# Voting Among Three Candidates

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(joint work with Chris Dong & Dominik Peters)

Workshop on Social Choice, Game Theory, and Mechanism Design
In Celebration of Hervé Moulin's 75th Birthday
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DSS
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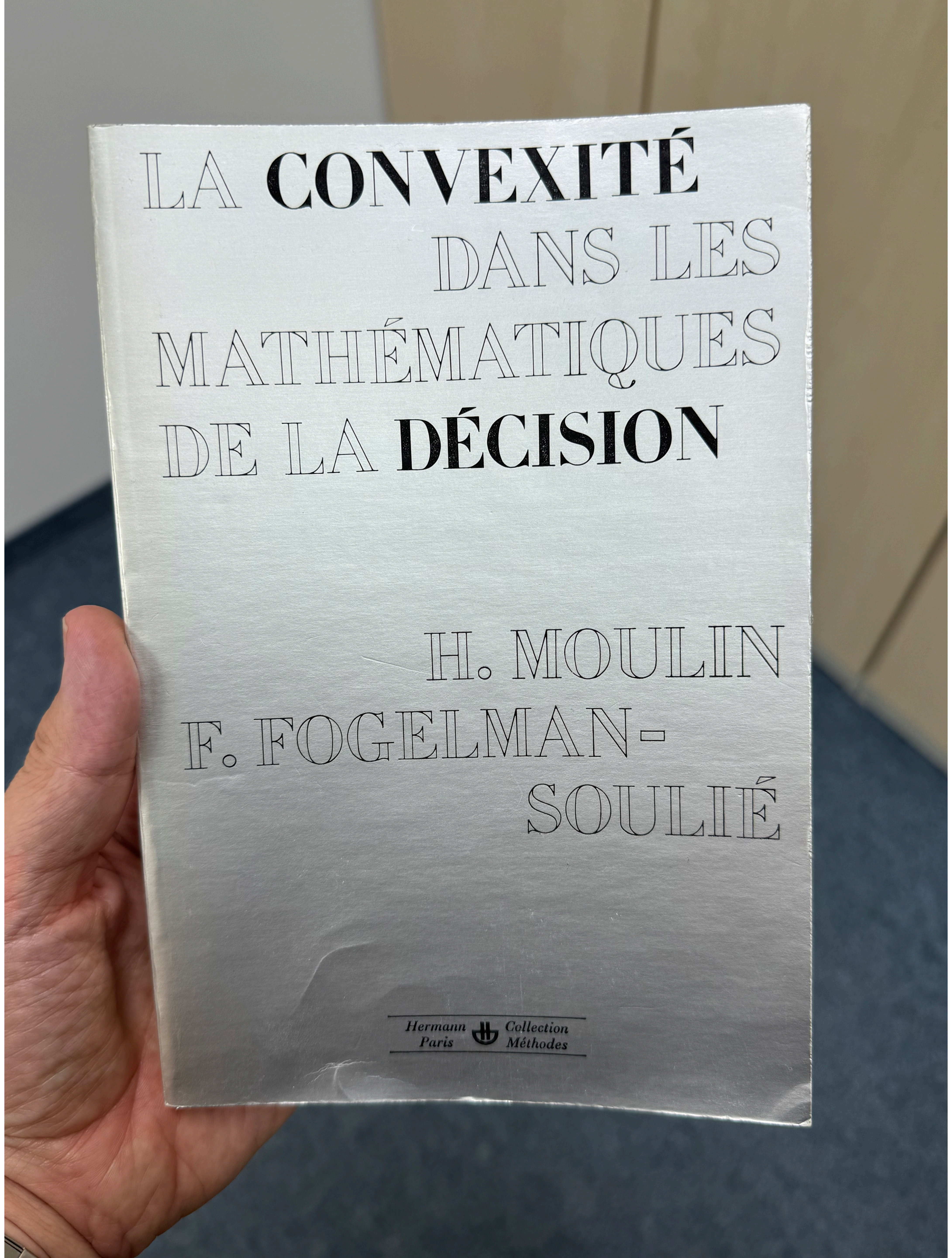



au Professeur G. Debreu.

en respectueux hommage

d'une ancienne invitée de Berkeley

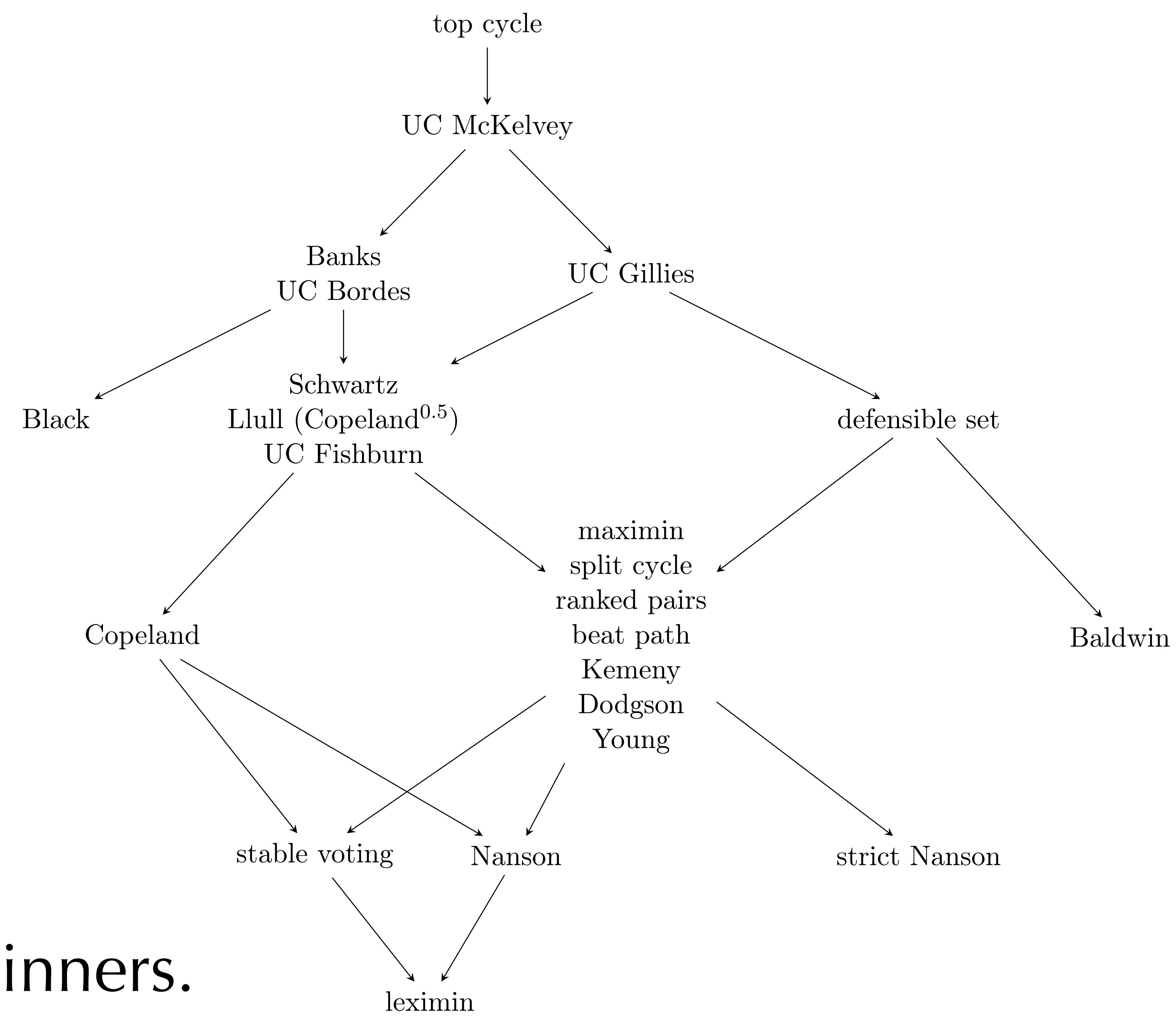
# LA CONVEXITÉ
# DANS LES MATHÉMATIQUES
# DE LA DÉCISION

Avec mes amitiés

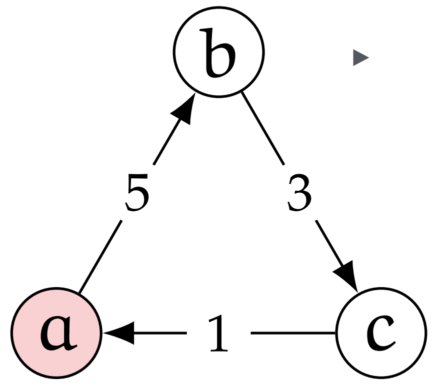
# Voting

- Consider $n$ voters who have strict preferences over candidates.
- A voting rule maps these preferences to a non-empty subset of candidates.
  - Resolute voting rules always return a single candidate.
- Two candidates
  - Simple and natural rules satisfy virtually all desirable properties.
- More than two candidates
  - significant challenges and inevitable tradeoffs (e.g., Arrow, 1951; Gibbard, 1973; Satterthwaite, 1975; Young & Levenglick, 1978; Moulin; 1988)
  - plethora of voting rules
- Does identifying a suitable rule become easier when focusing on the case of *exactly three candidates*?

# Condorcet Extensions




- A Condorcet winner is preferred to both other candidates by some majority of voters.
  - Any other candidate can be overthrown by a coordinated majority.
- Condorcet extensions select a Condorcet winner whenever one exists.
  - Condorcet extensions have appealing strategic properties.
- Black (1948): return Condorcet winner, otherwise Borda winners.
- Maximin: return candidates whose minimal majority margin is maximal.




  - *When the three [pairwise majority] views cannot exist together [because of a cycle], the adopted view results from the two [pairwise majority views] that are most probable [i.e., have the largest majority].*

(Condorcet, 1785, p. 125)

- Nanson (1883): repeatedly delete all candidates whose Borda score is not above average.
- Leximin: break tie between maximin winners by maximizing second-lowest margins.

# Reinforcement

- Condorcet extensions have faced criticism due to their vulnerability to variable-electorate paradoxes, namely
  - the reinforcement paradox (Young and Levenglick, 1978) and
  - the no-show paradox (Moulin, 1988).
- Reinforcement (Young, 1974): Candidates who win in two disjoint electorates should be precisely the winners in the union of these electorates.
- Theorem (Young and Levenglick, 1978):
  Every Condorcet extension violates reinforcement when $n \geq 13$.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**Proposition:** (Young [1978]) a Condorcet Consistent voting rule must violate the Reinforcement axiom at some profile of preferences
note: the known proof of statement $i$) requires 13 voters or more
open question: what is the smallest number of voters for which the statement holds?
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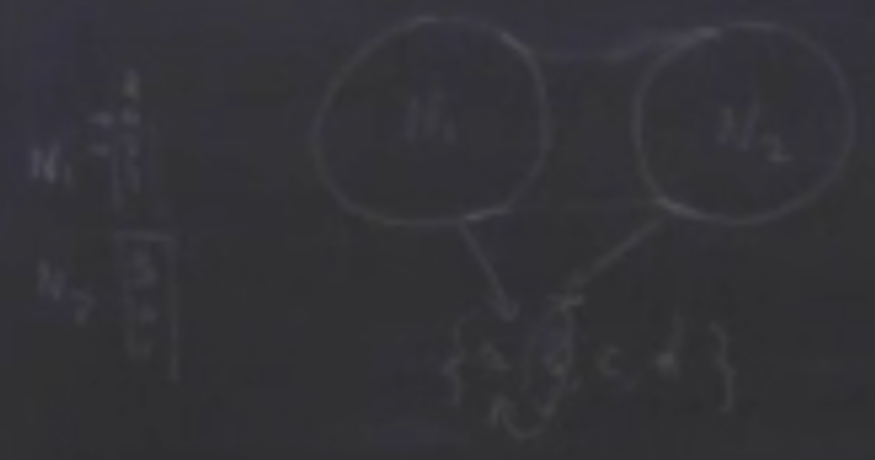

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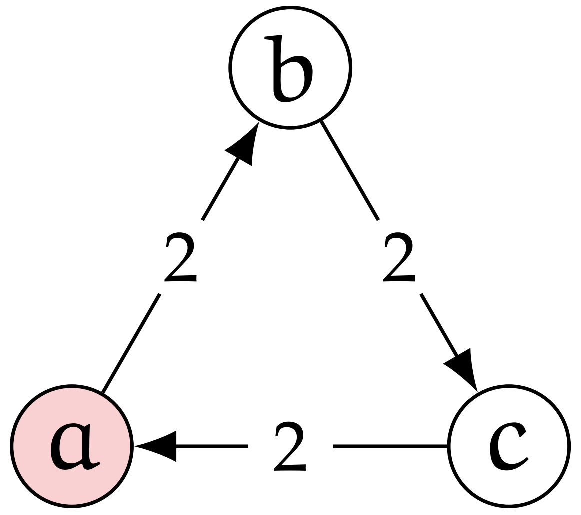
# Reinforcement Results

- Theorem: Every Condorcet extension violates reinforcement iff $n \geq 8$.
  - computer-aided proof argues over hundreds of profiles
  - much simpler proofs when assuming anonymity or letting $n \geq 9$
  - artificial refinement of maximin satisfies reinforcement when $n \leq 7$, anonymity, participation, and monotonicity
- Theorem: Every anonymous and neutral Condorcet extension violates reinforcement iff $n \geq 5$.
  - Black's rule and leximin satisfy reinforcement when $n \leq 4$.
  - Scoring rule with score vector (3,1,0) satisfies Condorcet-consistency when $n \leq 4$ (and reinforcement, participation, and monotonicity).
    - Only non-trivial scoring rule to always return the Condorcet winner (among other candidates) when $n \leq 6$.

| 2 | 2 | 2 |
|---|---|---|
| a | b | c |
| b | c | a |
| c | a | b |

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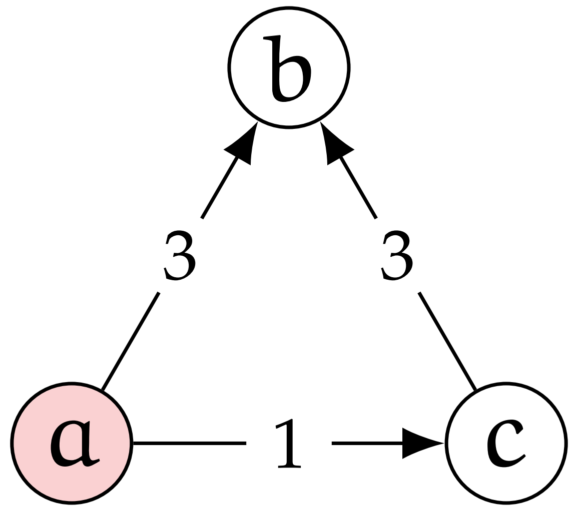
| 1 | 2 |
|---|---|
| c | a |
| a | c |
| b | b |




Condorcet

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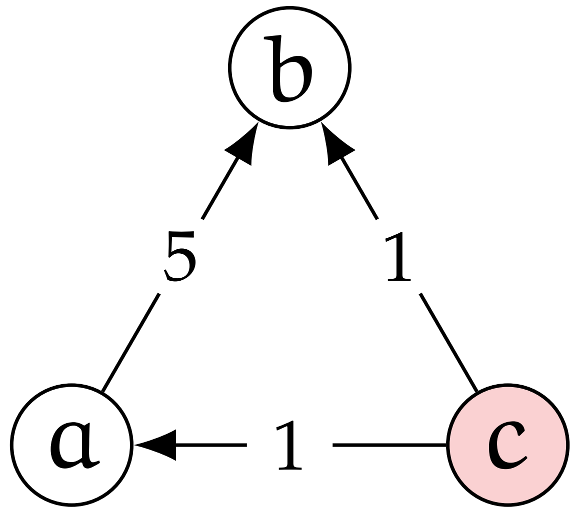
| 2 | 2 | 3 | 2 |
|---|---|---|---|
| a | b | c | a |
| b | c | a | c |
| c | a | b | b |



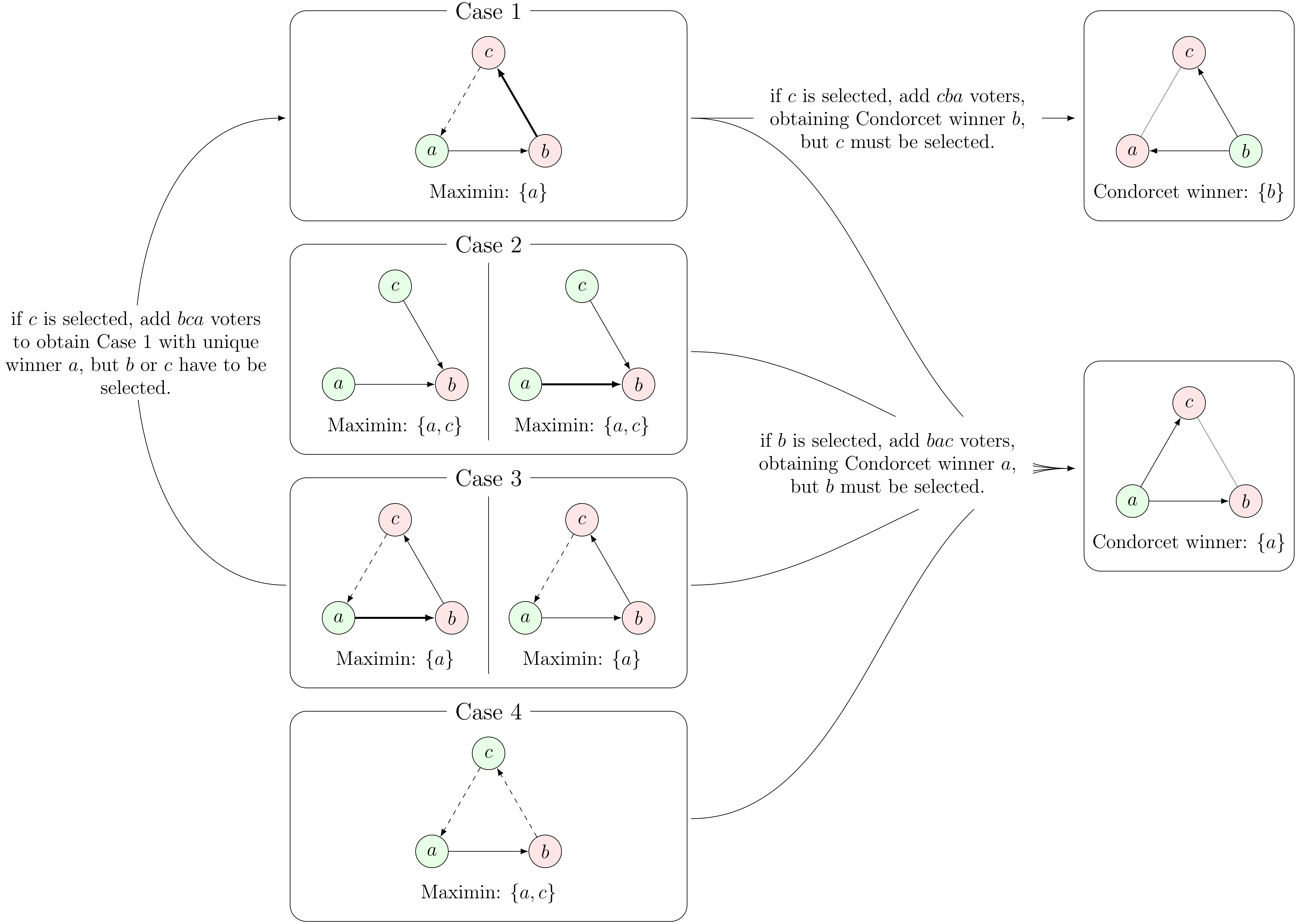

Condorcet

# Participation

- Participation (Brams & Fishburn, 1983): Voters should never be better off by abstaining from an election.
- Theorem (Moulin, 1988): For four candidates, every resolute Condorcet extension violates participation when $n \geq 25$.
  - Maximin with fixed tie-breaking order satisfies participation for three candidates.
  - Theorem (Brandt et al., 2017, Special Issue for Hervé Moulin's 65th Birthday): For four candidates, every resolute Condorcet extension violates participation iff $n \geq 12$.
- Theorem (Jimeno et al., 2009): For five candidates, every Condorcet extension violates optimist participation when $n \geq 27$.
  - Theorem (Brandt et al., 2017): For four candidates, no Condorcet extension satisfies optimist participation iff $n \geq 17$.

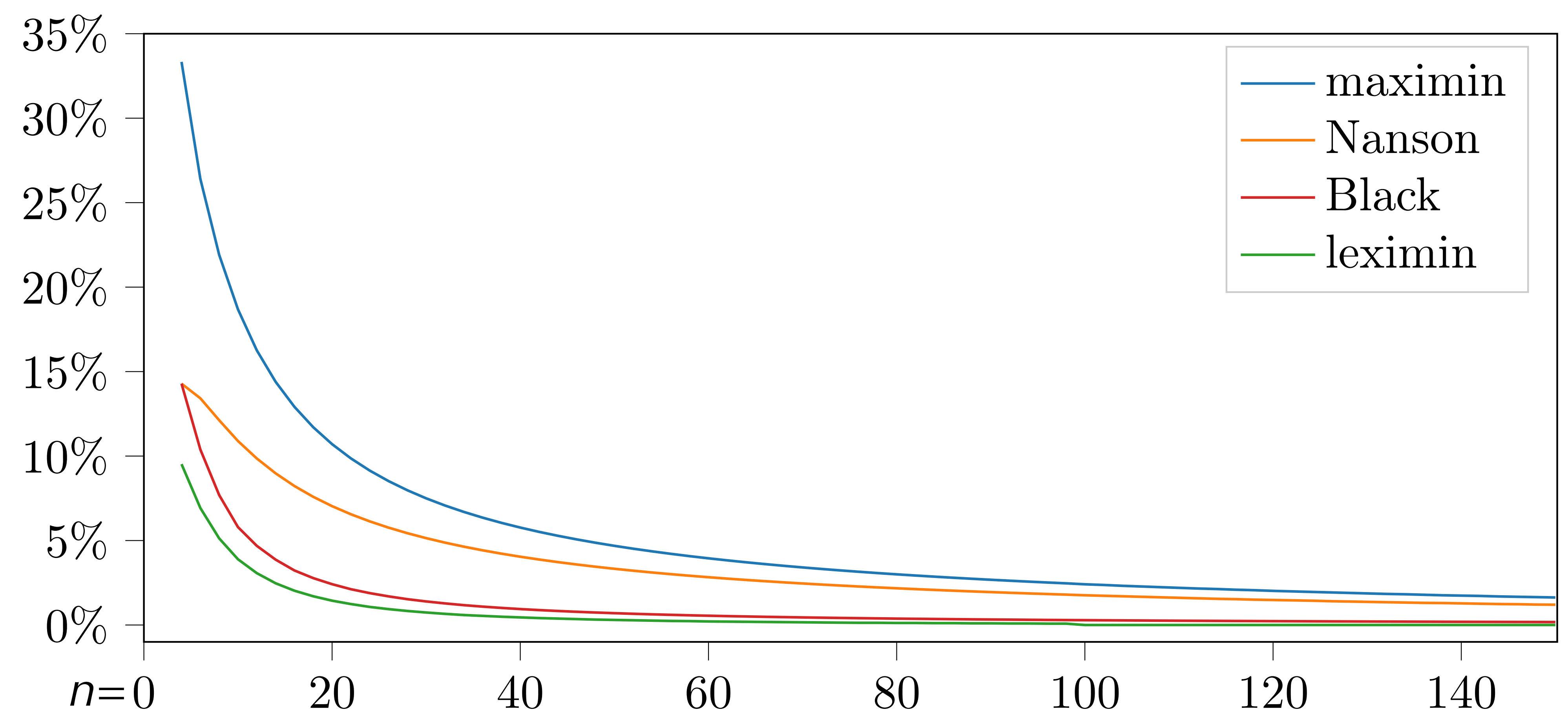
# Participation Results

- Theorem: Every homogeneous Condorcet extension that satisfies optimist participation is a refinement of maximin.
  - Corollary: Every resolute and homogeneous Condorcet extension that satisfies participation is a refinement of maximin.
- Theorem: Maximin is the only homogeneous and continuous Condorcet extension that satisfies optimist participation.
- Theorem: Nanson's rule is the only homogeneous, neutral, and pairwise strong Condorcet extension that satisfies optimist participation and tie-break positive responsiveness.
- Theorem: Leximin is the only homogeneous, neutral, and pairwise Condorcet extension that satisfies optimist participation and positive responsiveness.



Case 1
Maximin: $\{a\}$
Case 2
Maximin: $\{a, c\}$
Maximin: $\{a, c\}$
Case 3
Maximin: $\{a\}$
Maximin: $\{a\}$
Case 4
Maximin: $\{a, c\}$
if $c$ is selected, add $bca$ voters to obtain Case 1 with unique winner $a$, but $b$ or $c$ have to be selected.
if $c$ is selected, add $cba$ voters, obtaining Condorcet winner $b$, but $c$ must be selected.
if $b$ is selected, add $bac$ voters, obtaining Condorcet winner $a$, but $b$ must be selected.
Condorcet winner: $\{b\}$
Condorcet winner: $\{a\}$

# Conclusion




Fraction of anonymous profiles in which rules return more than one winner

- Maximin and two of its refinements—Nanson's rule and leximin—are particularly robust to common criticisms of Condorcet extensions.
  - characterized by their immunity to the no-show paradox
  - suitable for real-world elections with three candidates
  - Nurmi (1989), Felsenthal & Nurmi (2018), and Lepelley & Smaoui (2019) also argue in favor of maximin.
  - Studies on the frequency of voting paradoxes using computer simulations and Ehrhart theory show that maximin also does well for *large* numbers of voters and three candidates (Courtin et al., 2014; Plassmann and Tideman, 2014; Heilmaier, 2020)

a b ~~c~~