Exploring the No-Show Paradox for Condorcet Extensions

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Abstract An important and surprising phenomenon in voting theory is the *No-Show Paradox (NSP)*, which occurs if a voter is better off by abstaining from an election. While it is known that certain voting rules suffer from this paradox in principle, the extent to which it is of practical concern is not well understood. We aim at filling this gap by analyzing the likelihood of the NSP for six Condorcet extensions (Black's rule, Baldwin's rule, Nanson's rule, MaxiMin, Tideman's rule, and Copeland's rule) under various preference models using Ehrhart theory as well as extensive computer simulations. We find that, for few alternatives, the probability of the NSP is rather small (less than 4% for four alternatives and all considered preference models, except for Copeland's rule). As the number of alternatives increases, the NSP becomes much more likely and which rule is most susceptible to abstention strongly depends on the underlying distribution of preferences.

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1 Introduction

Voting theory has shown that every voting rule can result in outcomes that seem undesirable. An important research question is how often these phenomena—known as voting paradoxes—occur and how relevant they are for real-world elections. In this chapter, we employ sophisticated analytical and experimental methods to assess the frequency of the No-Show Paradox (NSP), which occurs if a voter is better off by abstaining from an election (Fishburn and Brams, 1983). The question we address goes back to Fishburn and Brams (1983), who write that "although probabilities of paradoxes have been estimated in other settings, we know of no attempts to assess the likelihoods of the paradoxes of preferential voting discussed above, and would propose this as an interesting possibility for investigation. Is it indeed true that serious flaws in preferential voting such as the No-Show Paradox [. . .] are sufficiently rare as to cause no practical concern?" It is well-known that all Condorcet extensions, a large class of attractive voting rules, suffer from the NSP and this is often used as an argument against Condorcet extensions. Our analysis covers six Condorcet extensions: Black's rule, Baldwin's rule, Nanson's rule, MaxiMin, Tideman's rule, and Copeland's rule.

In principle, quantitative results on voting paradoxes can be obtained via three different approaches. The analytical approach uses theoretical models to quantify paradoxes based on certain assumptions about the voters' preferences such as the *impartial anonymous culture (IAC)* model, in which every anonymous preference profile is equally likely. Analytical results usually tend to be quite hard to obtain and are limited to simple—and often unrealistic—assumptions. The experimental approach uses computer simulations based on underlying stochastic models of how the preference profiles are distributed. Experimental results have less general validity than analytical results, but can be obtained for arbitrary distributions of preferences. Finally, the empirical approach is based on evaluating real-world data to analyze how frequently paradoxes actually occur or how frequently they would have occurred if certain voting rules had been used for the given preferences. Unfortunately, only very limited real-world data for elections is available.

We analytically study the NSP under the assumption of IAC via *Ehrhart theory*, which goes back to the French mathematician Eugène Ehrhart (Ehrhart, 1962). The idea of Ehrhart theory is to model the space of all preference profiles as a discrete simplex and then count the number of integer points inside of the polytope defined by the paradox in question. The number of these integer points can be described by so-called quasi- or Ehrhart-polynomials, which can be computed with the help of computers. The computation of the quasi-polynomials that arise in our context is computationally very demanding, because the dimension of the polytopes grows super-exponentially in the number of alternatives and was only made possible by recent advances of the computer algebra system Normaliz (Bruns et al., 2019a). We complement these results by very elaborate simulations using four common preference models in addition to IAC (IC, urn, spatial, and Mallows). In contrast to existing results, our analysis goes well beyond three alternatives.

2 Related Work

The NSP was first observed by Fishburn and Brams (1983) for a voting rule called *single-transferrable vote (STV)*. Moulin (1988) later proved that all Condorcet extensions are prone to the NSP; the corresponding bound on the number of voters was recently tightened by Brandt et al. (2017). Similar results were obtained for weak preferences and stronger versions of the paradox (Pérez, 2001; Duddy, 2014). The NSP was also transferred to other settings including set-valued voting rules (see, e.g., Jimeno et al., 2009; Pérez et al., 2010; Pérez et al., 2015; Brandl et al., 2019a), probabilistic voting rules (see, e.g., Brandl et al., 2015, 2019b) and random assignment rules (Brandl et al., 2017).[1]

The frequency of the NSP was first studied by Ray (1986), who, in line with Fishburn and Brams's classic paper, analyzed situations where STV can be manipulated in elections with three alternatives. A similar goal was pursued by Lepelley and Merlin (2000) who quantified occurrences of the NSP assuming preferences are distributed according to IC or IAC. However, in contrast to the present approach, Lepelley and Merlin employed different statistical techniques to estimate the likelihood of multiple variants of the paradox and focused on score-based runoff rules in elections with three alternatives. In a recent paper, this setting was revisited by Kamwa et al. (2018) who focused on *single-peaked* preferences, where alternatives can be ordered on a one-dimensional axis and voters' preferences are determined by proximity to their optimal point on this axis. Under this assumption, they found that multiple scoring runoff rules do not suffer from any variant of the NSP while for others, e.g., plurality runoff, the probabilities of a paradox to occur are significantly lower compared to the unrestricted domain.

The general idea to quantify voting paradoxes via IAC has been around since the formal introduction of this preference model by Gehrlein and Fishburn (1976) (see, e.g., Lepelley et al., 1996; Le Breton et al., 2016; Lepelley et al., 2018). Still, it took a good 30 years until the connection to Ehrhart theory (Ehrhart, 1962) was established by Lepelley et al. (2008). We refer to Gehrlein and Lepelley (2011, 2017) for a more profound explanation of all details and an overview of results subsequently achieved (see also, e.g., Wilson and Pritchard, 2007; Schürmann, 2013; Le Breton et al., 2016). The step from three to four alternatives, i.e., from six to 24 dimensions, was only made possible through recent advances in computer algebra systems by De Loera et al. (2012) and Bruns and Söger (2015). Brandt et al. (2016b) used a framework similar to ours to study the frequency of two single-profile paradoxes (the Condorcet Loser Paradox and the Agenda Contraction Paradox). In a recent paper, Bruns et al. (2019b) also made use of the possibility to analyze situations with four alternatives and looked at the Condorcet efficiency of plurality and plurality with runoff as well as the structure of majority graphs and varying Borda paradoxes.

Plassmann and Tideman (2014) conducted computer simulations for various voting rules and paradoxes under a modified spatial model in the three-alternative

[1] Interestingly, when considering set-valued or probabilistic voting rules, there are Condorcet extensions immune to the NSP under suitable assumptions (Brandl et al., 2019a,b).

case. To the best of our knowledge, this is—apart from Brandt et al. (2016b) and Bruns et al. (2019b)—the only study of Condorcet extensions from a quantitative angle.

3 Preliminaries

Let A be a set of m alternatives and $N = \{1, \ldots, n\}$ a set of voters. We assume that every agent $i \in N$ is endowed with a preference relation $\succ_i$ over the alternatives A. More formally, $\succ_i$ is a complete, asymmetric and transitive binary relation, $\succ_i \in A \times A$, which gives a strict ranking over the alternatives. If $x \succ_i y$, we say that i prefers x to y.

A *preference profile* $\succ$ is a tuple consisting of one preference relation per voter, i.e., $\succ = (\succ_1, \ldots, \succ_n)$. By $\succ_{-i}$ we denote the preference profile resulting of voter i abstaining the election, $\succ_{-i} = (\succ_1, \ldots, \succ_{i-1}, \succ_{i+1}, \ldots, \succ_n)$. For most purposes, however, the ordering within the tuple of preference relations is irrelevant and one can alternatively consider multisets of preference relations, so-called *anonymous preference profiles*.

For two alternatives $x, y \in A$ and a preference profile $\succ$ we define the *majority margin* $g_{xy}(\succ)$ as

$$g_{xy}(\succ) = |\{i \in N : x \succ_i y\}| - |\{i \in N : y \succ_i x\}|.$$

Whenever $\succ$ is clear from the context we only write g_{xy}. A *voting rule* is a function f mapping a preference profile $\succ$ to a single alternative, $f(\succ) \in A$.

3.1 Condorcet Extensions

Alternative $x \in A$ is a *Condorcet winner* if it beats all other alternatives in pairwise majority comparisons, i.e., $g_{xy} > 0$ for all $y \in A \setminus \{x\}$. Similarly, x is a *weak Condorcet winner* if it beats or ties all other alternatives, i.e., $g_{xy} \geq 0$ for all $y \in A$. If a voting rule always selects the Condorcet winner whenever one exists, it is called a *Condorcet extension*. A *weak Condorcet extension* returns a weak Condorcet winner whenever (at least) one exists. Clearly, every weak Condorcet extension is a Condorcet extension. A wide variety of Condorcet extensions has been studied in the literature (see, e.g., Fishburn, 1977; Brandt et al., 2016a). In this chapter, we consider six Condorcet extensions: Black's rule, Baldwin's rule, Nanson's rule, MaxiMin, Tideman's rule, and Copeland's rule. The main criteria for selecting these rules were *discriminability* (in order to minimize the influence of lexicographic tie-breaking), *simplicity* (to allow for Ehrhart analysis and because voters generally prefer 'simpler' rules), and *efficient computability* (to enable rigorous and comprehensive simulations).[2] In the following, we briefly define the rules.

Black's rule (Black, 1958) selects the Condorcet winner whenever one exists and otherwise returns a winner according to Borda's rule, where each voter assigns $m-1$ points to his most preferred alternatives, $m-2$ points to his second most preferred alternative, etc., and an alternative with highest accumulated score wins (Borda's rule itself is no Condorcet extension). For the formal definition below, we use affinely equivalent Borda scores based on majority margins.

$$f_{\text{Black}}(\succ) \in \begin{cases} x & \text{if } x \text{ is a Condorcet winner in } \succ \\ \arg\max_{x\in A} \sum_{y\in A} g_{xy} & \text{otherwise.} \end{cases}$$

Baldwin's rule (Baldwin, 1926) proceeds in multiple rounds. In each round, we drop all alternatives with the lowest Borda score and then continue with the reduced preference profile, which is used to calculate updated scores. If multiple—but not all—alternatives are tied last, we delete all of them. Baldwin's rule chooses one of the alternatives that remains when no more alternative can be removed.

Nanson's rule (Nanson, 1883; Niou, 1987) is similar to Baldwin's rule in so far as it also focuses on the Borda scores and gradually eliminates alternatives. However, in contrast to before, we now remove all alternatives with average or below-average Borda score in every round. Nanson's rule returns an alternative out of those that remain when all alternatives have identical score.

The *MaxiMin* rule (Black, 1958), which is also known as the Simpson-Kramer method (Simpson, 1969; Kramer, 1977), looks at the worst pairwise majority comparison for each alternative. It then returns an alternative with maximal such score, formally

$$f_{\text{MaxiMin}}(\succ) \in \arg\max_{x\in A} \min_{y\in A\setminus\{x\}} g_{xy}.$$

Tideman's rule (Tideman, 1987) focuses on the sum of all pairwise majority defeats. It yields an alternative where this sum is closest to zero in terms of absolute value, i.e.,

$$f_{\text{Tideman}}(\succ) \in \arg\max_{x\in A} \sum_{y\in A} \min(0, g_{xy}).^3$$

Copeland's rule (Copeland, 1951) only relies on the signs of the majority margins. It chooses an alternative where the number of majority wins plus half the number of majority draws is maximal:

$$f_{\text{Copeland}}(\succ) \in \arg\max_{x\in A} |\{y\in A : g_{xy} > 0\}| + 1/2\,|\{y \in A : g_{xy} = 0\}|$$

[2] Note that other discriminating Condorcet extensions such as Kemeny's rule, Dodgson's rule, and Young's rule are NP-hard to compute (see, e.g., Brandt et al., 2016a).

[3] Tideman's rule is arguably the least well-known voting rule presented here. It was proposed to efficiently approximate Dodgson's rule and is not to be confused with *ranked pairs* which is sometimes also called Tideman's rule. Also note that the 'dual' rule returning alternatives for which the sum of weighted pairwise majority *wins* is maximal is not a Condorcet extension.

In order to obtain well-defined voting rules we employ alphabetic tie-breaking for all rules defined above. Note that the actual tie-breaking ordering does not influence our results as long as this ordering is fixed. This is not the case if we would allow for tie-breaking based on the preference profile or the choice set. All presented voting rules can be computed in polynomial time and do not rely on the exact preference profile $\succ$ but only on the majority margins that can conveniently be represented by a skew-symmetric matrix or a weighted directed graph.

In order to illustrate these definitions, consider an example with seven voters and four alternatives given by the preference profile and the matrix of pairwise majority margins below. The preference profile is given as a table where a column with header k represents a group of k voters with preferences given in decreasing order.

3	3	1
a	d	b
c	c	d
b	b	a
d	a	c

$$(g_{xy})_{x,y\in A} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -5 & 1 \\ -1 & 5 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix} \end{array}$$

In the absence of a Condorcet winner, Black's rule relies on the Borda scores which can be computed to be $s(\succ) = (10, 9, 12, 11)$ or, affinely equivalent, $(-1, -3, 3, 1)$ when determining them based on the majority margins only. Hence, $f_{\text{Black}}(\succ) = c$.

Having the lowest Borda score, b consequently is the first alternative to be eliminated when applying Baldwin's rule. After dropping c next, we have a strict majority in favor of d against a and thus $f_{\text{Baldwin}}(\succ) = d$.

In the first round of Nanson's rule, we eliminate a and b since both alternatives have a Borda score which is below average. Thereafter, we obtain a strict majority for d against c, meaning d has higher Borda score and it follows $f_{\text{Nanson}}(\succ) = d$.

For MaxiMin, we analyze all alternatives' worst pairwise majority comparison and see that a, c, and d are tied with -1. Due to alphabetic tie-breaking we have $f_{\text{MaxiMin}}(\succ) = a$.

Tideman's rule counts the sum of all pairwise majority defeats, which we find to be 2, 5, 2, and 1 for a, b, c, and d, respectively. The alternative with minimal sum is chosen, hence, $f_{\text{Tideman}}(\succ) = d$.

Lastly, Copeland's rule selects an alternative based on the number of pairwise majority wins and here breaks the tie between b and d alphabetically leading to $f_{\text{Copeland}}(\succ) = b$.

3.2 Strategic Abstention

A voting rule f is *manipulable by strategic abstention* if there exist some N, A, and $\succ$ such that for some $i \in N$, $f(\succ_{-i}) \succ_i f(\succ)$. Given an occurrence of manipulability by strategic abstention, f is said to suffer from the *No-Show Paradox* (NSP) (for N, A, $\succ$). Slightly abusing notation, we also say that $\succ$ is prone to the NSP whenever f, N, and A are clear from the context. All rules defined here are Condorcet extensions and therefore manipulable by strategic abstention. Occurrences of the NSP for Black's, Baldwin's, and Copeland's rule require three alternatives while four alternatives are needed for MaxiMin as well as Nanson's and Tideman's rule.

It is interesting to note that whenever a Condorcet winner exists, no *weak Condorcet extension* allows for manipulation by strategic abstention by a single voter. To see this, assume alternative x is the Condorcet winner, i.e., x wins in a pairwise majority comparison against all other alternatives. While some of these strict majority preferences might turn to indifferences if voter i abstains from the election procedure, this can only happen for comparisons to alternatives less preferred than x according to $>_i$. Hence, every alternative strictly more preferred than x still loses at least the pairwise majority comparison against x, which remains a weak Condorcet winner. We deduce that irrespective of other possible weak Condorcet winners and the underlying tie-breaking, no alternative preferred to x can be chosen. Of the rules defined above, MaxiMin and Tideman's rule are weak Condorcet extensions.[4]

3.3 Stochastic Preference Models

When analyzing properties of voting rules, it is a common approach to sample preferences according to some underlying model. Various concepts to model preferences have been introduced over the years; we refer to Critchlow et al. (1991) and Marden (1995) for a detailed discussion. We focus on three parameter-free models, *impartial culture (IC)* where each voter's preferences are drawn uniformly at random, *impartial anonymous culture (IAC)* where anonymous preference profiles are drawn uniformly at random, and the two-dimensional *spatial model* where we uniformly sample points in the unit square and their proximity determines the voters' preferences. Furthermore, we consider two model families that allow to simulate different degrees of voter correlation. These are the *urn model* (Berg, 1985) with parameter 10 (i.e., whenever a preference relation is drawn, 10 copies of the same relation are added to the urn) and *Mallows' model* (Mallows, 1957) with $\phi = 0.8$. These parameters induce stronger voter correlation than IC and IAC, which is widely considered to be more realistic.

4 Quantifying the No-Show Paradox

The goal in this chapter is to quantify the frequency of the NSP, i.e., to investigate for how many preference profiles a voter is incentivized to abstain from an election.

[4] For both MaxiMin and Tideman's rule, this holds by the observation that a weak Condorcet winner does not lose any pairwise majority comparison. Black's rule fails to be a weak Condorcet extension by definition; a counterexample for Baldwin's, Nanson's, and Copeland's rule is given by Fishburn (1977).

In order to achieve this goal, we employ an exact analysis via Ehrhart Theory and experimental analysis via sampled preference profiles.

4.1 Exact Analysis via Ehrhart Theory

The imminent strength of exact analysis is that it gives reliable theoretical results. On the downside, precise computation is only feasible for very simple preference models and for small values of m. We focus on IAC and make use of Ehrhart theory.

First, note that an anonymous preference profile is completely specified by the number of voters sharing each of the $m!$ possible rankings on m alternatives. Hence, we can uniquely represent an anonymous profile by an integer point x in a space of $m!$ dimensions. We interpret x_i as the number of voters who share ranking i, where rankings are ordered lexicographically. For example, when there are three voters, an anonymous profile is of the following type.

x_1	x_2	x_3	x_4	x_5	x_6
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

For fixed m, our goal is to describe all profiles that are prone to the NSP by using linear (in)equalities that describe a polytope P_n.[5] Given that this is possible, the fraction of profiles prone to the NSP can be computed by dividing the number of integer points contained in P_n by the total number of profiles for n voters, i.e., the number of integer points x satisfying $x_i \geq 0$ for all $1 \leq i \leq m!$ and $\sum_{1 \leq i \leq m!} x_i = n$.

While the latter number is known to be $\binom{m!+n-1}{m!-1}$, the former can be determined using Ehrhart theory. Ehrhart (1962) shows that it can be found by so-called *Ehrhart-* or *quasi-polynomials* f—a collection of q polynomials f_i of degree d such that $f(n) = f_i(n)$ if $n \equiv i \mod q$. Obtaining f is possible via computer programs like LattE (De Loera et al., 2004) or Normaliz (Bruns et al., 2019a).

Copeland's Rule

In order to illustrate this method, first consider Copeland's rule in elections with three alternatives under IAC. For the modeling we need to give linear constraints in terms of voter types—or equivalently majority margins—that describe polytopes containing all profiles prone to the NSP.

We first distinguish between the six possible manipulations from x to y, $x \neq y \in A = \{a, b, c\}$. A case-by-case analysis shows that, due to alphabetic tie-breaking, only manipulations from a to b or c and from either b or c to a are

[5] More precisely, P_n is a *dilated polytope* depending on n, $P_n = nP = \{n\mathbf{x} \colon \mathbf{x} \in P\}$.

possible. In particular, for each of these cases, there is exactly one voting situation admitting an occurrence of the NSP. We find that we can specify the respective profiles using one polytope each:

$$\begin{array}{llllr} g_{ba} \geq 2, & g_{ac} \geq 1, & g_{cb} = 1, & x_6 \geq 1 & (P_1) \\ g_{ca} \geq 2, & g_{ab} \geq 1, & g_{bc} = 1, & x_4 \geq 1 & (P_2) \\ g_{ac} \geq 2, & g_{ba} \geq 1, & g_{bc} = 0, & x_1 \geq 1 & (P_3) \\ g_{ab} \geq 1, & g_{ca} \geq 1, & g_{bc} = 0, & x_2 \geq 1 & (P_4) \end{array}$$

For the sake of readability we here omit but implicitly assume that the total number of voters present is n and there is a nonnegative number of voters per voter type. Polytope P_1, for example, describes a manipulation from a to b. The first three (in)equalities ensure that there is no Condorcet winner in the current profile and that a wins by tie-breaking. The first inequality ensures that b beats a even after the manipulator leaves, while the equality in the third column ensures that b will become a Condorcet winner. The last column demands the presence of a voter of type x_6, the only type able to manipulate. Note that P_1 and P_2 require n to be odd while every profile contained in P_3 or P_4 contains an even number of voters. The total number of anonymous preference profiles admitting a manipulation by abstention is given by the number of integer points contained in polytopes P_1 to P_4.

Black's Rule

When considering different rules or a larger number of alternatives, we find that the number of polytopes as well as the number of linear constraints defining them grows rapidly. Black's rule, for instance, can only be manipulated from a Condorcet winner to a Borda winner or *vice versa*. This distinction is also one of voter parity: a manipulation away from a Condorcet winner is possible for odd n, while n is required to be even in the converse case. In contrast to before, Black's rule allows for a manipulation between any pair of alternatives regardless of n. Hence, we obtain a total of 12 polytopes, one for every possible manipulation and parity of n. The polytopes for even n look as follows.

$$\begin{array}{llll} g_{ab} + g_{ac} \geq g_{ba} + g_{bc}, & g_{ba} \geq 1, & x_6 \geq 1, & \\ g_{ab} + g_{ac} \geq g_{ca} + g_{cb}, & g_{bc} = 0 & & (P_1) \end{array}$$

$$\begin{array}{llll} g_{ab} + g_{ac} \geq g_{ba} + g_{bc}, & g_{ca} \geq 1, & x_4 \geq 1, & \\ g_{ab} + g_{ac} \geq g_{ca} + g_{cb}, & g_{cb} = 0 & & (P_2) \end{array}$$

$$\begin{array}{llll} g_{ba} + g_{bc} \geq g_{ab} + g_{ac} + 1, & g_{ab} \geq 1, & x_5 \geq 1, & \\ g_{ba} + g_{bc} \geq g_{ca} + g_{cb}, & g_{ac} = 0 & & (P_3) \end{array}$$

$$
\begin{aligned}
&g_{ba} + g_{bc} \geq g_{ab} + g_{ac} + 1, && g_{cb} \geq 1, && x_2 \geq 1, \\
&g_{ba} + g_{bc} \geq g_{ca} + g_{cb}, && g_{ca} = 0 &&
\end{aligned} \tag{P_4}
$$

$$
\begin{aligned}
&g_{ca} + g_{cb} \geq g_{ab} + g_{ac} + 1, && g_{ac} \geq 1, && x_3 \geq 1, \\
&g_{ca} + g_{cb} \geq g_{ba} + g_{bc} + 1, && g_{ab} = 0 &&
\end{aligned} \tag{P_5}
$$

$$
\begin{aligned}
&g_{ca} + g_{cb} \geq g_{ab} + g_{ac} + 1, && g_{bc} \geq 1, && x_1 \geq 1, \\
&g_{ca} + g_{cb} \geq g_{ba} + g_{bc} + 1, && g_{ba} = 0 &&
\end{aligned} \tag{P_6}
$$

Polytope P_1, for example, describes a manipulation from a to b (when n is even). The inequalities in the left column model that a currently is the Borda winner. The (in)equalities in the second column guarantee that a manipulator can make b Condorcet winner by abstaining as well as that with him being present, there is no Condorcet winner. The last column demands the presence of a voter of type x_6, the only type able to manipulate.

MaxiMin

When moving to MaxiMin and four alternatives, determining the necessary polytopes becomes tedious. Since alphabetic tie-breaking rules out most symmetries, we need 168 disjoint polytopes of varying sizes to encompass all profiles prone to the NSP. Each of these is defined by 8 to 10 constraints, not counting the total number of voters and nonnegative number per type.

Recall the definition of MaxiMin from Section 3 and assume $f_{\text{MaxiMin}} = x$. For the NSP to occur, two intrinsic conditions have to be satisfied: *(i)* There is a voter i such that $f_{\text{MaxiMin}}(\succ_{-i}) = y \neq x$ and *(ii)* for voter i, we have $y \succ_i x$. We find that for $A = \{a, b, c, d\}$, conditions *(i)* and *(ii)* entail that manipulation from a to b is only possible for $\succ_i$: c, b, a, d and $\succ_j$: d, b, a, c. It can be shown that no instance exists in which both voter types can influence the outcome in their favor. For the sake of this example, let us focus on $\succ_i$.

A first analysis shows that a's highest defeat has to be against d while b's highest defeat necessarily is against c with $g_{ad} = g_{bc}$,[6] and any other defeat of b lower by at least two. This gives rise to a first set of essential constraints.[7]

$$
\begin{aligned}
&g_{ad} = g_{bc}, && g_{ad} \leq 0, \\
&g_{ab} \geq g_{ad}, && g_{ba} \geq g_{ad} + 2 \\
&x_i \geq 1
\end{aligned} \tag{basis}
$$

[6] Theoretically, we only require $g_{ad} - 1 \leq g_{bc} \leq g_{ad}$. As either all g_{xy} are even or all g_{xy} are odd, this collapses to $g_{ad} = g_{bc}$.

[7] Some inequalities are omitted to remove redundancies when taken together with later constraints.

At this point, we distinguish between $g_{cd} = 0$, $g_{cd} \leq -1$, and $g_{cd} \geq 1$. In case $g_{cd} = 0$, we trivially only need bounds on the defeats of c against a and d against b:

$$g_{cd} = 0, \qquad g_{ca} \leq g_{ad}, \qquad g_{db} \leq g_{ad} \tag{A}$$

If $g_{cd} \leq -1$, c's highest defeat could be against a, d, or both. We consequently need a case distinction to accommodate for these possibilities.

$$g_{cd} \leq -1, \qquad g_{db} \leq g_{ab} \tag{B}$$

$$g_{cd} \leq g_{ad}, \qquad g_{ca} \leq g_{ad} \tag{B.1}$$

$$g_{cd} \leq g_{ad}, \qquad g_{ca} \geq g_{ad} + 1, \qquad g_{ac} \geq g_{ad} \tag{B.2}$$

$$g_{cd} \geq g_{ad} + 1, \qquad g_{ca} \leq g_{ad} \tag{B.3}$$

For $g_{cd} \geq 1$ and an almost symmetric reasoning with reversed arguments for c and d we obtain (C), (C.1), (C.2), and (C.3).

Finally, the total set of profiles admitting a manipulation from a to b by i can be described by seven polytopes making use of the constraints developed above. We obtain

- P_1 = (basis) + (A),
- P_2 = (basis) + (B) + (B.1), P_3 = (basis) + (B) + (B.2), P_4 = (basis) + (B) + (B.3),
- P_5 = (basis)+(C)+(C.1), P_6 = (basis)+(C)+(C.2), and P_7 = (basis)+(C)+(C.3).[8]

As we are interested in not only voter type $>_i$ but also $>_j$ and equivalently not only manipulations from a to b but also all different combinations, we need to undergo a similar reasoning 24 times. This amounts to a total of 168 disjoint polytopes to encompass all profiles prone to the NSP. We remark that even though manipulation instances are roughly in line for all 24 types of voters, there are no exact symmetries that allow for reducing the number of polytopes. This is due mostly to lexicographic—i.e., non-symmetric—tie-breaking and the required presence of a certain voter type in the electorate. Both effects diminish as n grows but discrepancies between different types of manipulators are significant up to lower three-digit n.

This approach of modeling profiles prone to the NSP is substantially more involved than using Ehrhart theory for single-profile paradoxes such as the Condorcet Loser Paradox because of three reasons.

(i) An occurrence of the NSP requires the presence of a certain type of voter.
(ii) Preference profiles for which different types of voters are able to manipulate must be counted only once.[9]
(iii) Possible manipulations not only rely on the winning alternative itself but on all majority margins that have to adhere to different constraints.

[8] We choose this informal notation for the sake of readability. It is to be understood in a way that P_1 is the polytope described by (in)equalities labelled (basis) as well as (A). We additionally assume for all polytopes that the sum of voters per type adds up to n and each type consists of a nonnegative number of voters.

[9] This effect is only relevant when there are at least four alternatives.

4.2 Experimental Analysis

In contrast to exact analysis, the experimental approach relies on simulations to grasp the development of different phenomena under varying conditions. On the upside, this usually allows for results for more complex problems or a larger scale of parameters, both of which might be prohibitive for exact calculations. At the same time, however, we find that we need a huge number of simulations per setting to get sound estimates which in turn often requires a high-performance computer and a lot of time. Also, there remains the risk that even a vast amount of simulations fails to capture one specific, possibly crucial, effect.

Regarding the pivotal question of our chapter, the frequency of the NSP for various voting rules, we sample preference profiles for different combinations of n and m using the modeling assumptions explained in Section 3. Our simulations were conducted on XeonE5-2697 v3 multi-core processors with 2 GB memory per job. The total runtime easily accumulates to thirty years on a single-core processor.

5 Results and Discussion

In this section we present our results obtained by both exact analysis and computer simulations.

5.1 Analytical Results under IAC

We first focus on Copeland's rule with three alternatives, as our modeling in Section 4.1 allows for an exact analysis of the NSP. In particular, we compute the following Ehrhart-polynomial $f(n)$ with period $q = 2$:

$$f_0(n) = {}^{1}\!/_{192}\, n^4 - {}^{1}\!/_{48}\, n^3 - {}^{1}\!/_{48}\, n^2 + {}^{1}\!/_{12}\, n$$
$$f_1(n) = {}^{1}\!/_{192}\, n^4 - {}^{5}\!/_{96}\, n^2 + {}^{3}\!/_{64}$$

Recall that $f(n) = f_i(n)$ if $n \equiv i \mod q$. Consequently, the fraction of profiles that admit a manipulation by strategic abstention is given by

$$\frac{f_0(n)}{\binom{n+5}{5}} \quad \text{if } n \text{ is even and} \qquad \frac{f_1(n)}{\binom{n+5}{5}} \quad \text{if } n \text{ is odd.}$$

This frequency of the NSP for Copeland's rule and $m = 3$ is plotted in Figure 1, together with results obtained by computer simulations.

With respect to Black's rule and $m = 3$, we obtain an Ehrhart-polynomial with slightly larger period $q = 6$. Once more, we can explicitly give $f(n)$ which looks as follows:

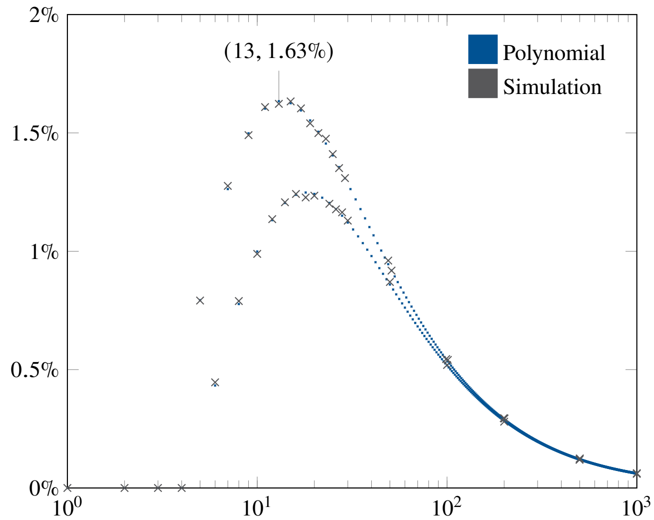

Fig. 1 Fraction of profiles prone to the NSP for Copeland's rule and $m = 3$. The alternating parity of the number of voters has a significant effect on the occurrence of the paradox and gives the appearance of two separate curves.

$$
\begin{aligned}
f_0(n) &= {}^1/_{192}\, n^4 - {}^5/_{48}\, n^2 \\
f_1(n) &= {}^1/_{192}\, n^4 - {}^1/_{48}\, n^3 - {}^7/_{96}\, n^2 + {}^3/_{16}\, n - {}^{19}/_{192} \\
f_2(n) &= {}^1/_{192}\, n^4 - {}^5/_{48}\, n^2 + {}^1/_3 \\
f_3(n) &= {}^1/_{192}\, n^4 - {}^1/_{48}\, n^3 - {}^7/_{96}\, n^2 + {}^3/_{16}\, n + {}^{15}/_{64} \\
f_4(n) &= {}^1/_{192}\, n^4 - {}^5/_{48}\, n^2 + {}^1/_3 \\
f_5(n) &= {}^1/_{192}\, n^4 - {}^1/_{48}\, n^3 - {}^7/_{96}\, n^2 + {}^3/_{16}\, n + {}^{15}/_{64}
\end{aligned}
$$

The fraction of profiles prone to the NSP for Black's rule and $m = 3$ is visualized in Figure 2.

Similar connections between analytical and experimental results for MaxiMin can be observed in Figure 3. Note that, while we are able to explicitly give the Ehrhart-polynomials for Copeland's and Black's rule and $m = 3$ here, this is not possible for MaxiMin and $m = 4$ due to space constraints. The corresponding polynomial $f(n)$ has a period of $q = 55\,440$, i.e., it consists of 55 440 different polynomials. We deduce that no two points in the MaxiMin chart of Figure 3 are computed via the same polynomial, which makes the regularity of the curve even more remarkable.

A couple of points come to mind when closely studying these graphs. First, we note that the results obtained by simulation almost perfectly match the exact calculations, which can be seen as strong evidence for the correctness of both. On the one hand, it confirms our modeling via polytopes, and at the same time highlights that we are running a sufficiently large number of simulations. While this does not bear definite

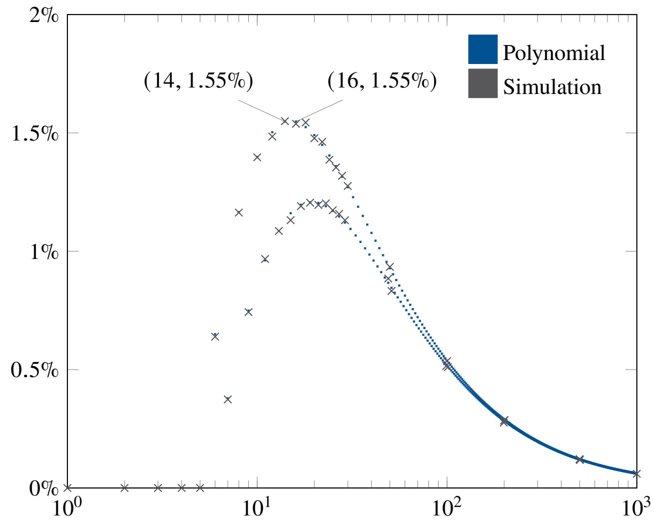

Fig. 2 Fraction of profiles prone to the NSP for Black's rule and $m = 3$.

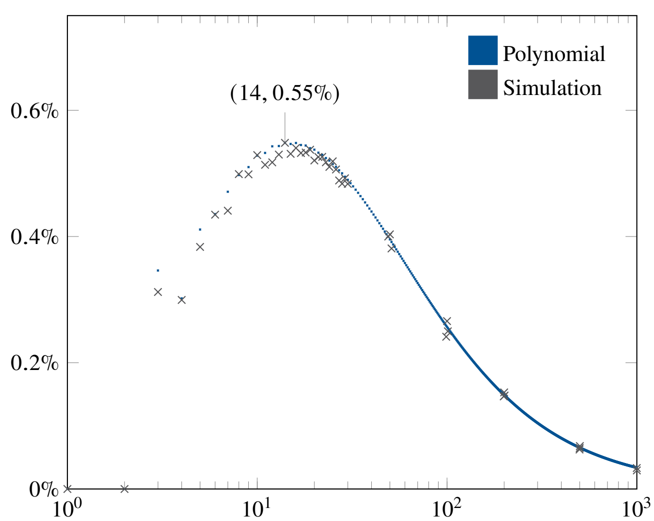

Fig. 3 Fraction of profiles prone to the NSP for MaxiMin and $m = 4$.

testimony to the correctness for larger m, we highlight that our implementation is both generic (with respect to m and n) and not particularly complex, which minimizes the risk of errors. We additionally believe that the perfect smoothness of Figure 4 together with the fact that the NSP is independent of n, m, and the underlying voting rule strongly suggests that our experimental results are sound and reliable.

We see that for Black's rule the maximum is attained at 14 and 16 voters with 1.55% of all profiles, for Copeland's rule the maximum is at 13 voters with 1.63% of all profiles, while for MaxiMin and $m = 4$ it is at 14 voters with 0.55% of all profiles. Hence, we can argue that for elections with very few alternatives, the NSP seems to hardly cause a problem, independent of the number of voters or the voting rule considered. Strikingly, the maxima occur at roughly the same number of voters, with this number varying between being even or odd. Also observe that Black's and Copeland's rule are more sensitive to the parity of n than MaxiMin.

Furthermore, we note that the probability for the NSP to occur converges to zero as n goes to ∞; this holds true for all voting rules considered and all fixed m. Intuitively, this is to be expected as for larger electorates, a single voter's power to sway the result diminishes. This first idea can be confirmed by considering the respective modeling via polytopes. Each modeling will contain at least one equality constraint, e.g., in the third column of our modeling of Copeland's rule in Section 4.1. Consequently, the polytopes describing profiles for which a manipulation is possible are of dimension at most $m! - 1$. By Ehrhart (1962), this means that the number of those profiles can be described by a polynomial of n of degree at most $m! - 1$. The total number of profiles, on the other hand, can equivalently be determined via a polynomial of degree $m!$. Hence, the fraction of profiles prone to the NSP is upper-bounded by $O(1/n)$. Following the intuitive argument, similar behavior is to be expected for all reasonable preference models and voting rules.

For $m = 4$, determining the Ehrhart polynomials for both Black's as well as Tideman's rule proved to be infeasible, even when using a custom-tailored version of Normaliz and employing a high-performance cluster.[10] Copeland's rule unfortunately causes problems even earlier: for four alternatives the modeling via linear (in)equalities quickly becomes very challenging due to the rule only caring about unweighted majority comparisons. For all rules, $m \geq 5$ appears to be out of scope for years to come.

5.2 Experimental Results under IAC

In this section, we rely on simulations to grasp how often the NSP can occur for different combinations of n and m up to 50 voters and 30 alternatives. Our results can be found in Figure 4 and allow for the following observations to be made.

To begin with, the relatively low fraction of profiles prone to the NSP for Copeland's rule, Black's rule, and MaxiMin with a small number of alternatives increases as m grows. This increase is quite dramatic for Copeland's rule and MaxiMin. In particular, for only 20 alternatives and both rules, a rough quarter of all profiles admit manipulation by abstention for a medium count of voters. This number is too large to discard the NSP as merely a theoretical problem. Black's rule, on the other hand, remains stable on a comparatively moderate level. Felsenthal and Nurmi

[10] For Black's rule, we find that the polynomial would be of period $q \approx 2.7 \cdot 10^7$ corresponding to a mid two-digit GB file size.

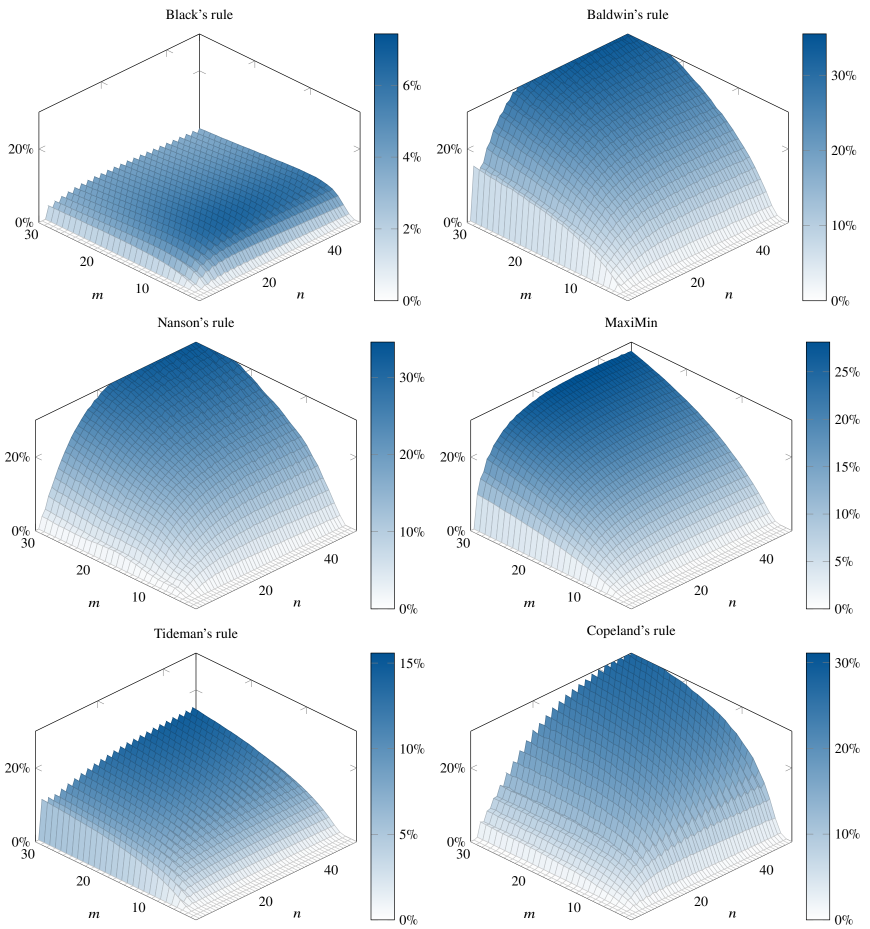

Fig. 4 Fraction of profiles prone to the NSP for different rules and increasing n and m

(2018) argue in favor of Nanson's rule as it is—in contrast to the related Baldwin's rule—not prone to the NSP for three alternatives. We show that this difference between the two rules becomes moot for larger numbers of alternatives: the fractions of profiles allowing for a manipulation are on a roughly identical, severely high level.[11] This shows that voting rules based on Borda scores do not necessarily fare better with respect to the NSP.

When examining Baldwin's rule in Figure 4, the ridge at $n = 3$ immediately catches the observer's eye.

[11] Felsenthal and Nurmi (2018) also show that none of the two rules fares strictly better than the other. Indeed, there are profiles where a manipulation is possible according to Baldwin's rule but not using Nanson's rule and *vice versa*.

We conjecture this unique behavior of Baldwin's rule is due to preference profiles similar in structure to the one depicted on the right. In case voter 3 places sufficiently many alternatives over x, x is going to be eliminated on the way causing y to eventually be chosen. Then again, if voter 3 abstains, x is always going to be selected as long as it beats y in the tie-breaking order. Note that x and y can be chosen almost freely, all other alternatives placed virtually arbitrarily, and many profiles only similar in structure also work.

1	1	1
x	y	$\vdots$
y		x
$\vdots$	$\vdots$	$\vdots$
		y
	x	$\vdots$

Especially when considering Black's, Tideman's, and Copeland's rule, we see that the parity of n crucially influences the results. However, the parity of n does not affect the fractions in a consistent way: higher fractions occur for Black's and Copeland's rule when n is even, in contrast to Tideman's rule where this happens when n is odd. For Black's rule, this is most probably due to the fact that there are more suitable profiles close to having a Condorcet winner ($g_{xy} = 0$) than profiles close to not having one ($g_{xy} = 1$).[12] Copeland score's are integers when the number of voters is odd and half-integers when the number of voters is even. Hence, differences between alternatives are potentially more distinct for an odd number of voters which we assume makes manipulations harder to achieve. For Tideman's rule, we currently lack a convincing explanation for the observed behavior, mostly because it is hard to intuitively grasp when exactly a preference profile is manipulable.

Regarding Baldwin's and Nanson's rule as well as MaxiMin, the parity of n seems to have little effect on the numbers. More detailed analysis shows that at least for MaxiMin this appearance is deceptive: when manipulating towards an alphabetically preferred alternative, fractions are higher for even n, while the contrary holds for manipulations towards an alphabetically less preferred alternative. In sum, these two effects approximately cancel each other out.

The flawless smoothness and regularity of all plots in Figure 4 are due to 10^6 runs per data point. This large number allows for all 95% confidence intervals to be smaller than 0.2%. Our simulations took 35 to 48 single-core hours for *each* data point and there are 1 500 data points per plot.

5.3 Comparing Different Preference Models

In order to get an impression of the frequency of the NSP under different preference models, we fix the number of alternatives to be $m = 4$ or $m = 30$ and sample 10^6

[12] For Black's rule, manipulation is only possible either towards or away from a Condorcet winner since Borda's rule is immune to strategic abstention and manipulation is impossible from Condorcet winner to Condorcet winner.

profiles for increasing n up to 1 000 or 200, respectively.[13] Figure 5 gives the fraction of profiles prone to the NSP.

A close inspection of these graphs allows for multiple conclusions. First, we see that in particular Black's rule shows a severe dependency on the parity of n. For better illustration, we depict two lines per preference model to highlight this effect; which line stands for odd and which for even n is easiest checked using their corresponding point of intersection with the x-axis, which is either 1, 2, or 3 throughout. Apart from explanations given earlier, it is not completely clear why differences are more prominent for some voting rules, why we sometimes see higher percentages for odd n and other times for even n, or why for some instances there is a large discrepancy for one preference model but hardly any for another.

IC and IAC are often criticized for being unrealistic and only giving worst-case estimates (see, e.g., Tsetlin et al., 2003; Regenwetter et al., 2006). This criticism is generally confirmed by our experiments, which show that the highest fractions of profiles is prone to the NSP when the sampling is done according to IC or IAC. A notable exception is Black's rule for 30 alternatives, where a different effect prevails: for many alternatives and comparably few voters, situations in which a Condorcet winner (almost) exists appear less frequently under IC or IAC than under the other preference models. In absence thereof, Black's rule collapses to Borda's rule, which is immune to the NSP. Note that were we to conduct a dual experiment with fixed n and increasing m, the fraction of profiles prone to the NSP using Black's rule and IC or IAC would converge to zero for similar reasons.

We moreover see that IC, IAC, and the urn model exhibit identical behavior for $m = 30$. The second and fourth column of Figure 5 therefore seem to only feature three preference models, even though all five are depicted. This may be surprising at first but is to be expected since IC and IAC can equivalently be seen as urn models with parameters 0 and 1, respectively. For $30! \approx 2.7 \cdot 10^{32}$ voter types and a comparatively small n the difference between parameters 0, 1, and 10 is simply too small for a visible difference.

The large conceptual similarities between Baldwin's and Nanson's rule are also reflected in the corresponding charts. Apart from the peak at $n = 3$ for Baldwin's rule, both look almost identical for all preference models with the small difference being that Nanson's rule appears to feature a slightly lower manipulability. Fewer rounds for winner determination thus do not seem to come at a cost with respect to the NSP.

Finally, Copeland's, Baldwin's, and Nanson's rule as well as MaxiMin to a lesser extent appear to fare exceptionally bad with respect to the NSP and IC, IAC, and the urn model. At the same time, none of these rules exhibits overly conspicuous behavior for the spatial and Mallows' model. This suggests that the risk of a possible manipulation is reduced by structural similarities in the individual preferences compared to a greater likelihood for very diverse rankings. Though generally in line with expectations, we currently do not have a profound explanation for the magnitude of

[13] For increasing m the computations quickly become very demanding. The values for $m = 30$ and $n \geq 99$ are determined with 50 000 runs each only. The size of all 95% confidence intervals is, however, still within 0.5%.

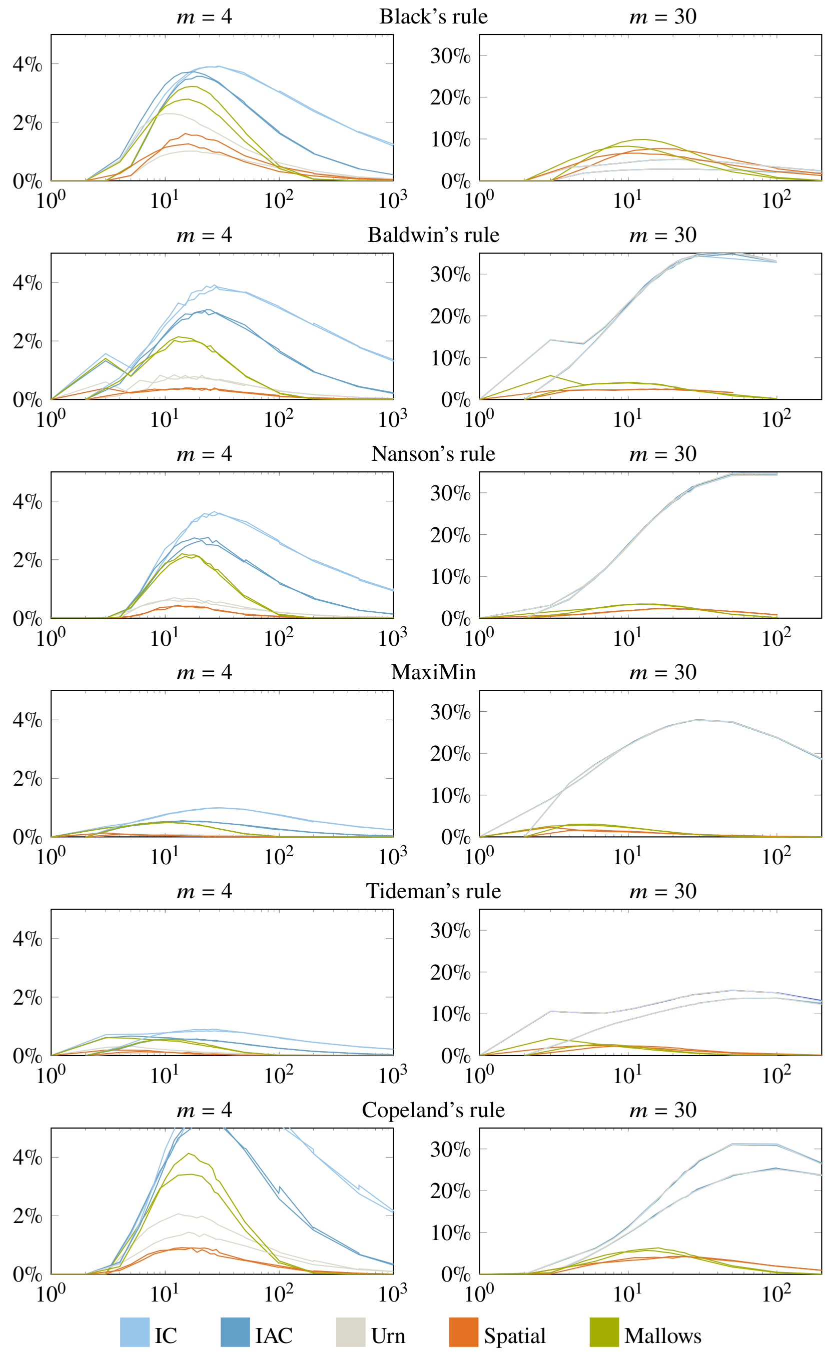

Fig. 5 Profiles prone to the NSP for different rules, fixed m, and increasing n on the x-axis; two lines per preference model depending on the parity of n; IC, IAC and the urn model collapse for $m = 30$, resulting in a bluish grey line

this effect. For Copeland's rule, it is plausible to assume that its particularly bad performance results from using less information, i.e., among all considered rules, Copeland's rule is the only one whose outcome only depends on unweighted majority comparisons.

The maximal fraction of total profiles prone to the NSP for $m = 4$, $m = 30$, different voting rules, preference models, and varying values of n is given in Table 1. Among other things, we for instance note that the maxima constantly occur for a higher number of voters for IC (26 to 51 voters) than for Mallows' model (3 to 17 voters), a fact probably due to an increasing (expected) structure under Mallows' model and larger n.

	m	IC	IAC	Spatial	Urn	Mallows
Black	4	3.92 (29)	3.73 (18)	1.62 (15)	2.30 (10)	3.23 (17)
	30	5.12 (22)	5.12 (22)	7.70 (17)	5.14 (20)	9.90 (13)
Baldwin	4	3.92 (27)	3.07 (23)	0.40 (15)	0.84 (12)	2.14 (13)
	30	35.4 (49)	35.4 (51)	2.54 (21)	35.7 (49)	5.73 (3)
Nanson	4	3.64 (27)	2.76 (24)	0.44 (13)	0.68 (16)	2.20 (14)
	30	34.9 (51)	34.8 (51)	2.38 (21)	34.7 (99)	3.40 (12)
MaxiMin	4	1.00 (30)	0.56 (14)	0.14 (3)	0.13 (3)	0.50 (10)
	30	28.0 (30)	28.0 (30)	2.31 (3)	28.0 (30)	3.01 (6)
Tideman	4	0.80 (26)	0.67 (5)	0.19 (5)	0.32 (5)	0.62 (3)
	30	15.6 (51)	15.6 (49)	2.42 (7)	15.6 (49)	4.12 (3)
Copeland	4	6.96 (29)	5.54 (20)	0.91 (14)	2.07 (13)	4.13 (16)
	30	31.2 (50)	31.0 (50)	4.28 (21)	31.1 (50)	6.33 (16)

Table 1 Maximal percentage of total profiles prone to the NSP for different combinations of voting rules and preference models with $m = 4$ or $m = 30$; the number of voters n for which the maximum occurs attached in parentheses

5.4 Empirical Analysis

We have also analyzed the NSP for empirical data obtained from real-world elections. Unfortunately, such data is generally relatively sparse and imprecise and often only fragmentarily available. A check of all 315 strict profiles contained in the PrefLib library (Mattei and Walsh, 2013) for occurrences of the NSP shows that two profiles admit a manipulation by abstention when Black's rule is used, one profile for each Copeland's, Baldwin's, and Nanson's rule, and that no manipulation is possible for

MaxiMin as well as Tideman's rule.[14] While this suggests a low susceptibility to the NSP in real-world elections, much more data would be required to allow for meaningful conclusions.

6 Conclusion

We analyzed the likelihood of the NSP for six Condorcet extensions (Black's, Baldwin's, and Nanson's rule, MaxiMin, and Tideman's as well as Copeland's rule) under various preference models using Ehrhart theory as well as extensive computer simulations and some empirical data. Our main results are as follows.

- When there are few alternatives, the probability of the NSP is almost negligible (when $m = 4$, less than 1% for MaxiMin and Tideman's rule, less than 4% for Black's, Baldwin's, and Nanson's rule, and less than 7% for Copeland's rule under all considered preference models).
- When there are 30 alternatives and preferences are modeled using IC, IAC, and the urn model, Black's rule is least susceptible to the NSP (< 6%), followed by Tideman's rule (< 16%), MaxiMin (< 29%), Copeland's rule (< 32%) Nanson's rule (< 35%) , and Baldwin's rule (< 36%).
- For 30 alternatives and the spatial and Mallows' model, this ordering is roughly reversed. MaxiMin and Nanson's rule are least susceptible (< 4%), followed by Tideman's rule (< 5%), Baldwin's rule (< 6%), Copeland's rule (< 7%), and Black's rule (< 10%).
- The parity of the number of voters significantly influences the manipulability of Black's, Tideman's, and Copeland's rule. Black's and Copeland's rule are more manipulable for an even number of voters whereas MaxiMin is more manipulable for an odd number of voters (under the IAC assumption).
- Whenever analysis via Ehrhart theory is feasible, the results are perfectly aligned with our simulation results, highlighting the accuracy of the experimental setup.
- Only four (out of 315) strict preference profiles in the PrefLib database are manipulable by strategic abstention (manipulations only occur for Black's, Baldwin's, Nanson's, and Copeland's rule, but not for MaxiMin and Tideman's rule).

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[14] For instance the profile allowing for a manipulation under Copeland's rule is immune to the NSP for all other rules. It features 10 alternatives and 30 voters. Baldwin's and Nanson's rule exhibit the NSP for the same profile.

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