+(A,B)).
\end{aligned}$$

Hence, the expected social welfare of the partition computed by WGDY is

$$\mathbb{E}_\sigma\left[\mathcal{SW}(\mathit{WGDY}(G,\sigma))\right] \geq \frac{1}{|\Pi(n)|}\sum_{(A,B)\in\Pi(n)} \frac{2}{n}w\left(E^+(A,B)\right) \geq \frac{1}{2}\frac{2}{n}w\left(E^+\right). \tag{1}$$

In the first inequality, we have used that each partition in $\Pi(n)$ is realized with equal probability, and in the second inequality, we have used that an edge in $E^+ = \{e \in \binom{n}{2} \colon w(e) > 0\}$ is in $E^+(A,B)$ for at least half of the partitions in Π. The latter holds because there are $n^2/4$ edges between A and B and $n(n-1)/2$ edges in total.

Additionally, the social welfare of any partition is at most $2w(E^+)$. Combining this with Equation (1), we obtain

$$c_{\mathit{WGDY}} \geq \frac{\frac{w(E^+)}{n}}{2w(E^+)} = \frac{1}{2n} \in \Omega\left(\frac{1}{n}\right).$$

◀

Next, we show that we can still use a variant of this algorithm even if we do not know n. The idea is to run WGDY on an exponentially growing number of agents repeatedly, possibly not finishing the last iteration.

▶ **Definition 7** (Iterated waiting algorithm). *The* iterated waiting algorithm *(IWA) alternates between placing 2^i agents in singleton coalitions and then assigning 2^i agents to coalitions with the previous 2^i agents greedily. The parameter is set to $i = 0$ at the start and is increased by 1 after 2^{i+1} agents arrive.*

The lower bound in the analysis follows, because IWA completes an execution of WGDY for a sufficiently large number of agents. For the upper bound, we can once again use the game in Figure 1. The analysis is even more involved as in the proof of Theorem 6 because we need to consider dependencies of the distributions of the agents between the phases of IWA.

▶ **Theorem 8.** *IWA has a competitive ratio of $c \in \Theta\left(\frac{1}{n}\right)$.*

5 Deterministic Model under Free Dissolution

In this section, we consider the model with deterministic arrivals under free dissolution. For bipartite online matching instances, where one side of the agents is present offline, Feldman

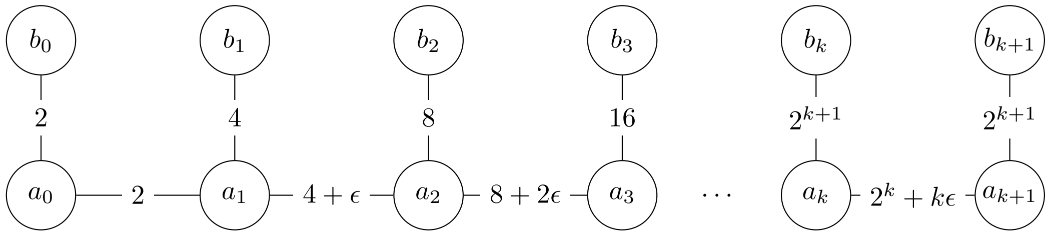

Figure 2 Family of instances for tightness of competitive ratio in Theorem 11.

et al. [13] show that GDY, i.e., transitioning to best coalitions within $\mathcal{A}^D(\pi, i)$ achieves a competitive ratio of $\frac{1}{2}$. However, if all vertices arrive online, then GDY does not even achieve a constant competitive ratio.

▶ **Proposition 9.** *In the deterministic model with free dissolution, GDY has a competitive ratio of* $\Theta(\frac{1}{n})$ *in the matching domain.*

The non-zero edges in construction in the previous proposition even induce *bipartite* graphs, and it is therefore essential for the result by Feldman et al. [13] that one side of the agents is present offline. Proposition 9 indicates that even achieving a constant competitive ratio is a non-trivial task for an online matching algorithm for weighted graphs. In fact, as we will see in Section 6, the usual result of achieving a competitive ratio of $1/2$ with some algorithm [20, 13, 24] is impossible. Still, we can modify the greedy algorithm to achieve a constant competitive ratio.

▶ **Definition 10** (Dissolution threshold algorithm)**.** *Given a matching* π *and a newly arriving vertex* i*, the* dissolution threshold algorithm *(DTA) is the greedy algorithm for the available matchings* $\mathcal{A}(\pi, i) = \{(\pi - j) \cup \{\{i, j\}\} : \{j\} \in \pi\} \cup \{(\pi \setminus \{C\}) \cup \{\{i, j\}, \{k\}\} : C = \{j, k\} \in \pi, w(i, j) \geq 2w(j, k)\}$.

In words, DTA only dissolves a matched pair if the weight of the new edge compared to the weight of the dissolved edge is larger by at least a factor of 2.

▶ **Theorem 11.** *In the deterministic model with free dissolution, DTA has a competitive ratio of* $\frac{1}{6}$ *in the matching domain.*

Proof. We show first that $c_{DTA} \leq \frac{1}{6}$. To this end, we define a family of instances $(G^{k,\epsilon})_{k\geq 1, \epsilon > 0}$ with a sufficiently large gap between the social welfare of the algorithmic and optimal solutions. The construction is depicted in Figure 2.

Let $G^k = (N^k, w^k)$ where $N^k = \{a_i, b_i : 0 \leq i \leq k+1\}$. The edge weights are given by $w(a_i, b_i) = 2^{i+1}$ for $0 \leq i \leq k$, $w(a_{k+1}, b_{k+1}) = 2^{k+1}$, and $w(a_i, a_{i+1}) = 2^i + i\epsilon$ for $0 \leq i \leq k$. All other weights are set to 0. The arrival order σ^k is $(a_0, a_1, b_0, a_2, b_1, \dots, a_{k+1}, b_k, b_{k+1})$.

Consider an execution of DTA for $(G^{k,\epsilon}, \sigma^k)$. It is easy to see that $DTA(G^{k,\epsilon}, \sigma^k) = \{\{a_k, a_{k+1}\}\}$, hence $\mathcal{SW}(DTA(G^{k,\epsilon})) = 2^k + k\epsilon$. On the other hand, the maximum weight matching for $G^{k,\epsilon}$ is $M^*(G^{k,\epsilon}) = \{\{a_i, b_i\} : 0 \leq i \leq k+1\}$ with $\mathcal{SW}(M^*(G^{k,\epsilon})) = 2^{k+1} + 2^{k+2} - 1$. Hence,

$$c_{DTA} = \inf_{G,\sigma} \frac{\mathcal{SW}(DTA(G,\sigma))}{\mathcal{SW}(M^*(G,\sigma))} \leq \inf_{k\geq 1, \epsilon > 0} \frac{\mathcal{SW}(DTA(G^{k,\epsilon}, \sigma^k))}{\mathcal{SW}(M^*(G^{k,\epsilon}))}$$
$$= \inf_{k\geq 1, \epsilon>0} \frac{2^k + k\epsilon}{2^{k+1} + 2^{k+2} - 1} = \frac{1}{6}.$$

Let M be the matching produced by *DTA* and let M^* be a maximum weight matching. Without loss of generality, we may assume that all edges in M^* have positive weight. Further let $F \subseteq E$ be the set of edges that were formed by *DTA* at some point, i.e., F consists of the edges in M as well as all edges that have been dissolved. The key idea is to consider the relation induced by the replacement of edges. More precisely, given two edges $e, e' \in F$, we say that e dominates e' with respect to replacement, written as $e \succ_R e'$, if there exists a chain of edges $e_0, \dots, e_j$ such that $e_0 = e'$, $e_j = e$ and for $i \in [j]$, the formation of e_i has lead to the dissolution of e_{i-1}. Note that M consists precisely of the maximal elements in F with respect to $\succ_R$.

We define a function $\mu\colon M^* \to 2^F$ as follows. Let $m = \{a, b\} \in M^*$ and assume that b arrives after a. For a set of edges $F' \subseteq F$, we define $\max_{\succ_R}(F') = \{f \in F'\colon \nexists f' \in F \text{ with } f' \succ_R f\}$.

If, at the arrival of b, a is already matched with c and $2w(a,c) \geq w(a,b)$, then $\mu(m) = \max_{\succ_R}\{f \in F\colon f \succ_R \{a,c\}, a \in f\}$. If, at the arrival of b, a is already matched with c and $2w(a,c) < w(a,b)$, then b is matched with some d and we define $\mu(m) = \max_{\succ_R}(\{f \in F\colon f \succ_R \{a,c\}, a \in f\} \cup \{f \in F\colon f \succ_R \{b,d\}, b \in f\})$. If at the arrival of b, a is unmatched, then b is matched with some d and we define $\mu(m) = \max_{\succ_R}\{f \in F\colon f \succ_R \{b,d\}, b \in f\}$. Note that μ always maps to a non-empty set. Indeed, if a is unmatched or matched to an agent c with $2w(a,c) < w(a,b)$, then b will certainly be matched because matching with a is an eligible option. Note that μ is a set-valued function, but for all $m \in M^*$, it holds that $|\mu(m)| \leq 2$. Moreover, the only case where $\mu(m)$ contains two edges is if we are in the second case and the edge $\{a, b\}$ is not created.

Bounding the weight of M^* proceeds in three steps. First, we bound the weights of edges in M^* by their image set in F. Then, we show that each edge in F is only in the image of few edges. Lastly, we bound the weight accumulated by edges dominated with respect to $\succ_R$.

To illustrate the proof, it is useful to consider the example in Figure 2 from the first part of the proof. There, $F = \{\{a_i, a_{i+1}\}\colon 0 \leq i \leq k\}$. Moreover, the maximum weight matching is $M^* = \{\{a_i, b_i\}\colon 0 \leq i \leq k+1\}$. For $0 \leq i \leq k$ we have $\mu(\{a_i, b_i\}) = \{\{a_i, a_{i+1}\}\}$ according to the first case in the definition of μ. In addition, $\mu(\{a_{k+1}, b_{k+1}\}) = \{\{a_k, a_{k+1}\}\}$, also according to the first case in the definition of μ.

▷ Claim 12. For all $m \in M^*$, it holds that $w(m) \leq 2w(\mu(m))$.

Proof. Let $m = \{a, b\} \in M^*$ and assume that b arrives after a. Assume first that, at the arrival of b, a is already matched with c and $2w(a,c) \geq w(a,b)$. Then, for every edge $f \in F$ with $f \succ_R \{a,c\}$ and $a \in f$, there exists a sequence of edges $e_0, \dots, e_j$ such that $e_0 = e$, $e_j = f$ and for $i \in [j]$, e_i has replaced e_{i-1}. Hence, $w(e) = w(e_0) \leq w(e_1) \leq \dots \leq w(e_j) = w(f)$. If follows that $w(m) \leq 2w(a,c) \leq 2w(\mu(m))$.

Assume next that, at the arrival of b, a is already matched with c and $2w(a,c) < w(a,b)$. Then, b will be matched with an agent d such that $w(b,d) \geq w(a,b) - 2w(a,c)$. As in the first case, $w(a,c) \leq w(\max_{\succ_R}\{f \in F\colon f \succ_R \{a,c\}, a \in f\})$ and $w(b,d) \leq w(\max_{\succ_R}\{f \in F\colon f \succ_R \{b,d\}, b \in f\})$.

If $d = a$, then $w(m) = w(b,d) \leq w(\mu(m)) \leq 2w(\mu(m))$. Otherwise, i.e., if $d \neq a$, then $\{a,c\} \neq \{b,d\}$ and therefore also $\max_{\succ_R}\{f \in F\colon f \succ_R \{a,c\}, a \in f\} \neq \max_{\succ_R}\{f \in F\colon f \succ_R \{b,d\}, b \in f\}$. Hence, $w(m) \leq w(b,d) + 2w(a,c) \leq 2w(\mu(m))$.

Finally, assume that a is unmatched. Then, b will be matched with an agent d such that $w(b,d) \geq w(a,b)$. As before, $w(b,d) \leq w(\max_{\succ_R}\{f \in F\colon f \succ_R \{b,d\}, b \in f\})$, and we conclude $w(m) \leq w(b,d) \leq w(\mu(m)) \leq 2w(\mu(m))$. This completes the proof of the claim. ◁

Next we bound the number of edges that map to the same edge. Given an edge $e \in F$, define $\mu^{-1}(e) := \{m \in M^* \colon \mu(m) = e\}$.

▷ Claim 13. If $e \in M$, then $|\mu^{-1}(e)| \leq 2$. If $e \in F \setminus M$, then $|\mu^{-1}(e)| \leq 1$.

Proof. By definition of μ, for every $m \in M^*$ and $m' \in \mu(m)$, $m' \cap m \neq \emptyset$. Let $e \in F$. Since M^* is a matching, there can be at most one edge in M^* with a non-empty intersection with each of the two endpoints of e. Hence, $|\mu^{-1}(e)| \leq 2$ and the first part of the claim holds.

Now, let $e = \{a, b\} \in F \setminus M$ and assume for contraction that there exist edges $m, m' \in M^*$ with $m \neq m'$, $e \cap m = \{a\}$, $e \cap m' = \{b\}$, and $e \in \mu(m) \cap \mu(m')$. Since, $e \notin M$, the edge e was replaced during the algorithm by another edge e'. By definition of DTA, it holds that $e \cap e' \neq \emptyset$, say $e \cap e' = \{a\}$. Moreover, $e' \succ_R e$, contradicting that $e \in \mu(m)$. Hence, $|\mu^{-1}(e)| \leq 1$. ◁

It remains to bound the weight of replaced edges. For this, we introduce the following notation. For $e \in M$, define $F_e = \{e' \in F \colon e \succ_R e'\}$.

▷ Claim 14. For all $e \in M$, it holds that $w(F_e) \leq w(e)$.

Proof. Let $e \in M$. Since each edge can only replace a single other edge, F_e is of the form $\{e_1, \dots, e_j\}$ with for some $j \geq 0$ such that, for all $i \in [j-1]$, e_i has replaced e_{i+1}, and e has replaced e_1. Moreover, since a replacement only happens upon a sufficiently large weight improvement, we know that, for all $i \in [j-1]$, $w(e_i) \geq 2w(e_{i+1})$, and $w(e) \geq 2w(e_1)$. Hence, for all $i \in [j]$, it holds that $w(e_i) \leq 2^{-i}w(e)$. Hence, $w(F_e) = \sum_{i=1}^{j} w(e_i) \leq w(e) \sum_{i=1}^{j} 2^{-i} \leq w(e)$. ◁

Combining all three claims, we compute

$$w(M^*) = \sum_{m \in M^*} w(m) \overset{\text{Claim 12}}{\leq} \sum_{m \in M^*} 2w(\mu(m)) \overset{\text{Claim 13}}{\leq} 4 \sum_{e \in M} w(e) + 2 \sum_{e \in F \setminus M} w(e)$$
$$\overset{\text{Claim 14}}{\leq} 4 \sum_{e \in M} w(e) + 2 \sum_{e \in M} w(e) = 6w(M).$$

◀

The next two lemmas let us apply DTA as an online coalition formation algorithm.

▶ **Lemma 15.** *Let $G = (N, w)$ be a complete weighted graph and M^* a maximum weight matching of G. Then, $w(M^*) \geq \frac{1}{n} w(E^+)$, where $E^+ = \{e \in \binom{N}{2} \colon w(e) > 0\}$.*

▶ **Lemma 16.** *Let ALG be a c-competitive algorithm for online matching. Then, ALG is $\frac{c}{n}$-competitive for online coalition formation.*

Proof. Let (G, σ) be an arbitrary instance of online coalition formation. Let M^* be a maximum weight matching for the underlying weighted graph G, and let π^* be a partition maximizing social welfare. Let $E^+ = \{e \in E \colon w(e) > 0\}$ be the set of positive edges. Then,

$$\mathcal{SW}(ALG(G, \sigma)) \geq c\, 2\, w(M^*) \geq c\, 2\, \frac{1}{n}\, w(E^+) \geq \frac{c}{n}\, \mathcal{SW}(\pi^*).$$

The first inequality holds because ALG is a c-competitive algorithm in the matching domain. The second inequality follows from Lemma 15. The last inequality holds because twice the sum of positive edges is an upper bound for the social welfare of any partition. ◀

As a consequence, we can apply *DTA* in the coalition formation domain to obtain a $\Theta(1/n)$-competitive algorithm under free dissolution.

▶ **Corollary 17.** *DTA is* $\frac{1}{6n}$*-competitive for coalition formation in the deterministic model under free dissolution.*

6 Boundaries for Optimal Algorithms

In both of our models, we have found algorithms with a competitive ratio of $\Theta\left(\frac{1}{n}\right)$. This raises the question, whether it is possible to achieve even better algorithms, for instance, with a constant competitive ratio. While we leave the ultimate answer to this question open, in this section, we give some insight why the optimal algorithm may be hard to find. We start with upper bounds for the performance of any algorithm in our two settings.

▶ **Proposition 18.** *In the random arrival model, no online coalition formation algorithm achieves a competitive ratio of more than* $\frac{1}{2}$*.*

▶ **Proposition 19.** *In the deterministic arrival model, no online coalition formation algorithm achieves a competitive ratio of more than* $\frac{1}{3}$ *under free dissolution.*

A similar bound holds for the online matching setting. As we have already discussed in Section 5, our next result is a surprising contrast to the usual possibility of achieving a competitive ratio of $1/2$ in many related settings [20, 13, 24]. Interestingly, once again, our construction only uses bipartite instances. Hence, the crucial property why we cannot achieve a better competitive ratio is that *all* agents arrive online.

▶ **Proposition 20.** *In the deterministic arrival model, no online matching algorithm achieves a competitive ratio of more than* ψ *under free dissolution, where* $\psi = \frac{2}{3+\sqrt{5}} \approx 0.382$*.*

Finally, we want to discuss general obstacles for finding classes of instances on which all algorithms perform poorly. Interestingly, the performance of *GDY*, *WGDY*, and *IWA* is bounded by the same class of instances, namely the instances depicted in Figure 1. This raises the question whether this instance is a general worst-case instance. Our next result shows that this is not the case, unless there exists an algorithm that achieves a constant competitive ratio whenever the number of agents is known. Indeed, if there exists some highly valuable edge, then we can make use of an optimal stopping algorithm to achieve a good competitive ratio. To this end, we show how to apply the odds algorithm [6]. The details are discussed in Appendix B.

▶ **Proposition 21.** *Let* I *be a set of ASHGs and* $\lambda \in (0, 1]$ *a constant such that for each ASHG in* I *given by* $G = (N, w)$*, the maximum weight edge* $e_{\max}$ *has a weight* $w(e_{\max}) \geq \lambda \cdot \pi^*(G)$*, where* $\pi^*(G)$ *maximizes social welfare. Then, there exists an online coalition formation algorithm ALG with* $c_{ALG} \in \Theta(1)$ *on* I *in the random arrival model with known* n*.*

7 Conclusion

We have considered two models of online coalition formation that both facilitate the existence of good algorithms compared to a deterministic arrival model. First, we have diminished the power of the adversary by considering a random arrival of agents. Second, we have increased the capabilities of an algorithm by allowing coalition dissolution. Both models allow for algorithms that achieve a competitive ratio of $\Theta(\frac{1}{n})$. Interestingly, this precisely gets rid of weight dependencies of the best algorithm in the deterministic model.

In both approaches, matchings play an important role. In the random arrival model, we present an algorithm whose output dominates the weight of a randomly created matching. Hence, matchings occur implicitly in the analysis of the algorithm. Under free dissolution, our coalition formation algorithm is itself a matching algorithm. The key challenge is to achieve a constant competitive ratio for online matching under the most general model, where all agents arrive online, and input graphs are weighted and possibly non-bipartite. The idea of our algorithm is to enhance the greedy algorithm by adding a threshold for the improvement in social welfare whenever dissolving a coalition (or edge).

Our work offers several promising directions for future research. First, while we have some indication that it is hard to obtain algorithms with a competitive ratio better than $\Theta(\frac{1}{n})$, we leave open the question whether there are algorithms obtaining a constant competitive ratio. Moreover, it would be interesting to find the optimal online matching algorithm for a fully online model with weighted non-bipartite input instance. Finally, there are many other classes of coalition formation games, such as fractional hedonic games [1] or ordinal classes of hedonic games. Considering a random arrival model or coalition dissolution for these games might lead to intriguing discoveries.

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A Omitted Proofs

In this appendix, we provide all proofs missing in the main paper.

Throughout the appendix, we use the following notation for a fixed arrival order σ. Given an agent $x \in N$ and arrival time $k \in [n]$, we denote by $A^x_{\leq k}$, $A^x_{=k}$, and $A^x_{\geq k}$ the events that $\sigma^{-1}(x) \leq k$, $\sigma^{-1}(x) = k$ and $\sigma^{-1}(x) \geq k$, respectively.

A.1 Random Arrival Model

We start with the results in Section 4, and consider first the analysis of *RMA*.

▶ **Proposition 4.** *RMA has a competitive ratio of $\Theta\left(\frac{1}{n}\right)$ for matching instances under a random arrival order when n is known.*

Proof. Consider an instance given by $G = (N, w)$ such that there exists agents $a, b \in N$ with $w(a, b) = 1$, and $w(x, y) = 0$ if $\{x, y\} \neq \{a, b\}$. Then, the maximum weight matching has weight 1, but *RMA* has only a chance of

$$\frac{\binom{2}{1}\binom{n-2}{\frac{n}{2}-1}}{\binom{n}{\frac{n}{2}}} \cdot \frac{1}{n} = \frac{n}{2(n-1)} \cdot \frac{1}{n} = \Theta\left(\frac{1}{n}\right),$$

to have a and b in the same coalition. There, the first part of the product is the chance that a and b appear in different stages of the algorithm, and the factor of $1/n$ is the probability that they are matched conditioned on them arriving in different stages. Hence, the competitive ratio of *RMA* satisfies $O\left(\frac{1}{n}\right)$.

On the other hand, in an execution of *RMA*, every positive edge has a probability of $\frac{n}{2(n-1)} \cdot \frac{1}{n}$ to contribute to the computed matching. Let $E^+ = \{e \in \binom{n}{2} \colon w(e) > 0\}$. Hence, $\mathbb{E}_\sigma[RMA(G,\sigma)] = \sum_{e \in E^+} \frac{n}{2(n-1)} \cdot \frac{1}{n} w(e) \geq \frac{1}{2n} w(E^+)$. Since any matching is of weight at most $w(E^+)$, we obtain that

$$c_{RMA} \geq \frac{1}{2n} = \Omega\left(\frac{1}{n}\right).$$

Hence, the competitive ratio of *RMA* satisfies $\Theta\left(\frac{1}{n}\right)$. ◀

Next, we complete the analysis of *WGDY* by proving the upper bound for its performance.

▶ **Theorem 6.** *The competitive ratio of WGDY for ASHGs under a random arrival and known and even n is $\Theta\left(\frac{1}{n}\right)$.*

Proof of upper bound. We complete the proof by proving that *WGDY* has a competitive ratio of $O\left(\frac{1}{n}\right)$. To this end we consider again the ASHG given by $G^\epsilon = (N, w^\epsilon)$ in Figure 1, as defined in the proof of Theorem 2.

As in the proof of Theorem 2, in the limit case for small ϵ, the optimal solution has a social welfare of 1 and the social welfare of the partition computed by *WGDY* is equal to the probability of matching a with b.

We start by splitting the probability that $\{a,b\}$ form a coalition by differentiating whether a and b arrive in the same phase of the algorithm and which one arrives first. If they arrive in different phases, then, because of symmetry, the probability that the coalition $\{a,b\}$ forms is the same. If both of them arrive in the first or second phase, then $\{a,b\}$ does not form and thus the probability is 0. Indeed, in this case, in the first phase no coalitions are formed and in the second a, b are directly matched to some agent $x \in X$ or $y \in Y$, respectively because $n+1$ agents arrive in the first phase and thus, because of the pigeonhole principle, at least one agent of each of X and Y arrives in the first phase. Thus, we can without loss of generality assume that a arrives in the first phase and b in the second. The probability that a arrives in the first phase is $\frac{1}{2}$ and then the probability that b arrives in the second phase is $\frac{m+1}{2m+1}$. To shorten the notation we replace the conditions in conditional probabilities by "·" after their first appearance. We obtain

$$\begin{aligned}
c_{WGDY} &\leq \mathbb{P}_\sigma(\{a,b\} \in WGDY(G^\epsilon,\sigma)) \\
&= \mathbb{P}_\sigma(\{a,b\} \in WGDY(G^\epsilon,\sigma) \mid A^a_{\leq m+1}, A^b_{\geq m+2}) \mathbb{P}_\sigma(A^a_{\leq m+1}, A^b_{\geq m+2}) \\
&\quad + \mathbb{P}_\sigma(\{a,b\} \mid A^a_{\geq m+2}, A^b_{\leq m+1}) \mathbb{P}_\sigma(A^a_{\geq m+2}, A^b_{\leq m+1}) \\
&= 2\mathbb{P}_\sigma(\{a,b\} \in WGDY(G^\epsilon,\sigma) \mid A^a_{\leq m+1}, A^b_{\geq m+2}) \mathbb{P}_\sigma(A^a_{\leq m+1}, A^b_{\geq m+2}) \\
&= 2\mathbb{P}_\sigma(\{a,b\} \in WGDY(G^\epsilon,\sigma) \mid \cdot\,) \mathbb{P}_\sigma(A^b_{\geq m+2} \mid A^a_{\leq m+1}) \mathbb{P}_\sigma(A^a_{\leq m+1}) \\
&= 2\mathbb{P}_\sigma(\{a,b\} \in WGDY(G^\epsilon,\sigma) \mid \cdot\,) \frac{m+1}{2m+1} \frac{1}{2} \\
&= \frac{m+1}{2m+1} \mathbb{P}_\sigma(\{a,b\} \in WGDY(G^\epsilon,\sigma) \mid \cdot\,).
\end{aligned}$$

Next, we sum over the number of agents $y \in Y$ that arrive in the second phase. The probability that exactly i agents from Y arrive in the second phase can be computed using the hypergeometric distribution. We now know that b arrives in the second phase, as do i agents $y \in Y$. The probability that $\{a,b\}$ is formed is then equal to the probability that all agents that arrive before b are from Y. To compute this, we sum over the number of agents from Y that arrive before b and fix the position where b arrives. We can calculate the probability that many agents from Y arrive before b by using the hypergeometric distribution again. Furthermore, the probability that b arrives in a fixed position in the second half is $\frac{1}{m+1}$. Once we simplify this expression, we see that $WGDY \in O\left(\frac{1}{n}\right)$.

$$\begin{aligned}
c_{WGDY} &\leq \frac{m+1}{2m+1} \sum_{i=0}^{m} \mathbb{P}_\sigma\Big(\{a,b\} \in WGDY(G^\epsilon,\sigma) \;\Big|\; |\{y \in Y \colon A^y_{\leq m+1}\}| = i, \cdot\,\Big) \\
&\quad \cdot \mathbb{P}_\sigma\Big(|\{y \in Y \colon A^y_{\leq m+1}\}| = i \;\Big|\; \cdot\,\Big) \\
&= \frac{m+1}{2m+1} \sum_{i=0}^{m} \frac{\binom{m}{i}\binom{m}{m-i}}{\binom{2m}{m}} \mathbb{P}_\sigma(\{a,b\} \in WGDY(G^\epsilon,\sigma) \mid \cdot\,) \\
&= \frac{m+1}{2m+1} \sum_{i=0}^{m} \frac{\binom{m}{i}\binom{m}{m-i}}{\binom{2m}{m}} \sum_{j=0}^{i} \mathbb{P}_\sigma(\{a,b\} \in WGDY(G^\epsilon,\sigma) \mid A^b_{=m+2+j}, \cdot\,) \\
&\quad \cdot \mathbb{P}_\sigma(A^b_{=m+2+j} \mid \cdot\,) \\
&= \frac{m+1}{2m+1} \sum_{i=0}^{m} \frac{\binom{m}{i}\binom{m}{m-i}}{\binom{2m}{m}} \sum_{j=0}^{i} \mathbb{P}_\sigma(\{d \colon d \prec^\sigma b, A^d_{\geq m+2}\} \subseteq Y \mid \cdot\,) \\
&\quad \cdot \mathbb{P}_\sigma(\sigma^{-1}(b) = m+2+j \mid \cdot\,) \\
&= \frac{m+1}{2m+1} \sum_{i=0}^{m} \frac{\binom{m}{i}\binom{m}{m-i}}{\binom{2m}{m}} \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{m}{j}} \frac{1}{m+1} \\
&= \frac{1}{m+1} = \frac{2}{n} \in O\left(\frac{1}{n}\right)
\end{aligned}$$

◀

Next, we provide the details for the analysis of *IWA*.

▶ **Theorem 8.** *IWA has a competitive ratio of* $c \in \Theta\left(\frac{1}{n}\right)$.

Proof. Observe that, on each subproblem given by a certain i, *IWA* simply performs an execution of *WGDY* for a subset of 2^{i+1} agents. We use our insights gained about *WGDY* to give a guarantee on the performance of *IWA*.

Let i^* be the last index such that *IWA* has completed both phases. We claim that $2^{i^*+1} \geq \frac{n}{4}$. Assume for contradiction that $2^{i^*+1} < \frac{n}{4}$. Then, the number of agents that have arrived until the completion of iteration i^* is $\sum_{i=0}^{i^*} 2^{i+1} \leq 2 \cdot 2^{i^*+1} < \frac{n}{2}$. Hence, there are still 2^{i^*+2} agents left to complete another iteration, a contradiction.

Let $J \subseteq N$ be the random subset of agents in the last completed iteration. Let $E^+(J) = \{e \in \binom{J}{2} \colon w(e) > 0\}$. By the same computation as in the proof of Theorem 6 for deriving Equation (1), we obtain

$$\mathbb{E}_{\sigma\sim\Sigma(N)}\left[\mathcal{SW}(IWA(G,\sigma))\right] \geq \mathbb{E}_{\sigma\sim\Sigma(J)}\left[\mathcal{SW}(WGDY(G[J],\sigma))\right] \geq \frac{1}{n} w(E^+(J)).$$

Since the set J is a uniformly random subset of N of size 2^{i^*+1}, it follows that every edge in G is present in $G[J]$ with equal probability.

Hence, for $E^+ = \{e \in \binom{N}{2} \colon w(e) > 0\}$, for $n \geq 5$, it holds that

$$w(E^+(J)) \geq \frac{\binom{\frac{n}{4}}{2}}{\binom{n}{2}} w(E^+) = \frac{1}{16}\frac{n(n-4)}{n(n-1)} w(E^+) \geq \frac{1}{64} w(E^+).$$

For $n \leq 4$, the social welfare is equal to twice the weight of any positive edge with probability at least $1/6$, and therefore at least $\frac{1}{3}w(E^+)$.

Analogous to the proof of Theorem 6, we obtain that $c_{IWA} \geq \frac{1}{128n} \in \Omega\left(\frac{1}{n}\right)$.

We proceed with the upper bound for the competitive ratio. Once again, we consider the ASHG given by $G^\epsilon = (N, w^\epsilon)$ in Figure 1, as defined in the proof of Theorem 2. Similar to the proofs of Theorems 2 and 6, in the limit case for small ϵ, the optimal solution has a social welfare of 1 and the social welfare of the partition computed by IWA is equal to the probability of matching a and b. We only consider values of n such that $n = \sum_{i=0}^{k} 2^{i+1}$ for some $k \in \mathbb{N}$. This assures that IWA completes all iterations.

By definition, IWA performs $WGDY$ on 2^{i+1} agents in iteration i. The coalition $\{a, b\}$ can therefore only form if a and b arrive in the same iteration i. To compute the probability that this coalition forms in a given iteration i, we proceed analogously to the proof in Theorem 6. There is one major difference that complicates the necessary computations. The agents that arrive in iteration i can be any subset of N and thus we need to account for the exact number of agents from the sets X and Y. As a consequence, it is also possible that the coalition $\{a, b\}$ forms even if both agents arrive in the second phase of the same iteration. Indeed, even if a and b both do not arrive in the first phase, it can still happen that only agents from Y and a arrive before b arrives.

Let us consider a fixed iteration i. There are 2^i agents that arrive in the first phase of iteration i and 2^i agents that arrive in the second phase for a total of 2^{i+1} agents per iteration. The arrival time of the first agent from iteration i is $1 + \sum_{j=0}^{i-1} 2^{j+1} = 2^{i+1} - 1$ and of the last agent is $\sum_{j=0}^{i} 2^{j+1} = 2^{i+2} - 2$. To shorten the notation let $A^x_{i,1}$ and $A^x_{i,2}$ denote the events that agent x arrives in the first and second phase of iteration i, i.e., they represent the events $A^x_{\geq 2^{i+1}-1} \cap A^x_{\leq 2^{i+1}+2^i-2}$ and $A^x_{\geq 2^{i+1}+2^i-1} \cap A^x_{\leq 2^{i+2}-2}$, respectively. Because of symmetry, we combine the cases where a arrives before and after b to one case where a arrives before b. Furthermore, we estimate the case where both agents arrive in the second phase also with the case where a arrives in the first and b in the second phase. This is possible, since, whenever the coalition $\{a, b\}$ forms, all agents before and between a and b are from one of the two sets X and Y. We can then move one of them to the first phase and they would still form a coalition. Since the probability that both arrive in different phases is also larger than both arriving in the second phase we have an upper bound. We obtain

$$
\begin{aligned}
c_{IWA} \leq & \mathbb{P}_\sigma\left(\{a, b\} \in IWA\left(G^\epsilon, \sigma\right)\right) \\
= & \sum_{i=0}^{i^*} 2\mathbb{P}_\sigma\left(\{a, b\} \in IWA\left(G^\epsilon, \sigma\right) \mid A^a_{i,1}, A^b_{i,2}\right) \mathbb{P}_\sigma\left(A^a_{i,1}, A^b_{i,2}\right) \\
& + \mathbb{P}_\sigma\left(\{a, b\} \in IWA\left(G^\epsilon, \sigma\right) \mid A^a_{i,2}, A^b_{i,2}\right) \mathbb{P}_\sigma\left(A^a_{i,2}, A^b_{i,2}\right) \\
\leq & \sum_{i=0}^{i^*} 4\mathbb{P}_\sigma\left(\{a, b\} \in IWA\left(G^\epsilon, \sigma\right) \mid A^a_{i,1}, A^b_{i,2}\right) \mathbb{P}_\sigma\left(A^a_{i,1}, A^b_{i,2}\right) \\
= & \sum_{i=0}^{i^*} 4\mathbb{P}_\sigma\left(\{a, b\} \in IWA\left(G^\epsilon, \sigma\right) \mid A^a_{i,1}, A^b_{i,2}\right) \frac{2^i}{n} \frac{2^i}{n-1} \\
= & \sum_{i=0}^{i^*} \frac{2^{2i+2}}{n^2 - n} \mathbb{P}_\sigma\left(\{a, b\} \in IWA\left(G^\epsilon, \sigma\right) \mid A^a_{i,1}, A^b_{i,2}\right).
\end{aligned}
$$

Next, we compute $\mathbb{P}_\sigma\left(\{a, b\} \in IWA\left(G^\epsilon, \sigma\right) \mid A^a_{i,1}, A^b_{i,2}\right)$ for iteration i. First, we sum over the number of agents j from Y that arrive in the second phase of iteration i and then we sum over all relevant arrival times of b. These are in the range $[2^{i+1} + 2^i - 1, 2^{i+1} + 2^i - 1 + j]$

as in all other cases the probability is 0. Next, we compute the probability that the coalition $\{a, b\}$ forms using the hypergeometric distribution as we did in Theorem 6. We then simplify the expression and finally bound it from above by using the inequality[4] $\sum_{j=0}^{2^i-1} \frac{\binom{m}{j}\binom{m}{2^i-1-j}}{\binom{2m}{2^i-1}} \frac{1}{2^i-j} \leq \frac{1}{2^i}$ which holds for $m \geq 0$ and $m + 2 > 2^i$. Note that these conditions are fulfilled since $n = 2m + 2 \geq 2$, $n = \sum_{i=0}^{i^*} 2^{i+1} = 2\sum_{i=0}^{i^*} 2^i = 2(m+1)$ and thus also $\sum_{i=0}^{i^*} 2^i = m+1 < m+2$. For brevity, we shorten the conditions in the conditional probabilities by "$\cdot$" after their first appearance.

$$
\begin{aligned}
&\mathbb{P}_\sigma\left(\{a,b\} \in \mathit{IWA}\,(G^\epsilon,\sigma) \mid A^a_{i,1}, A^b_{i,2}\right)\\
&= \sum_{j=0}^{2^i-1} \mathbb{P}_\sigma\Big(\{a,b\} \in \mathit{IWA}\,(G^\epsilon,\sigma) \Big| |\{y \in Y : A^y_{i,2}\}| = j, \cdot\Big)\mathbb{P}_\sigma(|\{y \in Y : A^y_{i,2}\}| = j \mid \cdot)\\
&= \sum_{j=0}^{2^i-1} \frac{\binom{m}{j}\binom{m}{2^i-1-j}}{\binom{2m}{2^i-1}} \mathbb{P}_\sigma\left(\{a,b\} \in \mathit{IWA}\,(G^\epsilon,\sigma) \mid \cdot\right)\\
&= \sum_{j=0}^{2^i-1} \frac{\binom{m}{j}\binom{m}{2^i-1-j}}{\binom{2m}{2^i-1}} \sum_{k=0}^{j} \mathbb{P}_\sigma\left(\{a,b\} \in \mathit{IWA}\,(G^\epsilon,\sigma) \mid A^b_{=2^{i+1}+2^i-1+k}, \cdot\right)\mathbb{P}_\sigma(A^b_{=2^{i+1}+2^i-1+k} \mid \cdot)\\
&= \sum_{j=0}^{2^i-1} \frac{\binom{m}{j}\binom{m}{2^i-1-j}}{\binom{2m}{2^i-1}} \sum_{k=0}^{j} \frac{1}{2^i} \mathbb{P}_\sigma\left(\{a,b\} \in \mathit{IWA}\,(G^\epsilon,\sigma) \mid \cdot\right)\\
&= \sum_{j=0}^{2^i-1} \frac{\binom{m}{j}\binom{m}{2^i-1-j}}{\binom{2m}{2^i-1}} \sum_{k=0}^{j} \frac{1}{2^i} \mathbb{P}_\sigma\left(\{d \in N \setminus \{a,b\} : d \succ^\sigma b \wedge A^d_{i,2}\} \subseteq Y \mid \cdot\right)\\
&= \sum_{j=0}^{2^i-1} \frac{\binom{m}{j}\binom{m}{2^i-1-j}}{\binom{2m}{2^i-1}} \sum_{k=0}^{j} \frac{1}{2^i} \frac{\binom{j}{k}}{\binom{2^i-1}{k}}\\
&= \sum_{j=0}^{2^i-1} \frac{\binom{m}{j}\binom{m}{2^i-1-j}}{\binom{2m}{2^i-1}} \frac{1}{2^i-j} \leq \frac{1}{2^i}
\end{aligned}
$$

With the assumption $n = \sum_{i=0}^{i^*} 2^{i+1}$ and the two previous equations it follows that

$$
\begin{aligned}
c_{IWA} &\leq \sum_{i=0}^{i^*} \frac{2^{2i+2}}{n^2-n} \mathbb{P}_\sigma\left(\{a,b\} \in \mathit{IWA}\,(G^\epsilon,\sigma) \mid A^a_{i,1}, A^b_{i,2}\right)\\
&\leq \sum_{i=0}^{i^*} \frac{2^{2i+2}}{n^2-n} \frac{1}{2^i}\\
&= \frac{2}{n^2-n} \sum_{i=0}^{i^*} 2^{i+1}\\
&= \frac{2n}{n^2-n} = \frac{2}{n-1} \in O\left(\frac{1}{n}\right).
\end{aligned}
$$

[4] We omit the proof for this inequality as it is a long and tedious computation that adds nothing of value to the proof. The correctness of the inequality in the given value ranges can be verified using mathematical software.

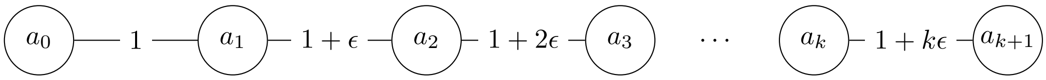

Figure 3 Family of instances for upper bound of the competitive ratio of *GDY* for matching instances under free dissolution.

We have shown that the competitive ratio is $\mathcal{O}\left(\frac{1}{n}\right)$. We conclude that *IWA* has a competitive ratio of $c_{IWA} \in \Theta\left(\frac{1}{n}\right)$ and this bound is tight. ◀

A.2 Deterministic Model under Free Dissolution

We continue with the proofs of results in Section 5.

▶ **Proposition 9.** *In the deterministic model with free dissolution, GDY has a competitive ratio of* $\Theta(\frac{1}{n})$ *in the matching domain.*

Proof. First, note that *GDY* will definitely match a pair of agents with the highest possible weight $W_{\max}$. Moreover, every matching has weight at most $\frac{n}{2}W_{\max}$, and therefore $c_{GDY} \in \Omega(\frac{1}{n})$.

For the upper bound, consider the family of instances depicted in Figure 3. Let $\epsilon > 0$ and $k \in \mathbb{N}$ even. Consider $G^{k,\epsilon} = (N^{k,\epsilon}, w^{k,\epsilon})$ where $N^{k,\epsilon} = \{a_i \colon 0 \le i \le k+1$ and, for $i \in [k+1]$, $w^{k,\epsilon}(a_{i-1}, a_i) = 1 + (i-1)\epsilon$. All other weights are set to 0. Let the arrival order be $(a_0, a_1, \dots, a_k, a_{k+1})$. Let $M^*(G^{k,\epsilon})$ be the maximum weight matching for the instance given by $G^{k,\epsilon}$.

Then, $M^*(G^{k,\epsilon}) = \left\{\{a_{2i}, a_{2i+1}\} \colon 0 \le i \le \frac{k}{2}\right\}$ and *GDY* outputs the matching $\{\{a_k, a_{k+1}\}\}$. Hence,

$$\inf_{\epsilon>0} \frac{\mathcal{SW}(GDY(G^{k,\epsilon}, \sigma^k))}{\mathcal{SW}(M^*(G^{k,\epsilon}, \sigma^k))} = \inf_{\epsilon>0} \frac{1+k\epsilon}{\frac{k}{2} + \frac{1}{4}k(k+2)\epsilon} = \frac{2}{k} \in \Theta\left(\frac{1}{n}\right).$$

◀

Next, we prove the lemma about maximum weight matchings.

▶ **Lemma 15.** *Let* $G = (N, w)$ *be a complete weighted graph and* M^* *a maximum weight matching of* G*. Then,* $w(M^*) \ge \frac{1}{n} w(E^+)$*, where* $E^+ = \{e \in \binom{N}{2} \colon w(e) > 0\}$*.*

Proof. Let $G = (N, w)$ be an arbitrary complete and weighted graph and M^* a maximum weight matching of G. Let $E^+ = \{e \in \binom{N}{2} \colon w(e) > 0\}$ be the set of edges of positive weight. We prove the assertion by induction over $n = |N|$. For $n = 1$ and $n = 2$, the statement is clearly true. Assume now that $n \ge 3$. If $M^* = \emptyset$, then $E^+ = \emptyset$ and the assertion is true. Hence, we may assume that $M^* \ne \emptyset$. Let $m = \{a, b\} \in M^*$ and define $E^+_m = \{e \in E^+ \colon e \cap m \ne \emptyset\}$. Consider $m' = \{c, d\} \in M^*$ with $m' \ne m$. Since M^* is a maximum weight matching, it holds that

$$w(m) + w(m') \ge \max(0, w(a, c)) + \max(0, w(b, d)) \tag{2}$$

and

$$w(m) + w(m') \ge \max(0, w(a, d)) + \max(0, w(b, c)). \tag{3}$$

Let $N_U \subseteq N$ and $N_M \subseteq N$ be the set of unmatched and matched vertices, respectively. Suppose that $N_U = \{x_1, \dots, x_k\}$. By maximality of M^*, for all $j \in [k]$, it holds that

$$w(m) \ge \max(0, w(a, x_j)) + \max(0, w(b, x_{j+1})).$$

There, we identify $x_{k+1} = x_1$. Hence,

$$\sum_{j \in [k]} \max(0, w(a, x_j)) + \max(0, w(b, x_j) \leq |N_U| w(m). \tag{4}$$

Combining Equations (2)–(4), we obtain

$$\begin{aligned} w(E_m^+) &= w(m) + \sum_{v \in N_M \setminus m} \max(0, w(a, v)) + \max(0, w(b, v)) \\ &\quad + \sum_{v \in N_U} \max(0, w(a, v)) + \max(0, w(b, v)) \\ &\leq w(m) + (|N_M| - 2) w(m) + 2 \sum_{m' \in M^* \setminus \{m\}} w(m') + |N_U| w(m) \\ &= (n-1) w(m) + 2 \sum_{m' \in M^* \setminus \{m\}} w(m'). \end{aligned} \tag{5}$$

We can apply induction for $G[N \setminus m]$ to obtain

$$(n-2) w(M^* \setminus \{m\}) \geq w(E^+ \setminus E_m^+). \tag{6}$$

Combining Equations (5) and (6) yields the assertion. ◀

A.3 Boundaries for Optimal Algorithms

We conclude this section with proving the general upper bounds for any algorithm in our respective settings considered in Section 6.

▶ **Proposition 18.** *In the random arrival model, no online coalition formation algorithm achieves a competitive ratio of more than* $\frac{1}{2}$.

Proof. Let ALG be any online coalition formation algorithm. Assume for contradiction that ALG is c-competitive for some $c > \frac{1}{2}$.

First, note that for any $x > 0$, if the the edge between the first two agents has a weight of x, then ALG has to form a coalition of size 2. Otherwise, the competitive ratio is 0 in the instance where just these two agents arrive.

Now, we define a family of instances similar to the instance in Figure 1, with the difference that the agents in the sets X and Y also have positive utilities. Let $\epsilon > 0$ be a sufficiently small number and k be a positive integer. Consider the ASHG given by $G^{k,\epsilon} = (N^{k,\epsilon}, w^{k,\epsilon})$, where $N^{k,\epsilon} = \{a, b\} \cup X \cup Y$ with $|X| = |Y| = k$. Hence, there are a total of $n = 2k+2$ agents. Utilities are given as $w^{k,\epsilon}(a, b) = 1$, $w^{k,\epsilon}(a, x_1) = w^{k,\epsilon}(b, y_1) = w^{k,\epsilon}(x_1, x_2) = w^{k,\epsilon}(y_1, y_2) = \epsilon$ for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. All other utilities are -1.

For ϵ sufficiently small, the social welfare is approximately maximized if the only non-singleton coalition is $\{a, b\}$.

Consider a random arrival order σ. If the first pair of agents are both from X or both from Y, then $\mathcal{SW}(ALG, \sigma) \leq n^2 \epsilon$.

In all other cases, i.e., for the other $(k+1)^2$ possibilities for the first two agents, the social welfare is at most 1. Hence,

$$\begin{aligned} c_{ALG} &\leq \inf_{k>0} \inf_{\epsilon>0} \mathbb{E}_\sigma \left[\mathcal{SW}(ALG(G^{k,\epsilon}, \sigma)) \right] \leq \inf_{k>0} \inf_{\epsilon>0} \frac{(k+1)^2 + \left[\binom{n}{2} - (k+1)^2 \right] n^2 \epsilon}{\binom{n}{2}} \\ &= \inf_{k>0} \inf_{\epsilon>0} \frac{(k+1)^2}{\frac{(2k+2)(2k+1)}{2}} + f(n)\epsilon = \inf_{k>0} \frac{k+1}{2k+1} = \frac{1}{2}. \end{aligned}$$

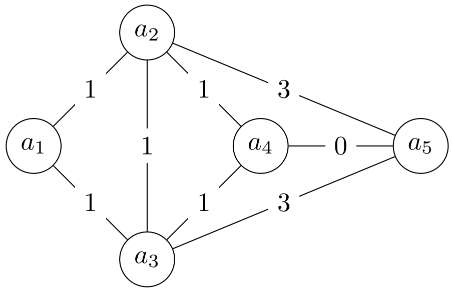

Figure 4 Instance for online coalition formation algorithm with free dissolution. The two missing edges have a weight of $w(a_1, a_4) = w(a_1, a_5) = -12$.

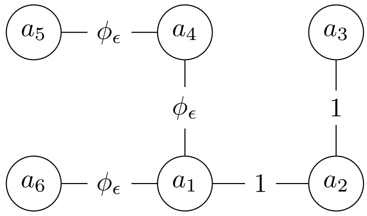

Figure 5 Instance for online matching algorithm with free dissolution. We have $\phi_\epsilon = \frac{1+\sqrt{5}}{2} + \epsilon$ and the missing edges have weight 0. The figure depicts the case in the proof where $b = a_1$.

There, $f(n)$ denotes a function only depending on n. The infimum is attained in the limit for k to infinity. ◀

▶ **Proposition 19.** *In the deterministic arrival model, no online coalition formation algorithm achieves a competitive ratio of more than $\frac{1}{3}$ under free dissolution.*

Proof. Let ALG be any online coalition formation algorithm with free dissolution. Assume for contradiction that ALG is c-competitive for some $c > \frac{1}{3}$.

Consider the ASHG given by $G = (N, w)$ depicted in Figure 4. The agents arrive in the order $(a_1, a_2, a_3, a_4, a_5)$. Since the algorithm is c-competitive for $c > \frac{1}{3}$, it has to maintain a coalition structure that has a social welfare of strictly more than $\frac{1}{3}$ of the current maximum social welfare. As a consequence the algorithm is forced to form the coalitions $\{\{a_1, a_2\}\}, \{\{a_1, a_2, a_3\}\}$, and $\{\{a_1, a_2, a_3\}, \{a_4\}\}$ after a_2, a_3, and a_4 arrive, respectively. If the algorithm forms a different partition at any of these steps, then it achieves a social welfare of at most $\frac{1}{3}$ of the current maximum social welfare. Thus, the adversary can stop and then $c_{ALG} \leq \frac{1}{3}$, a contradiction.

Finally, agent a_5 arrives. In the ASHG given by G, the maximum social welfare of 9 is achieved by the partition $\pi^* = \{\{a_1\}, \{a_2, a_3, a_4, a_5\}\}$. However, the partition before a_5 arrives is $\{\{a_1, a_2, a_3\}, \{a_4\}\}$. It is easy to see that the highest social welfare that can be achieved with free dissolution is 3. This contradicts the assumption that $c_{ALG} > \frac{1}{3}$. ◀

▶ **Proposition 20.** *In the deterministic arrival model, no online matching algorithm achieves a competitive ratio of more than ψ under free dissolution, where $\psi = \frac{2}{3+\sqrt{5}} \approx 0.382$.*

Proof. Let ALG be any online coalition formation algorithm with free dissolution. Let $\psi = \frac{2}{3+\sqrt{5}}$ and assume for contradiction that ALG is c-competitive for some $c > \psi$.

We provide an adversarial strategy along the game depicted in Figure 5. Since the algorithm is c-competitive, it has to maintain a coalition structure that has a social welfare of strictly more than a c-fraction of the current maximum social welfare.

First, two agents a_1 and a_2 arrive with $w(a_1, a_2) = 1$. Hence, ALG has to form the edge $\{a_1, a_2\}$. Now, an agent a_3 arrives with $w(a_1, a_3) = 0$ and $w(a_2, a_3) = 1$. Then, ALG only maintains a matching of positive weight if it leaves a_3 unmatched, or if it dissolves $\{a_1, a_2\}$ and forms $\{a_2, a_3\}$. Since the resulting situation is completely symmetric, we assume without loss of generality that ALG leaves a_3 unmatched.

Let $\epsilon > 0$ and consider the constant $\phi_\epsilon = \frac{1+\sqrt{5}}{2} + \epsilon$.[5] Next, an agent a_4 arrives with $w(a_1, a_4) = \phi_\epsilon$ and $w(a_2, a_4) = w(a_3, a_4) = 0$. Now, ALG only maintains a matching of positive weight if it leaves a_4 unmatched, or if it dissolves $\{a_1, a_2\}$ and forms $\{a_1, a_4\}$. In the former case, ALG creates a matching of weight 1, while the maximum weight matching is $M_1 = \{\{a_1, a_4\}, \{a_2, a_3\}\}$ with $w(M_1) = 1 + \phi_\epsilon$. This implies that $c_{ALG} \leq \frac{1}{1+\phi_\epsilon} = \frac{1}{1+\frac{1+\sqrt{5}}{2}+\epsilon} < \frac{1}{1+\frac{1+\sqrt{5}}{2}} = \frac{2}{3+\sqrt{5}} = \psi < c$, a contradiction. Hence, ALG has to dissolve $\{a_1, a_2\}$ and form $\{a_1, a_4\}$.

Next, an agent a_5 arrives with $w(a_4, a_5) = \phi_\epsilon$ and $w(a_1, a_5) = w(a_2, a_5) = w(a_3, a_5) = 0$. Similar to the situation after the arrival of the third agent, ALG has to leave a_4 unmatched, or to dissolve $\{a_1, a_4\}$ and form $\{a_4, a_5\}$. Let $b \in \{a_1, a_5\}$, such that ALG creates the matching $\{\{a_4, b\}\}$.

Now, an agent a_6 arrives with $w(b, a_6) = \phi_\epsilon$ and $w(x, a_6)$ for all $x \in \{a_i : i \in [5]\} \setminus \{b\}$. Hence, ALG will achieve a matching of weight at most ϕ_ϵ. For $b' \in \{a_1, a_4\}$ with $b' \neq b$, consider the matching $M_2 = \{\{a_2, a_3\}, \{b, a_6\}, \{b', a_4\}\}$. Then, $w(M_2) = 1 + 2\phi_\epsilon$. Hence,

$$c_{ALG} \leq \lim_{\epsilon \to 0} \frac{\phi_\epsilon}{1 + 2\phi_\epsilon} = \frac{\frac{1+\sqrt{5}}{2}}{1 + 1 + \sqrt{5}} = \frac{1 + \sqrt{5}}{2(2 + \sqrt{5})} = \frac{2}{3 + \sqrt{5}} = \psi < c.$$

This is our final contradiction, and hence such an algorithm cannot exist. ◀

B Optimal Stopping Algorithm for Known n

The goal of this section is to prove Proposition 21. Throughout the section, we assume that the number of arriving agents is known to the algorithm. We introduce a new algorithm that we will analyze under a random arrival order.[6]

The key idea is to interpret the computation of a partition in an online ASHG as an optimal stopping problem for independent events. Let $J_1, \dots, J_n$ be mutually independent indicator events. A time step k is called a *success time* if $J_k = 1$. In an *optimal stopping problem* the indicator events $J_1, \dots, J_n$ are observed sequentially and in each step the algorithm may stop the process. The goal is to stop at the last success time, i.e., the goal is to maximize the probability to stop at the last time step where the corresponding event happens.

An optimal algorithm for this problem is the odds strategy that stops at the first success time after a certain stopping time. The optimal stopping time s as well as the resulting success probability of the algorithm can be computed in sublinear time [6, Theorem 1]. Let $p_k = \mathbb{P}(J_k = 1)$ be the probability that the kth event happens and let $r_k = \frac{p_k}{1-p_k}$ be the so called *odds* of I_k. The odds algorithm then sums the odds in reverse order until their sum exceeds one at time step s or sets $s = 1$ otherwise. The algorithm then returns the optimal stopping time s. More precisely, if we are in the first case, then the algorithm returns the largest s such that $\sum_{i=s}^{n} r_i \geq 1$.

[5] To maintain games with rational weights, we can restrict attention to those ϵ where ϕ_ϵ is rational.

[6] Without further assumptions, it is not possible to extend this algorithm to unknown n [6, Chapter 4].

For $2 \leq k \leq n$, let J_k be the event "the maximum weight edge connected to the kth agent is strictly larger than all edges among the first $k-1$ agents". We show next that the events $J_2, \dots, J_n$ are mutually independent. This allows us to execute the odds algorithm.

We start with some notation. Let $E_k = \{(a,b) \in E \colon A^a_{\leq k} \wedge A^b_{\leq k}\}$, $e_k = \arg\max_{e \in E_k} w(e)$, and $e_k = (a_k, b_k)$. The set E_k contains all edges between the first k agents, edge e_k is the maximal weight edge among those, and a_k and b_k are the agents that are connected by e_k. Event $J_k = 1$ if $A^{a_k}_{=k}$ or $A^{b_k}_{=k}$.

▶ **Lemma 22.** *Let an ASHG be given for which all edge weights are pairwise different. Consider indices $2 \leq k_1 < k_2 < \cdots < k_j \leq n$ for some $j \leq n$. Then, it holds that*

$$\mathbb{P}\left(\bigcap_{i=1}^{j} J_{k_i}\right) = \prod_{i=1}^{j} \frac{2}{k_i}. \tag{7}$$

In particular, the events $J_2, \dots, J_n$ are mutually independent.

Proof. Let $j \leq n$ and consider indices $2 \leq k_1 < k_2 < \cdots < k_j \leq n$.

We count the number of arrival orders where the events $J_{k_1}, J_{k_2}, \dots, J_{k_j}$ happen simultaneously and divide by the total number of arrival orders. For this we iterate over the agents in the reverse arrival order. Let $t \in [n]$ be the arrival time we are currently considering. There are two cases, either $\exists i \in [j]$ such that $t = k_i$ or $\forall i \in [j] \colon t \neq k_i$. In the first case, event J_{k_i} happens if $A^{a_{k_i}}_{=k_i}$ or $A^{b_{k_i}}_{=k_i}$, i.e., there are two choices, either a_{k_i} or b_{k_i} arrive at time t. In the second case, any of the remaining t alternatives can arrive. The indices $k_1, \dots, k_j$ index the first case. Let $1 \leq l_1 < l_2 < \cdots < l_{n-j} \leq n$ be the indices of the second case. The total number of arrival orders in which the events $J_{k_1}, J_{k_2}, \dots, J_{k_j}$ happen simultaneously is thus $\prod_{i=1}^{j} 2 \prod_{i=1}^{n-j} l_i$. We divide by the total number of arrival orders $n!$ and get

$$\mathbb{P}\left(\bigcap_{i=1}^{j} J_{k_i}\right) = \frac{\prod_{i=1}^{j} 2 \prod_{i=1}^{n-j} l_i}{n!} = \frac{\prod_{i=1}^{j} 2}{\prod_{i=1}^{j} k_i} = \prod_{i=1}^{j} \frac{2}{k_i}.$$

This proves Equation (7). As a consequence,

$$\mathbb{P}\left(\bigcap_{i=1}^{j} J_{k_i}\right) = \prod_{i=1}^{j} \frac{2}{k_i} = \prod_{i=1}^{j} \mathbb{P}(J_{k_i}).$$

The second equality follows from applying Equation (7) for single events. Hence, we have shown mutual independence of $J_2, \dots, J_n$. ◀

We consider the following algorithm.

▶ **Definition 23** (Maximum edge algorithm). *The* maximum edge algorithm *(MAXE) executes the odds algorithm [6] offline to compute an optimal stopping time s, where n and $p_k = \frac{2}{k}$ for all $k \in \{2, \dots, n\}$ is used as input. Next, the odds strategy is performed online, i.e., no coalition is formed until at least s agents arrived, then the first edge that is larger than all previously seen edges is matched if its weight is strictly larger than 0. If multiple such edges arrive in a step, then the one with highest weight is matched.*

The key insight of identifying *MAXE* with an optimal stopping problem is captured in the next lemma.

▶ **Lemma 24.** *If the odds algorithm for input length n and probabilities $p_k = \frac{2}{k}$ for all $k \in \{2, \dots, n\}$ stops with the last 1, then MAXE outputs a partition of social welfare $2w(e_{\max})$.*

Proof. The arrival of the largest edge corresponds to the last time k where J_k occurs. ◀

We now show that *MAXE* has a constant competitive ratio for every set of instances in which an algorithm that matches only one edge can achieve a constant competitive ratio.

▶ **Proposition 21.** *Let I be a set of ASHGs and $\lambda \in (0, 1]$ a constant such that for each ASHG in I given by $G = (N, w)$, the maximum weight edge $e_{\max}$ has a weight $w(e_{\max}) \geq \lambda \cdot \pi^*(G)$, where $\pi^*(G)$ maximizes social welfare. Then, there exists an online coalition formation algorithm ALG with $c_{ALG} \in \Theta(1)$ on I in the random arrival model with known n.*

Proof. Consider first the case of ASHGs given by graphs with pairwise different edge weights. By Lemma 22, we can apply the odds theorem [6, Theorem 1] for input length n and probabilities $p_k = \frac{2}{k}$ for all $k \in \{2, \dots, n\}$. Then, the odds algorithm has a probability of at least $\frac{1}{2e}$ to terminate successfully [6, Theorem 2]. Moreover, by Lemma 24, *MAXE* computes a partition of social welfare $2w(e_{\max})$ if the odds algorithm terminates successfully. Hence, since $w(e_{\max}) \geq \lambda \cdot \pi^*(G)$, we get that $c_{MAXE} \geq \frac{\lambda}{e} \in \Theta(1)$.

To obtain an algorithm without pairwise different edge weights, we can perturb every arriving edge by independently drawn random variables that select a perturbation from the interval $[-\epsilon, \epsilon]$ uniformly at random, where $\epsilon > 0$ is sufficiently small (this can be selected after observing the first positive edge and can be set to a fraction of this edge). Hence, the perturbed weights are pairwise different with probability 1. By the first part of the proof, *MAXE* computes a partition with social welfare at least $2(w(e_{\max}) - \epsilon)$, whenever the odds algorithm terminates successfully. ◀