The Metric Distortion of Randomized Social Choice Functions: C1 Maximal Lottery Rules and Simulations

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Abstract

The metric distortion of a randomized social choice function (RSCF) quantifies its worst-case approximation ratio of the optimal social cost when the voters' costs for alternatives are given by distances in a metric space. This notion has recently attracted significant attention as numerous RSCFs that aim to minimize the metric distortion have been suggested. However, such tailored voting rules have usually little appeal other than their low metric distortion. In this paper, we will thus study the metric distortion of well-established RSCFs. In more detail, we first show that C1 maximal lotteries, a well-known class of RSCFs, have a metric distortion of 4 and furthermore prove that this is optimal within the class of majoritarian RSCFs (which only depend on the majority relation). As second contribution, we perform extensive computer experiments on the metric distortion of established RSCFs to obtain insights into their average-case performance. These computer experiments are based on a new linear program for computing the metric distortion of a lottery on a given profile and reveal that some classical RSCFs perform almost as good as the best known RSCF with respect to the metric distortion on randomly sampled profiles.

1 Introduction

An important challenge in multi-agent systems is collective decision making: given the possibly conflicting preferences of a group of agents over some alternatives, a joint decision has to be made. To address this problem, researchers in the field of social choice theory try to identify desirable mechanisms to aggregate the agents' preferences. In more detail, social choice theory is mainly concerned with *social choice functions (SCFs)* and *randomized social choice functions (RSCFs)*, which formalize deterministic and randomized voting rules: an SCF maps the voters' preferences (expressed as linear rankings of the alternatives) to a single winner, and an RSCF returns a probability distribution over the alternatives from which the final winner will be chosen. Moreover, social choice theorists traditionally reason for or against specific voting rules by showing that they satisfy or fail desirable properties [Arrow *et al.*, 2011; Brandt *et al.*, 2016].

As an alternative to this classic approach, Procaccia and Rosenschein [2006] introduced the distortion of voting rules. The idea of this notion is that voters have latent cardinal utilities over the alternatives and voting rules should hence try to elect alternatives with high social welfare. However, SCFs and RSCFs do not have access to the voters' utilities, and the *distortion* of a voting rule thus quantifies the worst-case ratio between the (expected) social welfare of the elected alternative and that of the optimal alternative. A prominent variant of this problem has been suggested by Anshelevich *et al.* [2015]: in the metric distortion setting, voters and alternatives are located in a metric space and the distance between an alternative and a voter specifies the cost occurred to a voter when the alternative is elected. Voting rules should then try to select an alternative with low social cost but, since voters only report ordinal preferences, they can only approximate the optimal social cost. The *metric distortion* of an SCF (resp. RSCF) is hence the worst-case ratio between the (expected) social cost of the elected alternative and of the optimal alternative, where the worst-case is taken over all preference profiles and all metric spaces that are consistent with the given profile.

The metric distortion of SCFs and RSCFs has recently attained significant attention (see, e.g., the survey by Anshelevich *et al.* [2021]). In particular, after Anshelevich *et al.* [2015] and Anshelevich and Postl [2017] have shown that no SCF (resp. RSCF) has a metric distortion of less than 3 (resp. 2), numerous authors tried to find voting rules with minimal metric distortion (see, e.g., [Anshelevich *et al.*, 2018; Kempe, 2020; Kizilkaya and Kempe, 2022; Charikar *et al.*, 2023]). However, many of the suggested voting rules are specifically tailored to minimize the metric distortion and have otherwise little normative appeal. We thus find it noteworthy that some well-established voting rules also have a low metric distortion, in particular when considering RSCFs: for instance, the uniform random dictatorship, which is arguably the most prominent RSCF in the literature, has a metric distortion of 3 [Feldman *et al.*, 2016; Anshelevich and Postl, 2017]. As another example, it has recently been shown that C2 maximal lottery (C2ML) rules, another well-known class of RSCFs, also have a metric distortion of 3 [Charikar *et al.*, 2023]. Since such established voting rules satisfy numerous desirable properties, we find it worthwhile to study their metric distortion in more detail, even though voting rules with lower metric distortion are known.

Our Contribution. The goal of this paper is to enhance the understanding of the metric distortion of established RSCFs. We will contribute to this end in two ways: firstly, we investigate the metric distortion of C1 maximal lottery (C1ML) rules, a class of RSCFs that is well-known for satisfying weak forms of strategyproofness and being robust to small changes in the voters' preferences [Laffond *et al.*, 1993; Hoang, 2017; Brandl *et al.*, 2022]. C1ML rules intuitively choose randomized Condorcet winners: these rules return a lottery p such that, for every lottery q, it is at least as likely that a majority of the voters prefers an outcome drawn from p to an outcome drawn from q than vice versa. As our first result, we show that every C1ML rule has a metric distortion of 4 and give a lower bound on their metric distortion that converges exponentially fast to 4 when the number of alternatives m increases. We furthermore prove that the metric distortion of every majoritarian RSCF (i.e., every RSCF that can only access the majority relation to compute the winners) converges to 4 as m increases. Since C1ML rules are majoritarian, they thus minimize the metric distortion within this class of RSCFs when the numbers of alternatives is unbounded. Our results thus settle the gap on the optimal metric distortion of majoritarian RSCFs.

Secondly, we are also interested in moving past worst-case analyses for the metric distortion of RSCFs because the corresponding worst-case instances often seem unrealistic. To this end, we conduct the first extensive computer experiments on the metric distortion of four types of RSCFs: the uniform random dictatorship, C1 maximal lottery rules, C2 maximal lottery rules, and the RSCFs suggested by Charikar *et al.* [2023] (we refer to these RSCFs as CRWW rules) which have the best currently known metric distortion. In more detail, for each combination of $n \in \{1 + 4k \colon k \in \{1, \dots, 25\}\}$ and $m \in \{5, 8, 11\}$ and three different distributions on the voters' preferences, we sample 1000 preference profiles with n voters and m alternatives, compute the lotteries chosen by our RSCFs, and then compute the worst-case metric distortion for the given lotteries and profiles. Hence, our experiments give insights into the metric distortion of the considered RSCFs for an average-case profile. Our simulation shows that C1ML and C2ML rules perform very well on average-case profiles as they are only slightly worse than CRWW rules. In light of their normative appeal, this gives strong arguments for using a C1ML or C2ML rule instead of an RSCFs that is designed to minimize the metric distortion. To make our computer experiments possible, we also derive a new linear program for computing the metric distortion of a lottery for a given profile, which might be of independent interest.

Related Work. To put our results into perspective, we will next review the most relevant results in the literature and refer to the survey by Anshelevich *et al.* [2021] for more details. An overview of the upper and lower bounds for the metric distortion of various classes of voting rules is given in Table 1.

The study of the metric distortion of deterministic SCFs was initiated by Anshelevich *et al.* [2015] who have, e.g., shown that the Copeland rule has a metric distortion of 5 and that no SCF has a metric distortion of less than 3. Inspired by these results, numerous researchers have tried to find voting rules with a metric distortion of 3. To this end, the metric distortion of many known voting rules has been studied [Goel *et al.*, 2017; Skowron and Elkind, 2017; Anshelevich *et al.*, 2018; Anagnostides *et al.*, 2022] which, however, did not result in an SCF with a metric distortion of less than 5. It was thus only in a recent line of work that SCFs with an optimal metric distortion of 3 have been designed [Munagala and Wang, 2019; Kempe, 2020; Gkatzelis *et al.*, 2020; Kizilkaya and Kempe, 2022; Kizilkaya and Kempe, 2023]. Interestingly, the most recent papers in this line of work try to design normatively appealing SCFs with optimal metric distortion.

	RSCF		SCF	
	LB	UB	LB	UB
All	2.112	2.753	3	3
Tops-only	3	3	∞	∞
Pairwise	3	3	3	$2+\sqrt{5}$
Majoritarian	$\mathbf{4}$	$\mathbf{4}$	5	5

Table 1: Overview over the best known upper and lower bounds on the metric distortion in various classes of voting rules. Each row together with the labels "RSCF" and "SCF" determines a class of voting rules. The columns labeled by "LB" and "UB" show the best known lower and upper bounds for the metric distortion of rules within the given class when there is an unbounded number of alternatives. The bold numbers are shown in this paper.

As an alternative approach to minimize the metric distortion, researchers also started to study RSCFs. In particular, Anshelevich and Postl [2017] have shown that no RSCF has a metric distortion of less than 2 and that the uniform random dictatorship has a metric distortion of 3. Moreover, Gross *et al.* [2017] have proven that that all tops-only RSCFs (i.e., all RSCFs that can only access the voters' favorite alternatives) have a metric distortion of at least $3 - \frac{2}{m}$ when there are m alternatives. Similarly, Charikar *et al.* [2023] have shown that C2 maximal lottery rules have a metric distortion of 3 and it is known that all pairwise RSCFs (i.e., all RSCFs that can only access the numbers of voters that prefer x to y for all pairs of alternative x, y) have a metric distortion of at least $3 - \frac{2}{m}$ [Goel *et al.*, 2017]. Thus, the uniform random dictatorship minimizes the metric distortion within the class of tops-only RSCFs and C2 maximal lottery rules within the class of pairwise RSCFs when the number of alternatives is unbounded. We note that these results are analogous to our results on C1 maximal lottery rules and emphasize the important role of well-known RSCFs in the metric distortion literature.

Finally, a number of further RSCFs have been suggested and analyzed with respect to their metric distortion setting (e.g., [Gross *et al.*, 2017; Fain *et al.*, 2019; Gkatzelis *et al.*, 2020]), but none of these guarantees a metric distortion of strictly less than 3 when the number of alternatives is unbounded. It was hence only very recently that both the upper and lower bound of the metric distortion of RSCFs has been improved: Charikar and Ramakrishnan [2022] have shown that every RSCF has a metric distortion of at least 2.112 and Charikar *et al.* [2023] designed the CRWW rules with a metric distortion of 2.753.

2 Model

Let $V_n = \{v_1, \dots, v_n\}$ denote a finite set of $n \geq 1$ voters and let $X_m = \{x_1, \dots, x_m\}$ denote a finite set of $m \geq 1$ alternatives. We suppose that every voter $v \in V_n$ reports a *preference relation* $\succ_v$, which is formally a complete, transitive, and anti-symmetric binary relation over X_m. The set of all preference relations on X_m is denoted by $\mathcal{R}(X_m)$. A *preference profile* R assigns every voter $i \in N$ to a preference relation $\succ_i$ and the set of all preference profiles over an electorate V_n and a set of alternatives X_m is given by $\mathcal{R}(X_m)^{V_n}$. In this paper, we will allow for both varying sets of voters and alternatives. The set of all preference profiles is hence given by $\mathcal{R}^* = \bigcup_{n,m\in\mathbb{N}} R(X_m)^{V_n}$. Moreover, we denote by $\mathcal{R}^*_m$ the set of all profiles on m alternatives, i.e., $\mathcal{R}^*_m = \bigcup_{n\in\mathbb{N}} R(X_m)^{V_n}$. Given a profile R, we will denote by V_R and X_R the sets of voters and alternatives that are present in the profile R, and by n_R and m_R the sizes of these sets.

Next, we will associate preference profiles with additional notation to facilitate the definition of voting rules. We thus define $t_v(R)$ as the top alternative of voter v in the profile R, i.e., $x = t_v(R)$ is the alternative such that $x \succ_v y$ for all $y \in X_R \setminus \{x\}$. Furthermore, we let the *support* $n_{xy}(R) = |\{v \in V_R \colon x \succ_v y\}|$ denote the number of voters who prefer x to y in R. Finally, the *majority relation* $\succsim_R$ of a profile R is defined by $x \succsim_R y$ if and only if $n_{xy}(R) \geq n_{yx}(R)$. That is, $x \succsim_R y$ if at least as many voters prefer x to y than vice versa. Following the literature, $\succ_R$ denotes the strict part of $\succsim_R$ (i.e., $x \succ_R y$ iff $x \succsim_R y$ and not $y \succsim_R x$) and $\sim_R$ the indifference part (i.e., $x \sim_R y$ iff $x \succsim_R y$ and $y \succsim_R x$).

2.1 Randomized Social Choice Functions

The study object of this paper are randomized social choice functions which are voting rules that may use chance to determine the winner of the election. To make this more formal, we define *lotteries* as probability distributions over the alternatives: a lottery p over a set of alternatives X_R is a function of the type $X_R \to [0, 1]$ such that $\sum_{x\in X_R} p(x) = 1$. We furthermore denote by $\Delta(X_R)$ the set of all lotteries over X_R. A *randomized social choice function (RSCF)* f is then a function that map every preference profile $R \in \mathcal{R}^*$ to a lottery $p \in \Delta(X_R)$. We denote by $f(R, x)$ the probability assigned to alternative x in the profile R.

We next introduce four (classes of) RSCFs:

Uniform random dictatorship. The uniform random dictatorship f_{RD} picks a voter $v \in V_R$ uniformly at random and implements her favorite alternative as the winner of the election. More formally, $f_{RD}(R, x) = \frac{|\{v\in V_R \colon t_v(R)=x\}|}{n_R}$ for every profile $R \in \mathcal{R}^*$ and alternative $x \in X_R$.

C2ML rules. C2 maximal lottery (C2ML) rules, which have been suggested by Fishburn [1984] and recently promoted by, e.g., Brandl *et al.* [2016], compute a randomized Condorcet winner: these rules select a lottery p such that, for all lotteries q, the expected number of voters that prefer the outcome chosen from p to the outcome chosen from q is at least as large as the expected number of voters that prefer the outcome chosen from q to the outcome chosen from p. To formalize this, we extend the support $n_{xy}(R)$ to lotteries p, q by defining $n_{pq}(R) = \sum_{x,y\in A} p(x)q(y)n_{xy}(R)$. Then, the set of C2 maximal lotteries is given by $\mathit{C2ML}(R) = \{p \in \Delta(X_R) \colon \forall q \in \Delta(X_R) \colon n_{pq}(R) \geq n_{qp}(R)\}$. We note that the set the of C2 maximal lotteries is always non-empty by the minimax theorem and almost always a singleton [Laffond *et al.*, 1997; Le Breton, 2005]. Finally, an RSCF is a C2ML rule if $f(R) \in \mathit{C2ML}(R)$ for every profile $R \in \mathcal{R}^*$.

C1ML rules. C1 maximal lottery (C1ML) rules, which go back to Fishburn [1984], also choose a randomized Condorcet winner but in a different sense: C1ML rules select a lottery p such that, for all lotteries q, it is at least as likely that a majority prefers the outcome chosen from p to an outcome chosen from q than vice versa. To formalize this, we extend the majority relation to lotteries p, q by defining $p \succsim_R q$ if and only if $\mathbb{P}_{x\sim p,y\sim q}[x \succ_R y] = \sum_{x,y\in A\colon x\succ_R y} p(x)q(y) \geq \sum_{x,y\in A\colon x\succ_R y} p(y)q(x) = \mathbb{P}_{x\sim p,y\sim q}[y \succ_R x]$. The set of set of C1 maximal lotteries is then $\mathit{C1ML}(R) = \{p \in \Delta(X_R) \colon \forall q \in \Delta(X_R) \colon p \succsim_R q\}$. Just as for C2 maximal lotteries, this set is always non-empty and almost always a singleton. In particular, if the number of voters is odd, there are unique C1 and C2 maximal lotteries. An RSCF is a C1ML rule if $f(R) \in \mathit{C1ML}(R)$ for all profiles $R \in \mathcal{R}^*$.

CRWW rules. Finally, we introduce the RSCFs suggested by Charikar *et al.* [2023], which we refer to as CRWW rules. As a subroutine, these rules rely on another RSCF called $f_{\beta-radius}$. To define this RSCF, we say x β-covers y in a profile R for some $\beta \in [0, 1]$ if $n_{xy}(R) \geq \beta n_R$ and $n_{zx}(R) \geq \beta n_R$ implies $n_{zy} \geq \beta n_R$ for all $z \in X_R$. Moreover, we define $U_\beta(R)$ as the set of alternatives that are not β-covered by any other alternative in R and $R|_{U_\beta(R)}$ as the profile that arises from R by removing all alternatives that are not in $U_\beta(R)$. Then, $f_{\beta-radius}$ computes the uniform random dictatorship on $R|_{U_\beta(R)}$, i.e., $f_{\beta-radius}(R) = f_{RD}(R|_{U_\beta(R)})$. Based on this subroutine, constants $B = 0.876353$, $p = \frac{1}{1+\int_{0.5}^{B} \frac{1}{1-x^2}dx} \approx 0.552327$, and the distribution $\rho(\beta) = \frac{p}{(1-p)(1-\beta^2)}$ on the interval $(\frac{1}{2}, B)$, CCRW rules are defined as follows: with probability p, we execute a C2ML rule and with probability $1 - p$, we sample a value $\beta \in (0.5, B)$ from the distribution $\rho(\beta)$ and return $f_{\beta-radius}(R)$. Hence, an RSCF f is a CCRW rule if there is a C2ML rule f' such that $f(R) = pf'(R) + (1 - p)\int_{0.5}^{B} \rho(\beta)f_{\beta-radius}(R)d\beta$ for all profiles $R \in \mathcal{R}^*$.

We note that the uniform random dictatorship f_{RD}, C2ML rules, and C1ML rules are well-known in the social choice literature. For example, f_{RD} is known to be strategyproof [Gibbard, 1977], whereas both C2ML rules and C1ML rules satisfy strong agenda consistency conditions [Brandl *et al.*, 2016]. By contrast, CCRW rules are designed to minimize the metric distortion and have otherwise little normative appeal. Moreover, we note that f_{RD}, C2ML rules, and C1ML rules belong to important classes of RSCFs: f_{RD} is a *tops-only* RSCF as it only accesses the voters' top alternatives $t_v(R)$, C2ML rules are *pairwise* as they only access the supports $n_{xy}(R)$ for all $x, y \in X_R$, and C1ML rule are *majoritarian* as they only access the majority relation $\succsim_R$ to compute the wining lottery. In more detail, an RSCF f is majoritarian if $f(R) = f(R')$ for all profiles $R, R' \in \mathcal{R}^*$ with $\succsim_R = \succsim_{R'}$.

2.2 Metric Distortion

In order to assess the quality of RSCFs, we analyze their metric distortion in this paper. The idea of this approach is that voters and alternatives are embedded in a metric space and that the distance between a voter v and an alternative x specifies the disutility that voter v experiences when alternative x is selected. Following the utilitarian approach, the optimal alternative is then the one that minimizes the total distance to all voters. However, since voters only report their ordinal preferences over the alternatives instead of their cardinal disutilities, we cannot simply determine the best alternative. The goal of metric distortion is then to select a lottery that approximates the optimal alternative well for every metric space that is consistent with the given preference profile.

To formalize this, we call a function $d : (V_R \cup X_R)^2 \to \mathbb{R}_{\geq 0}$ a *metric* if it satisfies for all $x, y, z \in V_R \cup X_R$ that *i)* $d(x, x) = 0$, *ii)* $d(x, y) = d(y, x)$, and *iii)* $d(x, z) \leq d(x, y) + d(y, z)$. We note that some definitions of metrics also require that $d(x, y) > 0$ if $x \neq y$, but the literature on metric distortion typically omits this condition since it does not affect the results. The distance $d(v, x)$ states the cost experienced by voter v when alternative x is selected. The *social cost* of a alternative x is thus $sc(x, d) = \sum_{v \in V_R} d(v, x)$ and the social cost of lottery p is $sc(p, d) = \sum_{x \in X_R} p(x)sc(x, d)$. Finally, a metric d is *consistent* with a profile R if $x \succ_v y$ implies $d(v, x) \leq d(v, y)$ for all voters $v \in V_R$ and alternatives $x, y \in X_R$ and we denote by $D(R)$ the set of metrics that are consistent with R.

Given a profile R, the goal of metric distortion is to find a lottery whose social cost is close to the optimal social cost for all metric spaces that are consistent with R. We hence define the metric distortion of a lottery p in a profile R as $dist(p, R) = \max_{d \in D(R)} \frac{sc(p,d)}{\min_{y \in A} sc(y,d)}$. Note that $\min_{y \in A} sc(y, d)$ might be 0; in this case, we set $dist(p, R) = \infty$ if $sc(p, d) > 0$ and $dist(p, R) = 1$ if $sc(p, d) = 0$. To simplify the presentation of our results, we will use that $\infty > x$ for all $x \in \mathbb{R}$ and $y + z\infty = \infty$ for all $y \in \mathbb{R}$, $z \in \mathbb{R}_{>0}$. Next, the *metric distortion* $dist(f)$ of an RSCF f is the worst-case distortion over all possible profiles, i.e., $dist(f) = \sup_{R \in \mathcal{R}^*} dist(f(R), R)$. To allow for a more fine-grained analysis, we further define $dist_m(f) = \sup_{R \in \mathcal{R}^*_m} dist(f(R), R)$ as the metric distortion of f when only profiles on m alternatives are considered. We note that $dist(f) = \infty$ and $dist_m(f) = \infty$ if the respective suprema are unbounded.

We recall here that the uniform random dictatorship f_{RD}, C2ML rules f_{C2ML}, and CRWW rules f_{CRWW} have a metric distortion of $dist(f_{RD}) = 3$, $dist(f_{C2ML}) = 3$, and $dist(f_{CCRW}) \leq 2.753$, respectively (i.e., the metric distortion of these RSCFs corresponds to the first three entries in the second column of Table 1). By contrast, the metric distortion of C1ML rules is unknown.

3 Analysis of C1 Maximal Lottery Rules

As our first contribution, we will show that C1ML rules have a metric distortion of 4, and that no other majoritarian RSCF has a lower metric distortion when the number of alternatives is unbounded. Due to space constraints, we defer all proofs but the one of Theorem 1 to the supplementary material.

To prove of our results, we first show a strong relation between the metric distortion of majoritarian RSCFs and distances in the majority relation. To this end, we define the *majority distance* $md(x, y, \succsim_R)$ as the length of the shortest path from x to y in the majority relation $\succsim_R$. In particular, $md(x, x, \succsim_R) = 0$, $md(x, y, \succsim_R) = 1$ if $x \succsim_R y$, and $md(x, y, \succsim_R) = \infty$ if there is no path from x to y in $\succsim_R$. We extend this notion also to lotteries by defining $md(p, y, \succsim_R) = \sum_{x \in A} p(x) md(x, y, \succsim_R)$ and note that $md(p, y, \succsim_R) = \infty$ if there is $x \in X_R$ with $p(x) > 0$ and $md(x, y, \succsim_R) = \infty$.

Proposition 1. *It holds for all majoritarian RSCFs f and preference profiles R that*

1) $dist(f(R), R) \leq 1 + 2 \max_{x \in X_R} md(f(R), x, \succsim_R)$.

2) $dist_m(f) \geq 1 + 2 \max_{x \in X_R} md(f(R), x, \succsim_R)$.

Proof Sketch. For Claim 1), we first note that there is nothing to show if $\max_{x \in X_R} md(f(R), x, \succsim_R) = \infty$ and we hence suppose that $md(f(R), x, \succsim_R) < \infty$ for all $x \in X_R$. We then prove that $sc(x, d) \leq (1 + 2md(x, y, \succsim_R))sc(y, d)$ for all $x, y \in X_R$ and $d \in D(R)$ by an induction on the majority distance between x and y. This insight implies Claim 1) as $dist(f(R), R) = \max_{d \in D(R)} \frac{\sum_{x \in X_R} f(R,x) sc(x,d)}{\min_{y \in X_R} sc(y,d)}$. For Claim 2), we show that there is for every $\epsilon > 0$ a preference profile R^ϵ and a metric space $d \in D(R^\epsilon)$ such that $\succsim_{R^\epsilon} = \succsim_R$ and $\frac{sc(f(R),d)}{\min_{y \in X_R} sc(y,d)} \geq 4 - \epsilon$. Since $f(R^\epsilon) = f(R)$ as f is majoritarian, we infer that $dist_m(f) = 4$ by letting ϵ go to 0. □

We note that related claims have been shown by Anshelevich *et al.* [2018] and Kempe [2020], but these results lack the lower bound given in 2). Based on Proposition 1, we will next compute the metric distortion of C1ML rules.

Theorem 1. *It holds for all C1ML rules f that $dist(f) \leq 4$ and that $dist_m(f) \geq 4 - (\frac{1}{3})^{\frac{\lceil m-3 \rceil}{2}}$ for all $m \geq 3$. Thus, $dist(f) = 4$ for every C1ML rule f.*

Proof. To prove this theorem, we will show that $dist(f) \leq 4$ and that $dist_m(f) \geq 4 - (\frac{1}{3})^{\lfloor \frac{m-3}{2} \rfloor}$ for every $m \geq 3$.

Upper bound: Let f denote an C1ML rule, let $R \in \mathcal{R}^*$ denote a profile, and define $p = f(R)$. It follows from a result by Dutta and Laslier [1999] that $p(x) > 0$ implies $md(x, y, \succsim_R) \leq 2$ for all $x, y \in X_R$. Based on this insight, we will next show that $md(p, z, \succsim_R) \leq \frac{3}{2}$ for all $z \in X_R$ as Claim 1) of Proposition 1 then proves that $dist(p, R) \leq 4$. We thus fix an alternative $z \in X_R$ and let q denote the lottery with $q(z) = 1$. Further, we define $X^+ = \{x \in X_R : x \succ_R z\}$ and $X^- = \{x \in X_R : z \succ_R x\}$. By the definition of C1ML rules, it holds that $p \succsim_R q$, which implies that $\sum_{x \in X^+} p(x) \geq \sum_{x \in X^-} p(x)$ as $q(z) = 1$. This means that $\sum_{x \in X^-} p(x) \leq \frac{1}{2}$. Next, $md(x, z, \succsim_R) = 1$ if $x \succsim_R z$ and $md(x, z, \succsim_R) = 2$ if $z \succ_R x$ due to our previous observation. Therefore, we infer that $md(p, z, \succsim_R) \leq \sum_{x \in X_R : x \succsim_R z} p(x) + 2 \sum_{x \in X_R : y \succ_R x} p(x) = 1 - \sum_{x \in X^-} p(x) + 2 \sum_{x \in X^-} p(x) \leq \frac{3}{2}$. Finally, Claim 1) of Proposition 1 shows that $dist(p, R) \leq 4$.

Lower bound: For proving our lower bound, we recall that C1ML rules only depend on the majority relation and that there is a unique maximal lottery if the majority relation is strict [Laffond *et al.*, 1997]. Moreover, by McGarvey's construction (1953), there is for every complete relation $\succsim$ on X_m a profile R with $\succsim_R = \succsim$. Due to Claim 2) of Proposition 1, we can hence show the lower bound by constructing a complete and anti-symmetric relation $\succsim^*$ for every X_m with $m \geq 3$ such that $md(p, x, \succsim^*) = \frac{3}{2} - \frac{1}{2} \cdot (\frac{1}{3})^{\lfloor \frac{m-3}{2} \rfloor}$, where p is the unique C1 maximal lottery of a profile R with $\succsim_R = \succsim^*$. We thus suppose first that $m \geq 3$ is odd and consider the following relation $\succsim^*$ on X_m: for all odd $k < m$ and all $j \geq k+2$, it holds that $x_{k+1} \succ^* x_k$, $x_k \succ^* x_j$, and $x_j \succ^* x_{k+1}$. It can be checked that the unique C1 maximal lottery p for this relation is defined by $p(x_k) = p(x_{k+1}) = (\frac{1}{3})^{\frac{k+1}{2}}$ for odd $k < m$ and $p(x_m) = (\frac{1}{3})^{\frac{m-1}{2}}$. This means that $\sum_{x \in X^o} p(x) = \sum_{x \in X^e} p(x_k) = \frac{1}{2} - \frac{1}{2}p(x_m)$ for the sets $X^o = \{x_k \in X_m \colon k \in \{1, 3, \dots, m-2\}\}$ and $X^e = \{x_k \in X_m \colon k \in \{2, 4, \dots, m-1\}\}$. Next, will compute $md(p, x_m, \succsim^*)$ and note for this that $md(x_k, x_m, \succsim^*) = 1$ and $md(x_{k+1}, x_m, \succsim^*) = 2$ for all odd $k < m$. Hence, we derive that $md(p, x_m, \succsim^*) = \sum_{x \in X^o} p(x) + 2\sum_{x \in X^e} p(x) = 3\left(\frac{1}{2} - \frac{1}{2}p(x_m)\right) = \frac{3}{2} - \frac{1}{2} \cdot (\frac{1}{3})^{\frac{m-3}{2}}$. Proposition 1 then shows that $dist_m(f) \geq 4 - (\frac{1}{3})^{\frac{m-3}{2}}$. Finally, to extend this result to even m, we add a new alternative to $\succsim^*$ that loses all majority comparisons. Every C1ML will assign probability 0 to this alternative and it does hence not affect our analysis. $\square$

A natural follow-up question of Theorem 1 is whether a majoritarian RSCF can have a lower metric distortion than 4. As we show next, this cannot be the case: the metric distortion of every such rule converges to 4 as m increases.

Theorem 2. *It holds for every majoritarian RSCF f that $dist_m(f) \geq 4 - \frac{3}{m}$ if $m \geq 3$ is odd and $dist_m(f) \geq 4 - \frac{3}{m-1}$ if $m \geq 3$ is even. Thus, $dist(f) \geq 4$.*

Proof sketch. In this sketch, we assume that $m \geq 3$ is odd as the case of even m is similar. To prove the theorem in this case, we will use Claim 2) of Proposition 1 and hence construct a profile R such that $\max_{x \in X_R} md(p, x, \succsim_R) \geq \frac{3}{2} - \frac{3}{2m}$ for every lottery p. Next, McGarvey's theorem (1953) allows us again to focus on complete relations. The theorem then follows by proving that $\max_{x \in X_R} md(p, x, \succsim) \geq \frac{3}{2} - \frac{3}{2m}$ for all lotteries p and the "cyclic" relation $\succsim$ given by $x_i \succ x_{i +_m k}$ for all $i \in \{1, \dots, m\}$, $k \in \{1, \dots, \frac{m-1}{2}\}$ (where $i +_m k = i + k$ if $i + k \leq m$ and $i +_m k = i + k - m$ else). $\square$

Remark 1. The upper bound of Theorem 1 is tight as there are C1ML rules f with $dist(f) = 4$. To see this, consider a profile R with $X_R = \{a, b, c\}$ and $a \succ_R b$, $b \succ_R c$ and $c \sim_R a$ and the lottery p given by $p(a) = p(c) = \frac{1}{2}$. Since p is C1 maximal in R and $md(p, b, \succsim_R) = \frac{3}{2}$, Proposition 1 shows that $dist(f) = 4$ for all C1ML rules f with $f(R) = p$. By contrast, the lower bound in Theorem 1 is not tight: it can be shown that every C1ML rule has a metric distortion of at least $4 - 3\gamma_m$, where γ_m denotes the minimal non-zero probability that a C1ML rule assigns to an alternative in a profile with m alternatives and an odd number of voters. However, the values γ_m are not well-understood [Fisher and Ryan, 1995], so we cannot use them to improve our lower bound.

Remark 2. Proposition 1 also identifies the majoritarian RSCFs that minimize $dist_m(f)$ for a fixed number of alternatives m: this RSCF f chooses for every profile R the lottery p that minimizes $\max_{x \in X_R} md(p, x, \succsim_M)$. Based on a computer-aided approach, we have shown that that this RSCF satisfies $dist_m(f) = 4 - \frac{3}{m}$ for all odd $m \leq 9$, which proves that the lower bound in Theorem 2 is tight in these cases.

4 Simulations

As our second contribution, we conduct extensive computer experiments to gain insights into the average-case metric distortion of the RSCFs defined in Section 2.1. In the following, we hence explain the set-up of these experiments (cf. Sections 4.1 and 4.2) and discuss their results (cf. Section 4.3).

4.1 Setup

For our experiments, we sample 1000 preference profiles with n voters and m alternatives according to three distributions over the voters' preference for every pair $(m, n) \in \{5, 8, 11\} \times \{1 + 4k \colon k \in \{1, \dots, 25\}\}$. For every preference profile R, we then compute the lotteries $f(R)$ selected by the uniform random dictatorship, C2ML rules, C1ML rules, and CWRR rules and the respective metric distortions $dist(f(R), R)$. We note that, since the numbers of voters n is always odd in our experiments, there are unique C1 and C2 maximal lotteries, so we do not have to worry about tie-breaking issues for these RSCFs. We repeat our experiment for three different probability distributions on the voters' preferences to take the effect of these distributions into account and finally plot in Figure 1 the average metric distortion over the 1000 profiles for all RSCFs, distributions, and combinations of m and n. In particular, we consider the following three distributions over the voters' preferences, which are chosen to cover large areas of the "map of elections" [Szufa *et al.*, 2020; Boehmer *et al.*, 2021].

Impartial Culture (IC). In this model, each voter is assigned a preference relation independently and uniformly at random. Hence, for each voter $v \in V_n$ and preference relation $\succ \in \mathcal{R}(X_m)$, the probability that $\succ$ is assigned to v is $\frac{1}{m!}$.

t-Euclidean Model (tEM). In this model, we assign voters and alternatives independently and uniformly at random to points in the t-dimensional cube $[-1, 1]^t$. The voters' preference relation are then given by their distances to the alternatives: a voter v prefers alternative x to alternative y if $|p_v - p_x|_2 < |p_i - p_y|_2$ where p_v, p_x, and p_y denote the points of v, x, and y in the t-dimensional cube. In our experiments, we use this model with for $t = 3$.

Mallow's Model (ϕMM). Mallow's model [Mallows, 1957] is parameterized by a parameter $\phi \in [0, 1]$ and preference relation $\succ$, and introduces a bias towards a common preference relation. In more detail, for every voter v and every preference relation $\succ'$, the probability that voter v is assigned the preference relation $\succ'$ is $\frac{\phi^{|\succ \setminus \succ'|}}{Z}$ (where $Z = \sum_{\hat{\succ} \in \mathcal{R}(X_m)} \phi^{|\succ \setminus \hat{\succ}|}$ and $\succ \setminus \succ' = \{(x, y) \colon x \succ y \wedge y \succ' x\}$). We use Mallow's model for the parameters $\phi = 0.5$ and $\succ = x_1 \succ x_2 \succ \dots \succ x_m$.

4.2 Computing the Metric Distortion

The main challenge for our experiments is to compute the metric distortion $dist(p, R)$ for a given profile R and lottery p. To this end, we first note that it suffices to compute the term $dist(p, R, x) = \max_{d\in D(R)} \frac{sc(p,d)}{sc(x,d)}$ for every alternative x because $dist(p, R) = \max_{x\in X_R} dist(p, R, x)$. Moreover, the term $\frac{sc(p,d)}{sc(x,d)}$ is invariant under scaling d, so we can assume that $sc(x, d) = 1$. Hence, we only need to find for every alternative x the metric d_x that maximizes $sc(p, d_x)$ subject to $d_x \in D(R)$ and $sc(x, d_x) = 1$. While this can be done by linear programs (LPs) that use the distances $d(x, v)$ as variables and encode that $d \in D(R)$ and $sc(x, d) = 1$, this straightforward approach is too slow for our experiments as we need $\mathcal{O}((n + m)^3)$ constraints to formalize the triangle inequalities for metrics.

To derive a more efficient method to compute $dist(p, R, x)$, we will use the idea of biased metrics because Charikar and Ramakrishnan [2022] show that the metric distortion of a lottery p for a profile R can be computed by only considering these metrics. To define these metrics, we let $\succeq_v$ denote the relation given by $x \succeq_v y$ if and only if $x \succ_v y$ or $x = y$ for all $x, y \in X_R$. Then, a metric d is *biased* for a profile R if there is an alternative $x^* \in X_R$ and a function $t : X_R \to \mathbb{R}_{\geq 0}$ such that *(i)* $t(x^*) = 0$, *(ii)* $d(x^*, v) = \frac{1}{2} \max_{x,y\in X_R\colon x\succeq_v y} t(x) - t(y)$ for all $v \in V_R$, and *(iii)* $d(x, v) = d(x_{i^*}, v) + \min_{y\in X_R\colon x\succeq_v y} t(y)$ for all $v \in V_R$ and all $x \in X_R \setminus \{x^*\}$. Unfortunately, due to the maximum and minimum in the definition of these metrics, we cannot directly use them to compute $dist(p, R)$. However, we can use the idea of biased metrics to derive a linear program to efficiently compute $dist(p, R, x_{i^*})$. In more detail, the following LP (called LP 1), which uses variables $d(x, v)$ and $t(x)$ for all $x \in X_R$ and $v \in V_R$, computes $dist(p, R, x^*)$ for every lottery p, profile R, and alternative x^*.

$$
\begin{aligned}
\max\ & \sum_{x\in X_R} p(x) \sum_{v\in V_R} d(x, v) \\
\text{s.t. }\ & t(x^*) = 0 \\
& t(x) \geq 0 && \forall x\in X_R \\
& d(x^*,v) \geq \tfrac{1}{2}(t(x) - t(y)) && \forall v\in V_R,\ x,y\in X_R\colon x \succeq_v y \\
& d(x,v) \leq d(x^*,v) + t(y) && \forall v\in V_R,\ x,y\in X_R\colon x \succeq_v y \\
& d(x,v) + d(x^*,v) \geq t(x) && \forall v\in V_R,\ x\in X_R \\
& \textstyle\sum_{v\in V_R} d(x^*,v) = 1
\end{aligned}
\tag{LP 1}
$$

Proposition 2. *Fix a lottery p, a profile R, and an alternative x^*. If the optimal objective value o^*_{LP} of LP 1 is bounded, then $dist(p, R, x^*) = o^*_{LP}$ and $dist(p, R, x^*) = \infty$ else.*

Proof sketch. Let R denote a profile, p a lottery, and x^* an alternative. First, we will show that $dist(p, R, x^*) \geq o_{LP}$ for the objective value o_{LP} of every feasible solution of LP 1. To prove this, we derive from an arbitrary feasible solution of LP 1 with objective value o_{LP} a metric $d \in D(R)$ such that $\frac{sc(p,d)}{sc(x^*,d)} \geq o_{LP}$. This implies that $dist(p, R, x^*) \geq o^*_{LP}$ if the optimal value o^*_{LP} of LP 1 is bounded and $dist(p, R, x^*) = \infty$ otherwise. Next, we will show that $dist(p, R, x^*) \leq o^*_{LP}$. For this, we prove that there is a biased metric $d \in D(R)$ that maximizes $\frac{sc(p,d)}{sc(x^*,d)}$ and then construct a feasible solution d_{LP} of LP 1 with objective value $\frac{sc(p,d)}{sc(x^*,d)}$ based on d. □

Given a profile R on n voters and m alternatives, LP 1 has $\mathcal{O}(nm^2)$ constraints and it is thus very fast to construct and solve this LP. In particular, even for profiles with 101 voters and 11 alternatives, we can compute the metric distortion of a lottery in a few seconds based on LP 1.

4.3 Simulation Results

Finally, we present and discuss our simulation results: Figure 1 contains a plot for every distribution and all values of $m \in \{5, 8, 11\}$ that shows the average metric distortion for all four considered RSCFs and all $n \in \{1 + 4k\colon k \in \{1, \dots, 25\}\}$. We first observe that, in all experiments, the average metric distortion over the sampled profiles is for all considered RSCFs much smaller than their worst-case metric distortion, thus indicating that such worst-case bounds are too pessimistic for more realistic profiles. Secondly, the average metric distortion of C1ML and C2ML rules is very similar, even though the worst-case metric distortion is 3 for C2ML rules and 4 of C1ML rules. This further demonstrates that worst-case bounds give only limited insights into the average-case performance of RSCFs, which emphasizes the value of our computer experiments. Finally, we note that the CRWW rule has almost always the best average metric distortion, but the C1ML and C2ML rules are often only slightly worse.

Next, the average metric distortions of our RSCFs strongly depend on the underlying distribution over the voters' preferences as well as the numbers of voters n and alternatives m. In particular, under the IC model, the average metric distortion of the uniform random dictatorship decreases for all values of m as n increases. We explain this phenomenon as follows: as the number of voters increases, it becomes more and more likely in the IC model that each alternative is top-ranked by roughly the same number of voters and that all alternatives are equally "good" in the drawn preference profile. In such profiles, the uniform random dictatorship f_{RD} assigns probability close to $\frac{1}{m}$ to all alternatives, which results in a metric distortion close to 2. In the supplementary material, we even prove that the expected metric distortion of f_{RD} converges to 2 in the IC model as n increases. By contrast, the average metric distortion of C2ML rules and C1ML rules under the IC model is largely constant in n but decreases as m increases. The reason for this is that C1ML and C2ML rules often only randomize over few alternatives (see [Brandl *et al.*, 2022] for this claim), even though alternatives are roughly equally good. Then, it can be shown (see the supplementary material) that, as the number of voters increases, the expected metric distortion of C1ML and C2ML rules converges approximately to $2 + \frac{1}{m-1}$ in the IC model, which explains very well the values observed in our experiments. Finally, for the CRWW rule, similar observations as for the C1ML and C2ML rule applies, but the effect is mitigated as we mix the C2ML rule with an RSCF related to the uniform random dictatorship.

By contrast, for both the Euclidean model (for $t = 3$) and Mallow's model (for $\phi = \frac{1}{2}$), the average metric distortion of the uniform random dictatorship is roughly constant in the number of voters and by far the largest among the tested

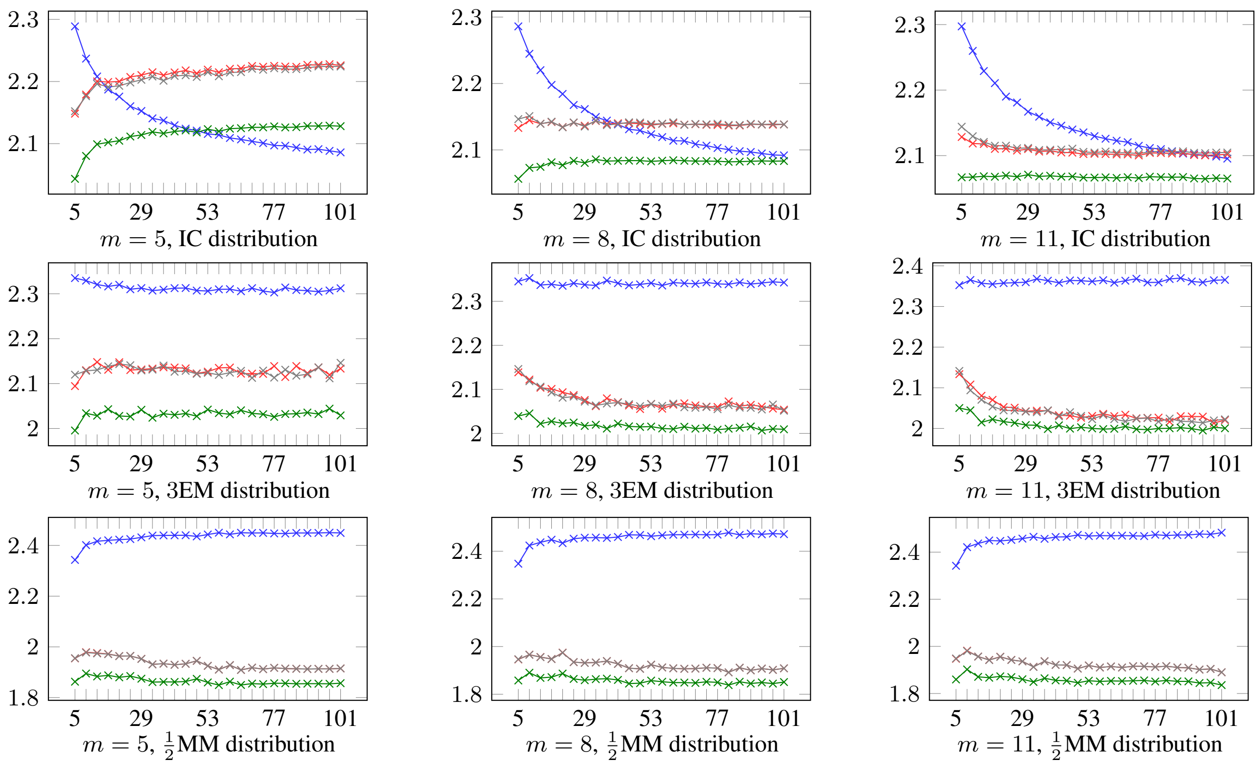

Figure 1: Results of our computer experiments. For each number of alternatives $m \in \{5, 8, 11\}$ and each distribution over the voters' preferences (IC, 3EM, $\frac{1}{2}MM$), there is a plot that shows the average metric distortion (y-axis) of the uniform random dictatorship (blue), the C2ML rule (red), the C1ML rule (grey), and the CCRW rule (green) subject to the number of voters $n \in \{1 + 4k \colon k \in \{1, \ldots, 25\}\}$ (x-axis).

RSCFs. The reason for this is that in these models, the supports $n_{xy}(R)$ between alternatives are likely to be large and there are thus often very strong or very weak alternatives in a sampled preference profile. However, regardless of the numbers of voters, f_{RD} cannot identify such alternatives as it only queries the voters' top alternatives and has thus a rather high average metric distortion. By contrast, the C1ML rule, the C2ML rule, and the CRWW rule take the supports $n_{xy}(R)$ into account and have therefore a significantly lower metric distortion for the Euclidean model and Mallow's model. For instance, if there is an alternative x such that $n_{xy}(R)$ is significantly larger than $\frac{n}{2}$ for all $y \in X_R \setminus \{x\}$, the C1ML and C2ML rules will elect x uniquely, which guarantees a low metric distortion. Moreover, the fact that the average metric distortion of these rules is under Mallow's model even smaller than under the Euclidean model indicates that the average metric distortion of these RSCFs becomes better when the supports $n_{xy}(R)$ increase since these values are under Mallow's model (with $\phi = \frac{1}{2}$) typically larger than in the Euclidean model (with $t = 3$). Also, even if there are such strong alternatives, it seems beneficial to put probabilities on other alternatives as demonstrated by fact that CRWW rule still has the smallest metric distortion. Finally, we note that in the Euclidean model and in Mallow's model, the average metric distortion of the CRWW rule and both the C1ML and C2ML rules are very similar, thus demonstrating that the latter are attractive RSCFs in terms of metric distortion on real-world profiles.

5 Conclusion

In this paper, we study the metric distortion of randomized social choice functions that are well-known in the literature, namely the uniform random dictatorship, C2 maximal lottery (C2ML) rules and C1 maximal lottery (C1ML) rules. In more detail, we first show that every C1ML rule has a metric distortion of at most 4, and that the metric distortion of every majoritarian RSCF (which only depend on the majority relation) converges to 4 as m increases. Hence, C1ML rules have the optimal metric distortion within the class of majoritarian RSCFs when the number of alternatives is unbounded. Secondly, we conduct extensive computer experiments on the metric distortion of all three aforementioned rules as well as the RSCF suggested by Charikar *et al.* [2023] (which currenlty is the best known RSCF in terms of metric distortion) to gain insights into the average-case metric distortion of these rules. These experiments reveal that while, the rule by Charikar *et al.* [2023] also has the best average metric distortion, C1ML rules and C2ML rules are only slightly worse. This gives a strong argument for the usage of the latter rules as they additionally satisfy numerous desirable properties.

Furthermore, our work offers several direction for future work. In particular, we believe that it is interesting to conduct similar computer experiments for further rules. Moreover, our approach also allows to compute the metric distortion of an RSCF on large profiles and it thus seems appealing to analyze the metric distortion of RSCFs on real-world profiles.

References

[Anagnostides *et al.*, 2022] Ioannis Anagnostides, Dimitris Fotakis, and Panagiotis Patsilinakos. Dimensionality and coordination in voting: The distortion of stv. In *Proceedings of the 36th AAAI Conference on Artificial Intelligence (AAAI)*, pages 4776–4784, 2022.

[Anshelevich and Postl, 2017] E. Anshelevich and J. Postl. Randomized social choice functions under metric preferences. *Journal of Artificial Intelligence Research*, 58:797–827, 2017.

[Anshelevich *et al.*, 2015] E. Anshelevich, O. Bhardwaj, and J. Postl. Approximating optimal social choice under metric preferences. In *Proceedings of the 29th AAAI Conference on Artificial Intelligence (AAAI)*, pages 777–783, 2015.

[Anshelevich *et al.*, 2018] E. Anshelevich, O. Bhardwaj, Edith Elkind, J. Postl, and Piotr Skowron. Approximating optimal social choice under metric preferences. *Aritificial Intelligence*, 264:27–51, 2018.

[Anshelevich *et al.*, 2021] E. Anshelevich, A. Filos-Ratsikas, Nisarg Shah, and A. A. Voudouris. Distortion in social choice problems: The first 15 years and beyond. In *Proceedings of the 30th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 4294–4301, 2021.

[Arrow *et al.*, 2011] K. J. Arrow, Amartya Sen, and K. Suzumura, editors. *Handbook of Social Choice and Welfare*, volume 2. North-Holland, 2011.

[Boehmer *et al.*, 2021] Niclas Boehmer, Robert Bredereck, Piotr Faliszewski, Rolf Niedermeier, and Stanislaw Szufa. Putting a compass on the map of elections. In *Proceedings of the 30th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 59–65, 2021.

[Brandl *et al.*, 2016] Florian Brandl, Felix Brandt, and Hans Georg Seedig. Consistent probabilistic social choice. *Econometrica*, 84(5):1839–1880, 2016.

[Brandl *et al.*, 2022] Florian Brandl, Felix Brandt, and C. Stricker. An analytical and experimental comparison of maximal lottery schemes. *Social Choice and Welfare*, 58(1):5–38, 2022.

[Brandt *et al.*, 2016] Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D. Procaccia. Introduction to computational social choice. In Felix Brandt, Vincent Conitzer, U. Endriss, J. Lang, and Ariel D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 1. Cambridge University Press, 2016.

[Charikar and Ramakrishnan, 2022] M. Charikar and P. Ramakrishnan. Metric distortion bounds for randomized social choice. *Proceedings of the 33th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2986–3004, 2022.

[Charikar *et al.*, 2023] Moses Charikar, Prasanna Ramakrishnan, Kangning Wang, and Hongxun Wu. Breaking the metric voting distortion barrier. Technical report, https://arxiv.org/abs/2306.17838, 2023.

[Dutta and Laslier, 1999] Bhaskar Dutta and Jean-François Laslier. Comparison functions and choice correspondences. *Social Choice and Welfare*, 16(4):513–532, 1999.

[Fain *et al.*, 2019] B. Fain, A. Goel, K. Munagala, and N. Prabhu. Random dictators with a random referee: Constant sample complexity for social choice. In *Proceedings of the 33rd AAAI Conference on Artificial Intelligence (AAAI)*, pages 1893–1900, 2019.

[Feldman *et al.*, 2016] M. Feldman, A. Fiat, and I. Golomb. On voting and facility location. In *Proceedings of the 17th ACM Conference on Economics and Computation (ACM-EC)*, pages 269–286, 2016.

[Fishburn, 1984] Peter C. Fishburn. Probabilistic social choice based on simple voting comparisons. *Review of Economic Studies*, 51(4):683–692, 1984.

[Fisher and Ryan, 1995] D. C. Fisher and J. Ryan. Tournament games and Condorcet voting. *Linear Algebra and its Applications*, 217:87–100, 1995.

[Gibbard, 1977] A. Gibbard. Manipulation of schemes that mix voting with chance. *Econometrica*, 45(3):665–681, 1977.

[Gkatzelis *et al.*, 2020] V. Gkatzelis, D. Halpern, and Nisarg Shah. Resolving the optimal metric distortion conjecture. In *Proceedings of the 61st Symposium on Foundations of Computer Science (FOCS)*, pages 1427–1438, 2020.

[Goel *et al.*, 2017] A. Goel, A. K. Krishnaswamy, and K. Munagala. Metric distortion of social choice rules: Lower bounds and fairness properties. In *Proceedings of the 18th ACM Conference on Economics and Computation (ACM-EC)*, pages 287–304, 2017.

[Gross *et al.*, 2017] S. Gross, E. Anshelevich, and Lirong Xia. Vote until two of you agree: Mechanisms with small distortion and sample complexity. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI)*, pages 544–550, 2017.

[Hoang, 2017] L. N. Hoang. Strategy-proofness of the randomized Condorcet voting system. *Social Choice and Welfare*, 48(3):679–701, 2017.

[Kempe, 2020] David Kempe. An analysis framework for metric voting based on lp duality. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI)*, pages 2079–2086, 2020.

[Kizilkaya and Kempe, 2022] F. E. Kizilkaya and D. Kempe. Plurality veto: A simple voting rule achieving optimal metric distortion. In *Proceedings of the 31th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 349–355, 2022.

[Kizilkaya and Kempe, 2023] Fatih Erdem Kizilkaya and David Kempe. Generalized veto core and a practical voting rule with optimal metric distortion. Technical report, https://arxiv.org/pdf/2305.19632.pdf, 2023.

[Laffond *et al.*, 1993] Gilbert Laffond, Jean-François Laslier, and Michel Le Breton. The bipartisan set of a tournament game. *Games and Economic Behavior*, 5(1):182–201, 1993.

[Laffond *et al.*, 1997] Gilbert Laffond, Jean-François Laslier, and Michel Le Breton. A theorem on symmetric two-player zero-sum games. *Journal of Economic Theory*, 72(2):426–431, 1997.

[Le Breton, 2005] Michel Le Breton. On the uniqueness of equilibrium in symmetric two-player zero-sum games with integer payoffs. *Économie publique*, 17(2):187–195, 2005.

[Mallows, 1957] C. L. Mallows. Non-null ranking models. *Biometrika*, 44(1/2):114–130, 1957.

[McGarvey, 1953] D. C. McGarvey. A theorem on the construction of voting paradoxes. *Econometrica*, 21(4):608–610, 1953.

[Munagala and Wang, 2019] K. Munagala and K. Wang. Improved metric distortion for deterministic social choice rules. In *Proceedings of the 20th ACM Conference on Economics and Computation (ACM-EC)*, pages 245–262, 2019.

[Procaccia and Rosenschein, 2006] Ariel D. Procaccia and Jeffrey S. Rosenschein. The distortion of cardinal preferences in voting. In *Cooperative Information Agents X*, pages 317–331. Springer, 2006.

[Skowron and Elkind, 2017] Piotr Skowron and Edith Elkind. Social choice under metric preferences: scoring rules and STV. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI)*, pages 706–712, 2017.

[Szufa *et al.*, 2020] S. Szufa, Piotr Faliszewski, Piotr Skowron, Arkadii Slinko, and Nimrod Talmon. Drawing a map of elections in the space of statistical cultures. In *Proceedings of the 19th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 1341–1349, 2020.

A Omitted Proofs

Here, we present the proofs omitted from the main body. We start by showing Proposition 1.

Proposition 1. *It holds for all majoritarian RSCFs f and preference profiles R that*

1) $dist(f(R), R) \leq 1 + 2\max_{x\in X_R} md(f(R), x, \succsim_R)$.

2) $dist_m(f) \geq 1 + 2\max_{x\in X_R} md(f(R), x, \succsim_R)$.

Proof. Let f denote a majoritarian RSCF, R an arbitrary profile, and $\succsim_R$ the corresponding majority relation. We will show the two claims of this proposition independently.

Proof of 1): Our first goal is to show that $dist(f(R), R) \leq 1 + 2\max_{x\in X_R} md(f(R), x, \succsim_R)$. To this end, we first note that, if $\max_{x\in X_R} md(f(R), x, \succsim_R) = \infty$, there is nothing to show as $dist(f(R), R) \leq 1 + 2\max_{x\in X_R} md(f(R), x, \succsim_R) = \infty$ holds trivially in this case. We hence assume that $md(f(R), x, \succsim_R) < \infty$ for all $x \in X_R$, and we will show that $sc(x, d) \leq (1 + 2md(x, y, \succsim_R))sc(y, d)$ for every metric $d \in D(R)$ and all alternatives $x, y \in X_R$ such that $md(x, y, \succsim_R) \neq \infty$. Since $f(R, x) > 0$ implies that $\max_{y\in X_R} md(x, y, \succsim_R) < \infty$, it then follows that $\frac{\sum_{x\in X_R} f(R,x)sc(x,d)}{sc(y,d)} \leq \frac{\sum_{x\in A} f(R,x)(1+2md(x,y,\succsim_R))sc(y,d)}{sc(y,d)} = 1 + 2md(f(R), y, \succsim_R)$ for all metrics $d \in D(R)$, so $dist(f(R), R) \leq 1 + 2\max_{x\in X_R} md(f(R), x, \succsim_R)$.

To prove that $sc(x, d) \leq (1 + 2md(x, y, \succsim_R))sc(y, d)$ for all alternatives $x, y \in X_R$ with $md(x, y, \succsim_R) \neq \infty$ and all metrics $d \in D(R)$, we proceed by induction on the majority distance between x and y in $\succsim_R$. First, if $md(x, y, \succsim_R) = 0$, then it clearly holds that $\frac{sc(x,d)}{sc(y,d)} = 1$ as $md(x, y, \succsim_R) = 0$ only holds if $x = y$. Next, we assume for the induction hypothesis that there is some $k \in \mathbb{N}$ such that $sc(x', d) \leq (1 + 2md(x', y', \succsim_R))sc(y', d)$ for all metrics $d \in D(R)$ and alternatives $x', y' \in X_R$ with $md(x', y', \succsim_R) \leq k$. For the induction step, we consider two alternatives $x, y \in X_R$ with $md(x, y, \succsim_R) = k + 1$ and an arbitrary metric $d \in D(R)$. Our goal is to show that $sc(x, d) \leq (1 + 2(k + 1))sc(y, d)$. To this end, let z denote the successor of x on the shortest path from x to y in $\succsim_R$, which means that $x \succsim_R z$ and $md(z, y, \succsim_R) = k$. By the induction hypothesis, we can thus conclude that $sc(z, d) \leq (1 + 2k)sc(y, d)$. Next, we partition the voters $v \in N_R$ in two sets $N_{xz} = \{v \in N_R \colon x \succ_v z\}$ and $N_{zx} = \{v \in N_R \colon z \succ_v x\}$. Since $d \in D(R)$, it follows for all voters $v \in N_{xz}$ that $d(v, x) \leq d(v, z)$. Moreover, using the triangle inequality, we can show the following inequality for the voters $v \in N_{zx}$, where v' is a voter in N_{xz}.

$$\begin{aligned} d(v, x) &\leq d(v, y) + d(y, v') + d(v', x) \\ &\leq d(v, y) + d(y, v') + d(v', z) \\ &\leq d(v, y) + d(y, v') + d(v', y) + d(y, v) + d(v, z) \\ &= 2d(v, y) + 2(j, v') + d(v, z) \end{aligned}$$

Finally, we observe that $|N_{xz}| \geq |N_{zx}|$ since $x \succsim_R z$, so there is an injective function t from N_{zx} to N_{xz}. Putting everything together, we infer the following inequality.

$$\begin{aligned} \sum_{v\in N_R} d(v, x) &= \sum_{v\in N_{xz}} d(v, x) + \sum_{v\in N_{zx}} d(v, x) \\ &\leq \sum_{v\in N_{xz}} d(v, z) \\ &\quad + \sum_{v\in N_{zx}} 2d(v, y) + 2d(t(v), y) + d(v, z) \\ &\leq \sum_{v\in N_R} d(v, z) + 2d(v, y) \\ &= sc(z, d) + 2sc(y, d) \\ &\leq (1 + 2(k + 1))sc(y, d) \end{aligned}$$

The first inequality follows from our bounds on $d(v, x)$ for $v \in N_{xz}$ and $v \in N_{zx}$, the second one simply reorganizes the terms and uses that t is an injective function, and the last inequality follows by the induction hypothesis. This inequality proves the induction step, so it follows that $sc(x, d) \leq (1 + 2md(x, y, \succsim_R))sc(y, d)$ for all alternatives $x, y \in X_R$ with $md(x, y, \succsim_R) < \infty$ and metrics $d \in D(R)$. This completes the proof of this lemma.

Proof of 2): As second point, we will show that $dist_m(f) \geq 1 + 2\max_{x\in X_R} md(f(R), x, \succsim_R)$. To this end, we use a case distinction with respect to whether $\max_{x\in X_R} md(f(R), x, \succsim_R) < \infty$ or $\max_{x\in X_R} md(f(R), x, \succsim_R) = \infty$.

Case 1: First, we suppose that $md(f(R), x, \succsim_R) < \infty$ for every alternative $x \in X_R$ and show that $dist_m(f) \geq 1+2\max_{x\in X_R} md(f(R), x, \succsim_R)$. For this, we fix an arbitrary alternative $x^* \in X_R$; we will construct a family of profiles R^ϵ (where ϵ is a parameter in $(0, 1)$) such that $\succsim_R = \succsim_{R^\epsilon}$ for every $\epsilon \in (0, 1)$ and $\lim_{\epsilon\to 0} dist(f(R^\epsilon), R^\epsilon) = 1 + 2md(f(R), x^*, \succsim_R)$. To this end, let $D^k = \{x \in X_R \colon md(x, x^*, \succsim_R) = k\}$ denote the set of alternatives that has a majority distance of k to x^*. Moreover, we define $D^0 = \{x^*\}$ and $D^m = \{y \in X_R \colon md(y, x^*, \succsim_R) = \infty\}$ denotes the set of alternatives that have no path to x^* in $\succsim_R$. We note that $x \succ_R y$ for all $x \in D^j$, $y \in D^{j'}$ such that $j + 2 \leq j' < m$ as otherwise, y would have a path to x^* of length $j + 1 \leq j'$ by going to x. Furthermore, $x \succ_R y$ for all $x \in X_R \setminus D^m$, $y \in D^m$ as there is a path from x to x^* in $\succsim_R$, but no such path exists for y. Based on this observation, we construct the following profile R^ϵ for $\epsilon \in (0, 1)$, where $D^i \succ_v D^j$ denotes that voter v prefers all alternatives in D^i to all alternatives in D^j:

1. There is a set of voters I_1 such that $|I_1| = \lceil \frac{1}{\epsilon} \rceil$ and $D^0 \succ_v D^2 \succ_v D^1 \succ_v D^4 \succ_v D^3 \succ_v D^6 \succ_v D^5 \succ_v \dots \succ_v D^m$ for each $v \in I_1$. The alternatives within each set D^i are ordered lexicographically.
2. There is a set of voters I_2 such that $|I_2| = \lceil \frac{1}{\epsilon} \rceil$ and $D^1 \succ_i D^0 \succ_v D^3 \succ_v D^2 \succ_v D^5 \succ_v D^4 \succ_v \dots \succ_v D^m$ for each $v \in I_2$. The alternatives within each set D^i are ordered inverse lexicographically.
3. For each pair of alternatives x, y such that $x \succ_R y$ and $x \in D^j$, $y \in D^{j'}$ for $|j - j'| \leq 1$, we add two voters v, v' with preferences $x \succ_v y \succ_v z_1 \succ_v \dots \succ_v z_{m-2}$ and $z_{m-2} \succ_{v'} \dots \succ_{v'} z_1 \succ_{v'} x \succ_{v'} y$. The set of these voters is called I_3 and we note that $|I_3| \leq m(m - 1)$.

We first note that the profile R^ϵ has indeed the same majority relation as R: the voters in I_1 and I_2 together enforce that a majority of voters prefers every alternative in D^j to every alternative in $D^{j'}$ for all $j \in \mathbb{N}$, $j' \in \mathbb{N} \cup \{\infty\}$ with $j + 2 \leq j'$ and cancel each other out with respect to the majority comparison between every other pair of alternatives. Hence, the voters in I^3 set these majority comparisons in the same way as in $\succsim_R$, so $\succsim_R = \succsim_{R^\epsilon}$.

Next, we define the following (partial) metric d that is consistent with R^ϵ:

$$d(v,x) = \begin{cases} 2\lceil \frac{k}{2} \rceil & \text{if } v \in I_1 \text{ and } x \in D^k \\ 1 + 2\lfloor \frac{k}{2} \rfloor & \text{if } v \in I_2 \text{ and } x \in D^k \\ m & \text{if } v \in I_3 \end{cases}$$

It can be checked that d can be extended to a full metric on $N_{R^\epsilon} \cup X_{R^\epsilon}$. For instance, we may assume that the voters and alternatives are placed in a two-dimensional space such that every alternative $x \in D^k$ lies at $(-k, 0)$ if k is even and at $(k+1, 0)$ if k is odd. Moreover, the voters $i \in I_1$ all lie at $(0,0)$, the voters $i \in I_2$ lie at $(1,0)$, and the voters $i \in I_3$ lie at $(0,m)$. Then, d corresponds to the $|\cdot|_\infty$ norm, which is known to be a metric.

Finally, we can compute the social cost of our alternatives and the distortion of f. To this end, we note that $sc(y,d) = 2\lceil \frac{k}{2} \rceil |I_1| + (1 + 2\lfloor \frac{k}{2} \rfloor)|I_2| + m|I_3| = (2k+1)\lceil \frac{1}{\epsilon} \rceil + m|I_3|$ for every alternative $y \in D^k$ and every k. In particular, this means that $sc(x^*, d) = \lceil \frac{1}{\epsilon} \rceil + m|I_3|$. Moreover, it holds that $f(R^\epsilon) = f(R)$ since $\succsim_R = \succsim_{R^\epsilon}$ and f is majoritarian. Next, because $md(f(R), x, \succsim_R) < \infty$ for all $x \in X_R$, we can compute for every $\epsilon \in (0,1)$ that

$$\begin{aligned} dist_m(f) &\geq dist(f(R^\epsilon), R^\epsilon) \\ &\geq \frac{\sum_{y \in X_R} f(R,y)((1 + 2md(y,x,\succsim_R))\lceil \frac{1}{\epsilon} \rceil + m|I_3|}{\lceil \frac{1}{\epsilon} \rceil + m|I_3|} \\ &= \frac{1 + 2md(f(R),x,\succsim_R)\lceil \frac{1}{\epsilon} \rceil + m|I_3|}{\lceil \frac{1}{\epsilon} \rceil + m|I_3|}. \end{aligned}$$

It is easy to see that, when ϵ goes to 0, the right side converges to $1 + 2md(f(R), x, \succsim_R)$ as $m|I_3|$ is a constant. Finally, since x is chosen arbitrarily, we thus infer that $dist_m \geq 1 + 2\max_{x \in X_R} md(f(R), x, \succsim_R)$.

Case 2: As the second case, we assume that $\max_{x \in X_R} md(f(R), x, \succsim_R) = \infty$ and we will show that $dist_m(f) = \infty$, too. To this end, we let x denote an alternative such that $md(f(R), x, \succsim_R) = \infty$ and we define the sets $B = \{y \in A\colon md(y, x, \succsim_R) < \infty\}$ and $C = \{y \in A\colon md(y, x, \succsim_R = \infty\}$. By the definition of the sets B and C, it holds that $y \succ_R z$ for all $y \in B$ and $z \in C$. We will next use this observation to construct a profile R' with $\succsim_R = \succsim_{R'}$ such that f has unbounded distortion in R'. To this end, we use a variant of McGarvey's construction McGarvey [1953]: for all alternatives pairs of alternatives $y, z \in B$ or $y, z \in C$ with $y \succ_R z$, we add two voters who *i)* both prefer all alternatives in B to all alternatives in C, *ii)* both prefer y to z, and *iii)* order all remaining pairs of alternatives exactly inverse. It can be checked that each pair of voters only ensures that $y \succ_R z$ for its respective pair of alternatives y, z, and that $x' \succ_R y'$ for all $x' \in B$, $y' \in C$. Hence, it is easy to check that $\succsim_R = \succsim'_R$, which implies that $f(R') = f(R)$ as f is majoritarian. Finally, consider the metric $d \in D(R')$ given by $d(v,x) = 0$ and $d(v,y) = 1$ for all $v \in V_{R'}$, $x \in B$, $y \in C$. It is easy to check that every alternative $y \in B$ has a social cost $sc(y,d) = 0$. By contrast, $sc(f(R'), d) = sc(f(R), d) > 0$ as $f(R, z) > 0$ for some alternative $z \in C$. Hence, $dist(f(R'), R') = \infty$, which proves this case. □

Next, we turn to the proof of Theorem 2

Theorem 2. *It holds for every majoritarian RSCF f that $dist_m(f) \geq 4 - \frac{3}{m}$ if $m \geq 3$ is odd and $dist_m(f) \geq 4 - \frac{3}{m-1}$ if $m \geq 3$ is even. Thus, $dist(f) \geq 4$.*

Proof. To prove this result, we will rely on Claim 2) of Proposition 1 and thus aim to construct a profile R such that every lottery p has a large expected majority distance $md(p, x, \succsim_R)$ for some alternative x. To this end, we note that is suffices to construct a suitable complete relation $\succsim$ on X_m as we can find for every such relation a profile R with $\succsim_R = \succsim$ [McGarvey, 1953].

We first focus on the case that $m \geq 3$ is odd and consider in this case the "cyclic" majority relation defined by $x_i \succ x_{i+_m k}$ for all $i \in \{1, \dots, m\}$ and $k \in \{1, \dots, \frac{m-1}{2}\}$, where $i +_m k = i + k$ if $i + k \leq m$ and $i +_m k = i + k - m$ if $i + k > m$. Our goal is to show that $\max_{x \in A} md(p, x, \succsim) \geq \frac{3}{2} - \frac{3}{2m}$ as Claim 2) in Proposition 1 then implies the theorem. We thus assume for contradiction that there is a lottery p such that $\max_{x \in A} md(p, x, \succsim) < \frac{3}{2} - \frac{3}{2m}$. Moreover, we define the lotteries p^k by $p^k(x_i) = p(x_{i +_m k})$ for all $i, k \in \{1, \dots, m\}$ and first aim to show that $\max_{x \in A} md(p^k, x, \succsim) < \frac{3}{2} - \frac{3}{2m}$, too. For this, we note that the symmetry of $\succsim$ implies that $md(x_i, x_j, \succsim) = md(x_{i +_m k}, x_{j +_m k}, \succsim)$ for all $i, j, k \in \{1, \dots, m\}$. Consequently, it holds that $md(p^k, x_i, \succsim) = md(p, x_{i +_m k}, \succsim)$ as $p^k(x_j) = p(x_{j +_m k})$ and $md(x_j, x_i, \succsim) = md(x_{j +_m k}, x_{i +_m k}, \succsim)$ for all $x_j \in X_m$. This implies that $\max_{x \in A} md(p^k, x, \succsim) = \max_{x \in A} md(p, x, \succsim)$. Finally, we consider the lottery p^* defined by $p^*(x) = \frac{1}{m} \sum_{k \in \{1, \dots, m\}} p^k(x)$ for all $x \in X_m$ and observe that $md(p^*, x_i, \succsim) = \frac{1}{m} \sum_{k \in \{1, \dots, m\}} md(p^k, x_i, \succsim) < \frac{3}{2} - \frac{3}{2m}$ for all x_i. However, $p^*(x_i) = \frac{1}{m} \sum_{k \in \{1, \dots, m\}} p^k(x_i) = \frac{1}{m} \sum_{k \in \{1, \dots, m\}} p(x_{i +_m k}) = \frac{1}{m}$ for all x_i. Since $md(x_1, x_j, \succsim) = 1$ for all $j \in \{2, \dots, \frac{m+1}{2}\}$ and $md(x_1, x_j, \succsim) = 2$ for all $j \in \{\frac{m+3}{2}, \dots, m\}$, we can thus compute that $md(p^*, x_1, \succsim) = \frac{1}{m} \sum_{x_i \in X_R} md(x_i, x_1, \succsim) = \frac{m-1}{2m} + \frac{2(m-1)}{2m} = \frac{3}{2} - \frac{3}{2m}$. This contradicts that $md(p^*, x_i, \succsim) < \frac{3}{2} - \frac{3}{2m}$ for all x_i, so the initial assumption that there is a lottery p with $\max_{x \in A} md(p, x, \succsim) < \frac{3}{2} - \frac{3}{2m}$ is wrong. Hence, $\max_{x \in A} md(p, x, \succsim) \geq \frac{3}{2} - \frac{3}{2m}$ for every lottery p and Proposition 1 shows the theorem for odd $m \geq 3$.

Finally, to extend the result also to even m, we can add an alternative x^* that loses all majority comparisons. Based on Claim 2) in Proposition 1, the metric distortion of a majoritarian RSCF is unbounded if it assigns positive probability to

x^*. On the other side, we can apply the same analysis as for the case that m is odd if $p(x^*) = 0$ and hence infer our lower bound. □

Finally, we will present the proof of Proposition 2.

Proposition 2. *Fix a lottery p, a profile R, and an alternative x^*. If the optimal objective value o^*_{LP} of LP 1 is bounded, then $dist(p, R, x^*) = o^*_{LP}$ and $dist(p, R, x^*) = \infty$ else.*

Proof. Let R denote an arbitrary profile, p a lottery, and x^* denote an arbitrary alternative. We will prove the proposition in two steps: we first show that that $dist(p, R, x^*) \geq o_{LP}$ for the objective value o_{LP} of every feasible solution of LP 1 and then that $dist(p, R, x^*) \leq o^*_{LP}$ where o_{LP^*} denotes the optimal objective value of LP 1 if this value is bounded and $o^*_{LP} = \infty$. From the first insight, it follows immediately that $dist(p, R, x^*) = \infty$ if LP 1 is unbounded as we can find for every $x \in \mathbb{R}$ a feasible solution with higher objective value. On the other hand, combining the first and the second insight imply that $dist(p, R, x^*) = o^*_{LP}$ if the optimal objective value of LP 1 is bounded.

Claim 1: $dist(p, R, x_{i^*}) \geq o_{LP}$ **for the objective value o_{LP} of every feasible solution of LP 1.**

Let d_{LP}, t_{LP} denote a feasible solution of LP 1 and let o_{LP} denote its objective value. To prove that $dist(p, R, x^*) \geq o_{LP}$, we will infer a metric $d \in D(R)$ that satisfies $d(x, v) = d_{LP}(x, v)$ for all $x \in X_R$, $v \in V_R$. Since $\sum_{v \in V_R} d_{LP}(x^*, v) = 1$, we can then infer that

$$o_{LP} = \sum_{x \in X_R} p(x) \sum_{v \in V_R} d_{LP}(x, v) = \frac{sc(p, d)}{sc(x^*, d)} \leq \max_{d \in D(R)} \frac{sc(p, d)}{sc(x^*, d)} = dist(p, R, x^*).$$

Towards proving this claim, we will first construct another feasible solution d'_{LP}, t'_{LP} with corresponding objective value o'_{LP} that satisfies that $d'_{LP}(x, v) \geq d(x^*, v)$ for all $x \in X_R$, $v \in V_R$ and $o'_{LP} \geq o_{LP}$. Now, if d_{LP} satisfies these conditions, we can simply set $d'_{LP} = d_{LP}$ and $t'_{LP} = t_{LP}$. We thus assume that there is an alternative x and a voter v such that $d_{LP}(x, v) < d_{LP}(x^*, v)$. In this case, we consider the solution $\bar{d}_{LP}$ derived from d_{LP} by setting $\bar{d}_{LP}(x, v) = d_{LP}(x^*, v)$. First, it is easy to verify that $\bar{d}_{LP}$ combined with the function $\bar{t}_{LP} = t_{LP}$ is still a feasible solution. Indeed, the only upper bounds on $\bar{d}_{LP}(x, v)$ are of the form $\bar{d}_{LP}(x, v) \leq \bar{d}_{LP}(x^*, v) + t(y)$, which are true since $\bar{d}_{LP}(x, v) = \bar{d}_{LP}(x^*, v)$ and $t(y) \geq 0$. Moreover, it is straightforward that increasing the value of $d_{LP}(x, v)$ does not decrease the objective value. Hence, $\bar{o}_{LP} \geq o_{LP}$, and by repeating this step, we will arrive at a feasible solution d'_{LP}, t'_{LP} such that $d'_{LP}(x, v) \geq d'_{LP}(x^*, v)$ for all alternatives $x \in X_R$ and voters $v \in V_R$.

As second step, we will again construct a feasible solution d''_{LP}, t''_{LP} of LP 1 such that $o''_{LP} \geq o_{LP}$ and $d''_{LP}(x, v) \leq d''_{LP}(y, v)$ for all voters $v \in V_R$ and alternatives $x, y \in X_R$ with $x \succ_v y$. If d'_{LP} satisfies this condition, we are immediately done and we hence suppose that there is a voter v and two distinct alternatives x, y such that $x \succ_j y$ and $d'_{LP}(x, v) > d'_{LP}(y, v)$. Note first that this is not possible if $y = x^*$ because the fourth condition of LP 1 ensures in this case that $d'_{LP}(x, v) \leq d'_{LP}(x^*, v) + t'_{LP}(x^*) = d'_{LP}(x^*, v)$. We hence assume from now on that $y \neq x^*$. In this case, we consider the solution $\bar{d}_{LP}$, $\bar{t}_{LP}$ derived from d'_{LP}, t'_{LP} by setting $\bar{d}_{LP}(y, v) = d'_{LP}(x, v)$. First, we note that this solution is feasible as the only upper bounds on $\bar{d}_{LP}(y, v)$ are given by $\bar{d}_{LP}(y, v) \leq \bar{d}_{LP}(x^*, v) + \bar{t}(y) = d'_{LP}(x^*, v) + t'_{LP}(z)$ for $z \in X_R$ with $z \succeq_v y$. Moreover, it holds that $\bar{d}_{LP}(x, v) = d'_{LP}(x, v) \leq d'_{LP}(x_{i^*}, v) + t(z)$ for all $z \in X_R$ with $x \succeq_v z$ since d'_{LP}, t'_{LP} is a feasible solution of LP 1. Finally, since $x \succeq_v y$, it therefore follows that $\bar{d}_{LP}$ is a feasible solution, too. Moreover, it is again straightforward that we did not decrease the objective value because we only increased the value of variables. Now, by repeating this step, it is easy to see that we will eventually arrive at a feasible solution d''_{LP} and $t''_{LP} = t_{LP}$ such that $o''_{LP} \geq o'_{LP}$ and $d''_{LP}(x, v) \leq d''(y, v)$ for all $v \in V_R$ and $x, y \in X_R$ with $x \succ_v y$. Moreover, d''_{LP} still satisfies that $d''_{LP}(x, v) \geq d''_{LP}(x^*, v)$ for all $v \in V_R$ and $x_i \in X_R$ as we only increase distances for alternatives other than x^*.

Finally, based on the solution d''_{LP}, t''_{LP}, we will construct a metric d that satisfies all our criteria. In particular, we define:

1. $d(x, v) = d(v, x) = d''_{LP}(x, v)$ for all $x \in X_R$ and $v \in V_R$.
2. $d(x, x) = 0$ for all $x \in X_R$ and $d(v, v) = 0$ for all $v \in V_R$.
3. $d(x, y) = \min_{v \in V_R} d''_{LP}(x, v) + d^*_{LP}(y, v)$ for all distinct $x, y \in X_R$.
4. $d(v, w) = \min_{x \in x_R} d''_{LP}(x, v) + d''_{LP}(x, w)$ for all distinct $v, w \in V_R$.

By its definition, it is straightforward that d is symmetric and that $d(z, z) = 0$ for all $z \in X_R \cup V_R$. Moreover, because d''_{LP} is consistent with R, the same holds for d. Hence, we only need to verify the triangle inequality, for which we start by an auxiliary observation: we will show that $d(x, v) \leq d(x, w) + d(y, w) + d(y, v)$ for all $x, y \in X_R$, $v, w \in V_R$. By the definition of d, this is equivalent to proving the same for d''_{LP}. We thus observe that

$$\begin{aligned} d''_{LP}(x, v) &\leq d''_{LP}(x^*, v) + t''_{LP}(x) \\ &\leq d''_{LP}(x^*, v) + d''(x^*, w) + d''(x, w) \\ &\leq d''_{LP}(y, v) + d''_{LP}(y, w) + d''_{LP}(x, w). \end{aligned}$$

The first and second inequality directly use the third and fifth constraint of our LP. The last inequality uses that, by construction of d''_{LP}, it holds that $d''_{LP}(x^*, v) \leq d''_{LP}(y, v)$ and $d''_{LP}(x^*, w) \leq d''_{LP}(y, w)$.

Finally, we are ready to show that d satisfies the triangle inequality. To this end, consider three distinct elements $x, y, z \in X_R \cup V_R$. We will show that $d(x, z) \leq d(x, y) + d(y, z)$ by considering three cases:

- $x, y, z \in X_R$: Let $v, w \in V_R$ denote the voters that minimize $d(x, v) + d(v, y)$ and $d(y, w) + d(w, z)$, respectively. By our auxiliary claim, it holds that $d(x, z) = \min_{v' \in V_R} d(x, v') + d(v', z) \leq d(x, v) + d(z, v) \leq d(x, v) + d(z, w) + d(w, y) + d(y, v) =$

$\min_{v'\in V_R} d(x,v') + d(v',y) + \min_{v'\in V_R} d(y,v') + d(v',z) = d(x,y) + d(y,z)$. An analogous argument works if $x,y,z \in V_R$.

- $x, y \in X_R$, $z \in V_R$: Let v denote the voter that minimizes $d(x,v) + d(v,y)$. By our auxiliary claim, it holds that $d(x,z) \leq d(x,v) + d(v,y) + d(y,z) = d(x,y) + d(y,z)$. The cases that $y, z \in X_R$, $x \in V_R$; $x, y \in V_R$, $z \in X_R$; and $y, z \in V_R$, $x \in X_R$ are symmetric.
- $x, z \in X_R$, $y \in V_R$: It holds that $d(x,z) = \min_{v\in N} d(x,v) + d(v,y) \leq d(x,y) + d(y,v)$. The case that $x, z \in V_R$, $y \in X_R$ is symmetric.

This proves that d is indeed a metric that is consistent with R. We can therefore conclude that $dist(p, R, x^*) \geq sc(p,d) \frac{sc(x^*,d)}{=} o''_{LP} \geq o_{LP}$ holds for all feasible solutions d_{LP}, t_{LP} with objective value o_{LP}.

Claim 2: $dist(p, R, x_{i^*}) \leq o^*_{LP}$ **where** o^*_{LP} **is the optimal objective value off LP 1.**

We will next show that $dist(p, R, x_{i^*}) \leq o^*_{LP}$. To this end, we note that this is trivial if $o^*_{LP} = \infty$, so we focus on the case that the optimal objective value of LP 1 is bounded. To this end, let $d \in D(R)$ denote a metric that maximizes $\frac{sc(p,d)}{sc(x^*,d)}$. We will next construct a biased metric $d^* \in D(R)$ that satisfies $\frac{sc(p,d^*)}{sc(x^*,d^*)} \geq \frac{sc(p,d)}{sc(x^*,d)}$. As second step, we will then derive a feasible solution d_{LP}, t_{LP} of LP 1 with objective value $o_{LP} = \frac{sc(p,d^*)}{sc(x^*,d^*)}$. This clearly proves the claim.

Following the proof of Charikar and Ramakrishnan [2022], we define the function $t(x)$ for all X_R by $t(x) = d(x, x^*)$. The biased metric d^* is then defined by

$$d^*(x^*, v) = \frac{1}{2} \max_{x,y\in X_R \colon x \succeq_v y} t(x) - t(y)$$
$$d^*(x, v) = d^*(x^*, v) + \min_{y\in X_R \colon x\succeq_v y} t(y).$$

We first note that d^* can be extended to a metric that is consistent with R due to Proposition 5.1 of Charikar and Ramakrishnan [2022]. Hence, it only remains to show that $\frac{sc(p,d^*)}{sc(x^*,d^*)} \geq \frac{sc(p,d)}{sc(x^*,d)}$. To this end, we will show that $sc(x^*, d^*) \leq sc(x^*, d)$ and $sc(x, d^*) - sc(x^*, d^*) \geq sc(x, d) - sc(x^*, d)$. This shows $\frac{sc(p,d^*)}{sc(x_{i^*},d^*)} \geq \frac{sc(p,d)}{sc(x^*,d)}$ as demonstrated by the following inequality.

$$\begin{aligned} \frac{sc(p,d^*)}{sc(x^*,d^*)} - 1 &= \frac{\sum_{x\in X_R} p(x)(sc(x,d^*) - sc(x^*,d^*))}{sc(x^*,d^*)} \\ &\geq \frac{\sum_{x\in X_R} p(x)(sc(x,d) - sc(x^*,d)}{sc(x^*,d))} \\ &= \frac{sc(p,d)}{sc(x^*,d)} - 1 \end{aligned}$$

We first show that $sc(x^*, d^*) \leq sc(x^*, d)$. To this end, we observe (analogous to Charikar and Ramakrishnan [2022] in Proposition 5.2) that $d(x, x^*) \leq d(x, v) + d(v, x^*) \leq d(y, v) + d(v, x^*) \leq d(y, x^*) + 2d(v, x^*)$ for all voters v and alternatives x, y with $x \succeq_v y$. Hence, $d(v, x^*) \geq \frac{1}{2} \max_{x,y\in X_R \colon x\succeq_v xy} t(x) - t(y) = d^*(v, x^*)$. Clearly, this implies that $sc(x^*, d^*) \leq sc(x^*, d)$, thus proving our claim. Secondly, we need to prove that $sc(x, d^*) - sc(x^*, d^*) \geq sc(x, d) - sc(x^*, d)$ for all $x \in X_R$. Since the inequality clearly holds for x^*, we assume that $x \neq x^*$. Following again the ideas of Charikar and Ramakrishnan [2022], we observe that $d(x, v) \leq d(y, v) \leq d(y, x^*) + d(x^*, v)$ for all voters v and alternatives x, y with $x \succeq_v y$. Hence, $d(x, v) - d(x^*, v) \leq \min_{y\in X_R \colon x\succeq_v y} d(y, x^*) = \min_{y\in X_R \colon x\succeq_v y} t(y) = d^*(x, v) - d^*(x^*, v)$. We thus conclude that $sc(x, d^*) - sc(x^*, d^*) \geq sc(x, d) - sc(x^*, d)$. Therefore, it follows indeed that $\frac{sc(p,d^*)}{sc(x^*,d^*)} \geq \frac{sc(p,d)}{sc(x^*,d)}$.

We next proceed with a case distinction with respect to whether $sc(x^*, d^*) = 0$ or $sc(x^*, d^*) > 0$. First, we consider the case that $sc(x^*, d^*) > 0$. In this case, we aim to construct a feasible solution d_{LP}, t_{LP} of LP 1 with objective value $o_{LP} = \frac{sc(p,d^*)}{sc(x^*,d^*)}$. Now, to derive this solution, we first note that every biased metric $d \in D(R)$ (together with its inducing function t) satisfies the first four constraints of LP 1 by definition. Moreover, d also satisfies the fifth constraint since $d(x, v) + d(x^*, v) = 2d(x^*, v) + \min_{y\in X_r \colon x\succeq y} t(y) \geq t(x)$ for all $x \in X_R$, $v \in V_R$. The last inequality follows as $2(d^*, v) = \max_{x,y\in X_R \colon x\succeq_v y} t(x) - t(y) \geq t(x) - \min_{y\in X_r \colon x\succeq y} t(y)$. Furthermore, we note that, for every biased metric $d \in D(R)$, and $\ell \in \mathbb{R}_{>0}$, the function t^ℓ defined by $t^\ell(x) = \ell t(x)$ induces a biased metric $d^\ell \in D(R)$ with $sc(x^*, d^\ell) = \ell sc(x^*, d)$ and $sc(p, d^\ell) = \ell sc(p, d)$. Because $sc(x^*, d) \geq 0$, it is thus easy to check that the biased metric d^ℓ together with its defining function t^ℓ for $\ell = \frac{1}{sc(x^*,d)}$ defined a feasible solution to LP 1 with $o_{LP} = \frac{sc(p,d^\ell)}{sc(x^*,d^\ell)} = \frac{sc(p,d^*)}{sc(x^*,d^*)}$. Hence, it is in this case easy to check that $dist(p, R, x^*) = \frac{sc(p,d^*)}{sc(x^*,d^*)} = o_{LP} \leq o^*_{LP}$, where o^*_{LP} denotes the optimal objective value of LP 1.

For the second case, we suppose that $sc(x^*, d^*) = 0$. For this case, we make a further case distinction with respect to whether $sc(p, d^*) = 0$ or $sc(p, d^*) > 0$. First, suppose that $sc(p, d^*) = 0$, which means that $dist(p, R, x^*) = \frac{sc(p,d^*)}{sc(x^*,d^*} = 1$. To show that $dist(p, R, x^*) \leq o^*_{LP}$, it thus suffices to construct a feasible solution of LP 1 with objective value 1. To this end, consider the following solution: $d_{LP}(x, v) = \frac{1}{n_R}$ for all $x \in X_R$, $v \in V_R$ and $t_{LP}(x) = 0$ for all $x \in X_R$. It is easy to check that this is indeed a feasible solution and that $\sum_{x\in X_R} p(x) \sum_{v\in V_R} d(x, v) = \sum_{x\in X_R} p(x) = 1$, thus verifying our claim.

As last case, we assume that $sc(x^*, d^*) = 0$ and $sc(p, d^*) > 0$, which means that $dist(p, R, x^*) = \infty$. In this case, we need to show that the optimal objective value of LP 1 is unbounded. Towards this end, we note that, since $sc(x^*, d^*) = 0$, $d^*(x^*, v) = 0$ for all voters $v \in V_R$. Next, we consider again the function $t^\ell(x) = \ell \cdot t(x)$ for all $x \in X_R$, $\ell \in \mathbb{R}_{>0}$ and let d^ℓ denote the corresponding biased metric. Finally, we define the solutions d^ℓ_{LP}, t^ℓ_{LP} to LP 1 by *i)* $d^\ell_{LP}(x^*, v) = \frac{1}{n_R}$ for all $v \in V_R$, *ii)* $d^\ell_{LP}(x, v) = d^\ell(x, v)$ for all $x \in X_R \setminus \{x^*\}$, $v \in V_R$, and *iii)* $t^\ell_{LP} = t^\ell$. It can be checked that d^ℓ_{LP}, t^ℓ_{LP} is a feasible solution to LP 1: to this end, we recall that every biased metric satisfies the first five constraints of our LP. Now, to infer d^ℓ_{LP} from d^ℓ, we only increase the distance $d^\ell_{LP}(x^*, v)$ to $\frac{1}{n_R}$ for all $v \in N_R$. Since there is no upper bound on $d^\ell_{LP}(x^*, v)$, this does not violate any of the first five constraints and ensures that the last one is true. Finally, we note that there is an alternative y such that $p(y) > 0$ and $sc(y, d^*) > 0$ as $sc(p, d^*) > 0$. Consequently, the objective value of the solutions d^ℓ_{LP}, t^ℓ_{LP} is lower bounded by $\ell p(y) d(y, d^*)$. Letting ℓ go to infinity thus shows that the objective value of LP 1 is not bounded in this case. Hence, it holds in all cases that $dist(p, R, x^*) \leq o^*_{LP}$, where o^*_{LP} denotes the optimal objective value of LP 1 if it is bounded and ∞ otherwise. $\square$

B Metric Distortion under the IC Model

As last part of this paper, we will formally prove the statements about the expected metric distortion of the uniform random dictatorship, C1ML rules, and C2ML rules made in Section 4.3. In particular, we will show that, in the IC model, the expected metric distortion of f_{RD} converges to 2 as the number of voters goes to infinity, and the expected metric distortion of C1ML rules and C2ML rules will converge to approximately $2 + \frac{1}{m-1}$. To make these statements formal, we denote by $IC(m, n)$ the probability distribution over preference profiles on n voters and m alternatives of the independent culture model created. Then, we will prove the following statement for the uniform random dictatorship.

Proposition 3. *It holds for every $m \geq 3$ that $\lim_{n\to\infty} \mathbb{E}_{R\sim IC(m,n)}[dist(f_{RD}(R), R)] = 2$.*

Unfortunately, we are not able to prove a fully analogous statement for C1ML and C2ML rules. The problem in the analysis of these rules is that we did not manage to bound the probability that these rules select a lottery that randomizes over all alternatives. To make this more formal, let $supp(f(R)) = \{x \in X_R : f(R, x) > 0\}$ denote the set of alternatives that are assigned positive probability by f in R. While computer experiments (see [Brandl *et al.*, 2022]) show that the probability $\mathbb{P}_{R\sim IC(m,n)}[supp(f(R)) = X_R]$ is very small for C1ML and C2ML rules, we cannot bound it and therefore cannot compute a tight lower bound for the expected metric distortion of these rules. We thus give next a more general result that depends on this probability.

Proposition 4. *Let $m \geq 3$. It holds for every RSCF f with $dist_m(f) < \infty$ and $z = \liminf_{n\to\infty} \mathbb{P}_{R\sim IC(m,n)}[supp(f(R)) \neq X_R]$ that*

1) $\limsup_{n\to\infty} \mathbb{E}_{R\sim IC(m,n)}[dist(f(R), R)] \leq 2 + \frac{1}{m-1}$

2) $\liminf_{n\to\infty} \mathbb{E}_{R\sim IC(m,n)}[dist(f(R), R)] \geq 2 + \frac{z}{m-1}$.

In particular, we note that for C1ML rules and C2ML rules, this means that the expected metric distortion will approximately converge to $2 + \frac{1}{m-1}$ as the probability $\mathbb{P}_{R\sim IC(m,n)}[supp(f(R)) \neq X_R]$ has experimentally been shown to be very small for large n. Hence, this result explains our computer simulations under the IC model very well. We note that, curiously, Proposition 4 also entails that the expected metric distortion of *every* deterministic SCF with bounded distortion converges to $2 + \frac{1}{m-1}$ under the IC model.

We next turn to the proofs of these two propositions. To this end, we let $n_\succ(R) = |\{v \in V_R : \succ_v = \succ\}$ denote the number of voters that report the preference relation $\succ$ in the profile R. Moreover, we will subsequently show three auxiliary lemmas: first, we investigate the metric distortion of every lottery on every profile where all preference relations are reported by the same number of voters (cf. Lemmas 1 and 2). Clearly, under the IC model, we can expect that the output profile is very similar to such a profile is the number of voters is large. We hence prove in Lemma 3 that we can bound the metric distortion of such a profile R based on the metric distortion of the chosen lottery for a large subprofile.

In more detail, in our first lemma, we will identify a metric $d \in D(R)$ that satisfies $dist(p, R, x^*) = \frac{sc(p,d)}{sc(x^*,d)}$ for all profiles R in which all preference relations appear equally often, all lotteries p, and all alternatives $x^* \in X_R$. Surprisingly, we show that we can focus on a single type of metrics for this maximization problem: it always suffices to consider the biased metric $d \in D(R)$ given by the function t with $t(x^*) = 0$ and $t(x) = 2$ for all $x \in X_R \setminus \{x^*\}$. We note that this gives further evidence for the conjecture by Charikar and Ramakrishnan [2022] that this type of metric is the worst-case for *all* profiles.

Lemma 1. *Assume $m \geq 3$ and let $R \in \mathcal{R}^*_m$ denote a profile such that $n_\succ(R) = n_{\succ'}(R) > 0$ for all preference relations $\succ, \succ' \in \mathcal{R}(X_R)$. It holds for all lotteries $p \in \Delta(X_R)$ and alternatives $x^* \in X_R$ that $dist(p, R, x^*) = \frac{sc(p,d^*)}{sc(x^*,d^*)}$, where d^* denotes the biased metric induced by the function t with $t(x^*) = 0$ and $t(x) = 2$ for all $x \in X_R \setminus \{x^*\}$*

Proof. Let R denote a profile as specified by the lemma and consider a lottery p and an alternative x^*. If $p(x^*) = 1$, then $\frac{sc(p,d)}{sc(x^*,d)} = 1$ for every metric $d \in D(R)$, so we assume that $p(x^*) < 1$. In this case, let $\hat{d} \in D(R)$ denote the biased metric given by the function $\hat{t}$ with $\hat{t}(x^*) = 0$ and $\hat{t}(x) = \ell$ for all $x \in X_R \setminus \{x^*\}$, where ℓ is chosen such that $sc(x^*, \hat{d}) = 1$. First, $\hat{d}$ is indeed a valid metric in $D(R)$ due to Proposition 5.1 of Charikar and Ramakrishnan [2022]. Next, we note that $\frac{sc(p,\hat{d})}{sc(x^*,\hat{d})} = \frac{sc(p,d^*)}{sc(x^*,d^*)}$ for the metric d^* stated in the lemma as $\hat{d} = \alpha d^*$ for some $\alpha \in \mathbb{R}_{>0}$. Hence, we aim to show that $dist(p, R, x^*) = \frac{sc(p,\hat{d})}{sc(x^*,\hat{d})}$. For this, we will prove that that $dist(p_x, R, x^*) = \frac{sc(p_x,\hat{d})}{sc(x^*,\hat{d})}$ for every alternative $x \in X_R$ and lottery p_x with $p_x(x) = 1$. This implies the lemma because

$$\begin{aligned}
\frac{sc(p, \hat{d})}{sc(x^*, \hat{d})} &\leq dist(p, R, x^*) \\
&= \max_{d \in D(R)} \frac{sc(p, d)}{sc(x^*, d)} \\
&\leq \sum_{x \in X_R} p(x) dist(p_x, R, x^*) \\
&= \sum_{x \in X_R} p(x) \frac{sc(x, \hat{d})}{sc(x^*, \hat{d})}
\end{aligned}$$

$$= \frac{sc(p, \hat{d})}{sc(x^*, \hat{d})}.$$

Now, we first note that the claim trivially follows follows for the lottery p_{x^*} as $\frac{sc(p_{x^*},d)}{sc(x^*,d)} = 1$ for every metric $d \in D(R)$. We thus focus on an alternative $\hat{x} \in X_R \setminus \{x^*\}$ in the subsequent. To this end, we will show that $\hat{d}$, $\hat{t}$ correspond to an optimal solution of LP 1 for $dist(p_{\hat{x}}, R, x^*)$ as Proposition 2 then implies that $dist(p_{\hat{x}}, R, x^*) = \frac{sc(p_{\hat{x}},\hat{d})}{sc(x^*,\hat{d})}$. We therefore observe that it is easy to show that $\hat{d}$, $\hat{t}$ are a feasible solution for this linear program, so we will subsequently only prove that our solution is also optimal.

Step 1: Since we want to reason about the optimal solutions of LP 1 (for $dist(p_{\hat{x}}, R, x^*)$), we first prove that the optimal objective value of this linear program is bounded. To this end, let d_{LP}, t_{LP} denote a feasible solution to LP 1. We first note that $\sum_{v \in V_R} d_{LP}(x^*, v) = 1$ and hence $d_{LP}(x^*, v) \leq 1$ for all $v \in V_R$. Moreover, since every preference relation appears at least once in R, there is a voter v such that $\hat{x} \succ_v x^*$ and we can conclude by the first and third constraints that $1 \geq d_{LP}(x^*, v) \geq \frac{1}{2}(t_{LP}(\hat{x}) - t_{LP}(x^*)) = \frac{1}{2}t_{LP}(\hat{x})$. Hence, it holds that $t_{LP}(\hat{x}) \leq 2$. By the fourth constraint, we can next conclude that that $d_{LP}(\hat{x}, v) \leq d_{LP}(x^*, v) + t(\hat{x}) \leq 1 + 2 = 3$ for all $v \in V_R$. Finally, we can now compute that the objective value of any solution is at most $\sum_{x \in X_R} p_{\hat{x}}(x) \sum_{v \in V_R} d(x, v) = \sum_{v \in V_R} d(\hat{x}, v) \leq 3n_R$. Since this holds for every feasible solution of LP 1, its optimal objective value is indeed bounded.

Step 2: Let d^0_{LP}, t^0_{LP} denote an optimal solution of LP 1 and let o^0_{LP} denote its objective value. Our next goal is to construct an optimal solution d^1_{LP}, t^1_{LP} of LP 1 such that $t^1_{LP}(x) = t^1_{LP}(y)$ for all $x, y \in X_R \setminus \{x^*, \hat{x}\}$. For this, we denote by Π the set of permutations $\pi : X_R \to X_R$ such that $\pi(x^*) = x^*$ and $\pi(\hat{x}) = \hat{x}$. Moreover, given a permutation $\pi \in \Pi$, we let R^π denote the profile defined by $x \succ^\pi_v y$ iff $\pi(x) \succ_v \pi(y)$ for all $x, y \in X_R$ and $v \in V_R$. Finally, we define $d^\pi(x, v) = d^0_{LP}(\pi(x), v)$ and $t^\pi(x) = t^0_{LP}(\pi(x))$ for all $x \in X_R$ and $v \in V_R$. Since R^π, d^π, and t^π are all derived from R, d^0_{LP}, and t^0_{LP} by renaming the alternatives according to π, it can be checked that d^π and t^π constitute a feasible solution of LP 1 for (p_x, R^π, x^*) with objective value $o^\pi_{LP} = o^0_{LP}$. In particular, it is important here that $\pi(\hat{x}) = \hat{x}$ and $\pi(x^*) = x^*$ as these ensure that $t^\pi(x^*) = 0$ and $d^\pi(\hat{x}, v) = d^0_{LP}(\hat{x}, v)$ for all $v \in V_R$. Next, since all preference relations appear equally often in the profile R, the profile R^π equals R up to renaming the voters. Hence, there is another bijection $\tau : V_R \to V_{R^\pi}$ such that $\succ_v = \succ^\pi_{\tau(v)}$ for all voters $v \in V_R$. Based on this permutation, we define the functions $\bar{d}^\pi$ and $\bar{t}^\pi$ by $\bar{d}^\pi(x, v) = d^\pi(x, \tau(v))$ and $\bar{t}^\pi(x) = t^\pi(x)$ for all $x \in X_R$ and $v \in V_R$. Since we essentially only rename variables in this step, it follows that $\bar{d}^\pi$, $\bar{t}^\pi$ are a feasible solution to LP 1 for $dist(p_{\hat{x}}, R, x^*)$. Moreover, the objective value of this solution is $\bar{o}^\pi_{LP} = o^\pi_{LP} = o^0_{LP}$.

Next, we define the solution d^1_{LP}, t^1_{LP} by $d^1_{LP}(x, v) = \frac{1}{(m-2)!} \sum_{\pi \in \Pi} \bar{d}^\pi(x, v)$ and $t^1_{LP}(x) = \frac{1}{(m-2)!} \sum_{\pi \in \Pi} \bar{t}^\pi(x)$ for all $x \in X_R$ and $v \in V_R$. Since d^1_{LP}, t^1_{LP} is a convex combination of feasible solutions of LP 1, it is itself again feasible. Furthermore, for every $\pi \in \Pi$, it holds that $\bar{o}^\pi_{LP} = o^0_{LP}$, so the objective value of our new solution is $o^1_{LP} = o^0_{LP}$. In particular, this means that d^1_{LP}, t^1_{LP} is an optimal solution to LP 1 (for $dist(p_{\hat{x}}, R, x^*)$). Finally, we note that $t^1_{LP}(x) = \frac{1}{(m-2)!} \sum_{\pi \in \Pi} \bar{t}^\pi(x) = \frac{1}{m-2} \sum_{z \in X_R \setminus \{\hat{x}, x^*\}} t^0_{LP}(z) = \frac{1}{(m-2)!} \sum_{\pi \in \Pi} \bar{t}^\pi(y) = t^1_{LP}(y)$ for all $x, y \in X_R \setminus \{x^*, \hat{x}\}$. Thus, our new solution satisfies all our requirements.

Step 3: As third step, we will show that there is a biased metric $\bar{d}$ defined by a function $\bar{t}$ with $\bar{t}(x) = \bar{t}(y)$ for all $x, y \in X_R \setminus \{\hat{x}, x^*\}$ that constitutes an optimal solution to LP 1. For this, let d^1_{LP}, t^1_{LP} denote the optimal solution constructed in the last step. First, we note that for all $x \in X_R \setminus \{x^*\}$, $v \in V_R$ with $d^1_{LP}(x, v) < d^1_{LP}(x^*, v) + \min_{y \in X_R\colon x \succeq_v y} t^1_{LP}(y)$, we can simply increase the value of d^1_{LP} to $d^1_{LP}(x^*, v) + \min_{y \in X_R\colon x \succeq_v y} t(y)$ without violating any constraints. Moreover, increasing the value of $d^1_{LP}(x, v)$ does not reduce the objective value, so there is another optimal solution d^2_{LP}, t^2_{LP} with $d^2_{LP}(x^*, v) = d^1_{LP}(x^*, v)$ for all $v \in V_R$, $t^2_{LP}(x) = t^1_{LP}(x)$ for all $x \in X_R$, and $d^2_{LP}(x, v) = d^2_{LP}(x^*, v) + \min_{y \in X_R\colon x \succeq_v y} t^2_{LP}(y)$ for all $x \in X_R \setminus \{x^*\}$, $v \in V_R$.

Next, we want to ensure that $d^2_{LP}(x^*, v) = \frac{1}{2} \max_{x, y \in X_R\colon x \succeq_v} t^2_{LP}(x) - t^2_{LP}(y)$. To this end, we assume that there is a voter v^* such that $d^2_{LP}(x^*, v^*) > \frac{1}{2} \max_{x, y \in X_R\colon x \succeq_{v^*} y} t^2_{LP}(x) - t^2_{LP}(y)$. In this case, we define $\delta = d^2_{LP}(x^*, v^*) - \frac{1}{2} \max_{x, y \in X_R\colon x \succeq_{v^*} y} t^2_{LP}(x) - t^2_{LP}(y)$ and observe that $\delta < 1$ as $d^2_{LP}(x^*, v) < 1$ for all voters $v \in V_R$. Next, consider the solution $\tilde{d}$, $\tilde{t}$ derived from d^2_{LP} and t^2_{LP} by setting $\tilde{d}(x, v^*) = d^2_{LP}(x, v^*) - \delta$ for all $x \in X_R$. We first note that $\tilde{d}$, $\tilde{t}$ still satisfies the first four constraints. Moreover, it holds for all $x \in X_R$ that $\tilde{d}(x, v^*) = \tilde{d}(x^*, v^*) + \min_{y \in X_R\colon x \succeq_{v^*} y} \tilde{t}(y)$, so $\tilde{d}(x, v^*) + \tilde{d}(x^*, v^*) = 2\tilde{d}(x^*, v^*) + \min_{y \in X_R\colon x \succeq_{v^*} y} \tilde{t}(y) \geq \tilde{t}(x)$ because $2\tilde{d}(x^*, v^*) \geq t(x) - \min_{y \in X_R\colon x \succeq_{v^*} y} \tilde{t}(y)$. Hence, our new solution only violates the normalization condition of LP 1, and we can restore this by scaling all variables by the value $\frac{1}{1-\delta}$, i.e., $\tilde{d}'(x, v) = \frac{1}{1-\delta} \tilde{d}(x, v)$ and $\tilde{t}'(x) = \frac{1}{1-\delta} \tilde{t}(x)$ for all $x \in X_R$ and $v \in V_R$ while leaving the remaining conditions intact. Finally, we compute the objective value of our new solution $\tilde{d}'$, $\tilde{t}'$:

$$\begin{aligned}
\sum_{v \in V_R} \tilde{d}'(\hat{x}, v) &= \frac{1}{1-\delta} \sum_{v \in V_R} \tilde{d}(\hat{x}, v) \\
&= \frac{1}{1-\delta} \sum_{v \in v(R)} d^2_{LP}(\hat{x}, v) - \frac{\delta}{1-\delta} \\
&= \frac{1}{1-\delta} \sum_{v \in V_R} d^2_{LP}(x^*, v) + \min_{y \in X_R\colon \hat{x} \succeq_v y} t^2_{LP}(y) \\
&\quad - \frac{\delta}{1-\delta} \\
&= \frac{1}{1-\delta} \sum_{v \in V_R} \min_{y \in X_R\colon \hat{x} \succeq_v y} t^2_{LP}(y)
\end{aligned}$$

$$+\frac{1}{1-\delta}-\frac{\delta}{1-\delta}$$
$$\geq \sum_{v\in v(R)} d^2_{LP}(x^*,v)+\min_{y\in X_R\colon \hat{x}\succeq_v y} t^2_{LP}(y)$$
$$= o^2_{LP}.$$

Here, the first two inequalities use the definitions of $\tilde{d}'$ and $\tilde{d}$ respectively. Next, we apply that $d^2_{LP}(\hat{x},v)=d^2_{LP}(x^*,v)+\min_{y\in X_R\colon \hat{x}\succeq_v y} t^2_{LP}(y)$ for all $v\in V_R$. In the third step, we then use that $\sum_{v\in V_R} d^2_{LP}(x^*,v)=1$. The remaining steps are simple arithmetic changes. This inequality proves that our new solution $\tilde{d}', \tilde{t}'$ is an optimal solution to LP 1.

Finally, we can repeat this step until we arrive at an optimal solution d^3_{LP}, t^3_{LP} such that *i)* $d^3_{LP}(x^*,v)=\frac{1}{2}\max_{x,y\in X_R\colon x\succeq_v y} t^3_{LP}(x)-t^3_{LP}(y)$ for all $v\in V_R$, *ii)* $d^3_{LP}(x,v)=d^3_{LP}(x^*,v)+\min_{y\in X_R\colon x\succeq_v y} t(y)$ for all $x\in X_R$, $v\in V_R$, *iii)* $t^3_{LP}(x)=t^3_{LP}(y)$ for all $x,y\in X_R$, and *iv)* $t^3_{LP}(x^*)=0$ and $t^3_{LP}(\hat{x})\geq 0$. In particular, for the last points, we note that we only scale the values t^1_{LP} by some constants during our constructions, so we directly inherit this insight from d^1_{LP}. Therefore, d^3_{LP} is the biased metric $\bar{d}$ defined by $\bar{t}(x)=t^3_{LP}(x)$ for all $x\in X_R$.

Step 4: As last step, we will show that $sc(\hat{x},\hat{d})\geq sc(\hat{x},\bar{d})$ for the metric $\bar{d}$ constructed during the last step. This completes the proof of this lemma since it means that $\hat{d}$, $\hat{t}$ are an optimal solution to LP 1. To this end, we recall that the function $\hat{t}$ that defines $\hat{d}$ is specified by a single value $\ell\in\mathbb{R}_{>0}$: $\hat{t}(x^*)=0$ and $\hat{t}(x)=\ell$ for all $x\in X_R\setminus\{x^*\}$. Moreover, the function $\bar{t}$ that defines $\bar{d}$ is specified by two values ℓ_1 and ℓ_2: $\bar{t}(x^*)=0$, $\bar{t}(\hat{x})=\ell_1$ and $\bar{t}(x)=\ell_2$ for all $x\in X_R\setminus\{\hat{x},x^*\}$. If $\ell_1=\ell_2>0$, we are done and we thus suppose that $\ell_1\neq\ell_2$.

Now, first suppose that $\ell_1\leq\ell$. In this case, we first note that $sc(x^*,\hat{d})=sc(x^*,\bar{d})=1$ by construction, so we will focus on $sc(\hat{x},\hat{d})$ and $sc(\hat{x},\bar{d})$. Therefore, we observe that

$$sc(\hat{x},\hat{d})-sc(x^*,\hat{d})=\sum_{v\in V_R}\min_{y\in X_R\colon x\succeq_v y}\hat{t}(y)=\frac{n_R}{2}\ell$$

because half of the voters prefer x to x^* (which means that $\min_{y\in X_R\colon x\succeq_v y}\hat{t}(y)=0$) and the other half of the voters prefers x^* to x (which means that $\min_{y\in X_R\colon x\succeq_v y}\hat{t}(y)=\ell$). An analogous argument shows that $sc(\hat{x},\bar{d})-sc(x^*,\bar{d})\leq\frac{n_R}{2}\ell_1$. Finally, combining our insights implies that $sc(\hat{x},\hat{d})-1\geq sc(x^*,\hat{d})\geq sc(\hat{x},\bar{d})$, which shows that the lemma holds in this case.

We thus suppose next that $\ell_1>\ell$. As first point, we note in this case that $\ell_2<\ell$. Indeed, if $\ell_1>\ell$ and $\ell_2\geq\ell$, then $\hat{d}(x^*,v)\leq\bar{d}(x^*,v)$ for all $v\in V_R$, and the inequality is strict for all voters that rank x^* below $\hat{x}$. In more detail, it holds that $\hat{d}(x^*,v)=0\leq\bar{d}(x^*,v)$ for all voters v that top-rank x^* and $\hat{d}(x^*,v)=\frac{\ell}{2}\leq\frac{\min(\ell_1,\ell_2)}{2}\leq\bar{d}(x^*,v)$ for all other voters. Hence, $sc(x^*,\bar{d})>sc(x^*,\hat{d})=1$, which contradicts that $sc(x^*,\bar{d})=1$. So, we derive indeed that $\ell_2<\ell$.

We thus suppose that that $\ell_2<\ell<\ell_1$ and assume for contradiction that $sc(\hat{x},\hat{d})<(\hat{x},\bar{d})$. Since $sc(x^*,\hat{d})=sc(x^*,\bar{d})=1$, this assumption implies that $sc(\hat{x},\hat{d})-sc(x^*,\hat{d})<sc(\hat{x},\bar{d})-sc(x^*,\bar{d})$. We will thus compute the values of these differences and therefore recall that $sc(\hat{x},\hat{d})-sc(x^*,\hat{d})=\frac{n_R}{2}\ell$. Moreover, $\bar{d}(\hat{x},v)=\bar{d}(x^*,v)$ for all voters $v\in V_R$ with $\hat{x}\succ_v x^*$, $\bar{d}(\hat{x},v)=\bar{d}(x^*,v)+\ell_1$ for all voters $v\in V_R$ that bottom-rank $\hat{x}$, and $\bar{d}(\hat{x},v)=\bar{d}(x^*,v)+\ell_2$ for all remaining voters as these prefer $\hat{x}$ to some other alternative $x\neq x^*$. Since $\frac{n_R}{2}$ voters prefer $\hat{x}$ to x^*, $\frac{n_R}{m}$ voters bottom-rank $\hat{x}$ in R, there are $n_R(\frac{1}{2}-\frac{1}{n})$ voters in the last case. Consequently,

$$sc(\hat{x},\bar{d})-sc(x^*,\bar{d})=\frac{n_R}{m}\ell_1+(\frac{n_R}{2}-\frac{n_R}{m})\ell_2.$$

Because $sc(\hat{x},\hat{d})-sc(x^*,\hat{d})<sc(\hat{x},\bar{d})-sc(x^*,\bar{d})$, we conclude that

$$\begin{aligned}&\frac{n_R}{2}\ell<\frac{n_R}{m}\ell_1+(\frac{n_R}{2}-\frac{n_R}{m})\ell_2\\ \iff&(\frac{n_R}{2}-\frac{n_R}{m})(\ell-\ell_2)<\frac{n_R}{m}(\ell_1-\ell)\\ \iff&\frac{m-2}{2}(\ell-\ell_2)<\ell_1-\ell.\end{aligned}\tag{1}$$

To derive a contradiction, we next want to use that $sc(x^*,\hat{d})=sc(x^*,\bar{d})$. We hence observe that

$$1=sc(x^*,\hat{d})=n_R\cdot\frac{m-1}{m}\cdot\frac{\ell}{2}$$

as the $\frac{n_R}{m}$ voters who top-rank x^* satisfy $\hat{d}(x^*,v)=\max_{x,y\in X_R\colon x\succeq_v y}\hat{t}(x)-\hat{t}(y)=0$ and all other voters have $d(x^*,v)=\frac{\ell}{2}$.

Furthermore, to compute $sc(x^*,\bar{d})$, we will determine (lower bounds on) $\bar{d}(x^*,v)$ for every voter $v\in V_R$. To verify the subsequent values, it suffices to identify the pair of alternatives $x,y\in X_R$ with $x\succeq_v y$ that maximizes $\frac{1}{2}(\bar{t}(x)-\bar{t}(y))$ due to the definition of $\bar{d}(x^*,v)$.

- $\bar{d}(x^*,v)\geq 0$ for all voters v top-rank x^*. There are $\frac{n_R}{m}$ such voters.
- $\bar{d}(x^*,v)=\frac{\ell_1}{2}$ for all voters v that bottom-rank x^*. There are $\frac{n_R}{m}$ such voters.
- $\bar{d}(x^*,v)=\frac{\ell_1}{2}$ for all voters v that neither top-rank nor bottom-rank x^* and that prefer $\hat{x}$ to x^*. We note that there are $n_R\frac{m-2}{m}$ voters that neither top-rank nor bottom-rank x^* and exactly half of them prefer $\hat{x}$ to x^*. Hence, there are $\frac{n_R(m-2)}{2m}$ such voters.
- $\bar{d}(x^*,v)\geq\frac{\ell_2}{2}$ for all voters v that neither top-rank nor bottom-rank x^* and that prefer x^* to $\hat{x}$. The central observation for this is that these voters prefer an alternative x with $\bar{t}(x)=\ell_2$ to x^*. Analogous to the last case, there are $\frac{n_R(m-2)}{2m}$ such voters.

Finally, we can now lower bound $sc(x^*,\bar{d})$:

$$\begin{aligned}sc(x^*,\bar{d})&=\sum_{v\in V_R}\bar{d}(x^*,v)\\ &\geq\frac{n_R}{2}\left(\frac{1}{m}0+\frac{1}{m}\ell_1+\frac{m-2}{2m}\ell_1+\frac{m-2}{2m}\ell_2\right)\end{aligned}$$

$$= \frac{n_R}{2m}\left(\frac{m}{2}\ell_1 + \frac{m-2}{2}\ell_2\right)$$

On the other side, we have $sc(x^*, \bar{d}) = sc(x^*, \hat{d}) = 1$. Since $sc(x^*, \hat{d}) = \frac{n_R(m-1)}{2m}\ell$, we derive that

$$\begin{aligned} &\frac{n_R(m-1)}{2m}\ell \geq \frac{n_R}{2m}\left(\frac{m}{2}\ell_1 + \frac{m-2}{2}\ell_2\right) \\ \iff\ & \frac{m-2}{2}(\ell - \ell_2) \geq \frac{m}{2}(\ell_1 - \ell) \\ \iff\ & \frac{m-2}{m}(\ell - \ell_2) \geq \ell_1 - \ell. \end{aligned} \tag{2}$$

Finally, we get from Equations 1 and 2 that $\frac{m-2}{2}(\ell - \ell_2) < \ell_1 - \ell \leq \frac{m-2}{m}(\ell - \ell_2)$. This is a contradiction as $m \geq 3$, so the initial assumption that $\frac{sc(\hat{x},\hat{d})}{sc(x^*,\hat{d})} < \frac{sc(\hat{x},\bar{d})}{sc(x^*,\bar{d}}$ must have been wrong. We have now exhausted all cases and thus conclude that $(\hat{x}, \hat{d}) \geq sc(\hat{x}, \bar{d})$, which finally proves the lemma. $\square$

Due to Lemma 1, we can now compute the metric distortion of every lottery on a profile R with $n_\succ(R) = n_{\succ'}(R)$ for all $\succ, \succ' \in \mathcal{R}(X_R)$.

Lemma 2. *Assume $m \geq 3$ and let $R \in \mathcal{R}^*_m$ denote a profile such that $n_\succ(R) = n_{\succ'}(R) > 0$ for all preference relations $\succ, \succ' \in \mathcal{R}(X_R)$. It holds for every lottery $p \in \Delta(X_R)$ that $dist(p, R) = 2 + \frac{1}{m-1} - \frac{m}{m-1}\min_{x\in X_R} p(x)$.*

Proof. Let R denote a profile such that $n_\succ(R) = n_{\succ'}(R) > 0$ for all preference relations $\succ, \succ' \in \mathcal{R}(X_R)$ and consider an arbitrary lottery p. We will next compute $dist(p, R, x^*)$ for every alternative $x^* \in X_R$. To this end, we use that, by Lemma 1, $dist(p, R, x^*) = \frac{sc(p,d)}{sc(x^*,d)}$ for the biased metric d defined by the function t with $t(x^*) = 0$ and $t(x) = 2$ for all $x \in X_R \setminus \{x^*\}$. Next, we observe that $\frac{sc(p,d)}{sc(x^*,d)} = \sum_{x\in X_R} p(x)\frac{sc(x,d)}{sc(x^*,d)}$. We will thus compute the social cost of every alternative.

For x^*, we first note that $d(x^*, v) = 0$ for all voters that top-rank x^* and $d(x^*, v) = 1$ for all other voters. Hence, it is easy to infer that $sc(x^*, d) = \frac{n_R(m-1)}{m}$. By contrast, to compute the social cost of an alternative $x \in X_R \setminus \{x\}$, we need a more elaborate analysis of the distances $d(x, v)$:

- $d(x, v) = 2$ for all voters who top-rank x^*. There are $\frac{n_R}{m}$ such voters.
- $d(x, v) = 1$ for all voters who bottom-rank x^*. There are $\frac{n_R}{m}$ such voters.
- $d(x, v) = 1$ for all voters who do neither top-rank nor bottom-rank x^* and prefer x to x^*. There are $\frac{n_R(m-2)}{m}$ voters who do neither top-rank nor bottom-rank x^* and precisely half of them prefer x to x^*. Thus, there are $\frac{n_R(m-2)}{2m}$ such voters.
- $d(x, v) = 3$ for all voters who do neither top-rank nor bottom-rank x^* and prefer x^* to x. There are again $\frac{n_R(m-2)}{2m}$ such voters.

We can hence compute that

$$\begin{aligned} sc(x, d) &= \sum_{v\in V_R} d(x, v) \\ &= n_R\left(\frac{2}{m} + \frac{1}{m} + \frac{m-2}{2m} + \frac{3(m-2)}{2m}\right) \\ &= \frac{n_R}{m}(2m-1). \end{aligned}$$

It hence follows that $\frac{sc(x^*,d)}{sc(x^*,d)} = 1$ and $\frac{sc(x,d)}{sc(x^*,d)} = \frac{2m-1}{m-1} = 2 + \frac{1}{m-1}$. Moreover, we can now compute that $\frac{sc(p,d)}{sc(x^*,d)} = (1 - p(x^*))(2 + \frac{1}{m-1}) + p(x^*) = 2 + \frac{1}{m} - \frac{m}{m-1}p(x^*)$. Clearly, this function is decreasing in $p(x^*)$, so we derive that $dist(p, R) = 2 + \frac{1}{m-1} - \frac{m}{m-1}\min_{x\in X_R} p(x)$. $\square$

We note that, by Lemma 2, the optimal lottery p for a profile R with $n_\succ(R) = n_{\succ'}(R) > 0$ for all $\succ, \succ' \in \mathcal{R}(X_R)$, assigns probability $p(x) = \frac{1}{m}$ to all $x \in X_R$. In particular, this lottery achieves a metric distortion of 2 for R. By contrast, every lottery that assigns 0 to some alternative has a metric distortion of $2 + \frac{1}{m-1}$ in R.

To be able to use Lemma 2 in the analysis of the expected metric distortion of RSCFs, we observe that each preference relation will appear roughly equally often with high probability in a preference profile drawn from the IC distribution if the number of voters n is sufficiently large. However, we cannot expect to get precisely a profile where every preference relation appears equally often, and we thus give next a lemma that allows to bound the metric distortion of a lottery p in a profile R based on a large subprofile of R.

Lemma 3. *Let R be a profile and let $p \in \Delta(X_R)$ denote a lottery. Moreover, let R' denote a profile derived from R by choosing a subset of the voters $V_{R'} \subsetneq V_R$ and setting $\succ'_v = \succ_v$ for all $v \in V_{R'}$, and define $\alpha = 1 - \frac{|V_{R'}|}{|V_R|}$. If $dist(p, R) < \infty$ and $dist(p, R') < \infty$, then $dist(p, R) \leq dist(p, R') + \alpha(dist(p, R) + 1)$.*

Proof. Let R and R' denote two profiles as defined by the lemma and let $\alpha = 1 - \frac{|V_{R'}|}{|V_R|}$. Moreover, consider an arbitrary lottery p, let d denote the metric $d \in D(R)$ that maximizes $\frac{sc}{(}p, d)\min_{x\in X_R} sc(x, d)$, and let x^* denote an alternative with $sc(x^*, d) = \min_{x\in X_R} sc(x, d)$. Finally, we define the set $\bar{V}_R = V_R \setminus V_{R'}$ and note that $\alpha n_R = |\bar{V}_R|$. Our main goal is to bound $\sum_{v\in\bar{V}_R} d(x, v)$ for every alternative $x \in X_R$. To this end, we first note that $d(x, v) \leq d(x, x^*) + d(x^*, v)$ for every voter $v \in V_R$. Moreover, $d(x, x^*) \leq d(x, v) + d(v, x^*)$ for every voter $v \in V_R$, so $d(x, x^*) \leq \frac{1}{n_R}(sc(x, d) + sc(x^*, d))$. Combining these insights means that

$$\begin{aligned} \sum_{v\in\bar{V}_R} d(v, x) &\leq |\bar{V}_R| d(x, x^*) + \sum_{v\in\bar{V}_R} d(x^*, v) \\ &\leq \alpha(sc(x, d) + sc(x^*, d)) + \sum_{v\in\bar{V}_R} d(x^*, v). \end{aligned}$$

Hence, we can now compute that

$$dist(p, R) = \sum_{x\in X_R} p(x)\frac{sc(x, d)}{sc(x^*, d)}$$

$$= \sum_{x\in X_R} p(x) \frac{\sum_{v\in V_{R'}} d(x,v) + \sum_{v\in \bar{V}_R} d(x,v)}{sc(x^*,d)}$$
$$\leq \sum_{x\in X_R} p(x) \frac{\sum_{v\in V_{R'}} d(x,v) + \sum_{v\in \bar{V}_R} d(x^*,v)}{sc(x^*,d)}$$
$$+ \sum_{x\in X_R} p(x) \frac{\alpha(sc(x,d)+sc(x^*,d))}{sc(x^*,d)}$$
$$= \sum_{x\in X_R} p(x) \frac{\sum_{v\in V_{R'}} d(x,v) + \sum_{v\in \bar{V}_R} d(x^*,v)}{\sum_{v\in V'_R} d(x^*,v) + \sum_{v\in \bar{V}_R} d(x^*,v)}$$
$$+ \alpha(\frac{sc(p,d)}{sc(x^*,d)} + 1)$$
$$\leq \sum_{x\in X_R} p(x) \frac{\sum_{v\in V_{R'}} d(x,v)}{\sum_{v\in V'_R} d(x^*,v)} + \alpha(dist(p,R)+1)$$
$$\leq dist(p,R') + \alpha(dist(p,R)+1).$$

The first two equalities merely employ definitions. The next step uses our previously deduced upper bound for $\sum_{v\in \bar{V}_R} d(x,v)$. The forth step follows as $\frac{sc(x^*,d)}{sc(x^*,d)} = 1$ and $\sum_{x\in X_R} p(x)sc(x,d) = sc(p,d)$. Finally, we use that $\frac{a}{b} \geq \frac{a+x}{b+x}$ for all $a,b,x \in \mathbb{R}_{\geq 0}$. The last step uses that d is also a valid metric for R', so $dist(p,R') \geq \sum_{x\in X_R} p(x) \frac{\sum_{v\in V_{R'}} d(x,v)}{\sum_{v\in V'_R} d(x^*,v)}$. This completes the proof of this lemma. □

Based on our previous lemmas, we can finally compute the expected metric distortion of the uniform random dictatorship.

Proposition 3. *It holds for every* $m \geq 3$ *that* $\lim_{n\to\infty} \mathbb{E}_{R\sim IC(m,n)}[dist(f_{RD}(R),R)] = 2$.

Proof. Fix some number of voters and alternatives m and n such that n is significantly larger than $m!$ (i.e., such that all subsequent terms are well-defined). We will give lower and upper bounds on $\mathbb{E}_{R\sim IC(m,n)}[dist(f_{RD}(R),R)]$ that both converge to 2 as n goes to infinity. This then also implies that $\lim_{n\to\infty} \mathbb{E}_{R\sim IC(m,n)}[dist(f_{RD}(R),R)] = 2$. To this end, we denote by R from now on a random variable that is distributed according to $IC(m,n)$ and set $\alpha = \frac{1}{\sqrt[3]{n}}$. We furthermore define by T^α the set of profiles on n voters and m alternatives such that $n_\succ(R) > (1-\alpha)\frac{n}{m!}$ for all $\succ \in \mathcal{R}(X_R)$ and note that, by the law of total probability, it holds that

$$\begin{aligned}&\mathbb{E}[dist(f_{RD}(R),R)|]\\ &= \mathbb{P}[R\notin T^\alpha]\cdot \mathbb{E}[dist(f_{RD}(R),R)|R\notin T^\alpha]\\ &+ \mathbb{P}[R\in T^\alpha]\cdot \mathbb{E}[dist(f_{RD}(R),R)|R\in T^\alpha].\end{aligned}$$

Upper bound: For our upper bound, we note that $\mathbb{E}[dist(f_{RD}(R),R)|R \notin T^\alpha] \leq 3$ and $\mathbb{E}[dist(f_{RD}(R),R)|R\in T^\alpha] \leq 3$ as $dist(f_{RD}(R),R) \leq 3$ for all profiles R. Moreover, we note for a fixed preference relation $\succ_1$ that

$$\begin{aligned}\mathbb{P}[R\notin T^\alpha] &= \mathbb{P}[\exists \succ \in \mathcal{R}(X_R)\colon n_\succ(R) \leq (1-\alpha)\frac{n}{m!}]\\ &\leq m!\mathbb{P}[n_{\succ_1}(R) \leq (1-\alpha)\frac{n}{m!}]\\ &\leq m! e^{-\frac{\alpha^2}{2}\cdot\frac{n}{m!}}\\ &= m! e^{-\frac{\sqrt[3]{n}}{2m!}}.\end{aligned}$$

Here, the first inequality is simply the union bound and the second one a standard Chernoff bound.

In light of our discussion so far, it follows that $\mathbb{E}[dist(f_{RD}(R),R)] \leq 3m!e^{-\frac{\sqrt[3]{n}}{2m!}} + (1 - m!e^{-\frac{\sqrt[3]{n}}{2m!}})\mathbb{E}[dist(f_{RD}(R),R)|R \in T^\alpha]$. We hence aim to bound $\mathbb{E}[dist(f_{RD}(R),R)|R\in T^\alpha]$. For this, we observe that $f_{RD}(R,x) > (1-\alpha)\frac{1}{m}$ for all $x\in X_R$ and $R\in T^\alpha$. Next, let R' denote the subprofile of R such that each preference relation appears $\lceil (1-\alpha)\frac{n}{m!}\rceil$ times; such a subprofile exists as $R\in T^\alpha$. By Lemma 2, we hence have that $dist(f_{RD}(R),R') \leq 2 + \frac{1}{m-1} - \frac{m}{m-1}\cdot(1-\alpha)\frac{1}{m} = 2+\frac{\alpha}{m-1}$. By Lemma 3 and the fact that $dist(f_{RD}(R),R) \leq 3$ for all profiles R, we furthermore conclude for all $R\in T^\alpha$ that

$$\begin{aligned}dist(f_{RD}(R),R) &\leq dist(f_{RD}(R),R')\\ &\quad + \frac{n-|V_{R'}|}{n}(1+dist(f_{RD}(R),R))\\ &\leq 2+\frac{\alpha}{m-1} + 4\frac{n-m!\lceil(1-\alpha)\frac{n}{m!}\rceil}{n}\\ &\leq 2+\frac{\alpha}{m-1}+4\alpha\\ &= 2+\frac{1}{\sqrt[3]{n}}(4+\frac{1}{m-1}).\end{aligned}$$

We can now finally compute now that $\mathbb{E}[dist(f(R),R)]$:

$$\begin{aligned}&\mathbb{E}[dist(f_{RD}(R),R)|]\\ &\leq 3m!e^{-\frac{\sqrt[3]{n}}{2m!}} + (1-m!e^{-\frac{\sqrt[3]{n}}{2m!}})\mathbb{E}[dist(f_{RD}(R),R)|R\in T^\alpha]\\ &\leq 3m!e^{-\frac{\sqrt[3]{n}}{2m!}} + (1-m!e^{-\frac{\sqrt[3]{n}}{2m!}})(2+\frac{1}{\sqrt[3]{n}}(4+\frac{1}{m-1})).\end{aligned}$$

Finally, it is easy to check that this bound indeed converges to 2 as n goes to infinity.

Lower bound: For the lower bound, we first note that $dist(p,R) \geq 1$ for every lottery p and every profile R. It hence follows that $\mathbb{E}[dist(f_{RD}(R),R)] \geq \mathbb{P}[R\in T^\alpha]\cdot \mathbb{E}[dist(f_{RD}(R),R)|R\in T^\alpha]$. Moreover, it holds that $\mathbb{P}[R\in T^\alpha] = 1-\mathbb{P}[R\notin T^\alpha] \leq 1-m!e^{-\frac{\sqrt[3]{n}}{2m!}}$ due to the previously discussed Chernoff bound.

Hence, we next aim to find a good lower bound on $\mathbb{E}[dist(f_{RD}(R),R)|R\in T^\alpha]$. To this end, fix a profile $R\in T^\alpha$ and an alternative x^* and consider the biased metric $d\in D(R)$ induced by the function t with $t(x^*)=0$ and $t(x)=2$ for all $x\in X_R\setminus\{x^*\}$. By the definition of $dist(f_{RD}(R),R)$, it follows that $dist(f_{RD}(R),R) \geq \frac{sc(f_{RD}(R),d)}{sc(x^*,d)}$. We will next investigate $sc(f_{RD}(R),d)$ and $sc(x^*,d)$ in more detail. To this end, we note that $f_{RD}(R,x) \geq (1-\alpha)\frac{1}{m}$ for every $x\in X_R$ and that there is a subprofile R' of R such that every ballot appears exactly $\lceil(1-\alpha)\frac{1}{m}\rceil$ in this profile because $R\in T^\alpha$. Moreover, it holds that $\sum_{v\in V_{R'}} d(x^*,v) = \frac{n_{R'}(m-1)}{m}$ and that $\sum_{v\in V_{R'}} d(x,v) = \frac{n_{R'}(2m-1)}{m}$ (this follows analogously to the proof of Lemma 2). Consequently, we can compute that

$$
\begin{aligned}
sc(f_{RD}(R),d) &= \sum_{x\in X_R} f_{RD}(R,x)sc(x,d)\\
&\geq (1-\alpha)\frac{1}{m}\sum_{x\in X_R}\sum_{v\in V_{R'}} d(v,x)\\
&\geq (1-\alpha)\Big(\frac{1}{m}\cdot\frac{n_{R'}(m-1)}{m}\\
&\quad+\frac{m-1}{m}\cdot\frac{n_{R'}(2m-1)}{m}\Big)\\
&= (1-\alpha)\frac{2n_{R'}(m-1)}{m}\\
&\geq (1-\alpha)^2\frac{2n(m-1)}{m}.
\end{aligned}
$$

Next, we will give an upper bound on $sc(x^*,d)$. To this end, we first recall that $\sum_{v\in V_{R'}} d(x^*,v)=\frac{n_{R'}(m-1)}{m}$. Moreover, $d(x^*,v)\leq 1$ for all $v\in V_R$. Thus, $\sum_{v\in V_R} d(x^*,v)\leq \frac{n_{R'}(m-1)}{m}+(n-n_{R'})\leq(1-\alpha)n\frac{m-1}{m}+\alpha n$. We hence derive that

$$
\begin{aligned}
\frac{sc(f_{RD}(R),d)}{sc(x^*,d)} &\geq \frac{2(1-\alpha)^2 n\frac{m-1}{m}}{(1-\alpha)n\frac{m-1}{m}+\alpha n}\\
&= 2\frac{(1-\alpha)^2}{1-\alpha+\alpha\frac{m}{m-1}}\\
&= 2\frac{1-\frac{1}{\sqrt[3]{n}}}{1+\frac{m}{m-1}\frac{1}{\sqrt[3]{n}-1}}
\end{aligned}
$$

Finally, we can now give a lower bound for $\mathbb{E}[dist(f_{RD}(R),R)]$:

$$
\begin{aligned}
&\mathbb{E}[dist(f_{RD}(R),R)]\\
&\geq \mathbb{P}[R\in T^\alpha]\mathbb{E}[dist(f_{RD}(R),R)|R\in T^\alpha]\\
&\geq (1-m!e^{-\frac{\sqrt[3]{n}}{2m!}})\cdot 2\frac{1-\frac{1}{\sqrt[3]{n}}}{1+\frac{m}{m-1}\frac{1}{\sqrt[3]{n}-1}}.
\end{aligned}
$$

Finally, it is easy to see that the right hand side converges to 2 when n goes to infinity. Hence, combining our upper and lower bounds proves that $\lim_{n\to\infty}\mathbb{E}_{R\sim IC(m,n)}[dist(f_{RD}(R),R)]=2$. ☐

Finally, we prove Proposition 4 in a very similar way than Proposition 3.

Proposition 4. *Let $m\geq 3$. It holds for every RSCF f with $dist_m(f)<\infty$ and $z=\liminf_{n\to\infty}\mathbb{P}_{R\sim IC(m,n)}[supp(f(R))\neq X_R]$ that*

1) $\limsup_{n\to\infty}\mathbb{E}_{R\sim IC(m,n)}[dist(f(R),R)]\leq 2+\frac{1}{m-1}$

2) $\liminf_{n\to\infty}\mathbb{E}_{R\sim IC(m,n)}[dist(f(R),R)]\geq 2+\frac{z}{m-1}$.

Proof. Fix some number of voters n and alternatives m and consider an arbitrary RSCF f with $dist_m(f)<\infty$. Just as for Proposition 3, we will give lower and upper bounds on $\mathbb{E}_{R\sim IC(m,n)}[dist(f(R),R)]$ that converge to $2+\frac{z}{m-1}$ and $2+\frac{1}{m-1}$ respectively, thus proving the proposition. To facilitate the proof, we let R denote a random variable which is distributed according to $IC(m,n)$, and set $\alpha=\frac{1}{\sqrt[3]{n}}$ and $y=dist_m(f)$. Moreover, we define T^α as the set of profiles R' such that $n_\succ(R')>\alpha\frac{n}{m!}$ for all $\succ\in\mathcal{R}(X_m)$, and S as the set of profiles R' with $supp(f(R'))\neq X_m$.

Upper bound: For our upper bound, we again use the law of total probability to infer that

$$
\begin{aligned}
&\mathbb{E}[dist(f(R),R)|]\\
&\quad=\mathbb{P}[R\notin T^\alpha]\cdot\mathbb{E}[dist(f(R),R)|R\notin T^\alpha]\\
&\quad+\mathbb{P}[R\in T^\alpha]\cdot\mathbb{E}[dist(f(R),R)|R\in T^\alpha].
\end{aligned}
$$

Now, analogously to the proof of Proposition 3, we can bound this probability by

$$
\begin{aligned}
&\mathbb{E}[dist(f(R),R)|]\\
&\leq ym!e^{-\frac{\sqrt[3]{n}}{2m!}}+(1-m!e^{-\frac{\sqrt[3]{n}}{2m!}})\mathbb{E}[dist(f(R),R)|R\in T^\alpha].
\end{aligned}
$$

We hence aim to bound $\mathbb{E}[dist(f(R),R)|R\in T^\alpha]$ next. Towards this end, we note that, every profile $R\in T^\alpha$ has a suprofile R' such that $n_\succ(R')=\lceil(1-\alpha)\frac{n}{m!}\rceil$ for every $\succ\in\mathcal{R}(X_m)$. Now, by Lemma 2, it follows that $dist(f(R),R')\leq 2+\frac{1}{m-1}$. Applying Lemma 3 then shows that

$$
\begin{aligned}
dist(f(R),R)) &\leq 2+\frac{1}{m-1}+(1-\frac{|V_{R'}|}{n})(y+1)\\
&\leq 2+\frac{1}{m-1}+\alpha(y+1).
\end{aligned}
$$

Hence, we can now conclude that

$$
\begin{aligned}
&\mathbb{E}[dist(f(R),R)|]\\
&\quad\leq ym!e^{-\frac{\sqrt[3]{n}}{2m!}}+(1-m!e^{-\frac{\sqrt[3]{n}}{2m!}})(2+\frac{1}{m-1}+\alpha(y+1)).
\end{aligned}
$$

Taking the limit shows then that $\limsup_{n\to\infty}\mathbb{E}_{R\sim IC(m,n)}[dist(f(R),R)]\leq 2+\frac{1}{m-1}$.

Lower bound: For the lower bound, we note that

$$
\begin{aligned}
&\mathbb{E}[dist(f(R),R)]\\
&\quad\geq\mathbb{P}[R\in T^\alpha\setminus S]\cdot\mathbb{E}[dist(f(R),R)|R\in T^\alpha\setminus S]\\
&\quad+\mathbb{P}[R\in T^\alpha\cap S]\cdot\mathbb{E}[dist(f(R),R)|R\in T^\alpha\cap S].
\end{aligned}
$$

Next, it is simple to see that $\mathbb{P}[R\in T^\alpha\cap S]\geq 1-\mathbb{P}[R\notin T^\alpha]-\mathbb{P}[R\notin S]=\mathbb{R}[R\in S]-\mathbb{P}[R\notin T^\alpha]$. Moreover, it holds that $\mathbb{P}[R\notin T^\alpha]\leq m!e^{-\frac{\sqrt[3]{n}}{2m!}}$, so we have that

$$
\mathbb{P}[R\in T^\alpha\cap S]\geq\mathbb{P}[R\in S]-m!e^{-\frac{\sqrt[3]{n}}{2m!}}.
$$

Subsequently, we will derive lower bounds on our expectations and first analyze $\mathbb{E}[dist(f(R),R)|R\in T^\alpha\cap S]$. To this end, we first fix a profile $R\in T^\alpha\cap S$ and investigate $dist(f(R),R)$. Moreover, let x^* denote an alternative with $f(R,x^*)=0$ (which exists as $R\in S$) and consider the biased metric $d\in D(R)$ given by the function t with $t(x^*)=0$ and $t(x)=2$ for all $x\in X_R\setminus\{x^*\}$. We observe again that R has a subprofile R' such that $n_\succ(R')=\lceil(1-\alpha)\frac{n}{m!}\rceil$ (because $R\in T^\alpha$). Similar to the proof of Lemma 1, it is easy to show for all $x \in X_R \setminus \{x^*\}$ that $\sum_{v\in V_{R'}} d(x,v) = \frac{n_{R'}(2m-1)}{m}$ and that $\sum_{v\in V_{R'}} d(x^*,v) = \frac{n_{R'}(m-1)}{m}$. Since $f(R,x^*)=0$, we can compute that

$$\begin{aligned}sc(f(R),R) &= \sum_{x\in X_R} f(R,x)\sum_{v\in V_R} d(v,x)\\ &\geq \sum_{x\in X_R} f(R,x)\sum_{v\in V_{R'}} d(v,x)\\ &= \frac{n_{R'}(2m-1)}{m}\\ &\geq (1-\alpha)n\frac{2m-1}{m}.\end{aligned}$$

By contrast, we can infer that $sc(x^*,R) \leq \frac{n_{R'}(m-1)}{m} + (n-n_{R'}) \leq (1-\alpha)n\frac{m-1}{m} + \alpha n$. In particular, we note for this inequality that $n_{R'} = m!\lceil(1-\alpha)\frac{n}{m}!\rceil \geq (1-\alpha)n$ and that $d(v,x^*)\leq 1$ for all $v\in V_R$. We can now derive that

$$\begin{aligned}\frac{sc(f(R),R)}{sc(x^*,d)} &\geq \frac{(1-\alpha)n\frac{2m-1}{m}}{(1-\alpha)n\frac{m-1}{m}+\alpha n}\\ &= \frac{\frac{2m-1}{m-1}}{1+\frac{\alpha m}{(1-\alpha)(m-1)}}\\ &= (2+\frac{1}{m-1})\cdot\frac{1}{1+\frac{m}{m-1}\cdot\frac{1}{\sqrt[3]{n}-1}}.\end{aligned}$$

Finally, we can now conclude that $dist(f(R),R) \geq \frac{sc(f(R),R)}{sc(x^*,d)} \geq (2+\frac{1}{m-1})\cdot\frac{1}{1+\frac{m}{m-1}\cdot\frac{1}{\sqrt[3]{n}-1}}$ for all $R\in T^\alpha\cap S$. As a consequence, $\mathbb{E}[dist(f(R),R)|R\in T^\alpha\cap S] \geq (2+\frac{1}{m-1})\cdot\frac{1}{1+\frac{m}{m-1}\cdot\frac{1}{\sqrt[3]{n}-1}}$, too.

Next, we will bound $\mathbb{E}[dist(f(R),R)|R\in T^\alpha\setminus S]$. To this end, let $R\in T^\alpha\setminus S$, let $x^*\in X^R$ denote an alternative that minimizes $f(R,x^*)$, and let d denote the same biased metric as before. Since $R\in T^\alpha$, there is a subprofile R' that contains every ballot precisely $n_\succ(R') = \lceil(1-\alpha)\frac{n}{m!}\rceil$ times. Since $f(R,x^*)\leq\frac{1}{m}$, we can compute that

$$\begin{aligned}sc(f(R),d) &\geq \sum_{v\in V_{R'}}\sum_{x\in X_R} f(R,x)d(x,v)\\ &= (1-f(R,x^*))\frac{n_{R'}(2m-1)}{m}\\ &\quad + f(R,x^*)\frac{n_{R'}(m-1)}{m}\\ &\geq 2n_{R'}\frac{m-1}{m}\\ &\geq 2(1-\alpha)n\frac{m-1}{m}.\end{aligned}$$

Moreover, by our previous analysis, $sc(x^*,d) \leq (1-\alpha)n\frac{m-1}{m}+\alpha n$. Hence, we derive that

$$\begin{aligned}dist(f(R),R) &\geq \frac{sc(f(R),d)}{sc(x^*,d)}\\ &\geq \frac{2(1-\alpha)n\frac{m-1}{m}}{(1-\alpha)\frac{m-1}{m}n+\alpha n}\\ &\geq 2\frac{1}{1+\frac{m}{m-1}\cdot\frac{\alpha}{1-\alpha}}\\ &\geq 2\frac{1}{1+\frac{m}{m-1}\cdot\frac{1}{\sqrt[3]{n}-1}}.\end{aligned}$$

Since this holds for every $R\in T^\alpha$, we infer that $\mathbb{E}[dist(f(R),R)|R\in T^\alpha\setminus S] \geq 2\frac{1}{1+\frac{1}{\sqrt[3]{n}-1}}$. Finally, we can now put everything together:

$$\begin{aligned}&\mathbb{E}[dist(f(R),R)]\\ &\geq \mathbb{P}[R\in T^\alpha\setminus S]\cdot\mathbb{E}[dist(f(R),R)|R\in T^\alpha\setminus S]\\ &\quad+\mathbb{P}[R\in T^\alpha\cap S]\cdot\mathbb{E}[dist(f(R),R)|R\in T^\alpha\cap S]\\ &\geq \mathbb{P}[R\in T^\alpha\setminus S]\cdot 2\cdot\frac{1}{1+\frac{m}{m-1}\cdot\frac{1}{\sqrt[3]{n}-1}}\\ &\quad+\mathbb{P}[R\in T^\alpha\cap S]\cdot(2+\frac{1}{m-1})\cdot\frac{1}{1+\frac{m}{m-1}\cdot\frac{1}{\sqrt[3]{n}}}\\ &= \mathbb{P}[R\in T^\alpha]\cdot 2\cdot\frac{1}{1+\frac{m}{m-1}\cdot\frac{1}{\sqrt[3]{n}-1}}\\ &\quad+\mathbb{P}[R\in T^\alpha\cap S]\cdot\frac{1}{m-1}\cdot\frac{1}{1+\frac{m}{m-1}\cdot\frac{1}{\sqrt[3]{n}-1}}\\ &\geq \frac{1}{1+\frac{m}{m-1}\cdot\frac{1}{\sqrt[3]{n}-1}}\cdot 2\cdot(1-m!e^{-\frac{\sqrt[3]{n}}{2m!}})\\ &\quad+\frac{1}{1+\frac{m}{m-1}\cdot\frac{1}{\sqrt[3]{n}-1}}\cdot(\mathbb{P}[R\in S]-m!e^{-\frac{\sqrt[3]{n}}{2m!}})\cdot\frac{1}{m-1}.\end{aligned}$$

Now, it is easy to verify that

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{1+\frac{m}{m-1}\cdot\frac{1}{\sqrt[3]{n}-1}} = 1 \text{ and}\\ &\lim_{n\to\infty} m!e^{-\frac{\sqrt[3]{n}}{2m!}}\cdot(2+\frac{1}{m-1})\cdot\frac{1}{1+\frac{m}{m-1}\cdot\frac{1}{\sqrt[3]{n}}} = 0.\end{aligned}$$

It hence follows that $\liminf_{n\to\infty}\mathbb{E}_{R\sim IC(m,n)}[dist(f(R),R)] \geq 2+\frac{z}{m-1}$. □