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Master's Thesis

Local Rationalizability and Choice Consistency

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Zusammenfassung

In dieser Masterarbeit geht es um Konsistenzbedingungen in der Wahltheorie. Wir führen verallgemeinerte Versionen der Rationalisierbarkeit ein und stellen damit neue Charakterisierungen von verschiedenen Konsistenzbedingungen wie zum Beispiel α und γ auf. Dabei liegt der Schwerpunkt auf Expansionskonsistenz. Diese Resultate können wir verwenden, um klassische Resultate zu beweisen. Zudem definieren wir eigene Konsistenzbedingungen, welche wir dann analysieren. Im Fall von γ^+ können wir sogar eine Charakterisierung liefern und einen Bezug zu einer bereits existierenden Bedingung herstellen. Anschließend wenden wir die Hauptresultate auf die Sozialwahltheorie an und analysieren verschiedene Funktionen, die unsere Definitionen erfüllen. Neben einer Charakterisierung von SC untersuchen wir zudem ein probabilistisches Setting, in dem wir aus konvexen Mengen wählen.

Contents

1 Introduction and Related Work

1.1 Rankings and Choice

Choices are a vital part of our lives. Every day, we face an enormous number of decisions, some of which are very important. To make these decisions reliably, we humans use the concept of rankings. We consider ratings before we choose a restaurant. We have university rankings, which many students consider before applying to one. On the other side of the job market, large companies also prefer to hire graduates of these top-ranking universities. We rate and rank everything, from cars, apps and sports teams to stock. Whenever we make choices, these rankings play an important role. Of course, our choices do not only depend on what we wish for, but also on what is feasible. Not everyone that is a Tesla enthusiast can afford to buy one, since they are quite expensive. When we choose with respect to a ranking, we will always pick the highest ranking option which is feasible for us. While this way of decision-making and choosing for individuals is very useful, it is very hard to derive a ranking for a group or even a society. Nobel laureate Arrow (1951) showed, that combining individual rankings into a group-ranking comes with a severe trade-off, which violates reasonable notions of efficiency and independence. This forces us, as humankind, to overthink and analyze ranking-based choice. Which properties of it are vital, which can we live without?

1.2 Weakenings and the Importance of Expansion Consistency

One possibility to relax the standard model of choice theory is to weaken the notion of rankings. In a mathematical setting, rankings are defined as transitive relations. Hence, a reasonable approach is to weaken the notion of transitivity. Schwartz (1976) and Sen (1971)[1] analyzed relations, which, instead of being transitive, were only acyclic or quasi-transitive. Sadly, this still leads to weaker impossibilities for collective choice. Hence our search needs to continue. Another discovery by Sen (1969, 1977) and Bordes (1976) was that ranking-based choice consists of two parts. One is a contraction consistency condition named α. It states that if one would choose something in a large feasible set, then one should also choose it in all smaller feasible sets. The other one is the expansion consistency condition β^+. It states that under certain conditions, everything chosen in some small set should be chosen in a larger set too.

This idea of using α and expansion consistency conditions turned out to be very useful. Many relation-based choice concepts can be split up. It was identified that the contraction consistency condition α is the main culprit for many impossibilities involving relation-based choice. Hence, when relation-based choice is too restrictive of a concept, we can drop α and focus on expansion consistent choice.

To our knowledge, expansion consistency conditions are not as well understood as their contraction-counterparts. Our main goal in this thesis hence is to gain a better understanding of expansion consistency. After all, various forms of expansion consistency are possible in collective choice without severe trade-offs. We can only benefit from a better understanding of how consistent our choice can be in a group setting.

[1] While Sen is also a Nobel laureate, he was awarded the prize for his contributions to welfare economics.

1.3 Structure of the Thesis

The thesis is structured as follows:
In Section 2, we formally define choice functions and consistency conditions. Afterwards, we introduce our new notions of upwards rationalizability and local revealed preference in Section 3. In Section 4, we use our new notions to present three characterizations of expansion consistency conditions. Applying these, we obtain proofs for classical characterizations involving α. Instead of only dealing with existing consistency conditions, we propose a new expansion consistency condition and characterize it using PIP-transitivity and upwards rationalizability in Section 5. We then compare our new condition to an expansion consistency condition of Schwartz. A dual path is explored in Section 6, where we use our new notion of downwards rationalizability to characterize α. Further, we characterize transitive downwards rationalizability using a new relation. Many technical conditions can be found and analyzed in Section 7, concluding the part of this thesis which deals with classical choice theory. In Section 8, we formally introduce social choice functions. Three of these, which due to their definition are very reminiscent of upwards rationalizability, are then analyzed in Section 9. For the Split Cycle, we present a characterization using γ in Section 10. In a probabilistic choice setting, Section 11 examines whether we can generalize our characterization of γ when dealing with convex and compact feasible sets. At the end of the thesis in Section 12, the most important results are summarized. Further, since new questions arose, some open problems are formulated there.

2 Preliminaries

We introduce some well-known notions and results of choice theory.

2.1 Choice Functions and Degrees of Rationalizability

When an individual has to make decisions, their choice not only depends on what they desire the most, but also on what is feasible for them. For example, Carl wants to buy a car out of the following offers. He *has* to buy one, since he needs it to get to work.

Model	Horse Power	Price	...
Audi R8	540 HP	145,000€	
BMW X5	400 HP	65,000€	
Citroen C4	131 HP	24,000€	
Dacia Duster	257 HP	12,000€	

We say that the offers are the *alternatives* Carl chooses from. For simplicity, we will abbreviate them using a, b, c, d for the names of their brands. The universe of all alternatives is denoted by $U = \{a, b, c, d\}$. In a perfect world, Carl has no restrictions, so all alternatives are *feasible*. He can buy whatever car he likes the most, say the Audi. This we denote by $C(U) = \{a\}$. If Carl instead has a budget restriction of $80,000$€, then not all alternatives are feasible any more. The set of alternatives from which he can now choose is $A = \{b, c, d\}$. Say Carl in this case wants the BMW or the Dacia. He cannot decide which one he likes more, but he certainly prefers both to the Citroen. Then we write $C(A) = \{b, d\}$. This does *not* mean that Carl buys both cars. He rather has to make a decision. In reality, any tiebreaking can be used. Carl could for example flip a coin or let a friend decide for him. Carl's restrictions do not only have to concern his budget. If he needs the car to transport large objects frequently, then the loading volume is not allowed to be too small. In different scenarios, hence different subsets of U are considered as feasible sets. In each of these cases and out of each finite, non-empty subset, Carl can choose his favorite alternatives. This we can now formalize.

Definition 1 (Choice Function)**.** Let U be non-empty and countable. A *choice function* C maps each non-empty, finite $A \subseteq U$ to a non-empty subset of A. Non-empty, finite subsets of U will be called *feasible sets*. The set of all feasible sets is denoted by $\mathcal{F}(U) := \{A \subseteq U |\text{ A is a feasible set}\}$.

Sometimes, we can assume that the choosing entity is able to rank the alternatives. Say, for example, that Carl has the following ranking:

1 Audi R8

2 BMW X5, Dacia Duster

3 Citroen C4

Then his choices become very easy to summarize. If given a feasible set A, he will always choose the alternatives in A, which have the highest position with respect to the ranking. This is in line with the choices we have already discussed, for example $C(U) = \{a\}$.

Choice based on a ranking is not only logical and structured, but also gives us a large amount of insight into how the alternatives can be compared to each other. Ideally, we would wish for such ranking-based choice as often as possible. Sadly, sometimes it already is impossible to match the choices of individuals with any ranking. Furthermore, even if individuals are able to rank the alternatives, society as a whole very often is not able to do so. To understand these phenomena better, we start by formalizing rankings. There are two main properties: First, two alternatives x, y are always comparable to each other. They either tie, or one is strictly better than the other. This is called *completeness* of the ranking. If x has a higher position than y in the ranking, then we say that *x is strictly preferred to y*. If x has a position that is higher or equal to that of y, we say that *x is at least as good as y*. Second, rankings are *transitive*. This means that if the BMW is at least as good as the Dacia, and the Dacia is at least as good as the Citroen, then also the BMW is at least as good as the Citroen. In fact, these two properties *characterize* what we intuitively call rankings. As we have already observed, each ranking is complete and transitive. On the other hand, whenever we have a complete and transitive collection of comparisons on U, these together form a ranking. Formally, we describe rankings using relations.

Definition 2 (Relations, Transitivity, Completeness)**.** Let $R \subseteq U \times U$. We then say that R is a *relation on U*. Instead of $(x, y) \in R$, we write $x\,R\,y$ and say that *x is at least as good as y*. The strict part of the relation R will be denoted by P. This means we define $(x, y) \in P$ if and only if $x\,R\,y$ and not $y\,R\,x$ and say that *x is strictly preferred to y*. We write $x\,P\,y$ instead of $(x, y) \in P$. Similarly, we write $x\,I\,y$ if and only if $x\,R\,y$ and $y\,R\,x$. In this case, we say that the relation is *indifferent* between x and y
We say that R is *transitive*, if for all $x, y, z \in U$ we have that $x\,R\,y$ and $y\,R\,z$ implies $x\,R\,z$. We say that R is *complete* (on U), if for all $x, y \in U$ we have $x\,R\,y$ or $y\,R\,x$.

For our car example, Carl expressed that the BMW and the Dacia have the same rank. This means that the BMW is at least as good as the Dacia, and the Dacia is at least as good as the BMW. In short, we can write this as $b\,I\,d$. We also see that Carl ranks the Audi higher than the Citroen. This means that the Audi is at least as good as the Citroen, but *not* the other way round. In short, we write $a\,P\,c$. If we go through all pairs of alternatives, we can construct R completely.
Now we have to define what it means to have the highest rank within a feasible set A. Observe that the top listed alternatives are at least as good as any other alternative in A. Also, every alternative $y \in A$ that is not ranked highest in A has to be below some other $x \in A$, which is hence strictly preferred to y. We see that the top ranked alternatives, and only they, are maximal.

Definition 3 (Maximality)**.** Let R be a complete relation on U. Let $A \subseteq U$ be a feasible set. We say that an element is *maximal* in A with respect to the relation R, if it is not strictly dominated by any other element in A. The set of maximal elements is denoted by

$$\max_R A := \{x \in A \,|\, \forall y \in A : \neg(y\,P\,x)\} \tag{1}$$

$$= \{x \in A \,|\, \forall y \in A : x\,R\,y\} \tag{2}$$

where we obtain (2) by using completeness of R.

We can now formally describe ranking-based choice functions.

Definition 4 (Transitive Rationalizability)**.** Let C be a choice function. We say that C is *transitively rationalizable*, if there is a transitive, complete relation R on U such that for all feasible sets A

$$C(A) = \max_R A$$

In this case we say that C is *transitively rationalized by* R.

Note that Definition 3 does not require the relation to be transitive, which is important for our purposes. Transitivity of choice can easily be violated. In fact, Nobel laureate Arrow (1951) unveiled severe implications for social choice. When looking at decisions of groups, transitive rationalizability is clashing with other reasonable notions of efficiency, independence and fairness. In the cases where we cannot do without the latter notions, we as a group have to accept that a collective ranking is impossible. But how rational can choice as a group still be? One attempt is to weaken the notion of transitivity. Instead of demanding that our collective relation R is transitive, we could demand that only its strict part P is transitive. Transitivity of R implies transitivity of P, but not vice versa.

Definition 5 (Quasi-Transitivity)**.** Let R be a relation on U. Let its strict part be denoted by P. We say that R is *quasi-transitive* if and only if for all $x, y, z \in U$: $x\,P\,y$ and $y\,P\,z$ implies $x\,P\,z$.

Definition 6 (Quasi-Transitive Rationalizability)**.** Let C be a choice function. We say that C is *quasi-transitively rationalizable*, if there is a quasi-transitive, complete relation R on U such that for all feasible sets A

$$C(A) = \max_R A$$

In this case we say that C is *quasi-transitively rationalized by* R.

This definition allows us to speak of rationalizability of choice without the need of transitivity. Still, Gibbard (1969) discovered that the notion of quasi-transitivity allows for similar, worrying impossibilities. We hence once more attempt a weakening of the notion, this time to an absolutely minimal one. We remember that we always have to make some decision, which means that our choice set must not be empty. If we want our choice to be according to some relation R, then its strict part P hence must not contain cycles. Else, if asked to choose from the cycle we cannot find any maximal elements. On the other hand, if P contains no cycles, then in all feasible sets the set of maximal elements will be non-empty.

Definition 7 (Acyclicity)**.** Let R be a relation on U. We say that R is *acyclic*, if and only if for all $x_1, \dots, x_k \in U$: $x_1\,P\,x_2, \dots, x_{k-1}\,P\,x_k$ implies $x_1\,R\,x_k$.

Yet again we do not manage to escape dire consequences. Brown (1975), Banks (1995) and others proved impossibilities even for acyclicity instead of transitivity. We cannot weaken the notion further. It seems like for now, we are at a dead end road and need a different approach.

2.2 Consistency Conditions

To see whether Carl's choice is logical, we were focusing on relations. Instead, we could also analyze whether his choices are predictable: Say we know his choices on a few feasible sets. Can we now correctly predict his choices on other feasible sets? With this question we enter the realm of *consistency*.

2.2.1 Contraction

We recall the feasible set $A = \{b, c, d\}$, where Carl would choose $C(A) = \{b, d\}$. Sadly, the Dacia is sold before Carl makes a decision, hence the feasible set shrinks to $B = \{b, c\}$. How will he choose now? Based on the fact that Carl would choose the BMW in A, it seems plausible that he will also choose it in B. We formalize this thought process with the following definition.

Definition 8 (α, Chernoff, 1954)**.** Let C be a choice function. We say that C satisfies α, if and only if for all feasible sets $B \subseteq A$:

$$C(A) \cap B \subseteq C(B)$$

Since we demand that some of the elements chosen in A are also chosen in the *subset* B, α is called a *contraction consistency condition*. In the literature there exist more of these, but for our purposes α suffices due to its strength.

2.2.2 Expansion

Let us suppose that for the feasible set $A = \{a, b\}$, Carl chooses the Audi. He does the same when the feasible set is $B = \{\text{a, c}\}$. Now we want to know which elements Carl chooses in the feasible set $A \cup B = \{a, b, c\}$. Since he chose the Audi in both A and B, it seems plausible that he also chooses it in the union of these two sets. We formalize this reasoning.

Definition 9 (γ, Sen, 1971)**.** Let C be a choice function. We say that C satisfies γ, if and only if for all feasible sets A, B:

$$C(A) \cap C(B) \subseteq C(A \cup B)$$

Since we focus on elements chosen in some sets and assume that they will also be chosen in a certain *superset*, γ is an *expansion consistency condition*.
Just like in the case of contraction, we can look at feasible sets $B \subseteq A$ for expansion consistent behavior.
Let us again look at the example $A = \{b, c, d\}$, $B = \{b, c\}$, but this time we premise two things: First, Carl chooses the BMW in B. Second, we also premise that in A, Carl will not choose the Dacia. Then we see that *all* elements chosen in A are contained in B. In other words, the best cars in A can already be found in B. By expanding our feasible set from B to A, we only added an "uninteresting" car. Hence, we expect Carl to choose the BMW in A too. Again, we formalize our reasoning.

Definition 10 (Aizerman, Schwartz, 1976[2]). Let C be a choice function. We say that C satisfies *Aizerman* if and only if the following holds: Let A, B be feasible sets, such that $B \subseteq A$. If $C(A) \subseteq B$, then $C(B) \subseteq C(A)$.

Just like γ, Aizerman is an expansion consistency condition.
One last time, we look at the example $A = \{b, c, d\}$, $B = \{b, c\}$. We again premise that Carl would choose the BMW on the feasible set B. On the other hand, this time we only premise that in A he chooses some car from B, without knowing whether the Dacia is chosen or not. Then, we know that *some* element chosen in A is contained in B. In other words, the subset B already contains some car, which is deemed to be the best choice in the larger set A. One might argue that B in this sense is a set with strong alternatives. Since the BMW is chosen in B, we might expect Carl to choose the BMW in A too.

Definition 11 (β^+, Bordes, 1976). Let C be a choice function. We say that C satisfies β^+, if and only if for all feasible sets A, B with $B \subseteq A$:

$$C(A) \cap B \neq \emptyset \Rightarrow C(B) \subseteq C(A)$$

While α and γ seem to intuitively make sense, our reasoning for Aizerman and β^+ became more and more far fetched. How can we get a better grasp of these conditions?

2.3 The Link Between Consistency and Rationalizability

So far, we failed to escape the so-called Arrovian impossibilities of collective choice. Our first approach was to introduce different degrees of rationalizability. We then tried to replace rationalizability with consistency conditions, but these quickly became hard to grasp. Our way out of this dilemma is a beautiful discovery: Consistency and rationalizability are deeply intertwined. It will turn out that the following, well-known results can be obtained through the theory we develop.

Let C be a choice function. Then the following equivalences hold:

(i) C satisfies α and γ iff C is rationalizable (Sen, 1971)

(ii) C satisfies α, γ and Aizerman iff C is rationalized by a quasi-transitive relation (Schwartz, 1976)

(iii) C satisfies α, and β^+ iff C is rationalized by a transitive relation (Bordes, 1976)

The first above result states that demanding choice to be based on some acyclic relation is equivalent to demanding that choice is consistent with respect to α and γ. The link between consistency and rationalizability goes even deeper: As the second result shows, Aizerman can be combined with α and γ for a higher degree of rationalizability. By adding β^+, we further restrict the relations. In fact, we arrive at transitive rationalizability.
Since one can show that β^+ implies both Aizerman and γ, we do not need to mention the latter conditions in the third result.
Remark 1. It is well known and easy to check that β^+ implies γ and Aizerman. Furthermore, γ and Aizerman are logically independent but together do not imply β^+. We prove one of the claims, which we will use in a later proof.

[2]Aizerman is the "expansion-part" of Postulate 5* introduced by Chernoff (1954).

Lemma 1 (β^+ implies γ)**.** *Let C be a choice function which satisfies β^+. Then C satisfies γ.*

Proof. Let A, B be feasible sets, $x \in C(A) \cap C(B)$. By non-emptiness we know that $C(A \cup B) \cap A \neq \emptyset$ or $C(A \cup B) \cap B \neq \emptyset$. In either case, by β^+ it follows that $x \in C(A \cup B)$ $\square$

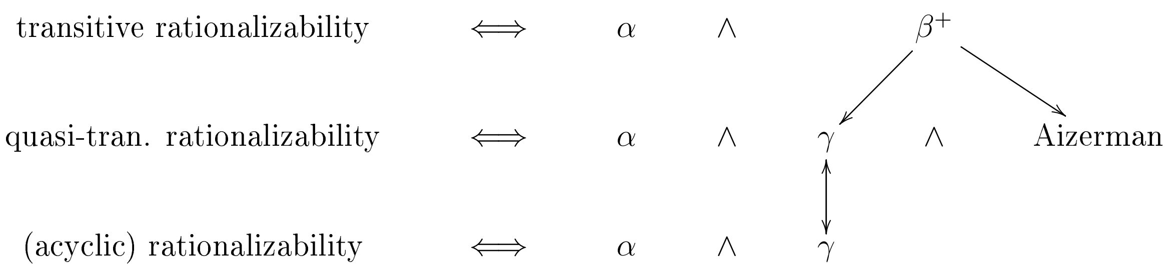
Given a rationalizable choice function C, how do we construct a rationalizing relation? The key idea is that C reveals the underlying preferences through its choices.

Definition 12 (Revealed Preference, Houthakker, 1950[3])**.** Let C be a choice function. We write $x\, R_C\, y$, if and only if there is some feasible set A containing y such that $x \in C(A)$. R_C is called the *revealed preference relation.*

In the case of rationalizability, it is already known that the revealed preference relation is a natural contender.

Proposition 1 (Sen, 1971)**.** *Let C be a choice function. Then it is rationalizable if and only if it is rationalized by its revealed preference relation.*

Now we have a connection between rationalizability and consistency conditions, but this link on its own still does not circumvent the Arrovian impossibilities. Sen (1977) discovered that the main culprit for the impossibilities is α. This discovery secures our escape from the impossibilities of collective choice: We can drop α and focus on expansion consistency only! The main objective of this thesis is thus set. Our goal is to gain a better understanding of expansion consistency without the need of assuming α.



[3]Samuelson (1938) introduced the notion of revealed preference in the field of economics. It was then translated to the setting of choice theory by Houthakker (1950).

3 Upwards Rationalizability, Local Revealed Preference

In this section, we introduce the main, new notions of this thesis. They generalize the existing concepts of rationalizability and revealed preference.

3.1 Definition of Upwards Rationalizability

Definition 13 (Upwards Rationalizability)**.** Let C be a choice function. We say that C is *upwards rationalizable* (UR), if there is a family of relations $(R^A)_A$ such that the following conditions hold :

(i) for all feasible sets A, $R^A \subseteq A \times A$ is acyclic and complete

(ii) for all feasible sets A:

$$C(A) = \max_{R^A} A$$

(iii) for all feasible sets A, B, such that $B \subseteq A$:

$$R^B \subseteq R^A$$

In this case, we say that C is *upwards rationalized by* $(R^A)_A$.

Remark 2. Upwards rationalizability is a generalization of rationalizability. If $R \subseteq U \times U$ rationalizes C, one can set $R^A := R \cap (A \times A)$.

The above definition states that the relations R^A inherit upwards. Often, if one starts by defining the strict parts P^A, it is easier to check that the strict parts inherit downwards, rather than also formulating R^A and then showing that the latter inherit upwards. We show rigorously that these two approaches are equivalent.

Lemma 2. *Let $(R^A)_A$ be a family of complete relations with strict parts $(P^A)_A$. Then for all A, B feasible with $B \subseteq A$ it holds that $R^B \subseteq R^A$ if and only if $x\,P^A\,y \Rightarrow x\,P^B\,y$ for all $x, y \in B$.*

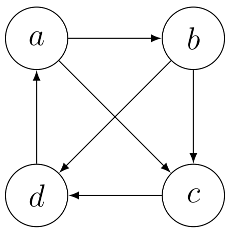
Proof. Let $R^B \subseteq R^A$. Let $x, y \in B$. If $\neg(x\,P^B\,y)$, then by completeness $y\,R^B\,x$. It follows that $y\,R^A\,x$. Hence $\neg(x\,P^A\,y)$. By contraposition, we obtain the wanted implication. Now, let $x\,P^A\,y \Rightarrow x\,P^B\,y$ for all $x, y \in B$ hold true. Then let $x\,R^B\,y$. We again use contraposition and conclude $\neg(y\,P^B\,x) \Rightarrow \neg(y\,P^A\,x) \;\Rightarrow x\,R^A\,y$. Since $x, y \in B$ were arbitrary, we obtain $R^B \subseteq R^A$. □

The above Lemma 2 implies, that in Definition 13 *(iii)* can be replaced by the following condition.

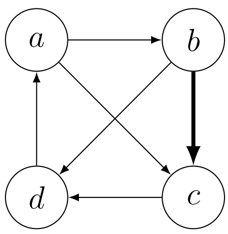
(iii′) $P^A \cap (B \times B) \subseteq P^B$

3.2 An Application of Upwards Rationalizability

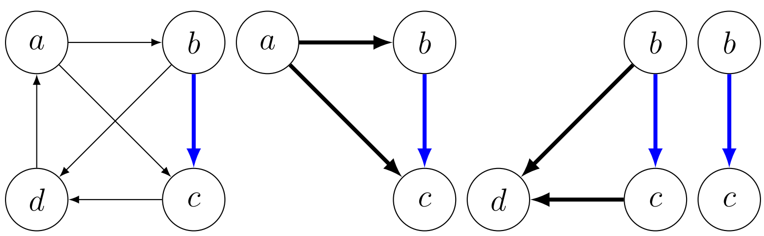
The just introduced notion might seem to be of abstract nature at first, but it stems from an idea that has been prevalent in social choice theory for a long time. Some well known, classical rules satisfy our definition of upwards rationalizability. For example, Good (1971), Smith (1973) and Bordes (1979) studied the Top Cycle rule, while Fishburn (1977) and Miller (1977) analyzed the Uncovered Set rule. Further, there are also modern social choice functions satisfying our definition, such as the Split Cycle rule proposed by Holliday and Pacuit (2020). All three rules are defined using relations R^A for each feasible set A and for all three rules, R^A inherits upwards. Remarkably, all three rules satisfy γ.
A rigorous introduction to social choice theory and an analysis of the three mentioned rules will follow in Section 8 and Section 9, after we present our main choice theoretical results.
To already get a better intuition for our definitions, we informally and quickly present the Uncovered Set rule (UC). We observe a round robin tournament, where the teams compete against each other in duels. The results are represented by arrows from winner to loser. We obtain a graph on $A = \{a, b, c, d\}$.



The idea of the Uncovered Set is to choose the winners of the tournament as follows: Some x is strictly better than some y (short $x\, P^A\, y$) if and only if it defeats y and all teams in A that y wins against. In our example tournament, we have $b\, P^A\, c$, which we denote using a thick edge.



Since there are no further strict preferences, we have $UC(A) = \{a, b, d\}$. Point (iii) of Definition 13, or rather point (iii'), holds true: On all feasible subsets $B \subseteq A$ containing b and c, we still have $b\, P^B\, c$. [4]



[4]Brandt and Fischer (2008) observed this property and called it "covering in subsets".

We also already see that upwards rationalizability is less restrictive than rationalizability: Even thought b is dominated by a on the set $B = \{a, b, c\}$, it still can be chosen in A, since the strict edge $a\,P^B\,b$ is allowed to become a weak edge $a\,I^A\,b$.

3.3 Local Revealed Preference

In our previous example, the so-called covering relation was used to define UC. In general, it seems much harder to tell whether a choice function satisfies upwards rationalizability, since we first need to construct a family of relations. Let a choice function C be given, for which we want to know whether it is UR. Assume for a second, that it was upwards rationalized by some family $(R^A)_A$. If we had $x\,P^A\,y$, then by Lemma 2 we would have $x\,P^B\,y$ for all feasible sets $B \subseteq A$ containing x and y. This means that y cannot be chosen on any of these subsets B. Omitting the assumption, we can formulate the strict part P_C^A of some relation as follows: Some x dominates some y with respect to the feasible set A strictly, if there is no feasible subset containing x in which y is chosen. When using contraposition, we obtain the relation as in the following Definition 14: Some x is at least as good as some y, if there is a subset $B \subseteq A$ which contains both and in which x is chosen.

Definition 14 (Local Revealed Preference)**.** Let C be a choice function, let A be a feasible set with $x, y \in A$. We write $x\,R_C^A\,y$ if and only if there is some feasible set $B \subseteq A$ containing y such that $x \in C(B)$. We call R_C^A the *local revealed preference relation* on A.

The main difference to Definition 12 is that we locally restrict our witness B to be a subset of A, while the revealed preference relation allows for arbitrary witnesses. We directly start with a lemma, which shows that the local revealed preference is a quite good guess. Further, we formally check that the contraposition has been done correctly. This is necessary on an intuitive level, since our original guess was using strict dominance, while our definition is using weak dominance.

Lemma 3. *For any choice function C, $(R_C^A)_A$ satisfies condition (i) and (iii) of Definition* 13. *Furthermore, following equivalence holds: $x\,P_C^A\,y \iff \forall B \subseteq A,\ x, y \in B : y \notin C(B)$. In words, y is strictly dominated by some x in A if and only if there is no subset of A which contains both elements and in which y is chosen.*

Proof. Completeness: Let $\neg(x\,R_C^A\,y)$. Then by definition $x \notin C(\{x, y\})$. By non-emptiness of choice sets, $y \in C(\{x, y\})$. Hence $y\,R_C^A\,x$.
Equivalence: Let $x\,P_C^A\,y$. Then especially $\neg(y\,R_C^A\,x)$. By definition, there hence is no $B \subseteq A$, such that $x, y \in B$ and $y \in C(B)$. Now, let x, y be given, such that the right hand side holds. Then, by definition $\neg(y\,R_C^A\,x)$. By completeness, $x\,R_C^A\,y$ and hence $x\,P_C^A\,y$.
Acyclicity: Let A be a feasible set, $x_1, \dots, x_k \in A$ such that $x_i\,P_C^A\,x_{i+1}$ for all $0 < i < k$. We know that $B := \{x_i | 1 \leq i \leq k\} \subseteq A$. For each $y \neq x_1$, there is some element $x \in B$ with $x\,P_C^A\,y$. Applying the just proven equivalence to $B \subseteq A$, it follows that $y \notin C(B)$. By non-emptiness we conclude $\{x_1\} = C(B)$. Hence B is a witness for $x_1\,R_C^A\,x_k$.
Definition 13 (*iii*): Let A, B be feasible sets such that $B \subseteq A$ and let $x, y \in B$, such that $x\,R_C^B\,y$. Now let $D \subseteq B$ be the witness containing y with $x \in C(D)$. Then of course $y \in D \subseteq A$ and still $x \in C(D)$. This by definition implies $x\,R_C^A\,y$. Hence $R_C^B \subseteq R_C^A$. □

Example 1. Let $U = \{a, b, c\}$. Let C be defined on all non-singleton subsets as follows:

A	$C(A)$
$\{a, b, c\}$	$\{a, b\}$
$\{a, b\}$	$\{a\}$
$\{b, c\}$	$\{b\}$
$\{a, c\}$	$\{a\}$

On $B := \{a, b\}$, we have $a\, P_C^B\, b$. On the other hand, we have $b\, R_C\, a$, since b is chosen on $A := \{a, b, c\}$. Hence we have $b \in \max_{R_C} B$, but $b \notin \max_{R_C^B} B$. Despite $a\, P_C^B\, b$, we still have $b\, I_C^A\, a$ and thus $b \in \max_{R_C^A} A$.
In summary, local revealed preference truly depends on the feasible set we examine. It is worth mentioning that $R_C^A \subseteq R_C^U = R_C$. Hence, the local revealed preference in general can only allow for less maximal elements in a feasible set A than its global, classical version. △

A quick observation is that under α, our restricted notion of revealed preference coincides with the classical one.

Lemma 4. *Let C satisfy α. Then $R_C^A = R_C \cap (A \times A)$ for all feasible sets A. (Especially, this implies $\max_{R_C^A} A = \max_{R_C} A$ for all feasible sets A.)*

Proof. By definition we have $R_C^A \subseteq R_C \cap (A \times A)$. Now let $a\, R_C\, b$ with $a, b \in A$. Then there is some witness $D \in \mathcal{F}(U)$ with $a \in C(D)$, $b \in D$. By α, we have $a \in C(\{a, b\})$. Since $\{a, b\} \subseteq A$, we have $x\, R_C^A\, y$. □

3.4 Local Revealed Preference and the γ-Hull

In this short subsection, we show that the concept of local revealed preference is relevant for expansion consistency, whether C satisfies γ or not. While the presented property is remarkable, it does not belong to our main results.

Definition 15 (Coarsenings, Refinements)**.** Let C, D be choice functions on U. We say that D is a *coarsening* of C, if $C(A) \subseteq D(A)$ for all feasible sets A.
Let $\mathcal{M}$ be a set of choice functions, let $C \in \mathcal{M}$. If every $D \in \mathcal{M}$ is a coarsening of C, we say that C is the finest choice function in $\mathcal{M}$.

It is clear that if such a finest choice function exists, it must be unique.
We now proceed as follows: First, we use an abstract definition to show the existence of a finest coarsening of C satisfying γ. Then, we show that it always chooses the maximal elements with respect to the local revealed preference relations.

Definition 16 (The γ-Hull)**.** Let C be a choice function. Set

$$\mathcal{M} := \{D \mid D \text{ is a choice function, satisfies } \gamma \text{ and } C(A) \subseteq D(A) \quad \forall A \in F(U)\}$$

as the set of all choice functions which satisfy γ and are coarsenings of C. It is non-empty, since the identity $TRIV(A) = A$ is always such a function. Then, for all feasible sets A, define the γ-*hull* of C as

$$\mathcal{H}(C)(A) := \bigcap_{D \in \mathcal{M}} D(A)$$

Lemma 5. *Let C be a choice function. Then $\mathcal{H} \coloneqq \mathcal{H}(C)$ is a well-defined choice function. Further, it is the finest coarsening of C satisfying γ.*

Proof. For all feasible A we know that $A \supseteq \mathcal{H}(A) \supseteq C(A) \neq \emptyset$. Hence $\mathcal{H}$ is a well-defined choice function. Let A, B be feasible sets and $x \in \mathcal{H}(A) \cap \mathcal{H}(B)$. Then for arbitrary $D \in \mathcal{M}$, we know that $x \in D(A) \cap D(B)$. Since D satisfies γ, we conclude that $x \in D(A \cup B)$. Since D was arbitrary, we conclude $x \in \mathcal{H}(A \cup B)$, hence $\mathcal{H}$ satisfies γ. Now, let C' be a coarsening of C satisfying γ. By definition of $\mathcal{M}$ we know that $C' \in \mathcal{M}$. Hence for all feasible sets A we observe that $\mathcal{H}(A) = \bigcap_{D \in \mathcal{M}} D(A) \subseteq C'(A)$. □

Lemma 6. *Let C be a choice function. Then $\mathcal{H}(A) = \max_{R_C^A} A$ for all feasible sets A.*

Proof. We abbreviate $\mathcal{G}(A) \coloneqq \max_{R_C^A} A$. Our goal is to show $\mathcal{H} = \mathcal{G}$.
We first show that the set inclusion from left to right holds. To do so we prove that $\mathcal{G}$ satisfies γ and is a coarsening of C. Let $x \in \mathcal{G}(A) \cap \mathcal{G}(B)$ for some feasible sets A, B. Now assume $x \notin \mathcal{G}(A \cup B)$. Then, by definition there has to be some $y \in A \cup B$ such that $y \, P_C^{A \cup B} \, x$. Without loss of generality say $y \in A$. Then also $y \, P_C^A \, x$ and hence $x \notin \mathcal{G}(A)$, a contradiction. Thus $\mathcal{G}$ satisfies γ. Let now $x \in C(A)$ for some feasible set A, let $y \in A$. Then, there is some $B \subseteq A$ such that $y \in B$ and $x \in C(B)$, namely $B \coloneqq A$. Hence $x \, R_C^A \, y$. Since y was arbitrary, x is maximal and hence $x \in \mathcal{G}(A)$. By Lemma 5, the inclusion $\mathcal{H} \subseteq \mathcal{G}$ holds.
We now show the other set inclusion. Let $x \in \mathcal{G}(A)$. Then, x is maximal. For all $y \in A$ we conclude that there exists some feasible set B_y such that $y \in B_y \subseteq A$ and $x \in C(B_y)$. For any $D \in \mathcal{M}$ we conclude $x \in C(B_y) \subseteq D(B_y)$ and hence $x \in D(\bigcup_{y \in A} B_y) = D(A)$ by γ. By definition of $\mathcal{H}$ it follows that $x \in \mathcal{H}(A)$. □

Proposition 2. *Let C be a choice function. Then $\mathcal{G}(A) \coloneqq \max_{R_C^A} A$ is the unique finest coarsening of C which satisfies γ.*

Proof. Combining Lemma 5 and Lemma 6, we obtain the desired statement. □

4 Three Characterizations

In this section, we present and prove three new characterizations, which form the basis of this thesis. They unveil that the concept of expansion consistency is deeply interwoven with the concept of relation based choice, even without involving contraction consistency.

4.1 Characterizing γ

Our first main result shows two things. On one hand, it is no surprise that rules like the Uncovered Set satisfy γ. On the other hand, and a bit more surprisingly, all choice functions satisfying γ can be represented with a family of relations as in Definition 13.

Theorem 1. *Let C be a choice function. Then the following are equivalent.*

(i) C satisfies γ

(ii) C is upwards rationalizable

Proof of Theorem 1. For "$(ii) \Rightarrow (i)$", let $(R^A)_A$ upwards rationalize C. Let $x \in C(A) \cap C(B)$. Then, assume $x \notin C(A \cup B) = \max_{R^{A\cup B}} A \cup B$. There has to be some $y \in A \cup B$, such that $y\, P^{A\cup B}\, x$. Without loss of generality say $y \in A$. Then, by $A \subseteq A \cup B$ it follows that $y\, P^A\, x$. Hence x is not maximal in A. By assumption we now conclude $x \notin \max_{R^A} A = C(A)$, a contradiction.
For "$(i) \Rightarrow (ii)$", we will use our natural candidate, the family of local revealed preference relations. Let A be a feasible set. By Lemma 3 we only need to show that $C(A) = \max_{R^A_C} A$. For the inclusion from left to right, we use contraposition. Let x not maximal. Then there is $y \in A$, such that $y\, P^A_C\, x$. Assume that $x \in C(A)$. But then there is some B such that $x \in C(B)$ and $y \in B$, namely $B := A$. Hence $x\, R^A_C\, y$, a contradiction. We conclude $x \notin C(A)$. For the inclusion from right to left, let x be maximal in A. Now, let $y \in A$ be given. By maximality of x and completeness of the relation, we know that $x\, R^A_C\, y$. Hence there is some $B_y \subseteq A$, such that x is chosen in it and y is contained in it. Since y was arbitrary and C satisfies γ, we know that

$$x \in C(B_y) \quad \forall y \in A \implies x \in C(\cup_{y\in A} B_y) = C(A)$$

Since we defined γ only for the union of two sets, formally one needs to use induction for the last implication. It is straight forward and omitted in this proof. □

In fact, we have additionally proven the following result. It validates the natural role of $(R^A_C)_A$ for upwards rationalizability.

Proposition 3. *Let C be a choice function. Then it is upwards rationalizable if and only if it is upwards rationalized by its family of local revealed preference relations.*

We observe a similarity between Theorem 1 and a result of Sen (1971), which was already mentioned in Section 2.3. By dropping α, we move from rationalizability to upwards rationalizability. In fact, we can obtain the classical result by applying ours. Since the former involves α, we need one additional observation.

Lemma 7. *Every rationalizable choice function satisfies α.*

Proof. Let R rationalize C, let $x \in C(A) \cap B$. Then $\forall y \in B : x \, R \, y$. Hence $x \in C(B)$. □

Technically speaking, Lemma 7 could also be obtained as a corollary of Theorem 6. The latter characterizes α using a new notion named downwards rationalizability, analogously defined to upwards rationalizability. We now have everything we need to formally state the classical result and prove it.

Corollary 1 (Sen, 1971)**.** *Let C be a choice function. Then the following are equivalent.*

(i) C satisfies α and γ

(ii) C is rationalizable

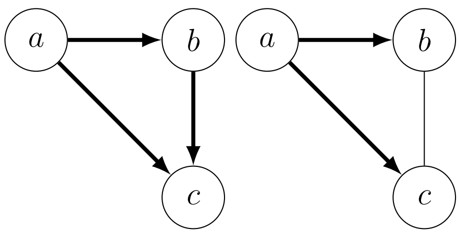
Proof of Corollary 1. Let C be rationalizable. By Theorem 1 and Lemma 7 it satisfies α and γ. Let C satisfy α and γ. Then by Theorem 1, Proposition 3 and Lemma 4 R_C rationalizes C. □

Another similarity to existing results is the key role of the revealed preference. We note that there is one difference when dropping α: While for rationalizability the relation is unique, for upwards rationalizability there can be multiple families satisfying Definition 13.

Example 2 (The family is in general not unique)**.** Let $U = \{a, b, c\}$ and let C be defined on the non-singleton sets as follows:

A	$C(A)$
$\{a, b, c\}$	$\{a\}$
$\{a, b\}$	$\{a\}$
$\{b, c\}$	$\{b\}$
$\{a, c\}$	$\{a\}$

Then, C satisfies γ and hence is upwards rationalizable. On singleton sets A, R^A is trivially defined. Further we have $a \, P^{\{a,b\}} \, b$, $a \, P^{\{a,c\}} \, c$ and $b \, P^{\{b,c\}} \, c$. On U however, there are multiple possibilities to define the relation:



For the two depicted possibilities, we can either have $b \, P^U \, c$, or $b \, I^U \, c$. In both cases, $(R^A)_A$ upwards rationalizes C. △

This ambiguity might seem unsatisfactory, especially since uniqueness of the rationalizing relation is given when additionally assuming α. Luckily, the problem can be quickly addressed by demanding inclusion minimality of all relations. Such a unique finest family always exists and consists of the local revealed preference relations. In the example above, the left version of R^U is equal to R^U_C.

Definition 17 (Finest Family)**.** Let $\mathcal{V}$ be a set of families of relations. Let $(R^A)_{A\in\mathcal{F}(U)}, (\tilde{R}^A)_{A\in\mathcal{F}(U)} \in \mathcal{V}$. We say that $(R^A)_A$ *is finer than* $(\tilde{R}^A)_A$, if $R^A \subseteq \tilde{R}^A$ for all $A \in \mathcal{F}(U)$. We say that $(R^A)_A$ is the *finest* family in $\mathcal{V}$ if and only if $(R^A)_A$ is finer than any other $(\tilde{R}^A)_A \in \mathcal{V}$.

Remark 3. If it exists, such a family is unique.

Proposition 4. *Let C satisfy γ. Then $(R_C^A)_A$ is the (unique) finest family of relations upwards rationalizing C.*

Proof. Let C be upwards rationalized by $(R^A)_A$. Let A be a feasible set, $x, y \in A$ s.th. $\neg(x\, R^A\, y)$. Then by completeness $y\, P^A\, x$. By Lemma 2 we have $y\, P^B\, x$ for all subsets B of A containing x, y. Hence $x \notin C(B)$ for all such B. It follows that $y\, P_C^A\, x$ and thus $\neg(x\, R_C^A\, y)$. □

Originally, we arrived at acyclicity by weakening the notion of transitivity. This was an attempt at escaping Arrow's impossibility. Since we escape it by dropping α, we can now demand that all relations are quasi-transitive or even transitive. Will this change affect the resulting choice functions?
Fascinatingly, being more restrictive with respect to the relations translates to demanding more strict expansion consistency conditions. Let us start with quasi-transitivity.

4.2 Characterizing γ and Aizerman

Definition 18 (Quasi-Transitive Upwards Rationalizability)**.** Let C be a choice function. We say that C is *quasi-transitively* UR, if there is a family of quasi-transitive relations $(R^A)_A$ which upwards rationalizes C. When we speak of a family of quasi-transitive relations $(R^A)_A$, we mean that for all A, the relation R^A is quasi-transitive.

This definition is more strict than Definition 13. For example, we will see in Section 9 that the Split Cycle rule is UR, but not quasi-transitively UR. On first sight, such a restriction only seems to be of abstract nature. Is there any reason why it should be considered at all? We remember from Section 2.3, that in the classical case, the restriction to quasi-transitivity yields the Aizerman condition. It seems that moving from acyclicity to quasi-transitivity does not change the degree of contraction consistency, but rather the degree of expansion consistency. The following result matches our intuition: We can obtain similar results when dropping α.

Theorem 2. *Let C be a choice function. Then the following are equivalent.*

(i) C satisfies γ and Aizerman

(ii) C is upwards rationalized by a family of quasi-transitive relations

Proof of Theorem 2. For the first implication, let C be a choice function satisfying γ and Aizerman. By Theorem 1 and Proposition 3, we already know that the local revealed preference relations upwards rationalize C. In addition, we now show that they are also quasi-transitive. Let A be a feasible set and $x, y, z \in A$ such that $x\, P_C^A\, y$ and $y\, P_C^A\, z$. We now need to show that $x\, P_C^A\, z$. For this, let an arbitrary $B \subseteq A$ be given, such that

$x, z \in B$. Note that if $y \in B$, then $yP_C^B z$ and hence $z \notin C(B)$. Else, set $A_y := B \cup \{y\}$. By $x, y \in A_y$ and Lemma 3 we have $y \notin C(A_y)$, which implies $C(A_y) \subseteq B \subseteq A_y$. Aizerman implies $C(B) \subseteq C(A_y)$. Since $A_y \subseteq A$ and $y, z \in A_y$, we have $z \notin C(A_y)$ by Lemma 3. Hence we can conclude $z \notin C(B)$. By Lemma 3 we obtain $xP_C^A z$.
For the other implication, let $(R^A)_A$ be a family of quasi-transitive preference relations which upwards rationalizes C. We already know that C satisfies γ by Theorem 1. For Aizerman, let A, B be feasible sets such that $C(A) \subseteq B \subseteq A$. We now need to show $C(B) \subseteq C(A)$. Let some $z \in B \setminus C(A)$ be given. Our goal is to show $z \notin C(B)$. There must be some $x_1 \in A$, such that $x_1 \, P^A \, z$. If $x_1 \in B$, then by Lemma 2 $x_1 \, P^B \, z$ and hence $z \notin C(B)$. Else, by $C(A) \subseteq B$, it must be that $x_1 \notin C(A)$. Hence there must be $x_2 \in A$ with $x_2 \, P^A \, x_1$. By quasi-transitivity of R^A we deduce that $x_2 \, P^A \, z$. Using induction, quasi-transitivity of R^A and finiteness of A, there must be some $x_l \in B$ with $x_l \, P^A \, z$. More formally, set $x_0 := z$. For any $k \in \mathbb{N}_0$ and any linearly ordered set $\{x_0, x_1, \ldots, x_k\} \subseteq A$ with $x_i \, P^A \, x_j \iff i > j$ and $x_k \notin C(A)$, there is some $x_{k+1} \in A$ with $x_{k+1} \, P^A \, x_i$ for all $i \leq k$. This is because x_k cannot be maximal in A, hence there must be some $x_{k+1} \in A$ with $x_{k+1} \, P^A \, x_k$. Then we only need to apply quasi-transitivity. As long as $x_{k+1} \notin B$, we know that $x_{k+1} \notin C(A)$. Hence we can reapply the above statement, but with $\{x_0, \ldots, x_{k+1}\}$. Iterating this process and using finiteness of A, we obtain that there must be some $l \in \mathbb{N}$ with $x_l \in B$ and $x_l \, P^A \, x_0$. Using Lemma 2 again, we obtain $x_l \, P^B \, z$. Hence $z \notin C(B)$. □

We have seen in the proof that if C is quasi-transitively upwards rationalizable, then all local revealed preference relations are quasi-transitive too.

Proposition 5. *Let C be a choice function. Then it is upwards rationalized by a family of quasi-transitive relations if and only if it is upwards rationalized by $(R_C^A)_A$ and all local revealed preference relations are quasi-transitive.*

Again, we can use our new result to obtain the classical one.

Corollary 2 (Schwartz, 1976)**.** *Let C be a choice function. Then the following are equivalent.*

(i) C satisfies α, γ and Aizerman

(ii) C is rationalized by a quasi-transitive relation

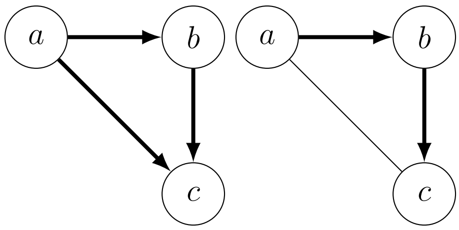
Proof of Corollary 2. Let C be quasi-transitively rationalizable. By Theorem 2 and Lemma 7 it satisfies α, γ and Aizerman. Let C satisfy α, γ and Aizerman. Then by Theorem 2, Lemma 4 and Proposition 5 we have that R_C is quasi-transitive and C is rationalized by R_C. □

We note that not all families which upwards rationalize C have to consist of quasi-transitive relations.

Example 3 (Not all families are quasi-transitive)**.** We again define C on the non-singleton sets as follows:

A	$C(A)$
$\{a,b,c\}$	$\{a\}$
$\{a,b\}$	$\{a\}$
$\{b,c\}$	$\{b\}$
$\{a,c\}$	$\{a\}$

Then, C satisfies γ and Aizerman. Hence it is quasi-transitively UR. On singleton sets A, R^A is trivially defined. Further we have $a\,P^{\{a,b\}}\,b$, $a\,P^{\{a,c\}}\,c$ and $b\,P^{\{b,c\}}\,c$. On U however, both following definitions of R^U are possible:



The left one coincides with R^U_C and is quasi-transitive. The right one is not. In both cases, C is upwards rationalized by $(R^A)_A$. △

4.3 Characterizing β^+

Now that we have examined acyclicity and quasi-transitivity, a natural next step is to look into transitivity.

Definition 19 (Transitive Upwards Rationalizability)**.** Let C be a choice function. We say that C is *transitively* UR, if there is a family of transitive relations $(R^A)_A$ which upwards rationalizes C. When we speak of a family of transitive relations $(R^A)_A$, we mean that for all A, the relation R^A is transitive.

Demanding transitivity is more restrictive than Definition 18. For example we will see in Section 9 that the Uncovered Set rule is quasi-transitively UR, but not transitively UR. Once more we remind ourselves of the classical results. There, moving from quasi-transitivity to transitivity yields β^+, which is stronger than both γ and Aizerman together. It seems that contraction consistency is not affected by moving from quasi-transitivity to transitivity. Instead, we again obtain a higher degree of expansion consistency. The following result again matches our intuition.

Theorem 3. *Let C be a choice function. Then the following are equivalent.*

(i) C satisfies β^+

(ii) C is upwards rationalized by a family of transitive relations

Proof of Theorem 3. First, let C satisfy β^+. By Lemma 1, Proposition 3 and Theorem 1, we know that C is upwards rationalized by its family of local revealed preference relations. Let A be a feasible set. We now show that R^A_C is transitive. Let x, y, z be given such that there are feasible sets $B, D \subseteq A$ with $x \in C(B)$, $y \in C(D)$, $y \in B$, $z \in D$. We now need to show that there is some feasible $E \subseteq A$ such that E contains z and x is chosen in it.

Set $E \coloneqq B \cup D$. Observe that by β^+ and $B \subseteq E$, if $C(E) \cap B \neq \emptyset$, then $x \in C(E)$. Assume now for contradiction that $C(E) \cap B = \emptyset$. By non-emptiness of $C(E)$, we have $C(E) \cap D \neq \emptyset$. Now, by β^+ it follows that $y \in C(D) \subseteq C(E)$. Hence the intersection cannot have been empty in the first place, a wanted contradiction.
For the other direction, let $(R^A)_A$ upwards rationalize C such that all relations are transitive. Now, let A, B be feasible sets such that $B \subseteq A$ and fix $y \in C(A) \cap B \neq \emptyset$. Let $x \in C(B)$. Then by completeness of R^B and maximality of x in B we obtain $x\, R^B\, y$. By upwards rationalizability $x\, R^A\, y$. By maximality of y in A we obtain $y\, R^A\, z$ for all $z \in A$. By transitivity we now obtain $x\, R^A\, z$ for all $z \in A$. Hence x is maximal in A and it follows that $x \in C(A)$. □

We have seen in the proof that if C is transitively upwards rationalizable, then the local revealed preference relations have to be transitive too.

Proposition 6. *Let C be a choice function. Then it is upwards rationalized by a family of transitive relations if and only if it is upwards rationalized by $(R_C^A)_A$ and all local revealed preference relations are transitive*[5]*.*

One more time, we can present a proof for a classical result.

Corollary 3 (Bordes, 1976)**.** *Let C be a choice function. Then the following are equivalent.*

(i) C satisfies α and β^+

(ii) C is rationalized by a transitive relation

Proof of Corollary 3. Let C be transitively rationalizable. By Theorem 3 and Lemma 7 it satisfies α and β^+. Let C satisfy α and β^+. Then by Theorem 3, Lemma 4 and Proposition 6 we have that R_C is transitive and C is rationalized by R_C. □

We note that just because C satisfies β^+ and is upwards rationalized by $(R^A)_A$, it does not have to be that all R^A are transitive. This can be seen in Example 3. There, the quasi-transitive version of R^U in fact is transitive.

[5]Interestingly, Bordes (1976) already observed that if C satisfies β^+, the revealed preference relation is transitive. We then only would need to apply that if C satisfies β^+, then the choice function restricted to $\mathcal{F}(A)$ also satisfies β^+.

5 PIP-Transitivity

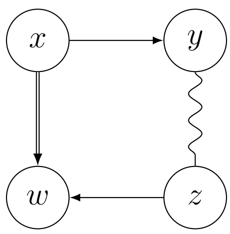
In this section, we present another main result of this thesis. We propose a new expansion consistency condition, which we call γ^+. We then use it to characterize PIP-transitive upwards rationalizability. Further, we compare it to a condition of Schwartz and give alternative formulations.

5.1 The Main Result

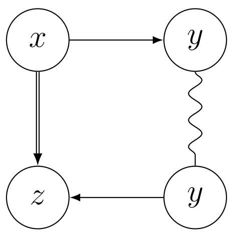
So far, we have dealt with three forms of rationalizability. In addition to (acyclic) rationalizability, we have touched on characterizations of quasi-transitive and transitive rationalizability. Between quasi-transitivity and transitivity, Schwartz (1976) further characterized a notion named PIP-transitivity. In some cases, this notion can better represent human behavior. Schwartz (1986) states that while transitivity is equivalent to representation by a utility function u, PIP-transitivity is equivalent to representation by a utility function u and a non-negative discriminatory function δ. The idea is that we only perceive some a to be strictly better than some b, if the increase in utility is noticeable for us, which is modelled by $u(a) > u(b) + \delta(b)$.

Definition 20 (PIP-Transitivity)**.** Let R be a relation on U. We say that R is PIP-transitive if and only if for all (not necessarily distinct) $x, y, z, w \in U$ the following holds. If $x\,P\,y$, $y\,I\,z$ and $z\,P\,w$, then $x\,P\,w$.

Graphically, we can represent the condition as follows:

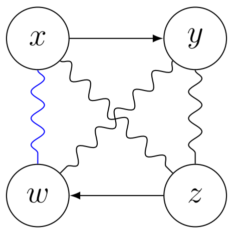


Here, the snake line encodes indifference, while the strict arrows encode strict preference. The strict double arrow from x to w means, that $x\,P\,w$ has to follow from the other arrows in the graph. The missing arrows, for example between x and z, mean that we do not need to further specify the relation. We quickly verify that PIP-transitivity implies quasi-transitivity. Let $x\,P\,y$, $y\,P\,z$. Then obviously $y\,I\,y$ and we obtain:



Further, transitivity implies PIP-transitivity. Let R be transitive with $x\,P\,y$, $y\,I\,z$ and $z\,P\,w$. We assume for contradiction $w\,R\,x$. Then by transitivity $w\,R\,z$, which is directly contradicting $z\,P\,w$.

Example 4 (Quasi-transitivity of R does not imply PIP-transitivity)**.** Let $U = \{x, y, z, w\}$. Further, let the relation R be given by the following graph.



The blue-colored edge violates PIP-transitivity. Still, the relation is quasi-transitive.

To characterize the just introduced notion, Schwartz used an expansion consistency condition, which he named $W4$. It has the same form as the conditions discussed in Section 7.2.

Definition 21 (W4, Schwartz, 1976)**.** Let C be a choice function. We say that C satisfies $W4$ if and only if the following statement holds true:

$$\text{Let } B \subseteq A \text{ be feasible sets.}$$
$$\text{If } C(A) \cap B \neq \emptyset,\ B \neq A \text{ and } C(A \setminus B) \not\subseteq C(A)$$
$$\text{then } C(B) \subseteq C(A)$$

How can we relate $W4$ to the already discussed expansion consistency conditions? Schwartz showed that $W4$ lies between Aizerman and β^+.

Proposition 7 (Schwartz, 1976)**.** *β^+ implies $W4$ implies Aizerman. The implications in the other direction do not hold.*

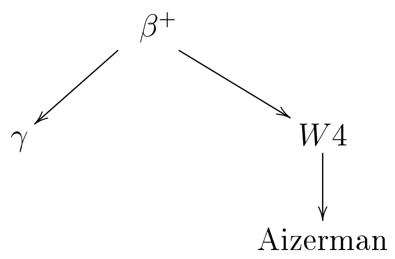
Since γ does not imply Aizerman, we directly see that γ does not imply $W4$. Conversely, $W4$ does not imply γ.

Example 5 ($W4$ does not imply γ)**.** Let $U = \{x, y, z\}$. We define C using the following table.

A	$C(A)$	condition
U	$U \setminus \{x\}$	—
A	A	$x \notin A$
A	x	$x \in A,\ A \neq U$

Since $x \in C(\{x, y\}) \cap C(\{x, z\})$, but $x \notin C(U)$, γ is violated. To see that $W4$ is satisfied, let A, B be feasible sets such that $B \subseteq A$. If $B = A$, then $W4$ is trivially satisfied. If $|B| = 1$, then $C(A) \cap B \neq \emptyset$ already implies that $C(B) \subseteq C(A)$. Hence, the last case we need to look at is $|B| = 2$, $A = U$. Let $C(A \setminus B) \not\subseteq C(A)$. Then $x \notin B$. By definition of C, hence $C(B) = B \subseteq A \setminus \{x\} = C(A)$. This construction works for all $|U| \geq 3$.

Graphically, we can represent the relation between the consistency conditions as follows:



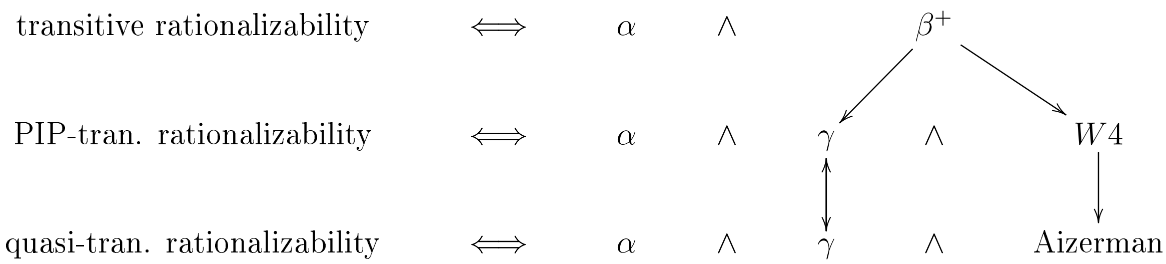
Now that we have a slightly better understanding of the condition, we examine Schwartz' characterization.

Theorem 4 (Schwartz, 1976)**.** *Let C be a choice function. Then the following are equivalent.*

(i) C satisfies α, γ and $W4$

(ii) C is rationalized by a PIP-transitive relation

Again we pause for a moment to put everything into context.



Curiously, the result of Schwartz cannot be reproduced directly when dropping α. While one can show that PIP-transitive upwards rationalizability does imply $W4$ and γ, the converse direction is not so clear. We are stuck and need a better understanding of PIP-transitivity. We hence propose a new expansion consistency condition[6].

Definition 22 (γ^+)**.** Let C be a choice function. We say that C satisfies γ^+ if and only if the following holds:
Let A, B be feasible sets. Then $C(A) \subseteq C(A \cup B)$ or $C(B) \subseteq C(A \cup B)$.

Remark 4. γ^+ implies γ, since $C(A) \cap C(B)$ is a subset of both $C(A)$ and $C(B)$.

We will now see that γ^+ plays a vital role for the concept of PIP-transitivity.

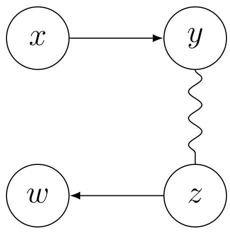
Theorem 5. *Let C be a choice function. Then the following are equivalent.*

(i) C satisfies γ^+

[6] γ^+ shares its name with a condition defined by Salant and Rubinstein (2008), but is weaker than the latter and does not have much in common with it. If not stated differently, by γ^+ we will always refer to Definition 22.

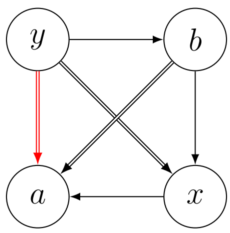
(ii) C is upwards rationalized by a family of PIP-transitive relations

Proof. First, let C satisfy γ^+. We already know that γ is satisfied, hence C is upwards rationalized by $(R_C^A)_A$. Let A be a feasible set. We now show that the local revealed preference relation R_C^A is PIP-transitive. Let $x\, P_C^A\, y$, $y\, I_C^A\, z$ and $z\, P_C^A\, w$.

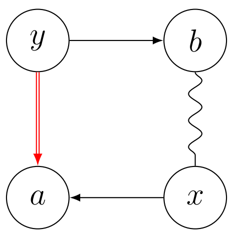


We now need to show that $x\, P_C^A\, w$. Assume for contradiction $w\, R_C^A\, x$. Then there exists some witness $W_{w,x} \subseteq A$ containing x with $w \in C(W_{w,x})$. Further, since $y\, I_C^A\, z$, there is some witness $W_{y,z} \subseteq A$ containing z with $y \in C(W_{y,z})$. Now we set $A' = W_{w,x}$, $B' = W_{y,z}$. By γ^+ we have that either $w \in C(A') \subseteq C(A' \cup B')$ or $y \in C(B') \subseteq C(A' \cup B')$. On the other hand we also have that $x, z \in A' \cup B' \subseteq A$. This contradicts either $x\, P_C^A\, y$ or $z\, P_C^A\, w$.

For the other direction, let $(R^A)_A$ upwards rationalize C and let all relations be PIP-transitive. Now, let A, B be feasible sets. Assume for contradiction, that neither $C(A)$, nor $C(B)$ is a subset of $C(A \cup B)$. Then, there are $a \in C(A) \setminus C(A \cup B)$ and $b \in C(B) \setminus C(A \cup B)$. Hence there is some $x \in A \cup B$ with $x\, P^{A \cup B}\, a$ and some $y \in A \cup B$ with $y\, P^{A \cup B}\, b$. Since $a \in C(A)$, it must be that $x \in B$. By $b \in C(B)$ we have that $b\, R^B\, x$. By upwards rationalizability, we more importantly have $b\, R^{A \cup B}\, x$. There now are two possibilities. First, it could be that $b\, P^{A \cup B}\, x$. By applying quasi-transitivity, we then obtain the following graph for $R^{A \cup B}$.



Else, we have $b\, I^{A \cup B}\, x$. We then obtain the following graph for $R^{A \cup B}$, on which we apply PIP-transitivity.



Since $b \in C(B)$, it must be that $y \in A$. By $a \in C(A)$, we have that $a\, R^A\, y$. By upwards rationalizability, we more importantly have $a\, R^{A \cup B}\, y$, a wanted contradiction to $y\, P^{A \cup B}\, a$ in both cases! □

Again we have shown more than we announced.

Proposition 8. *Let C be a choice function. Then it is upwards rationalized by a family of PIP-transitive relations if and only if it is upwards rationalized by $(R_C^A)_A$ and all local revealed preference relations are PIP-transitive.*

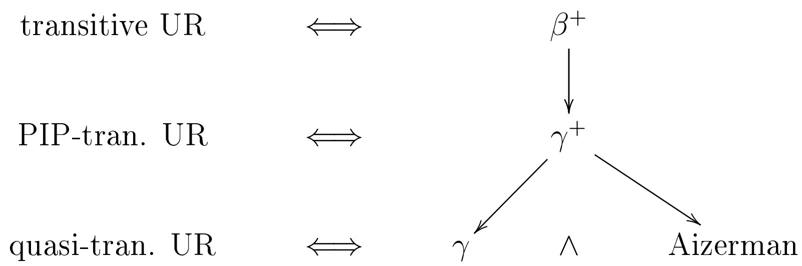
With the same argumentation as the previous times, we hence obtain the following corollary.

Corollary 4. *Let C be a choice function. Then the following are equivalent.*

(i) C satisfies α and γ^+

(ii) C is rationalized by a PIP-transitive relation

Proof. Apply Lemma 4. □



5.2 Further Comments

Our next goal is to show that $W4$ and γ are not equivalent to PIP-transitive upwards rationalizability. Further, we provide an alternative formulation of $W4$ in the style of γ^+. To reach our first goal, we start by providing an alternative formulation of γ^+.

Definition 23 ($W4^+$)**.** Let C be a choice function. We say that C satisfies $W4^+$ if and only if the following holds:

$$\text{Let } B \subseteq A \text{ be feasible sets.}$$
$$\text{If } C(A) \cap B \neq \emptyset,\ B \neq A \text{ and } \exists D,\ A \setminus B \subseteq D \subseteq A : C(D) \not\subseteq C(A)$$
$$\text{then } C(B) \subseteq C(A)$$

We now show that $W4^+$ and γ^+ are equivalent. We hence will know that γ^+ implies Schwartz' $W4$.

Lemma 8. *Let C be a choice function. Then the following are equivalent:*

(i) C satisfies γ^+

(ii) C satisfies $W4^+$

Proof. "$(i) \implies (ii)$": Let γ^+ be satisfied. Let $B \subset A$, $D \subseteq A$ be given such that $C(A) \cap B \neq \emptyset$, $A \setminus B \subseteq D$ and $C(D) \not\subseteq C(A)$. Set $\tilde{A} := B$, $\tilde{B} := D$. Then, $\tilde{A} \cup \tilde{B} = A$. Since $C(\tilde{B}) \not\subseteq C(\tilde{A} \cup \tilde{B})$, we can apply γ^+. It follows that $C(B) = C(\tilde{A}) \subseteq C(\tilde{A} \cup \tilde{B}) = C(A)$.

"$(ii) \implies (i)$": Let $W4^+$ be satisfied. Now let $\tilde{A}, \tilde{B}$ be two distinct feasible sets. Assume for contradiction that $C(\tilde{A}) \not\subseteq C(\tilde{A} \cup \tilde{B})$ and $C(\tilde{B}) \not\subseteq C(\tilde{A} \cup \tilde{B})$. By non-emptiness, we can without loss of generality assume $\tilde{B} \cap C(\tilde{A} \cup \tilde{B}) \neq \emptyset$. Set $A := \tilde{A} \cup \tilde{B}$. $B := \tilde{B}$, $D := \tilde{A}$. Then we have that $C(A) \cap B \neq \emptyset$, $A \setminus B \subseteq D \subseteq A$. and $C(D) \not\subseteq C(A)$. Hence we can apply $W4^+$, and obtain that $C(\tilde{B}) = C(B) \subseteq C(A) = C(\tilde{A} \cup \tilde{B})$, a wanted contradiction. □

We compare $W4$ and $W4^+$.

$$\begin{gathered} W4: \\ \text{Let } B \subseteq A \text{ be feasible sets.} \\ \text{If } C(A) \cap B \neq \emptyset,\ B \neq A \text{ and } C(A \setminus B) \not\subseteq C(A) \\ \text{then } C(B) \subseteq C(A) \end{gathered}$$

$$\begin{gathered} W4^+: \\ \text{Let } B \subseteq A \text{ be feasible sets.} \\ \text{If } C(A) \cap B \neq \emptyset,\ B \neq A \text{ and } \exists D,\ A \setminus B \subseteq D \subseteq A : C(D) \not\subseteq C(A) \\ \text{then } C(B) \subseteq C(A) \end{gathered}$$

First, we state the obvious.

Lemma 9. *$W4^+$ implies $W4$. Hence γ^+ implies $W4$.*

Proof. Let C satisfy $W4^+$. Now we show that $W4$ is satisfied. Let $B \subseteq A$ be feasible sets, such that $C(A) \cap B \neq \emptyset$, $B \neq A$ and $C(A \setminus B) \not\subseteq C(A)$. Then $\exists D,\ A \setminus B \subseteq D \subseteq A : C(D) \not\subseteq C(A)$, namely $D := A \setminus B$. Hence we can apply $W4^+$ and obtain $C(B) \subseteq C(A)$. By Lemma 8, γ^+ also implies $W4$. □

Our goal now is to figure out, whether $W4$ and γ are also equivalent to PIP-transitive upwards rationalizability. This would reproduce Theorem 4. Earlier on, we were stuck, but now we know exactly how to look for counterexamples: If we want to show that γ and $W4$ do not imply $W4^+$, our construction needs a witness D which is a *strict* superset of $A \setminus B$.

Example 6 (γ and $W4$ do not imply $W4^+$)**.** Let $U = \{a, b, c, d\}$. We define a choice function C as follows.

$$\begin{gathered} \underline{a}, b, \underline{c}, d \\ \underline{a}, b, \underline{c} \quad \underline{a}, \underline{c}, d \quad \underline{b}, \underline{c}, d \quad \underline{a}, b, \underline{d} \\ \underline{a}, b \quad \underline{a}, \underline{c} \quad b, \underline{c} \quad \underline{a}, d \quad \underline{b}, d \quad \underline{c}, d \end{gathered}$$

In the above table, all non-singleton feasible sets are listed without the set brackets. The underlined elements are the ones which are chosen. For example, we have $C(U) = \{a, c\}$. First we want to check that $W4$ is satisfied. To do so, we go through all pairs of $B \subset A$. Now we need to show that one of two statements must hold true. One option is that the consequent must be true, which means $C(B) \subseteq C(A)$. The other possibility is that the antecedent is false, which is especially the case if $C(A \setminus B) \subseteq C(A)$. To do so, we first fix **A** and then color in all $B \subset A$. We start with $A = U$.

$$\underline{\mathbf{a}},\mathbf{b},\underline{\mathbf{c}},\mathbf{d}$$
$$\underline{a},b,\underline{c} \quad \underline{a},\underline{c},d \quad \underline{b},\underline{c},d \quad \underline{a},b,\underline{d}$$
$$\underline{a},b \quad \underline{a},\underline{c} \quad b,\underline{c} \quad \underline{a},d \quad \underline{b},d \quad \underline{c},d$$

For $A = \{a, b, c\}$, $A = \{a, c, d\}$ and $A = \{b, c, d\}$, the task is easy and hence summed up in one table.

$$\underline{a},b,\underline{c},d$$
$$\underline{\mathbf{a}},\mathbf{b},\underline{\mathbf{c}} \quad \underline{\mathbf{a}},\underline{\mathbf{c}},\mathbf{d} \quad \underline{\mathbf{b}},\underline{\mathbf{c}},\mathbf{d} \quad \underline{a},b,\underline{d}$$
$$\underline{a},b \quad \underline{a},\underline{c} \quad b,\underline{c} \quad \underline{a},d \quad \underline{b},d \quad \underline{c},d$$

For $A = \{a, b, d\}$, we again color everything in.

$$\underline{a},b,\underline{c},d$$
$$\underline{a},b,\underline{c} \quad \underline{a},\underline{c},d \quad \underline{b},\underline{c},d \quad \underline{\mathbf{a}},\mathbf{b},\underline{\mathbf{d}}$$
$$\underline{a},b \quad \underline{a},\underline{c} \quad b,\underline{c} \quad \underline{a},d \quad \underline{b},d \quad \underline{c},d$$

For $|A| = 2$ we have $|B| = 1$. The statement now becomes trivial since $B = C(B)$ by non-emptiness. If $C(A) \cap B \neq \emptyset$, then $C(B) = B \subseteq C(A)$.
Now that we have shown that $W4$ is satisfied, we quickly show that γ is satisfied too. To do so we observe that a, c are chosen in every possible subset respectively. Hence if $x \in \{a, c\}$, and $x \in C(A) \cap C(B)$, then we also have $x \in C(A \cup B)$. Also, note that d is chosen in exactly one non-singleton set. Hence if $d \in C(A) \cap C(B)$, then we have $d \in C(A \cup B)$, since $A \cup B$ is now equal to A or B. b is chosen only in $\{b, c, d\}$, $\{b, d\}$ and $\{b\}$, which are linearly ordered using the subset relation. This implies that if we have $c \in C(A) \cap C(B)$, then $A \cup B$ is equal to A or B and trivially γ is satisfied.
To show that $W4^+$ is violated (and hence γ^+ and PIP-transitive upwards rationalizability are too violated), we color in **A**, B and D for which the implication does not hold true.

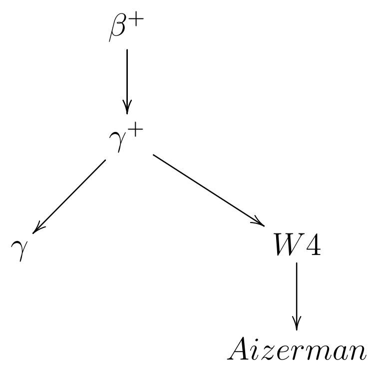
$$\underline{\mathbf{a}},\mathbf{b},\underline{\mathbf{c}},\mathbf{d}$$
$$\underline{a},b,\underline{c} \quad \underline{a},\underline{c},d \quad \underline{b},\underline{c},d \quad \underline{a},b,\underline{d}$$
$$\underline{a},b \quad \underline{a},\underline{c} \quad b,\underline{c} \quad \underline{a},d \quad \underline{b},d \quad \underline{c},d$$

Note that, as discussed before, D needs to be a true superset of $A \setminus B$. △

Now we know that γ^+ is stronger than γ and $W4$ together.

$$\text{PIP-tran. UR} \iff \gamma^+ \iff W4^+ \not\Leftarrow\!\!\Rightarrow \gamma \wedge W4$$

We can now visualize the relations between the most important expansion consistency conditions discussed in this thesis:



Further, we note that Theorem 4 is sharp in the sense that the characterization does not hold true if any of the axioms are omitted. There are examples on 3 alternatives for which $W4$ and γ are satisfied, but not α. (Obviously, expansion consistency does not imply contraction consistency.) For α and γ, we already know that they are equivalent to rationalizability. This is a strictly weaker notion than PIP-transitive rationalizability, as we can see using Example 4. Hence α and γ cannot imply $W4$. For α and $W4$, we need to show that they do not imply γ.

Example 7 (α and $W4$ do not imply γ)**.** Let C be defined by the following table, where the chosen elements are the underlined ones.

$$\begin{array}{cccccc} & & \underline{a},\underline{b},\underline{c},d & & & \\ \underline{a},\underline{b},\underline{c} & \underline{a},\underline{c},d & \underline{b},\underline{c},d & \underline{a},\underline{b},\underline{d} & & \\ \underline{a},\underline{b} & \underline{a},\underline{c} & \underline{b},\underline{c} & \underline{a},\underline{d} & \underline{b},\underline{d} & \underline{c},\underline{d} \end{array}$$

First we show that α is satisfied. We see that a, b, c are chosen whenever possible. Hence, when $x \in \{a, b, c\}$, $x \in C(A)$, $x \in B \subseteq A$, then also $x \in C(B)$. Only d is left, where we can easily check that if $d \in C(A)$, $d \in B \subseteq A$, then $d \in C(B)$. For clarity we highlight all non-singleton sets, in which d is chosen.

$$\begin{array}{cccccc} & & \underline{a},\underline{b},\underline{c},d & & & \\ \underline{a},\underline{b},\underline{c} & \underline{a},\underline{c},d & \underline{b},\underline{c},d & \underline{a},\underline{b},\underline{d} & & \\ \underline{a},\underline{b} & \underline{a},\underline{c} & \underline{b},\underline{c} & \underline{a},\underline{d} & \underline{b},\underline{d} & \underline{c},\underline{d} \end{array}$$

After showing that α is fulfilled, we now argue that $W4$ is satisfied. We argue as follows: Let $A \in \mathcal{F}(U)$, such that $|C(A)| + 1 \geq |A|$. Then for all $B \subseteq A$, we either have $C(B) \subseteq C(A)$ or $C(A \setminus B) \subseteq C(A)$. To prove this claim, we have to look at two cases. If $A = C(A)$, then trivially we obtain $C(B) \subseteq B \subseteq A \subseteq C(A)$. Else $C(A) = A \setminus \{x\}$. If $C(B) \not\subseteq C(A)$, then $x \in B$. Hence $C(A \setminus B) \subseteq A \setminus B \subseteq A \setminus \{x\} = C(A)$.
Further, γ is violated.

$$\begin{array}{cccccc} & & \underline{a},\underline{b},\underline{c},d & & & \\ \underline{a},\underline{b},\underline{c} & \underline{a},\underline{c},d & \underline{b},\underline{c},d & \underline{a},\underline{b},\underline{d} & & \\ \underline{a},\underline{b} & \underline{a},\underline{c} & \underline{b},\underline{c} & \underline{a},\underline{d} & \underline{b},\underline{d} & \underline{c},\underline{d} \end{array}$$

$\triangle$

We conclude this section by giving an alternative, slightly more intuitive formulation of $W4$. The idea stems from the color-coding of Example 6.

Definition 24 ($W4^*$)**.** Let C be a choice function. We say that C satisfies $W4^*$, if the following holds:
Let A, B be disjoint feasible sets. (This means that $A \cap B = \emptyset$.) Then $C(A) \subseteq C(A \cup B)$ or $C(B) \subseteq C(A \cup B)$.

Proposition 9. *Let C be a choice function. Then the following are equivalent:*

(i) C satisfies $W4$

(ii) C satisfies $W4^*$

Proof. First, let C satisfy $W4$. Let A^*, B^* be disjoint feasible sets, such that $C(A^*) \nsubseteq C(A^* \cup B^*)$. We can apply $W4$ using $A := A^* \cup B^*$ and $B := B^*$, since then by disjointness $A \setminus B = A^*$. This implies $C(B^*) = C(B) \subseteq C(A) = C(A^* \cup B^*)$.
Now, let C satisfy $W4^*$. Let two feasible sets $B \subset A$ be given, such that $C(A) \cap B \neq \emptyset$ and $C(A \setminus B) \nsubseteq C(A)$. We can apply $W4^*$ using $A^* := A \setminus B$ and $B^* := B$. By definition, they are disjoint and their union is A. Since by assumption $C(A^*) \nsubseteq C(A^* \cup B^*)$, it has to be that $C(B^*) \subseteq C(A^* \cup B^*)$. This is equivalent to $C(B) \subseteq C(A)$. □

With this formulation, we can quickly see why $W4$ is weaker than γ^+: It only considers disjoint feasible sets A, B, while the latter considers all pairs of feasible sets. Further we notice that γ is only a non-trivial condition for feasible sets A, B, which are not disjoint. Hence we also see why $W4$ and γ are independent of each other.

Everything in this section considered, the author would argue that γ^+ is a viable alternative to W4. First, it plays a more stringent role in the theory of upwards rationalizability. Second, one might argue that the restriction to disjoint feasible sets is unnecessary in the sense that allowing all sets seems more natural. Of course, there might be some bias of the author towards the condition he created, hence the reader is invited to form their own opinion.

6 α and Downwards Rationalizability

In this section, we present the last group of main results of this thesis. Instead of demanding that R^A inherits upwards, we instead demand that it inherits downwards. Intuitively, we expect to move from expansion consistency to contraction consistency. Indeed, we can use our notion of downwards rationalizability to characterize α. Furthermore, we strengthen our intuition that quasi-transitivity and transitivity are not dependent on contraction consistency. First, quasi-transitive downwards rationalizability is also equivalent to α. Second, we characterize transitive downwards rationalizability with new conditions which leave the realm of contraction consistency. These conditions are based on a relation which we call the competing relation.

6.1 Characterizing α

Definition 25 (Downwards Rationalizability)**.** Let C be a choice function. We say that C is *downwards rationalizable* (DR), if there is a family of relations $(R^A)_A$ such that the following conditions hold:

(i) For all feasible sets A, $R^A \subseteq A \times A$ is acyclic and complete

(ii) For all feasible sets A

$$C(A) = \max_{R^A} A$$

(iii) For all feasible sets A, B, such that $B \subseteq A$:

$$R^A \cap (B \times B) \subseteq R^B$$

In this case, we say that C is *downwards rationalized by* $(R^A)_A$.

Remark 5. Downwards rationalizability is obviously a generalization of rationalizability. For R rationalizing C, we can set $R^A := R|_{A\times A}$.
Further, condition (iii) can be replaced by the following condition:

(iii') $P^B \subseteq P^A$

With this definition, we obtain somewhat similar results to the case of expansion consistency. Downwards rationalizability is equivalent to α, but also equivalent to quasi-transitive downwards rationalizability.

Theorem 6. *Let C be a choice function. Then the following are equivalent.*

(i) C satisfies α

(ii) C is downwards rationalizable

(iii) C is downwards rationalized by a family of quasi-transitive relations

Proof of Theorem 6. The direction from (iii) to (ii) to (i) is straight forward. For the last implication, we show the claim by induction over $|U| = m$. Our induction hypothesis consists of three properties. For all choice functions C on a finite U satisfying α, there exists a family of relations $(R^A)_{\emptyset \neq A \subseteq U}$ with the following properties:

(a)

$$C(A) = \max_{R^A} A \quad \text{for all feasible A}$$

(b) For all feasible $B \subseteq A$ it holds that

$$R^A \cap (B \times B) \subseteq R^B$$

(c) R^A is complete and quasi-transitive for all A

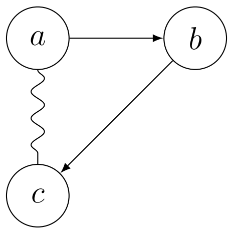
Let us start with $m = 1$, say $U = \{x\}$. Then by non-emptiness $C(U) = U$. We set $x \, R^U \, x$ and obtain a (family of) transitive relation(s), which fulfills properties (a) to (c).
Now, let $|U| = m + 1$. By non-emptiness, we can fix any $x \in C(U)$. Set $U' := U \setminus \{x\}$. By induction assumption, there is a family of relations $(R^A)_{\emptyset \neq A \subseteq U'}$ satisfying properties (a) to (c) on all subsets of U'. From now on we denote subsets of U' as A, B. For such sets we define $A^+ := A \cup \{x\}$. Now, for any feasible A we define R^{A^+} as follows:

- $y \, R^{A^+} z \iff y \, R^A \, z$ for all $y, z \in A$
- $x \, R^{A^+} y$ for all $y \in A^+$
- $y \, R^{A^+} x \iff y \in C(A^+)$ for all $y \in A$

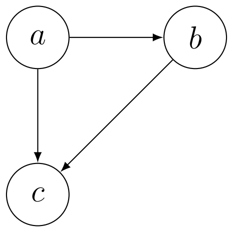
We need to show that all three properties are fulfilled for all subsets of U.
Let us start with (a). Observe that by α, $x \in C(A^+)$. This matches with $x \, R^{A^+} y$ for all $y \in A$. Now, let $y \in A$. If $y \in C(A^+)$, then by α we conclude that $y \in C(A)$. By induction hypothesis, it must be that $y \, R^A \, z$ for all $z \in A$. Hence $y \, R^{A^+} z$ for all such z. Furthermore, by definition we have $y \, R^{A^+} x$. Hence y is maximal in A^+. Otherwise $y \notin C(A^+)$. It follows that $x \, R^{A^+} y$ but $\neg(y \, R^{A^+} x)$, or in other words $x \, P^{A^+} y$.
Now we move on to (b). Let two feasible sets $B \subseteq A$ be given. (b) holds for (B, A) by induction hypothesis. Hence it now remains to look at (B, A^+) and (B^+, A^+). Let $y, z \in B$ with $y \, R^{A^+} z$. Then by definition $y \, R^A \, z$. By induction hypothesis again we have $y \, R^B \, z$. Thus we have shown the claim for (B, A^+). In addition, by definition we obtain $y \, R^{B^+} z$. For x, we trivially obtain $x \, R^{B^+} y$ for all $y \in B^+$. Hence, the only case which remains is $y \, R^{A^+} x$ for some $y \in B$. But then by definition $y \in C(A^+)$ and hence by α it follows that $y \in C(B^+)$. Again by definition, we obtain $y \, R^{B^+} x$.
Last, we check (c). Let A be a feasible set. R^A is complete and quasi-transitive by induction hypothesis. First, we check completeness of R^{A^+}. $x \, R^{A^+} y$ for all $y \in A^+$. Let $y, z \in A$, such that $\neg(y \, R^{A^+} z)$. Then by definition $\neg(y \, R^A \, z)$. We use completeness of R^A and obtain $z \, R^A \, y$. This by definition implies $z \, R^{A^+} y$. For quasi-transitivity, we only need to go through two cases since x by (a) can never be dominated strictly. Let $y, z, w \in A$. If $x \, P^{A^+} y$ and $y \, P^{A^+} z$, by (a) we obtain $z \notin C(A^+)$ and hence by definition $\neg(z \, R^{A^+} x)$, or in other words $x \, P^{A^+} z$. If on the other hand $y \, P^{A^+} z$ and $z \, P^{A^+} w$, by definition we obtain $y \, P^A \, z$, $z \, P^A \, w$. By induction hypothesis, we obtain $y \, P^A \, w$. Again by definition, we obtain $y \, P^{A^+} w$. □

Applying Theorem 6, we directly obtain Lemma 7.
While Theorem 6 does state that downwards rationalizability and quasi-transitive downwards rationalizability are equivalent, we need to be careful. Not every family of relations downwards rationalizing C has to be quasi-transitive, as we can see in the following example.

Example 8 (Not every downwards rationalizing relation is quasi-transitive)**.** Let $U = \{a, b, c\}$. We define C by setting $C(A) := A$ for all feasible sets, except for $C(U) := \{a\}$. For all $A \in \mathcal{F}(U)$, $A \neq U$, we set $R^A := A \times A$. We define R^U as follows:



It is easy to check that $(R^A)_A$ downwards rationalizes U. Further, R^U is not quasi-transitive. Instead, we could use $\bar{R}^U$:



With this replacement, we now have a family of quasi-transitive relations which downwards rationalizes U. △

Example 9 (Downwards rationalizability does not imply transitive downwards rationalizability)**.** There is a choice function C which satisfies α but is not downwards rationalized by any family of transitive relations.
Let $U = \{a, b, c, d\}$. Define C as follows:

A	$C(A)$
$\{a, b, c, d\}$	$\{c\}$
$\{a, b, c\}$	$\{a, c\}$
$\{a, b, d\}$	$\{b\}$
$\{d\}$	$\{d\}$
B	$B \cap \{a, b, c\}$

where the last row means that for all not previously listed feasible sets B, $C(B) = B \cap \{a, b, c\}$. It is easy to verify that C satisfies α. We assume for contradiction, that there is some family of transitive relations $(R^A)_A$ downwards rationalizing C. Now, we look at $A = \{a, b, c\}$ and $B = \{a, b, d\}$. It is clear that $a\, I^A\, c$. If further it were that $c\, I^A\, b$ or $a\, I^A\, b$, then by transitivity both would hold and b would be maximal, a contradiction. Hence we have $a\, P^A\, b$. Then it follows directly that $a\, P^U\, b$, hence $a\, R^B\, b$. Since b is chosen in B, by assumption we have $b\, R^B\, d$. By transitivity we have $a\, R^B\, d$. By maximality of a we have $a \in C(B)$, a contradiction. △

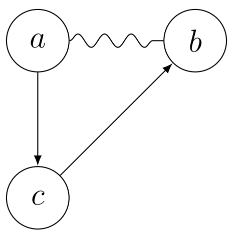
6.2 Uniqueness for Downwards Rationalizability

For α, things are less structured than for γ. While $(R^A_C)_A$ is the unique finest family which can upwards rationalize C, uniqueness is not given for downwards rationalizability, no matter whether we demand a finest or coarsest family. We start with a lemma, which will be used a few times in this section.

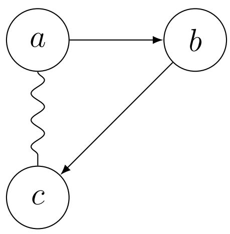
Lemma 10. *Let C be upwards or downwards rationalized by some family of relations $(R^A)_A$. Let $x, y \in A \in \mathcal{F}(U)$. If $x\, P^A\, y$, then $x\, R^B\, y$ for all $B \in \mathcal{F}(U)$ with $x, y \in B$.*

Proof. First, let (R^A) upwards rationalize C, let $x\, P^A\, y$ for some feasible sets A, B and $x, y \in A \cap B$. Then, it has to be that $x\, P^{\{x,y\}}\, y$ by Lemma 2. Especially, $x\, R^{\{x,y\}}\, y \implies x\, R^B\, y$.
Now, let (R^A) downwards rationalize C, let $x\, P^A\, y$ for some feasible sets A, B and $x, y \in A \cap B$. Then, we can analogously show that $x\, P^{A \cup B}\, y$. Especially, $x\, R^{A\cup B}\, y \implies x\, R^B\, y$. □

Example 10 (No unique coarsest relation)**.** Consider $U = \{a, b, c\}$, $C(U) = \{a\}$. For all $B \subset U$, we set $C(B) := B$. Then C satisfies α. It is then clear that for all such B we have $R^B = B \times B$. We look at the following relation R^U, where weak edges are denoted by wavy lines and strict edges are pointing towards the dominated alternative.



We see that there is no strictly coarser relation on U which yields the same choice set, since then either b or c has to be undominated. Also it is easy to verify that $(R^A)_{\emptyset \neq A \subseteq U}$ downwards rationalizes C. We also look at another relation $\bar{R}^U$, given by the following graph:

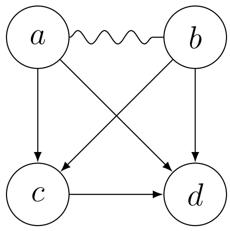


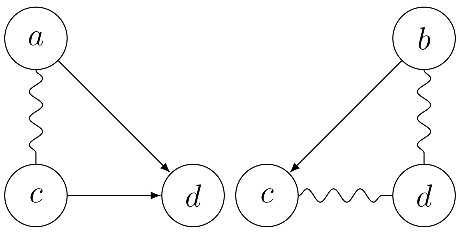
Again, any strictly coarser relation has b or c as additional maximal element. Also, it is easy to verify that $(\bar{R}^A)_{\emptyset \neq A \subseteq U}$ downwards rationalizes C (with $\bar{R}^B = B \times B$ for all strict subsets of U).
So both relations are inclusion maximal and downwards rationalize C, but they are not identical. △

Example 11 (No unique finest relation)**.** A unique finest relation downwards rationalizing C does not always have to exist.
Let $U = \{a, b, c, d\}$. Define C as follows:

A	$C(A)$
$\{a,b,c,d\}$	$\{a,b\}$
$\{a,c,d\}$	$\{a,c\}$
$\{b,c,d\}$	$\{b,d\}$
B	B

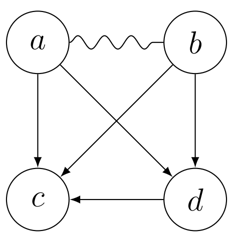
where the last row means that for all not previously listed feasible sets B, $C(B) = B$. One quickly verifies that C satisfies α. Again, it is clear that for all B of the last row we need $R^B := B \times B =: \bar{R}^B$. For the first family of relations $(R^A)_A$, we look at the following graphs:

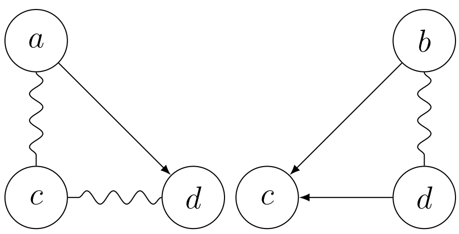




We observe for R^U and $R^{\{a,c,d\}}$ that each strictly finer, complete relation has one less maximal element. In the case of R^U, either a or b will be dominated. In the case of $R^{\{a,c,d\}}$, either a or c will be dominated. Now we look at $A = \{b, c, d\}$. Note that by $c\,P^{\{a,c,d\}}\,d$ we have to set $c\,R^A\,d$. Furthermore, if we make the edge $b\,I^A\,d$ strict, we lose one of the maximal elements b or d. Hence, there is no strictly finer relation than R^A such that the family still downwards rationalizes C. All in all, $(R^A)_{\emptyset \neq A \subseteq U}$ is an inclusion minimal family downwards rationalizing C.

For the second family of relations $\bar{R}^A$, we mirror our previous construction:





Analogously we observe for $\bar{R}^U$ and $\bar{R}^{\{b,c,d\}}$, that each strictly finer, complete relation has one less maximal element. In the case of $\bar{R}^U$, again either a or b will be dominated. In the case of $\bar{R}^{\{b,c,d\}}$, either b or d will be dominated. Now we look at $A = \{a, c, d\}$. Note that by $d\,\bar{R}^U\,c$, we have to set $d\,\bar{R}^A\,c$. The only other way to make this edge strict is to have $d\,\bar{P}^A\,c$, losing a maximal element. Furthermore, if we make the edge $a\,\bar{I}^A\,c$ strict, we also lose one of the maximal elements a or c. Hence, when fixing all other relations as done, there is no strictly finer relation than $\bar{R}^A$ such that the family downwards rationalizes C. All in all, $(\bar{R}^A)_{\emptyset \neq A \subseteq U}$ is another inclusion minimal family downwards rationalizing C. △

6.3 Transitive Downwards Rationalizability

6.3.1 The Main Result

To characterize transitive downwards rationalizability, we introduce a new relation on U. We remember that the revealed preference argues that some x is at least as good as some y, if there is *any* set containing y in which x is chosen. Sometimes, this makes alternatives tie even though one seems to be intuitively stronger than the other. For example, we examine the choice function C, which is defined by $C(A) = A$ for all feasible sets A, except for the following cases.

A	$C(A)$
$\{a, b, c, d\}$	$\{a\}$
$\{a, b, c\}$	$\{a\}$
$\{a, b, d\}$	$\{a\}$

The revealed preference states that b is at least as good as a, since the former is chosen in $\{a, b\}$. On the other hand, one could argue that a is in some sense strictly better than b, since in many sets a is chosen, while b is not. Furthermore, in all sets in which b is chosen, a is also chosen. Our argumentation results in the following relation.

Definition 26 (Competing Relation)**.** Let C be a choice function. We define its *competing relation* $\succsim$ on U as follows: $x \succsim y$ if and only if $y \in C(A) \implies x \in C(A)$ for all feasible sets $A \supseteq \{x, y\}$.

In our above example we have $a \succ b$, which matches our intuition. On first sight, our relation appears to have some theoretical drawbacks when compared to the revealed preference. For non-empty choice, the revealed preference relation is not only complete, but also acyclic. In general, neither must hold true for the competing relation.

Example 12. Let $|U| \geq 3$. Then the competing relation can be incomplete. Let C be defined by $C(A) = A$ for all feasible sets A, except for the following two cases.

A	$C(A)$
$\{a, b, c\}$	$\{a\}$
$\{a, b\}$	$\{b\}$

Clearly, a and b cannot be compared. Note that the relation is acyclic. △

Acyclicity is a bit more complicated.

Example 13. Let $U = \{a, b, c, x\}$. Then the competing relation can be cyclic. Let C be defined by $C(A) = A$ for all feasible sets A, except for the following cases.

A	$C(A)$
$\{a, b, x\}$	$\{a, x\}$
$\{b, c, x\}$	$\{b, x\}$
$\{a, c, x\}$	$\{c, x\}$
U	$\{x\}$

Let $\succsim$ denote the competing relation of C. Then it is clear that $a \succ b \succ c \succ a$, a cycle. On the other hand, we also have $x \succ a, b, c$, hence the relation is complete. △

Instead of viewing these two observations as a disadvantage of the competing relation, we could also argue that acyclicity and completeness of $\succsim$ mean that C satisfies a high degree of consistency.
As we saw in Example 9, α still allows that a is chosen while b is not in some set ($\{a, b, c\}$) and b is chosen while a is not in some other set ($\{a, b, d\}$). This violation of completeness results in transitive downwards rationalizability being impossible. With the addition of acyclicity and the help of two lemmas, we can present a characterization.

Lemma 11. *If C is downwards rationalized by a family of transitive relations, then $\succsim$ is acyclic.*

Proof. Let (R^A) be a family of transitive relations downwards rationalizing C. Assume for contradiction there are $x_1, \ldots, x_k$ with $x_1 \succ \cdots \succ x_k \succ x_{k+1} := x_1$. Let $i \leq k$. There must be a feasible set A_i with $x_{i+1} \in A_i \setminus C(A_i)$ and $x_i \in C(A_i)$. It follows that $x_i \, P^{A_i} x_{i+1}$, else by transitivity x_{i+1} would be chosen in A_i. Now, set $A := \cup_{i \leq k} A_i$, which is a feasible set. By downwards rationalizability, it has to be that $x_i \, P^A \, x_{i+1}$ for all $i \leq k$. Hence $x_1 \, P^A \ldots P^A \, x_k \, P^A \, x_1$, a wanted contradiction to R^A being acyclic (transitive). □

Lemma 12. *Let R be an acyclic and complete relation on a countable set U. Then there is a transitive and complete relation $R' \subseteq R$ on U.*

Proof. As always, we denote the strict part of R by P and the symmetric part by I. First, set $k := |\{(x, y) | \, x, y \in U, \, xIy, \, x \neq y\}|/2$. We deal with the case $k < \infty$ using induction. For $k = 0$, it is clear that R itself is antisymmetric and complete. For transitivity, let $x \, R \, y$, $y \, R \, z$. If the three are not pairwise distinct, then trivially $x \, R \, z$. Else, by antisymmetry it must be that $x \, P \, y$, $y \, P \, z$. By acyclicity it must hence be that $x \, R \, z$.
Now let $k > 0$. If R itself is transitive, we are done. Else, there must be pairwise distinct $x_1, \ldots, x_k \in U$ with $x_k \, P \, x_1$, but $x_i \, R \, x_{i+1}$ for all $i \leq k$, where we denote $x_{k+1} := x_1$. Since R is acyclic, there has to be $i < k$ with $x_i \, I \, x_{i+1}$. Assume now for contradiction, that for each such i the relation $R \setminus \{(x_i, x_{i+1})\}$ is no longer acyclic. This would mean that there is a strict path $x_i \, P \, \ldots \, P \, x_{i+1}$. For all other i we directly have $x_i \, P \, x_{i+1}$. Hence there is a cycle $x_1 \, P \, \ldots \, P \, x_k \, P \, x_1$, a wanted contradiction to acyclicity of R. Now let i be given, such that $x_i \, I \, x_{i+1}$ and $R^* := R \setminus \{(x_i, x_{i+1})\}$ is acyclic. Then R^* is still complete and we can use induction to obtain a relation $R' \subseteq R^* \subseteq R$, which is transitive and complete.
For $k = \infty$, it must be that $|U| = \infty$. We enumerate the elements of $U = \{x_1, \, x_2, \, \ldots\}$. Then, we can inductively build relations $R^{(k)} \subseteq R$, such that $R^{(k)}$ is complete and transitive on $\{x_1, \, \ldots, \, x_k\}$ using the complete and acyclic input $R^{(k-1)} \cup \{(x, y) \in R \, | \, x =$

$x_k \vee y = x_k\}$. Then, the lim inf of these relations is a transitive, complete relation on U. □

Theorem 7. *Let C be a choice function. The following are equivalent.*

(i) C is downwards rationalized by a family of transitive relations.

(ii) C satisfies α and its competing relation is complete and acyclic.

Proof. First, we look at $(i) \implies (ii)$. Lemma 11 ensures that the competing relation is acyclic. α is clearly satisfied. Assume for contradiction, that completeness is violated for x, y on the sets A, B. Then, by transitivity it has to be that $x\, P^A\, y$ and $y\, P^B\, x$, a contradiction to Lemma 10.
For the other direction, let (ii) be true. We want to show that there exists a family of relations $(R^A)_{A\in\mathcal{F}(U)}$ with the following properties:

(i)
$$C(A) = \max_{R^A} A \quad \text{for all feasible A}$$

(ii) For all feasible $B \subseteq A$ it holds that
$$R^A \cap (B \times B) \subseteq R^B$$

(iii) R^A is complete and transitive for all A

To do so, let $\succsim$ denote the competing relation, which has to be acyclic and complete by assumption. Using Lemma 12, we obtain a transitive and complete subrelation $\succsim^*$. For each feasible A and $x, y \in A$ we define:

$$x\, R^A\, y \iff x \succsim^* y \vee x \in C(A)$$

For (i), let $A \in \mathcal{F}(U)$. If $x \in C(A)$, then $x\, R^A\, y$ for all $y \in A$. If $x \in A \setminus C(A)$, then observe that by completeness of $\succsim$ for all $y \in C(A)$: $y \succ x$, which implies $y\, P^A\, x$.
For (ii), let $x, y \in B \subseteq A$ with $x\, R^A\, y$. If $x \in C(A)$, by α we have $x \in C(B)$, hence $x\, R^B\, y$. Else $x \succsim^* y$, hence also $x\, R^B\, y$.
For (iii), let $x\, R^A\, y$ and $y\, R^A\, z$ for some feasible set A with $x, y, z \in A$. If $x \in C(A)$, then trivially $x\, R^A\, z$. Else, it must be that $x \succsim^* y$. This implies $x \succsim y$ and hence $y \notin C(A)$. From this we deduce $y \succsim^* z$. Using transitivity of $\succsim^*$, we have $x \succsim^* z$, and hence $x\, R^A\, z$, which proves transitivity. Since $\succsim^*$ is complete on U, R^A is complete on A. □

6.3.2 Further Comments

Example 13 and Example 9 already show that even when α is satisfied, completeness and acyclicity of the competing relation do not imply each other. For the above characterization to be sharp, we only need to show that acyclicity and completeness of the competing relation do not imply α.

Example 14 (Completeness and acyclicity do not imply α)**.** Let $U = \{a, b, c\}$. We set $C(A) := A$ for all feasible sets, except for $C(\{a, b\}) := \{a\}$. Then the competing relation of C is complete and acyclic. Nonetheless, α is not satisfied. △

Neither completeness nor acyclicity are contraction consistency conditions. Are they related to expansion consistency? While completeness is implied by transitive upwards rationalizability, the latter does not imply acyclicity of the competing relation.

Proposition 10. *Let C be a choice function. If C is upwards rationalized by a transitive family of relations, then its competing relation is complete.*

Proof. Let $(R^A)_A$ be a family of transitive relations upwards rationalizing C. To check completeness of the competing relation, let $x, y \in A$ be given, such that $x \in C(A)$, $y \in A \setminus C(A)$. Let B be any other feasible set with $y \in C(B)$, $x \in B$. By transitivity and completeness of R^A, it has to be that $x\, P^A\, y$ (else $y \in C(A)$). By Lemma 10 we have $x\, R^B\, y$. By transitivity of R^B and maximality of y in B, we obtain $x\, R^B\, z$ for all $z \in B$, hence $x \in C(B)$. □

Example 15 (Transitive upwards rationalizability does not imply acyclicity of the competing relation)**.** Let $U = \{a, b, c, d\}$. We define $C(A) = A$ for all feasible sets A, except for the following ones.

A	$C(A)$	condition
a, b, d	a, d	—
b, c, d	b, d	—
c, a, d	c, d	—
a, b	a	—
b, c	b	—
a, c	c	—
A	d	$d \in A$, $\lvert A\rvert = 2$

C satisfies β^+. Further, the competing relation has a cycle $a \succ b \succ c \succ a$. △

Now, we have presented all main results of this thesis. From this point on, we embark on a sidequest to explore choice theoretical questions, as well as study applications of our theory.

6.3.3 Resoluteness$_2$ and Transitive Downwards Rationalizability

This subsection does not bear any new results. Instead, it shows where our attempts cross the path of already existing results. Namely, completeness of the competing relation is equivalent to a condition called *weak WARP*, which was used by Ehlers and Sprumont (2008). Together with two other consistency conditions and a weak form of resoluteness, it characterizes the Top Cycle rule (TC), which is known to satisfy β^+ and which we introduce in Section 9. Noticeably, transitive downwards rationalizability implies all of their axioms, except for resoluteness$_2$. We attempt to gain a better understanding of our new notion by applying their result and arriving at full rationalizability. Sadly, this is not due to transitivity, but rather due to α and resoluteness, as a result of Moulin (1985) suggests.

Definition 27 (WWARP, Jamison and Lau, 1973)**.** If $x, y \in U$ and there is $A \in \mathcal{F}(U)$, such that $x \in C(A)$ and $y \in A \setminus C(A)$, there is no $B \in \mathcal{F}(U)$ such that $y \in C(B)$ and $x \in B \setminus C(B)$.

Definition 28 (GCE, Weak α)**.** Let C be a choice function. We say that C satisfies *GCE* (or say that C is a *generalized Condorcet extension*) if and only if the following holds: If $x \in A$, and for all $y \in A \setminus \{x\}$ we have that $C(\{x,y\}) = \{x\}$, then $C(A) = \{x\}$.

We say that C satisfies *weak* α if and only if the following holds: For all $A \in \mathcal{F}(U)$, $|A| \geq 2$, we have that $C(A) \subseteq \bigcup_{x \in A} C(A \setminus \{x\})$.

Definition 29 (Resoluteness$_2$)**.** Let $|A| = 2$. Then $|C(A)| = 1$

Theorem 8 (Ehlers and Sprumont, 2008)**.** *Let C be a choice function. Then the following are equivalent.*

(i) C satisfies WWARP, GCE, weak α and resoluteness$_2$

(ii) There is some $R_N \in \mathcal{F}(U)$ with $|N|$ odd, such that $C(A) = TC(A, R_N)$ for all $A \in \mathcal{F}(U)$

Interestingly, transitive downwards rationalizability implies WWARP, GCE and weak α. Hence, all choice functions *resolute on two alternatives* and satisfying transitive downwards rationalizability are Top Cycle rules. Hence, they also satisfy β^+ and are transitively rationalizable.

Proposition 11. *Let C satisfy transitive downwards rationalizability. Then C satisfies WWARP, GCE and weak α.*

Proof. By Theorem 7, the competing relation of C is complete. This is by definition equivalent to C satisfying WWARP.
For GCE, let $x \in A$ and $C(\{x,y\}) = \{x\}$ for all $y \in A \setminus \{x\}$. Assume for contradiction that $C(A) \neq \{x\}$. Let $y \neq x$, $y \in C(A)$. Then by α $y \in C(\{x,y\})$, a contradiction.
For weak α, let $A \in \mathcal{F}(U)$, $|A| \geq 2$. Let $x, y \in A$ with $x \neq y$. Then we can apply α with $B_x := A \setminus \{x\}$ and $B_y := A \setminus \{y\}$. We obtain that $C(A) \setminus \{x\} \subseteq C(A \setminus \{x\})$ and $C(A) \setminus \{y\} \subseteq C(A \setminus \{y\})$. Hence we know that $C(A) \subseteq C(A \setminus \{x\}) \cup C(A \setminus \{y\}) \subseteq \bigcup_{z \in A} C(A \setminus \{z\})$. □

Moulin has already proven a similar statement without the need of transitivity.

Theorem 9 (Moulin, 1985)**.** *Let C be a resolute choice function, which means $|C(A)| = 1$ for all feasible A. Then, the following are equivalent.*

- *C satisfies α*
- *C satisfies α and γ*
- *C satisfies α and β^+*

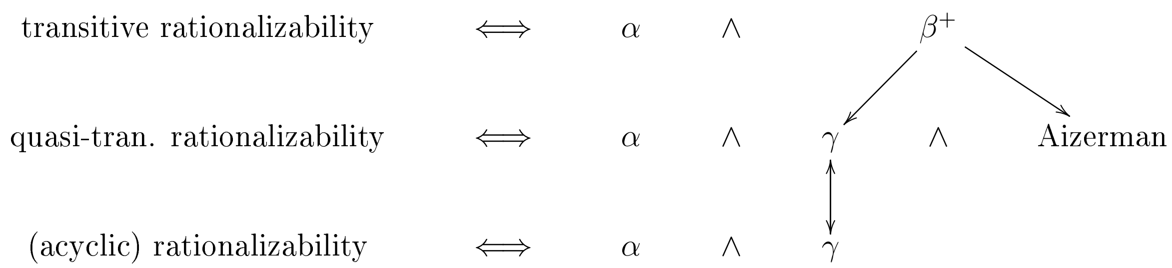
It is easy to show that α and resoluteness$_2$ imply resoluteness. Hence, we did not gain any knowledge about transitive downwards rationalizability. Still, the author hopes that this subsection grants the reader some insight into already existing work.

7 Search for New Consistency Conditions

In this section, we engage with the concept of creating expansion consistency conditions. Even though the author does not consider the following results to be as important as the previous ones, the process of reaching them made Section 5 possible. In fact, γ^+ was originally created as an attempt to answer the following question II. Further, the approach to formulate a condition clearly stronger than γ was only chosen after the following approaches of creating conditions clearly stronger than Aizerman.

7.1 Three Questions

The classical results can be summed up in a diagram:



We now ask ourselves three questions:

I Can we replace β^+ by γ and some expansion consistency condition X in the first row? Such X is not allowed to imply γ. Ideally, it should also be stronger than Aizerman.

II Can we find a single expansion consistency condition Y which can replace γ and Aizerman in the second row?

III Can we find a formulation of γ which is more similar to the formulations of other known expansion consistency conditions?

While attempting to answer them, we will define a few new expansion consistency conditions. Note that quasi-transitive rationalizability is equivalent to α and a notion called *stability*, which was introduced by Brandt et al. (2018). Hence, one technical way to answer question II is to merge the two conditions named $\hat{\alpha}_{\subseteq}$ (also known as Aizerman) and $\hat{\gamma}_{\subseteq}$ to a single condition.

7.2 A Large Class of Expansion Consistency Conditions

To the best of the author's knowledge, the term expansion consistency condition is not well-defined. Our understanding is that they are implications of the form "If something happens on certain subsets of A, then certain elements need to be within $C(A)$." Usually, expansion consistency conditions are all implied by β^+, just like contraction consistency conditions are implied by α.
When we look at different expansion consistency conditions used by Schwartz (1976), such as β^+ and Aizerman, we see that many of them can be written in the following way:

$$\begin{array}{c}\text{Let } B \subseteq A \text{ be feasible sets.}\\ \text{If } \boxed{}\\ \text{then } C(B) \subseteq C(A)\end{array}$$

If B is in some sense a reasonable subset of A, then everything chosen in B is also chosen in A. That is a bit abstract, so let us compare Aizerman, our imagined X and β^+.

Aizerman	X	β^+
	Let $B \subseteq A$ be feasible sets.	
If $C(A) \subseteq B$	If $*$	If $C(A) \cap B \neq \emptyset$
	then $C(B) \subseteq C(A)$	

Under Aizerman, B is only deemed a reasonable subset, if it contains *all* elements which are chosen in A.
Under β^+, B is already deemed a reasonable subset, if it contains *any* element which is chosen in A.
Our goal for X is to find a fitting notion of reasonability in between those two. It is easy to verify that if $C(A) \subseteq B$ implies $*$, then X implies Aizerman. Analogously, if $*$ implies $C(A) \cap B \neq \emptyset$, then β^+ implies X.

7.3 The Expansion Consistency Condition χ

The following property originated as an attempt to answer question I. It was found using the idea illustrated in Section 7.2. Sadly, we will see that χ is too strong in the sense that it implies γ. Nonetheless, its formulation is quite compact and it can function as a replacement for β^+ whenever α can be assumed.

Definition 30 (Property χ)**.** Let C be a choice function. We say that C satisfies χ if and only if the following holds:

$$\begin{array}{c}\text{Let } B \subseteq A \text{ be feasible sets.}\\ \text{If } |C(A) \cap B| \geq |C(A) \setminus B|\\ \text{then } C(B) \subseteq C(A)\end{array}$$

B is reasonable under χ, if it contains at least half the elements of $C(A)$. By non-emptiness of choice, containing all elements of $C(A)$ implies containing at least half of $C(A)$ implies containing any element of $C(A)$. By Section 7.2 we could now directly conclude that β^+ implies χ implies Aizerman. We will demonstrate both proofs for χ. In the future, we will omit similar proofs.

Lemma 13. *Let C satisy χ. Then C satisfies Aizerman.*

Proof. Let $B \subseteq A$ be given, such that $C(A) \subseteq B$. Then $|C(A) \setminus B| = 0$. Since the inequality in Definition 30 now must hold, we can apply χ and obtain $C(B) \subseteq C(A)$. □

Lemma 14. *Let C satisfy β^+. Then C satisfies χ.*

Proof. Let C satisfy β^+. Let $B \subseteq A$ be feasible sets, such that $|C(A) \cap B| \geq |C(A) \setminus B|$. Then, by non-emptiness of $C(A)$, it must be that $C(A) \cap B \neq \emptyset$. Else, we would have $|C(A)| = |C(A) \cap B| + |C(A) \setminus B| = 0 + 0$. Hence we can apply β^+ and obtain $C(B) \subseteq C(A)$. □

Example 16 (χ does not imply β^+)**.** Let $|U| = 4$. We define C as follows:

A	$C(A)$
$\{a,b,c,d\}$	$\{a,b,d\}$
$\{c,d\}$	$\{c\}$
$\{c\}$	$\{c\}$
$\{d\}$	$\{d\}$
A	$A \setminus \{c,d\}$

where the last column is meant for all $A \subseteq U$ which were not listed in the table before. Note that C is a well-defined choice function, since for all sets the choice is non-empty. C violates β^+, which we can see by setting $A := \{a,b,c,d\}$, $B := \{c,d\}$. Then $B \subseteq A$, $C(A) \cap B = \{d\} \neq \emptyset$, but $C(B) \nsubseteq C(A)$. To show that C satisfies χ, let $B \subseteq A$ be arbitrary feasible sets. If $A = B$, the set inclusion $C(B) \subseteq C(A)$ holds trivially. If $|B| = 1$, say $B = \{x\}$, then non-emptiness of C and $|C(A) \cap B| \geq |C(A) \setminus B|$ imply that $x \in C(A)$, hence $C(B) = B \subseteq C(A)$. Now let $|B| \geq 2, |A| > |B|$. First, if $B = \{c,d\}$, then non-emptiness of choice and $|C(A) \cap B| \geq |C(A) \setminus B|$ imply that we must choose c or d in A. Hence $A = \{a,b,c,d\}$. But the inequality "$1 \geq 2$" does not hold, hence χ cannot be violated. At last, let B be one of the not explicitly listed sets, $|B| \geq 2$. Then $C(B) = B \setminus \{c,d\}$. We see that since $|A| \geq 3$, the inclusion $A \setminus \{c,d\} \subseteq C(A)$ holds true. Hence $C(B) \subseteq C(A)$. △

The following result implies that χ is not an answer to question I, since it implies γ. After finding γ^+, the author realized that χ is even strong enough to imply γ^+.

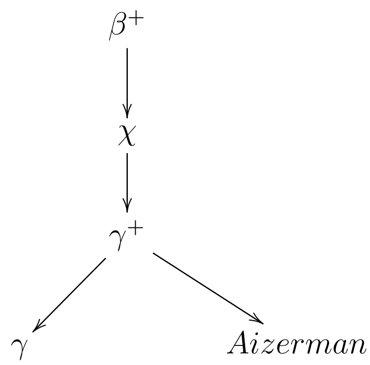
Lemma 15. *Let C satisfy χ. Then C satisfies γ^+.*

Proof. Let A, B be feasible sets. Then we use $A' := A \cup B$ and $B' := A$. If $|C(A') \cap B'| \geq |C(A') \setminus B'|$, then we can apply χ with A' and B' and obtain $C(A) \subseteq C(A \cup B)$. Else, $|C(A') \cap B'| < |C(A') \setminus B'|$. Altogether we obtain

$$|C(A') \setminus B| \leq |C(A') \cap B'| < |C(A') \setminus B'| \leq |C(A') \cap B|$$

In more detail, $C(A') \setminus B \subseteq C(A') \cap B'$, since every element that is not in B has to be in $A = B'$. Also, $C(A') \setminus B' \subseteq C(A') \cap B$, since every element that is not in A has to be in B. Hence we can apply χ to A' and B. We obtain $C(B) \subseteq C(A \cup B)$ □

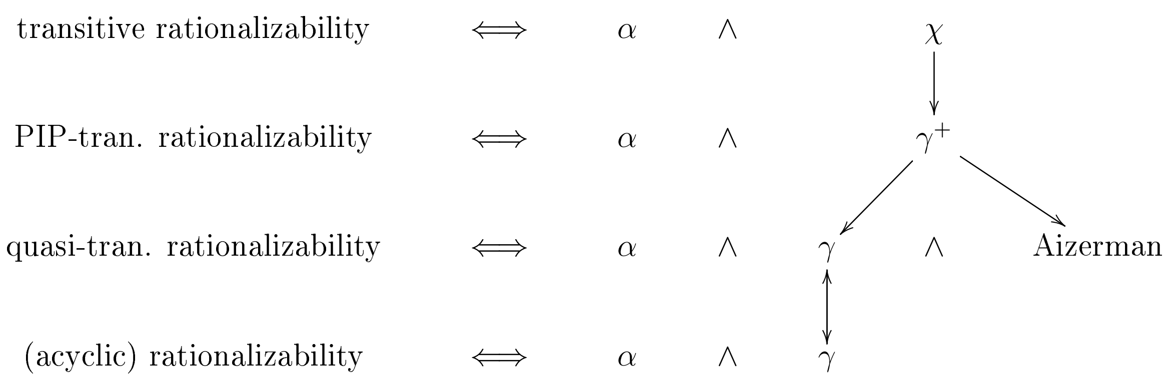
We obtain the following diagram:



Theorem 10. *Let C satisfy α and χ. Then C satisfies β^+.*

Proof. We now know that C satisfies α, γ, and Aizerman. By Corollary 2, C is rationalized by a quasi-transitive, complete preference relation R. We now show that it must be transitive. Let x, y, z be given, such that $x\,R\,y$, $y\,R\,z$. Then $x \in C(\{x, y\})$. Assume $z\,P\,x$ for contradiction. Set $A \coloneqq \{x, y, z\}$. Set $B \coloneqq \{x, y\}$. If $y \in C(A)$, then $|C(A) \cap B| = 1 \geq |C(A) \setminus B|$. By χ we obtain $x \in C(A)$, a contradiction. Else, it must be that $C(A) = \{z\}$. Set $B' \coloneqq \{y, z\}$ Since $|C(A) \cap B'| = 1 \geq 0 = |C(A) \setminus B|$, we apply χ and conclude that $y \in C(A)$, a contradiction. □

So in fact, we can replace β^+ by the cardinality-based χ when assuming α.



Corollary 5. *There are choice functions which satisfy γ^+, but not χ.*

Proof. This follows from Theorem 10 and Theorem 5, since PIP-transitive rationalizability is a strictly weaker notion than transitive rationalizability. □

7.4 Parameterized χ

Since χ with α already implies β^+, we attempt to answer question II by weakening χ slightly. This attempt also fails, but gives rise to a hierarchy of conditions which shows how Aizerman, χ and β^+ are related using a parameter. The main idea is that some $B \subseteq A$ is considered reasonable if and only if it contains at least a fraction p of the elements of $C(A)$.

Definition 31 (Parameterized χ)**.** For $p \in [0, 1]$ we say that a choice function C satisfies $\chi^{(p)}$, if the following holds:

Let $B \subseteq A$ be feasible sets.
If $|C(A) \cap B| \geq p|C(A)|$
then $C(B) \subseteq C(A)$

We say that C satisfies $\chi^{(\downarrow p)}$, if C satisfies $\chi^{(q)}$ for all $q \in (p, 1]$. This is equivalent to making the above inequality in the antecedent strict.

Remark 6. One can check that for $p, q \in [0, 1]$ with $p < q$, $\chi^{(p)}$ implies $\chi^{(\downarrow p)}$ implies $\chi^{(q)}$. Due to non-emptiness, it is easy to see that the only C satisfying $\chi^{(0)}$ is the trivial choice function with $C(A) = A$ for all feasible sets.

We first show that we parameterize between β^+ and Aizerman, with χ being in the middle.

Lemma 16. *The following equivalences hold:*

- $\chi^{(1)} \iff$ *Aizerman*
- $\chi^{(\frac{1}{2})} \iff \chi$
- $\chi^{(\downarrow 0)} \iff \beta^+$

Proof. For Aizerman, we observe that $(|C(A) \cap B| \geq |C(A)|) \iff (C(A) \subseteq B)$.
For χ, we observe that $(|C(A) \cap B| \geq \frac{1}{2}|C(A)|) \iff (|C(A) \cap B| \geq \frac{1}{2}(|C(A) \cap B| + |C(A) \setminus B|)) \iff (|C(A) \cap B| \geq |C(A) \setminus B|)$.
For β^+, we observe that $(\exists \varepsilon > 0 : |C(A) \cap B| \geq \varepsilon |C(A)|) \iff C(A) \cap B \neq \emptyset$. □

From Lemma 16, Corollary 3 and Theorem 10 we can directly conclude the following statement.

Lemma 17. *Let C be a choice function. Then for all $0 < p \leq \frac{1}{2}$ the following are equivalent.*

(i) C satisfies α and $\chi^{(p)}$

(ii) C is transitively rationalizable

Now we show that the $\chi^{(p)}$ truly form a hierarchy.

Lemma 18. *For all $p < q$, $\chi^{(q)}$ does not imply $\chi^{(p)}$ (for large enough U).*

Proof. Let $k, n \in \mathbb{N}$ be given such that $p < \frac{k}{n} < q$. Then we set $U := \{x_1, \ldots, x_n, y\}$ with $|U| = n + 1$. We define a choice function C using the following table, where we denote $D := \{x_1, \ldots, x_k, y\}$

A	$C(A)$	condition
U	$U \setminus \{y\}$	—
D	D	—
A	$A \setminus D$	for all $U \neq A \not\subseteq D$
A	$A \setminus \{y\}$	for all $\{y\} \neq A \subset D$
$\{y\}$	$\{y\}$	—

We now show that C fulfills $\chi^{(q)}$ but not $\chi^{(p)}$. For a violation of $\chi^{(p)}$, set $A := U$, $B := D$. We see that $|C(A) \cap B| = k$, $|C(A)| = n$. By assumption $k > pn$, but at the same time $y \in C(B) \setminus C(A)$.
Now we focus on $\chi^{(q)}$. First, let $A = U$. For all $B \neq D$, we have $C(B) \subseteq C(A)$. Using $|C(A) \cap D| = k$, $|C(A)| = n$ and $k < nq$ we see that there is no violation of $\chi^{(q)}$. For $A = D$, we trivially obtain $C(B) \subseteq B \subseteq A = C(A)$ for any B we need to consider. For $A \not\subseteq D$, $A \neq U$, let $|C(A) \cap B| \geq q|C(A)| > 0$ for $B \subseteq A$. Observe that then B cannot be a subset of D or D itself. Hence $C(B) = B \setminus D \subseteq A \setminus D = C(A)$. Finally, let $\{y\} \neq A \subset D$. For all $B \subseteq A$ satisfying the antecedent of Definition 31, we have that $C(B) = B \setminus \{y\} \subseteq A \setminus \{y\} = C(A)$. □

$\chi^{(\downarrow p)}$ is strictly weaker than $\chi^{(p)}$ if and only if p is rational.

Lemma 19. *Let $p \in [0,1]$ be rational. Then, for large enough U, $\chi^{(\downarrow p)}$ does not imply $\chi^{(p)}$. If $p \in [0,1]$ is irrational, $\chi^{(\downarrow p)}$ and $\chi^{(p)}$ are equivalent.*

Proof. First let p be irrational. Let C satisfy $\chi^{(\downarrow p)}$. Let feasible sets B, A with $B \subseteq A$ and $|C(A) \cap B| \geq p|C(A)|$ be given. Since the left hand side is a natural number and the right hand side is not, we have that $|C(A) \cap B| > p|C(A)|$. By $\chi^{(\downarrow p)}$, we have $C(B) \subseteq C(A)$. $p = 0$ has been dealt with, since not only *TRIV* satisfies β^+. Let $p = \frac{k}{n}$ with $k > 0$, $U \coloneqq \{x_1, \ldots, x_n, y\}$ such that $|U| = n + 1$. We again define the choice function C using the following table, where we denote $D \coloneqq \{x_1, \ldots, x_k, y\}$

A	$C(A)$	condition
U	$U \setminus \{y\}$	—
D	D	—
A	$A \setminus D$	for all other $A \not\subseteq D$
A	$A \setminus \{y\}$	for all $\{y\} \neq A \subset D$
$\{y\}$	$\{y\}$	—

We now show that C fulfills $\chi^{(\downarrow p)}$, but not $\chi^{(p)}$. For a violation of $\chi^{(p)}$, set $A \coloneqq U$, $B \coloneqq D$. We see that $|C(A) \cap B| = k$, $|C(A)| = n$. By assumption $k \geq pn$, but at the same time $y \in C(B) \setminus C(A)$.
Now we focus on $\chi^{(q)}$. First, let $A = U$. For all $B \neq D$, we have $C(B) \subseteq C(A)$. Using $|C(A) \cap D| = k$, $|C(A)| = n$ and $k = np < nq$ for all $q > p$, we see that there is no violation of $\chi^{(\downarrow p)}$ for the pair (U, D). For $A = D$, we trivially obtain $C(B) \subseteq B \subseteq A = C(A)$ for any B we need to consider. For $A \not\subseteq D$, $A \neq U$, let $|C(A) \cap B| > p|C(A)| > 0$ for $B \subseteq A$. Observe that then B cannot be a subset of D or D itself. Hence $C(B) = B \setminus D \subseteq A \setminus D = C(A)$. Finally, let $A \subset D$. For all $B \subseteq A$ satisfying the antecedent of Definition 31, we have that $C(B) = B \setminus \{y\} \subseteq A \setminus \{y\} = C(A)$. □

Can we find a p for which $\chi^{(p)}$ (or $\chi^{(\downarrow p)}$) is an answer to question II? χ itself is already too strong. We additionally show that any weakening of $\chi^{(\frac{1}{2})}$ is too weak. To do so, we only need to consider the strongest weakening of χ, which is $\chi^{(\downarrow \frac{1}{2})}$.

Example 17. α and $\chi^{(\downarrow \frac{1}{2})}$ do not imply (quasi-transitive) rationalizability.
Let $U = \{a, b, c\}$. We set $C(A) = A$ for all feasible sets, except for $C(U) = \{b, c\}$.
α is clearly satisfied. Furthermore, C is not rationalizable: Due to trivial choice on the sets of size 2, it would have to be that $x \, I \, y$ for all $x, y \in U$. But this contradicts $a \notin C(U)$. For checking that $\chi^{(\downarrow \frac{1}{2})}$ is satisfied, let A, B be feasible with $B \subseteq A$. If $A \neq U$, then $C(A) = A$ and hence trivially $C(B) \subseteq C(A)$. Let $A = U$. If $|C(A) \cap B| > \frac{1}{2}|C(A)| = 1$, then again trivially $C(B) \subseteq C(A)$. △

On the other hand, χ was also too strong for question I, so again our only chance is to weaken it. We now show that $\chi^{(\downarrow \frac{1}{2})}$ is too weak to imply β^+ with α and γ together.

Example 18 (α, γ and $\chi^{(\downarrow \frac{1}{2})}$ do not imply transitive rationalizability)**.** Let $U = \{a, b, c\}$. We define the choice function C on all non-singleton sets via the following table:

A	$C(A)$
$\{a, b, c\}$	$\{b, c\}$
$\{a, b\}$	$\{a, b\}$
$\{a, c\}$	$\{c\}$
$\{b, c\}$	$\{b, c\}$

One can easily check that C is rationalized by R, where $a\,I\,b$, $b\,I\,c$, $c\,P\,a$. Since R is quasi-transitive but not transitive, C satisfies α and γ, but not β^+. Now, we only need to check that for all $p > \frac{1}{2}$, $\chi^{(p)}$ is satisfied. Let $B \subseteq A$ be feasible sets. For $|A| \leq 2$, if $B \cap C(A) \neq \emptyset$, then $B = C(A)$ or $B = A$. Hence trivially $C(B) \subseteq C(A)$. Let $A = U$. $|C(A) \cap B| \geq p|C(A)| = 2p > 1$. Hence $\{a, b, c\} \supseteq B \supseteq C(A) = \{b, c\}$. Again $C(B) \subseteq C(A)$ △

7.5 Technically Correct

In this subsection, we present very technical answers to the three questions proposed. They show that answering the questions is difficult, but not impossible on a fundamental level. Even though the answers meet all our formal requirements, we still consider the questions to be open when demanding intuitively appealing conditions.
We first present an answer for question I.

Definition 32 (Artificial I)**.** Let C be a choice function. We say that C satisfies *Artificial I*, if the following holds:

Let $B \subseteq A$ be feasible sets.
If $C(A) \cap B \neq \emptyset$ and C restricted to $\mathcal{F}(A)$ satisfies γ
then $C(B) \subseteq C(A)$

Example 19 (Artificial I does not imply γ)**.** Let $U = \{a, b, c\}$. We define $C(A) = A$ for all feasible sets A, except for $C(U) = \{b, c\}$. Then, γ is violated, since $a \notin C(U)$. Further, Artificial I is satisfied: Let $B \subseteq A$ be two feasible sets. If $A \neq U$, we have $C(A) = A$ and hence trivially $C(B) \subseteq A = C(A)$. For $A = U$, we see that C restricted to subsets of U is C itself. Since C does not satisfy γ, the antecedent is not true and hence the implication correct.

Lemma 20. *Artificial I answers question I. This means that α, γ and Artificial I are equivalent to transitive rationalizability, while Artificial I is an expansion consistency condition and does not imply γ.*

Proof. By definition, Artificial I is an expansion consistency condition (even though arguably not intuitive). Further, we have already seen that it does not imply γ. We now can conclude the proof by showing that Artificial I and γ are equivalent to β^+. If β^+ is satisfied, we have that $C(B) \subseteq C(A)$ for all $B \subseteq A$ with $C(A) \cap B \neq \emptyset$. Hence Artificial I is satisfied by definition. Further, we already know that β^+ implies γ. For the other implication, assume that C satisfies γ and Artificial I. Now let A, B be feasible sets with $B \subseteq A$ and $C(A) \cap B \neq \emptyset$. Since C satisfies γ, we know that C restricted to subsets of any feasible set satisfies γ too. Hence, we can apply Artificial I and obtain $C(B) \subseteq C(A)$. □

Now, we present an answer for question III.

Definition 33 (Artificial III)**.** Let C be a choice function. We say that C satisfies *Artificial III*, if the following holds:

Let $B \subseteq A$ be feasible sets.
If there are feasible sets D, E with $D \cup E = A$ and $B \subseteq C(D) \cap C(E)$
then $C(B) \subseteq C(A)$

Lemma 21. *Artificial III answers question III. This means that it is an expansion consistency condition as discussed in Section 7.2 and equivalent to γ.*

Proof. We only need to show the equivalence. First, let C satisfy γ. Let $B \subseteq A$ and D, E be given as in Definition 33. By γ it has to be that $C(D) \cap C(E) \subseteq C(A)$. We hence obtain $C(B) \subseteq B \subseteq C(D) \cap C(E) \subseteq C(A)$.
For the other direction, let C satisfy Artificial III. Let $\tilde{A}, \tilde{B}$ be feasible sets. We set $D := \tilde{A}$, $E := \tilde{B}$ and $A := \tilde{A} \cup \tilde{B}$. Now let $b \in C(\tilde{A}) \cap C(\tilde{B})$. We set $B := \{b\}$. By applying Artificial III, we obtain that $C(B) \subseteq C(A)$. Since $\{b\} = B = C(B)$ and $b \in C(\tilde{A}) \cap C(\tilde{B})$ was arbitrary, we have that $C(\tilde{A}) \cap C(\tilde{B}) \subseteq C(\tilde{A} \cup \tilde{B})$. □

We could now define "Artificial II" by combining the antecedents of Aizerman and Artificial III using a logical OR symbol. This would by construction be equivalent to Aizerman and γ, hence answer question II.

7.6 Consistency of Transitivity

We characterize transitive upwards rationalizability using a new consistency condition. Note that we leave the realm of contraction and expansion consistency conditions, hence we do not provide an answer to question I.
We already know that if γ is satisfied, then C is upwards rationalized by $(R_C^A)_A$. Hence all we need to do is to ensure that R_C^A is transitive for all A.

Definition 34 (τ)**.** Let C be a choice function. We say that C satisfies τ, if for all $A, B \in \mathcal{F}(U)$ the following holds: If $A \cap C(B) \neq \emptyset$, then for all $x \in C(A)$, $y \in B$, there is some $D \subseteq A \cup B$ with $y \in D$ and $x \in C(D)$.

Lemma 22. *Let C be a choice function. Then the following are equivalent.*

(i) C satisfies τ

(ii) R_C^A is transitive for all $A \in \mathcal{F}(U)$

(iii) The γ-hull $\mathcal{H}(C)$ satisfies β^+

Proof. By Lemma 6 we have $\mathcal{H}(C)(A) = \max_{R_C^A} A$. Hence, by Theorem 3, (ii) implies (iii).
To show that (iii) implies (ii), we show $R_{\mathcal{H}(C)}^A = R_C^A$ for all feasible A. Then we only need to apply Theorem 3 and Proposition 6. By definition we have $C \subseteq \mathcal{H}(C)$, which implies the inclusion "$\supseteq$" using the same witness $B \subseteq A$. For the other inclusion, let $x\, R_{\mathcal{H}(C)}^A\, y$. Then there is some $B \subseteq A$, such that $y \in B$ and $x \in \mathcal{H}(C)(B)$. By Lemma 6 we know that $x\, R_C^B\, y$. By upwards inheritance of R_C^A we have $x\, R_C^A\, y$.
Next, we show that (i) implies (ii). Let C satisfy τ. Let A be a feasible set, such that $x\, R_C^A\, y$, $y\, R_C^A\, z$. Let $B, D \subseteq A$ be the witnesses for that. This means $x \in C(B)$, $y \in B$, $y \in C(D)$, $z \in D$. Then we can apply τ using B, D. Hence there is some $E \subseteq B \cup D \subseteq A$, such that $z \in E$, $x \in C(E)$. By definition we obtain $x\, R_C^A\, z$.
Conversely, let (ii) hold true, let A, B given as in Definition 34. Let $z \in C(B) \cap A$. Then we have $x\, R^{A\cup B}\, z$, $z\, R^{A\cup B}\, y$. Since $R_C^{A\cup B}$ is transitive, we have $x\, R^{A\cup B}\, y$. Hence (i) holds true.

□

Hence we have a condition that relates to question I.

Proposition 12. *Let C be a choice function. Then the following are equivalent.*

(i) C satisfies γ and τ

(ii) C is upwards rationalized by a family of transitive relations

Proof. Let (i) hold true. Then C is equal to the γ-hull of C. Further, the latter satisfies β^+ by Lemma 22. Hence (ii) holds true by Theorem 3.
Now let (ii) hold true. Then C satisfies β^+ by Theorem 3. Especially, it is equal to its γ-hull, which thus also satisfies β^+. By Lemma 22, C satisfies τ. Further, it also satisfies γ. □

We can try to form τ into an expansion consistency condition by demanding that D is equal to $A \cup B$, instead of D being an arbitrary subset. This is equivalent to a known condition proposed by Salant and Rubinstein (2008), which they call γ^+. (It has nothing to do with the γ^+ which we define in Section 5.) In the context of non-empty choice theory, their version of γ^+ is equivalent to β^+, hence too strong for our purposes.

8 A Formal Introduction to Social Choice

This section formally introduces majoritarian and pairwise social choice functions.

Definition 35 (Preference Profiles)**.** Let $N = \{1, \ldots, n\}$, U be countable. A *preference profile* R_N on U represents the voters' preferences over all alternatives in U. More formally, let $\mathcal{L}(U)$ be the set of all linear orderings on U. Then a preference profile is a mapping

$$R_N : N \to \mathcal{L}(U)$$

Set $\mathcal{R}(n, U)$ as the set of all preference profiles over U for $N = \{1, \ldots, n\}$. Then, set

$$\mathcal{R}(U) := \bigcup_{n \in \mathbb{N}} \mathcal{R}(n, U)$$

as the set of all preference profiles over U for any finite number of voters.
Let $R_N \in \mathcal{R}(U)$. When we duplicate all voters and their preferences, we denote the resulting preference profile as $2R_N \in \mathcal{R}(U)$. More formally, let $L = \{1, \ldots, 2n\}$. Then $2R_N := R_L$, where $R_L(n + i) = R_L(i) = R_N(i)$ for all $i \leq n$.

We can interpret preference profiles in two ways. We can either take them at face value and assume them to be the voters' true preferences, or we can view them as submitted ballots, which might differ from their true preferences for strategical reasons. While the second interpretation is both insightful and interesting, for this thesis we shall not deal with it. In the following sections, we think of the preference profiles as true preferences and try to choose reasonable winners based on that assumption.

Intuitively, we assume voting and choosing to be something simple. But if we think about it, what really is the best choice for an entire group? The social choice should depend on the feasible set and the preferences of the individuals. This we can formalize.

Definition 36 (Social Choice Function)**.** A *social choice function* on U is a mapping

$$S : \mathcal{F}(U) \times \mathcal{R}(U) \to \mathcal{F}(U)$$

such that $S(A, R_N) \subseteq A$ for all $R_N \in \mathcal{R}(U)$, $A \in \mathcal{F}(U)$. If R_N is clear from the context, we will sometimes abbreviate $S(A) = S(A, R_N)$.

This definition still allows for unreasonable functions. For example, we could always order the alternatives lexicographically and then choose the first one as winner, independent of the voters' preferences. We hence introduce a few axioms.

Definition 37 (Neutrality)**.** Let S be a social choice function. We say that S is *neutral*, if swapping the alternatives in the preference relation of each individual leads to swapping the alternatives chosen by society. More formally, let $R_N \in \mathcal{R}(U)$. Let $\pi : U \to U$ be a bijection. By $\pi(R_N)$ we denote the preference profile where the alternatives are renamed under π. By $\pi(A) \subseteq U$ we denote the image of $A \subseteq U$. Then it has to be that $S(\pi(A), \pi(R_N)) = \pi(S(A, R_N))$.

Just like for choice functions, we allow there to be more than one winner. This is inevitable, if we want to follow certain standards of fairness. For example, we look at $N = \{1, 2\}$, $A = \{a, b\}$ and the following preference profile R_N:

1	1
a	b
b	a

which depicts that 1 voter prefers a to b and 1 voter prefers b to a. If we want our choice to be neutral and anonymous (which can be defined analogously to neutrality), we cannot choose a single winner.

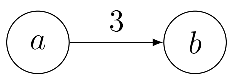
A widely used approach to choose winners is based on the concept of pairwise duels. For each pair of alternatives (x, y), we count how many people prefer x to y minus how many people prefer y to x. Based on these numbers we decide who wins. For example, let us look at the following anonymized preference profile.

3	1	1
a	c	b
b	a	c
c	b	a

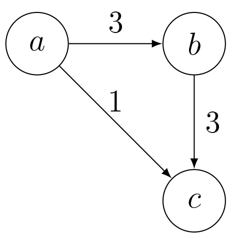
In this profile, 4 people prefer a to b, while 1 person prefers b to a.

3	1	1
a	c	b
b	a	c
c	b	a

The margin of this majority comparison is hence equal to 3. We can write this down compactly.



Doing this for all pairs of alternatives yields the following weighted graph.



Based on the graph, it seems reasonable to declare a as the winner. We now formalize this approach and introduce terminology.

Formally, we represent the graph by its weighted adjacency matrix.

$$M = \begin{pmatrix} 0 & 3 & 1 \\ -3 & 0 & 3 \\ -1 & -3 & 0 \end{pmatrix}$$

Definition 38 (Majority Margin, Cycles)**.** Let $R_N \in \mathcal{R}(U)$ be a preference profile. Let $\succ_i = R_N(i)$ denote voter i's preference relation. Then we set $m_R(x, y) = |\{i \in N|\, x \succ_i y\}| - |\{i \in N|\, y \succ_i x\}|$ as the *majority margin* of x over y. The (skew-symmetric) matrix of all majority margins will be denoted by $M_R := (m_R(x, y))_{x,y \in U}$.
If R_N is clear from the context, we can write $x \succ y$ instead of $m_R(x, y) > 0$ and $x \succsim y$ instead of $m_R(x, y) \geq 0$. We say that $(x_1, \ldots, x_k)$ form a *cycle*, if $x_i \succ x_{i+1}$ for all $i \leq k$, where we set $x_{k+1} := x_1$.

We see that each majority margin matrix induces a weighted, directed graph on U. The following result states that the converse direction only is possible with a restriction.

Theorem 11 (Debord, 1987)**.** *Let (U, M) define a weighted, directed graph on U, where $M \in \mathbb{Z}^{U \times U}$ is a skew-symmetric matrix. Then the following are equivalent.*

- *There is $R_N \in \mathcal{R}(U)$ with $M = M_R$*
- *all M_{xy} with $x \neq y \in U$ have the same parity*

Definition 39 (Pairwise[7] and Majoritarian[8] Social Choice Functions)**.** Let S be a social choice function. We say that S is *pairwise*, if S is neutral and only depends on the majority margins of the feasible alternatives. More formally, this means that $S(A, R_N) = S(A, R'_L)$ for all feasible A and all $R_N, R'_L \in \mathcal{R}(U)$ with $m_R(x, y) = m_{R'}(x, y)$ for all $x, y \in A$. If S is neutral and only depends on the sign of the majority margins, then we say that S is *majoritarian*. More formally, this means that $S(A, R_N) = S(A, R'_L)$ for all $R_N, R'_L \in \mathcal{R}(U)$ such that $m_R(x, y) > 0$ if and only if $m_{R'}(x, y) > 0$ for all $x, y \in A$.
When $M := M_R = M_{R'}$, the choice set of pairwise social choice functions does not depend on the underlying preference profile. Hence, we will sometimes use abuse of notation and write $S(A, M)$ when the skew-symmetric matrix M meets all requirements of Theorem 11.

Now that we have formally defined social choice functions, we only need to adapt our definition of consistency before we can start analyzing them.

Definition 40 (Conditions for Social Choice Functions)**.** Let $*$ be any condition which makes sense for choice functions, let S be a social choice function. We say that S satisfies $*$ if and only if $S(\,\cdot\,, R_N)$ satisfies $*$ for all $R_N \in \mathcal{R}$.

[7]See Young (1974) and Zwicker (1991)
[8]For example see Laslier (1997)

9 A Formal Analysis of 3 Social Choice Functions

We finally get back to where we started. We recall that our motivation for diving into the depths of choice theory was based on understanding and designing good voting rules. Now, we will use our newly gained knowledge to analyze social choice functions which satisfy various expansion consistency conditions. The examples we will go over are the Top Cycle rule, Uncovered Set rule and Split Cycle rule.
In the literature, sometimes instead of directly defining R^A (Top Cycle), only the strict part P^A will be defined (Uncovered Set, Split Cycle). We rigorously show that both approaches are equivalent.

Lemma 23. *Let T be an asymmetric, acyclic relation on A. Then we obtain a complete, acyclic relation R on A by defining $x\,R\,y \iff \neg(y\,T\,x)$. Furthermore, $T = P$, where P denotes the strict part of R.*

Proof. We first show completeness. Let $\neg(x\,R\,y)$. By definition of R this is equivalent to $y\,T\,x$. By asymmetry of T it follows that $\neg(x\,T\,y)$, hence $y\,R\,x$. To be formally correct, let P denote the strict part of R. In other words $x\,P\,y$ if and only if $x\,R\,y$ and not $y\,R\,x$. We now need to show that P is equal to T and hence acyclic. Let $x\,P\,y$. This especially implies $\neg(y\,R\,x)$, which by definition is equivalent to $x\,T\,y$. Let $x\,T\,y$. This, by definition of R, is equivalent to $\neg(y\,R\,x)$. Furthermore, by asymmetry $\neg(y\,T\,x)$. By definition of R this implies $x\,R\,y$. Again by definition of P, this implies $x\,P\,y$. □

Now, we can finally define our social choice functions.

9.1 Top Cycle Rule

Definition 41 (Top Cycle as Social Choice Function)**.** Let $R_N \in \mathcal{R}(U)$ be given, let $A \subseteq U$ be a feasible set. We write $x\,R^A\,y$ if $x = y$, or if $\exists y_0, \dots, y_k \in A : y_i \succsim y_{i+1}$ for all $i = 0, \dots, k-1$. Here, we denoted $y_0 = x$ and $y_k = y$. We then define

$$TC(A, R_N) \coloneqq \max_{R^A} A$$

Remark 7. The Top Cycle is by definition majoritarian. There are several ways to define the Top Cycle when the majority relation contains ties. The one above is also known as the Smith set (see Smith, 1973), which is not to be confused with the Schwartz set (see Schwartz, 1970). However, when the majority relation is strict, there is no ambiguity and all definitions coincide.

Proposition 13. *TC satisfies β^+ (and hence also γ and Aizerman).*

Proof. First, fix an arbitrary R_N. We now show that $(R^A)_A$ as defined above upwards rationalizes $TC(\,\cdot\,, R_N)$ and all relations are transitive. We then are done by Theorem 3. Let $A \in \mathcal{F}(U)$.
Transitivity (and hence acyclicity) of R^A: Let $x\,R^A\,y$, $y\,R^A\,z$ for some pairwise distinct $x, y, z \in A$. Then there are $y_1, \dots, y_k$, $z_1, \dots, z_l \in A$ such that $x \succsim y_1 \succsim \cdots \succsim y_{k-1} \succsim y \succsim z_1 \succsim \cdots \succsim z_{l-1} \succsim z$. Hence $x\,R^A\,z$.

Completeness of R^A: It is clear that $x\,I^A\,x$. Let $y \neq x$ with $\neg(y\,R^A\,x)$. Then it has to be that $x \succ y$. Hence $x\,R^A\,y$.
Choosing the maximal elements: By definition.
Upwards inheritance of R^B: Let $B \subseteq A$ be feasible sets such that $x\,R^B\,y$. Then there exist the required $y_1, \ldots, y_k \in B \subseteq A$ and hence $x\,R^A\,y$.

□

9.2 Uncovered Set Rule

Definition 42 (Uncovered Set as Social Choice Function)**.** Let $R_N \in \mathcal{R}(U)$ be given. Then we define the strict part of the covering relation on $A \in \mathcal{F}(U)$ as follows for $x \neq y$.

$$\begin{aligned} x\,C^A\,y :\iff \quad & x \succ y \text{ and} \\ & \forall z \in A: \quad y \succ z \implies x \succ z \text{ and} \\ & \forall z \in A: \quad y \succsim z \implies x \succsim z \end{aligned}$$

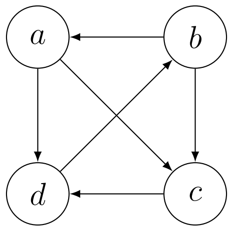
For asymmetry, let $x\,C^A\,y$. Then it has to be that $x \succ y$. Since $M_{y,y} = 0$, we have $\neg(y\,C^A\,x)$. For acyclicity, we show that C^A is transitive. Let $x\,C^A\,y$, $y\,C^A\,z$. Due to $y\,C^A\,z$, it has to be that $y \succ z$. Hence $x \succ z$. Let $w \in A$, such that $z \succ w$. Then $y \succ w$, hence $x \succ w$. If $z \succsim w$, we can use the same argumentation. Hence $x\,C^A\,z$.
We then define R^A as in Lemma 23. Finally, set

$$UC(A, R_N) := \max_{R^A} A$$

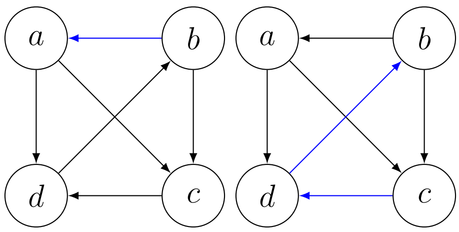
Remark 8. The Uncovered Set is by definition majoritarian. There are several ways to define the Uncovered Set when the majority relation contains ties. The one above was used by McKelvey (1986) and is hence known as the McKelvey Uncovered Set. For an analysis of several versions (including deep, Gilles, Bordes and McKelvey), see Duggan (2013). However, when the majority relation is antisymmetric, there is no ambiguity and all definitions coincide.

Proposition 14. *UC satisfies γ and Aizerman, but not β^+.*

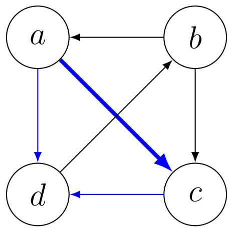
Proof. First, fix an arbitrary $R_N \in \mathcal{R}(U)$. We now show that $UC(\,\cdot\,, R_N)$ is upwards rationalized by (R^A) as defined above.
Acyclicity and completeness of R^A: By definition.
Choosing the maximal elements: By definition.
Upwards inheritance of R^B: By definition C^A satisfies downwards inheritance, which is equivalent (see Lemma 2).
Quasi-transitivity of R^A: Already shown in the definition.
We now show that $R^A_C = R^A$ for all feasible sets A. From Proposition 4 we know that $R^A_{UC} \subseteq R^A$. If, on the other hand, $x\,R^A\,y$, then $x \succsim y$ or there is some $z \in A$ such that $x \in UC(\{x, y, z\}) \implies x\,R^A_{UC}\,y$. If $x \succsim y$, then $x \in UC(\{x, y\})$, hence $x\,R^A_{UC}\,y$.
To show that R^A_{UC} is not always transitive, we look at the following graph:



Since $b \succ a$, we have that $b\, R^A\, a$. Since $c \succ d$ but $\neg(b \succ d)$, we have that $c\, R^A\, b$.



If R^A was transitive, we would have $c\, R^A\, a$. This is not the case, since $a\, C^A\, c$.



□

9.3 Split Cycle Rule

Definition 43 (Split Cycle as Social Choice Function)**.** Let $R_N \in \mathcal{R}(U)$ be given. We define the strict part of the splitting relation as follows.

$$x\, S^A\, y :\Longleftrightarrow x \succ y \,\wedge\, m_R(x,y) \text{ is not minimal in any cycle.}$$

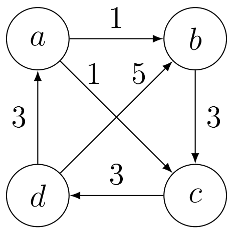
The latter means that for all cycles $(y = y_0, \ldots, x = y_k)$ in A, there is some $i < k$ such that $m_R(y_i, y_{i+1}) < m_R(x,y)$.
For asymmetry, let $x\, S^A\, y$. Then especially $x \succ y$, hence $\neg(y\, S^A\, x)$. For acyclicity, assume there is some cycle $x_0\, S^A\, x_1\, S^A\, \ldots\, S^A\, x_k\, S^A\, x_0$. Then, especially $x_0 \succ x_1 \succ \cdots \succ x_k \succ x_0$. But then one of the majority margins has to be minimal in this cycle, hence one of the edges cannot have been splitting, a wanted contradiction. We can now apply Lemma 23 and set

$$SC(A, R_N) \coloneqq \max_{R^A} A$$

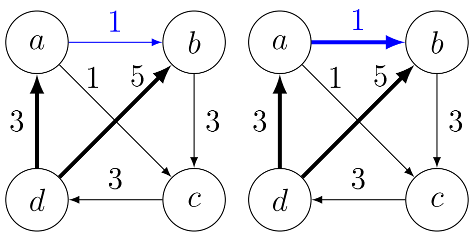
Remark 9. By definition, SC is pairwise.

Notably, the splitting relation R^A does not have to coincide with the local revealed preference relation. We give an example of size 4 with $A = \{a,b,c,d\}$.

Example 20 (Splitting relation can be larger than the local revealed preference relation).



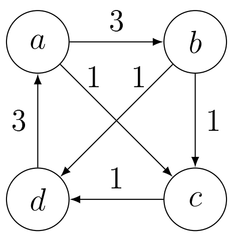
Here, the edge $a \succ b$ is not splitting. Still we have $a\, P^A_{SC}\, b$, since b is not chosen in any subset of A which contains a. We visualize the splitting relation on the left hand side and the local revealed preference relation on the right hand side:



△

Proposition 15. *The Split Cycle satisfies γ, but not Aizerman (and hence not β^+).*

Proof. Acyclicity and completeness of R^A: By definition.
Choosing the maximal elements: By definition.
Upwards inheritance of R^A: Let $B \subseteq A$, such that $x\, R^B\, y$. Then, either $x \succ y$ or there is some cycle $(x, y_1, \ldots, y_k = y)$ in B, such that $m_R(y, x) \leq m_R(y_i, y_{i+1})$ for all $i < k$. Since R_N is not dependent on the feasible set and $(x, y_1, \ldots, y_k = y)$ is a cycle in A, we have that $x\, R^A\, y$.
To show that SC does not satisfy Aizerman, we check that the local revealed preference relation is not always quasi-transitive.



We see that d splits a and a splits b. Hence it has to be that $d\, P^A_{SC}\, a$ and $a\, P^A_{SC}\, b$. Nonetheless, we see that $b \in SC(\{b, d\})$, hence $b\, R^A_{SC}\, d$. □

9.4 A General Observation

Now that we formally analyzed the three functions, we ask ourselves a more general question. How can we construct reasonable, pairwise social choice functions which satisfy γ? Our result is that we can obtain all such functions by choosing strict majority wins in an acyclic, downwards inheriting manner.

We demand that if the feasible set consists of only two alternatives, the winner of the duel should not lose the choice. In the context of voting, picture an election where the people decide between two alternatives. If one alternative wins the majority vote, it might be the optimal choice for the group as a whole. Maybe, the majority victory was not decisive enough and both alternatives tie. But what should not happen is that only the majority defeated alternative wins the election.

Definition 44 (Faithful Social Choice Functions)**.** We say that a social choice function S is *faithful*, if the following holds: Let R_N be given and $x, y \in U$ with $x \succ y$. Then $\{y\} \neq S(\{x,y\}, R_N)$.

Faithfulness is a simple and broadly accepted notion. It can also be seen as a non-probabilistic variant of a definition by Fishburn and Gehrlein (1977). In their setting, they allow for probability distributions over the preferences, as well as the outcomes. What both definitions have in common is that the function should be faithful to the favored alternative. As a sanity check, by definition TC, UC, SC all are faithful. We see that all strict edges of their relations are also majority edges. This has to be the case for all such functions, as we will see now.

Proposition 16. *Let S be a pairwise*[9] *social choice function satisfying γ. Let $R_N \in \mathcal{R}(U)$ and let $(R^A)_A$ be any family upwards rationalizing $S(\,\cdot\,, R_N)$. Then the following are equivalent:*

(i) If $x\, P^A\, y$ for any $x, y \in U$, $A \in \mathcal{F}(A)$, then $x \succ y$

(ii) S is faithful

Proof. First, let S be faithful. Let $x\, P^A\, y$. Then it has to be that $x\, P^{\{x,y\}}\, y$, hence $\{x\} = S(\{x,y\}, R_N)$. Since S is faithful, it cannot be that $y \succ x$. Since S is pairwise, it further cannot be that $x \sim y$. Hence $x \succ y$.
Now, let S not be faithful. Then there have to be $x \neq y$, such that $\{x\} = S(\{x,y\}, M)$, but $y \succ x$. It has to be that $x\, P^{\{x,y\}}\, y$, hence (i) is violated for $A = \{x,y\}$. □

Summarized, faithfulness is a weak, but desirable property. To construct faithful social choice functions satisfying γ, it suffices to focus on choosing acyclic subsets of the majority wins, such that they inherit downwards.

[9] For the result, it suffices that S always chooses the majority winner on two alternatives. Else, we only obtain that $x \succsim y$.

10 A Characterization of the Split Cycle Rule

Holliday and Pacuit (2020) have characterized the Split Cycle using 6 axioms. Remarkably, their setting too deals with variable relations. By virtue of one of their axioms, we obtain upwards inheriting relations. Hence, we modify their axioms and involve γ to present a characterization with the same proof idea.

Theorem 12. *The Split Cycle is the finest social choice function S satisfying all of the following properties:*

A1 Pairwiseness

A2 Doubling homogeneity: $x \notin S(A, R_N) \implies x \notin S(A, 2R_N)$

A3 γ

A4 Crucial defeat for pairwise social choice functions: If $x \notin S(A, M)$, then there is some majority edge $y \succ x$ in A such that if we strictly lower the majority margin of any other edges (or set them to zero) , then $x \notin S(B, M^*)$ for the changed majority margin matrix M^* and all feasible subsets $B \subseteq A$ where $y, x \in B$.[10]

Proof. One can easily verify that SC satisfies axioms $A1$ to $A3$. For $A4$, fix any majority margin matrix M, let A feasible, and let $x \in A \setminus SC(A, M)$. This means, that there is some $y \in A$ with $y\, S^A\, x$ with respect to M. By definition of the splitting relation, the margin $m(y, x)$ is not minimal in any cycle containing the edge $y \succ x$ with respect to M. Hence, after lowering or nulling any other edges resulting in the majority margin M^*, it will still not be minimal in any cycle containing itself and we obtain $y\, S^A\, x$ with respect to M^*. Hence, it especially holds that $x \notin SC(A, M^*)$. Further, since the strict part of the splitting relation inherits downwards, it has to be that $x \notin SC(B, M^*)$ for all feasible $B \subseteq A$ containing y.
Now let S be a social choice function satisfying axioms $A1$ to $A4$. Assume that there is some majority margin matrix M, some feasible set A and some $x \in A$ such that $x \in SC(A, M)$ but $x \notin S(A, M)$. By doubling homogeneity, we have $x \notin S(A, 2M)$. Let $y \succ x$ be the crucial defeat with respect to $2M$, which needs to exist by $A4$. By assumption it has to be that $\neg(y\, S^A\, x)$. Hence there must be some cycle $(y, x, x_2, \ldots, x_k)$ in A where the edge $y \succ x$ has minimal weight with respect to $2M$. Set $B := \{y, x, x_2, \ldots, x_k\} \subseteq A$. Now, we modify the graph induced by $2M$ to arrive at some M^* as follows: We lower all edges of the cycle $(y, x, x_2 \ldots x_k, y)$ until they have weight $m(y, x)$. All other edges are nullified. M^* is still a majority graph, since all edges have even weight. Since $y \succ x$ is a crucial defeat with respect to $2M$, it has to be that $x \notin S(B, M^*)$. By pairwiseness and symmetry of the alternatives, it must now be that $z \notin S(B, M^*)$ for all $z \in B$. This is a wanted contradiction to non-emptiness of S. □

To get a better feeling for the axioms used in this characterization, we compare our version to a setting more similar to the original characterization.

[10]We can formulate a non-pairwise version of A4, similar to B5. This is omitted, since the resulting axiom is more technical while arguably adding no value to the result.

Definition 45 (VCCR)**.** Let U be non-empty and countable. A *variable-election collective choice rule* (VCCR) maps each combination of feasible set and preference profile to a complete relation on the feasible set. Further, it is only dependent on the individual rankings restricted to the feasible set. More formally, let A be feasible. By $R^*(A)$ we denote the set of all complete relations on A. Set $R^* := \cup_A R^*(A)$. $V : \mathcal{F}(U) \times \mathcal{R}(U) \to R^*$ is called a VCCR, if $V(A, R_N) \in R^*(A)$ for all $A \in \mathcal{F}(U), R_N \in \mathcal{R}(U)$.[11] Further, we demand that $V(A, R_N) = V(A, R'_N)$ whenever R_N restricted to A is equal to R'_N restricted to A.[12]

Note that by the above definition, cyclic relations (and hence possibly empty choice sets) are not forbidden.

Definition 46 (Availability)**.** Let V be a VCCR. Then it satisfies *availability* if and only if $\max_{V(A,R_N)} A \neq \emptyset$ for all A and R_N.

We omit analogous definitions of neutrality and anonymity for VCCRs.

Theorem 13 (Holliday and Pacuit, 2020)**.** *The Split Cycle is the finest VCCR satisfying all of the following properties:*

B1 Anonymity and Neutrality

B2 Monotonicity$_2$: If x defeats y with respect to $(\{x, y\}, R_N)$ and R'_N is equal to R_N, except for one voter who only ranks x one spot higher than before, then x still defeats y with respect to $(\{x, y\}, R'_N)$.

B3 Doubling homogeneity: x defeats y with respect to $(A, R_N) \implies x$ defeats y with respect to $(A, 2R_N)$

B4 Neutral Reversal: If R_L is obtained from R_N by adding two voters with reversed preference relations, then $V(A, R_N) = V(A, R_L)$ for all feasible A.

B5 Coherent IIA: If x defeats y with respect to (A, R_N) and R^*_N is any preference profile such that $R^*_N|_{\{x,y\}} = R_N|_{\{x,y\}}$ and the majority margin matrix $M_{R^*_N}$ is obtained by nulling or weakening (zero or more) edges other than the one between x and y, then x defeats y with respect to (B, R^*_N) for all feasible $B \subseteq A$.

B6 Availability

As discussed before, $B6$ is covered by non-emptiness of social choice functions. Further, with $B5$ together it implies that $V(A, R_N)$ always has to be acyclic.
$B2$ is needed to give the collective choice a "direction". $B2$ in conjunction with $B1$ guarantees, that the majority winner is always chosen in sets of cardinality two. $A4$ guarantees the same, since the crucial defeat has to be a strict majority defeat.
$A2$ is analogously defined to $B3$. Both are only needed, so that we are allowed to delete

[11]In the original paper, the asymmetric, strict part is used instead of the complete, weak part of the relation for semantic reasons. Formally, both definitions are equivalent.

[12]In the original paper, VCCRs are defined for inputs of the form $R_N \in \mathcal{R}(A)$ for any feasible set A, not only $A = U$. Our version is equivalent and closer to our previous definitions, since we additionally specify the feasible set A.

all edges within a cycle.
Coherent IIA is intertwined with many different properties. $B5$ in conjunction with anonymity implies, that V for fixed N is only dependent on the majority margin matrix. By adding neutral reversal, it follows that when an alternative is not chosen for some R_N, then it is also not chosen for any R_L with the same majority margin matrix and $|L| \geq |N|$. Technically speaking, this combined with neutrality only implies one half of pairwiseness. To imply pairwiseness, we would need $B3$ to be an if and only if statement. Further, coherent IIA together with the definition of VCCRs implies upwards rationalizability: In $B5$ we can insert $R_N^* := R_N$ and use that we are allowed to lower zero majority margins. Hence, we obtain that the strict part of the relation has to inherit downwards. This observation motivates our version of the characterization and leads to $A3$. Finally, coherent IIA and monotonicity$_2$ imply a technical, non-pairwise version of crucial defeat, in which we specify the preference profiles instead of only working with matrices.

11 Generalization to Probabilistic Choice Functions

Our goal in this section is to reproduce our characterization of γ for the case where choice is made not only over alternatives, but also includes probability distributions over them. Naturally, one can identify an alternative a with the probability distribution p, for which $p(a) = 1$. We use the definitions of Brandl and Brandt (2020).

11.1 The Probabilistic Setting of Brandl and Brandt (2020)

11.1.1 Probabilistic Choice Functions

Definition 47 (Feasible Sets, Probability Distributions)**.** For this section, we demand that U is finite and non-empty. By Δ_U, or in short Δ, we denote the set of all probability distributions over U, which is the set of all $p \in \mathbb{R}^U_{\geq 0}$ with $\sum_{u\in U} p(u) = 1$. $\mathcal{F}(U)$ denotes the set of all non-empty, closed (with respect to any norm, say the Euclidean) and convex subsets of Δ. These are called *feasible* sets.

Definition 48 (Probabilistic Choice Functions)**.** A map $C : \mathcal{F}(U) \to \mathcal{F}(U)$ is called a *(probabilistic) choice function*, if it satisfies the following properties:

- C maps each feasible set X to a feasible subset of X
- C is (upper-hemi) continuous
- $C([p, q]) \in \{\{p\}, \{q\}, [p, q]\}$ for all $p, q \in \Delta$

where $[p, q] \coloneqq \mathrm{conv}(\{p, q\})$

Semantically speaking, one is offered to choose the best probability distributions among a closed, convex and non-empty subset. We assume that there can be ties, but that the chosen set is always convex (and non-empty).

Definition 49 (Probabilistic α)**.** Let C be a choice function. We say that C satisfies α, if for all feasible X, Y with $X \cap Y \neq \emptyset$

$$C(X) \cap Y \subseteq C(X \cap Y)$$

Definition 50 (Probabilistic γ)**.** Let C be a choice function. We say that C satisfies γ, if for all feasible X, Y

$$C(X) \cap C(Y) \subseteq C(\mathrm{conv}(X \cup Y))$$

11.1.2 Rationalizable Choice

Definition 51 (Continuity, Convexity)**.** Let $R \subseteq \Delta \times \Delta$ be a complete relation. By P we denote its strict part: $p\,P\,q :\iff \neg(q\,R\,p)$. By I we denote its symmetric part. For $p \in \Delta$, we denote its *lower contour* set by $L(p) \coloneqq \{q \in \Delta |\, p\,P\,q\}$. Similarly, $U(p) \coloneqq \{q \in \Delta |\, q\,P\,p\}$ and $I(p) \coloneqq \{q \in \Delta |\, p\,I\,q\}$. We say that R satisfies *continuity*, if for all $p \in \Delta$:

$$U(p), L(p) \text{ are open}$$

We say that R satisfies convexity, if for all $p \in \Delta$:

$$U(p), L(p), U(p) \cup I(p), L(p) \cup I(p) \text{ are convex}$$

Also, for feasible sets X define

$$\max_R X := \{p \in X | \, p \, R \, y \text{ for all } y \in X\}$$

We say that a choice function C is *rationalized by* a complete relation R, if $C(X) = \max_R X$ for all feasible sets X.

Remark 10. There are relations R, for which $\max_R X$ can be the empty set. However, this cannot be the case if R is continuous and convex (Sonnenschein, 1971). If a relation R is convex, then $I(p)$ will always be convex, since $I(p) = (L(p) \cup I(p)) \cap (U(p) \cup I(p))$. There are even better reasons as why to demand these axioms, as one can see in the next result.

Proposition 17 (Brandl and Brandt, 2020)**.** *A probabilistic choice function is rationalizable by a complete, continuous and convex relation R if and only if it satisfies α and γ.*

11.2 Probabilistic Upwards Rationalizability

Definition 52 (Probabilistic Upwards Rationalizability)**.** Let C be a choice function. We say that C is *upwards rationalizable*, if there is a family of relations $(R^X)_X$ such that the following conditions hold :

(i) for all feasible sets X, $R^X \subseteq X \times X$ is complete

(ii) for all feasible sets X:

$$C(X) = \max_{R^X} X$$

(iii) for all feasible sets X, Y, such that $Y \subseteq X$:

$$R^Y \subseteq R^X$$

(iv) Let X, Y feasible and $x \in X \cap Y$. If x is dominated in $\text{conv}(X \cup Y)$ by some z regarding $R^{\text{conv}(X \cup Y)}$, then there has to be some y in $X \cup Y$ which dominates x.

In this case we say that C is *upwards rationalized by* $(R^X)_X$.

Remark 11. The first three properties are analogous to the classical case, only dropping acyclicity. This is necessary, as Steinhaus and Trybula (1959) showed that preferences over lotteries can be complete, convex, continuous and cyclic at the same time. More precisely, they showed that preferences over lotteries can be cyclic, even when the preferences over degenerate lotteries are transitive. Also note that the family of relations now is uncountable, while in previous sections it was finite or countable.

Definition 53 (Probabilistic Local Revealed Preference)**.** For each feasible set X, $x, y \in X$, we write $x\, R_C^X\, y$ if and only if there is a feasible subset $Y \subseteq X$ such that $x \in C(Y)$ and $y \in Y$.

Theorem 14. *Let C be a probabilistic choice function. Then it satisfies γ if and only if it is upwards rationalizable.*

Proof. Let C satisfy γ. For this implication, we will use the shorthand $R^X := R_C^X$. We need to verify that all four properties of Definition 52 are satisfied. Let us start with completeness. Let X be a feasible set, $x, y \in X$ such that $\neg(xR^X y)$. Then $y \notin C([x, y])$ By Definition 48, it has to be that $\{x\} = C(\{x, y\})$, hence $xR^C y$. Next, we show that (ii) holds true. Let $x \in C(X)$. Then, for each $y \in X$, we can choose X as the witness: X is feasible, $y \in X$ and by assumption $x \in C(X)$. We conclude $xR^X y$ for all y, and hence x is maximal in X. On the other hand, let $x \in X$ be maximal regarding R^X. Fix any sequence of $(y_i)_i$ in X with the property that $\text{conv}(\{x, y_1, \ldots, y_k\}) \to X$. Then, by Definition 53 and maximality of x, for each i there is a feasible set with $y_i \in Y_i \subseteq X$, such that $x \in C(Y_i)$. Obviously $\text{conv}(\{x, y_1, \ldots, y_k\}) \subseteq \text{conv}(\cup_{i \le k} Y_i) \to X$. Also, by induction and γ it follows that $x \in C(\text{conv}(\cup_{i \le k} Y_i))$ for all k. By upper-hemi continuity of C we have that $x \in C(X)$. Upwards inheritance is straight forward. Let $Y \subseteq X$ be feasible sets, let $x\, R^Y\, y$. There is some witness $Z \subseteq Y$, for which of course $Z \subseteq X$. Hence by Definition 53 $x\, R^X\, y$. For (iv), let $x \in X \cap Y$, $y \in Z = \text{conv}(X \cup Y)$ with $yP^Z x$. Assume for contradiction, that for all $y \in X \cup Y$ $x\, R^Z\, y$. Now, fix any sequence $(y_i)_i$ in $X \cup Y$ such that $\text{conv}(\{x, y_1, \ldots, y_k\}) \to Z$. By assumption we have that $x\, R^Z\, y_i$ for all $i \in \mathbb{N}$. Hence, for each i there is a feasible set with $y_i \in Y_i \subseteq Z$ such that $x \in C(Y_i)$. Using induction and γ, this implies $x \in C(\text{conv}(\cup_{i \le k} Y_i))$ for all k. Furthermore $\text{conv}(\{x, y_1, \ldots, y_k\}) \subseteq \text{conv}(\cup_{i \le k} Y_i) \to Z$ Hence by continuity $x \in C(Z)$, which is a contradiction to $y\, P^Z\, x$.
For the other implication, let $(R^X)_X$ now denote any family of relations which upwards rationalizes C as in Definition 52. Let X, Y be feasible sets, $x \in C(X) \cap C(Y)$. This directly implies that $x\, R^X\, y$ for all $y \in X$ and $x\, R^Y\, y$ for all $y \in Y$. In both cases we can use property (iii) of Definition 52 and obtain that $x\, R^{\text{conv}(X \cup Y)}\, y$ for all $y \in X \cup Y$. Now, assume for contradiction that $z\, P^Z\, x$ for some $z \in Z = \text{conv}(X \cup Y)$. By (iv), there has to be a $y \in X \cup Y$ with $y\, P^Z\, x$. This is impossible, hence x is maximal regarding R^Z, which implies $x \in C(Z)$. □

Remark 12. It is open whether we can completely match Proposition 17. The local revealed preference relations are not always continuous or convex, but there could exist another family of relations which is. For a violation of convexity, consider an expected utility-maximizer with $u(a) > u(b) > u(c)$. On all subsets of Δ, we choose with respect to utility maximization, while on Δ we choose $[a, \frac{1}{2}(b + c)]$.

While it seems like we cannot guarantee continuity of the local revealed preference relations, we can guarantee the following.

Lemma 24. *Let C satisfy γ. Then for all feasible X, $p \in X$, $U(p)$ is open with respect to R_C^X.*

Proof. Let C satisfy γ, let X be a feasible set and $x \in X$. For this proof, again we abbreviate $R^X := R_C^X$. Assume that $U(x)$ is not open. Then, there is some $y \in X$ with

$y\,P^X\,x$ and a sequence $(y_i)_i$ in X such that $y_i \to y$ and $x\,R^X\,y_i$ for all i. But then for all i there are feasible subsets $Y_i \ni y_i$ such that $x \in C(Y_i)$. We set $X_n \coloneqq \mathrm{conv}(\cup_{i \le n} Y_i)$. Then by induction and γ it follows that $x \in C(X_n)$ for all n. Further, since $X_n \subseteq X_{n+1}$ and all are closed and convex, we have $X_n \to Z \coloneqq \overline{\cup_{n \in \mathbb{N}} X_n}$. Z is closed and convex, hence a feasible set. By continuity we have that $x \in C(Z)$. On the other hand also $y \in Z$, a contradiction to $y\,P^X\,x$ and $Z \subseteq X$. $\square$

12 Conclusion and Further Research

We arrive at the end of this thesis. Here, we summarize our results, as well as state open problems, which can be tackled in the future.

12.1 Conclusion

Arrovian impossibilities showed that for collective choice, rationalizability conflicts with reasonable notions of independence, fairness and efficiency. To retain the latter, we must violate rationalizability and especially α, leaving only expansion consistency conditions possible. For designing good collective choice functions, it is hence of importance to understand conditions such as γ, Aizerman and β^+.
We introduce the new notion of upwards rationalizability, which is based on well-known solution concepts such as TC and UC. Using it, we characterize the three previously mentioned expansion consistency conditions. These characterizations are then used to present proofs of classical characterizations. Further, we present a new expansion consistency condition named γ^+ and characterize it, as well as compare it to a condition introduced by Schwartz (1976). The latter is slightly weaker by restricting itself to disjoint feasible sets. We then define the notion of downwards rationalizability and use it to characterize α. After these main results, we define further expansion consistency conditions, analyze three social choice functions and examine probabilistic choice theory.

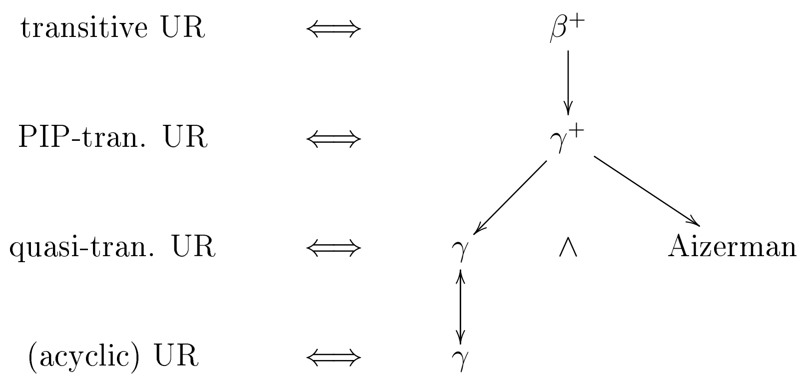


Figure 1: Our main characterizations of expansion consistency conditions.

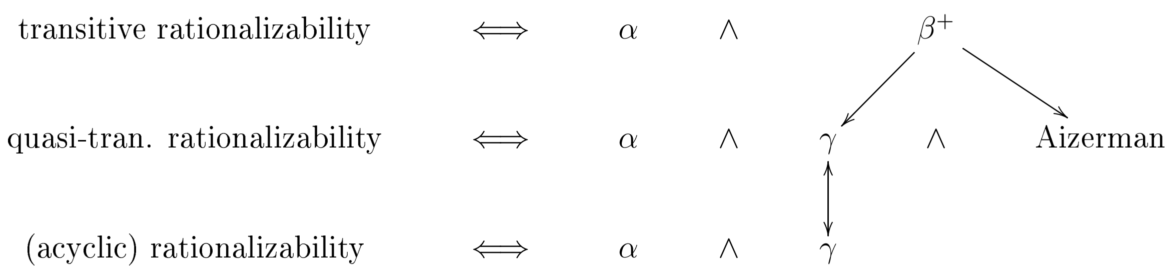


Figure 2: Three classical characterizations, which we prove using our own results.

12.2 Further Research

While the most important questions have been answered, new and interesting open problems have emerged.
Between transitivity and PIP-transitivity, there exists another notion named PIP+IPP-transitivity. We recall from Section 5, that PIP-transitive rationalizability is equivalent to representation by a utility function u and a non-negative discriminatory function δ. Additionally, PIP+IPP-transitive rationalizability demands that δ can be chosen as a constant. Schwartz (1976) characterized it using α and a consistency condition which he named $W3$. The goal now is to characterize PIP+IPP-transitive upwards rationalizability using expansion consistency conditions. $W3$ of course is a candidate for this task, but neither of the two implications were possible for the author. It seems like a new expansion consistency condition, stronger than γ^+, is needed.
For the characterization of the Split Cycle presented in Section 10, all axioms but "crucial defeat" are undisputed and used throughout the literature of social choice theory. On the other hand, "crucial defeat" itself is closely linked to upwards rationalizability and hence might be unintuitive to some readers. Is there a way to replace this axiom by more intuitive ones?
The reader might have noticed that our definition of probabilistic choice functions in Section 11 contains some technical assumptions. Such axioms usually only exist because they are needed somewhere in the proofs. Hence it would be appealing to weaken them if possible. In Definition 48, the "line axiom" is defined as follows: $C([x, y]) \in \{\{x\}, \{y\}, [x, y]\}$ for all $x, y \in \Delta$. We propose a weakening: $C([x, y]) \cap \{x, y\} \neq \emptyset$ for all $x, y \in \Delta$. Can the existing results of Brandl and Brandt (2020) be reproduced with this weakening? Further, it is unclear whether there always exists a family of continuous and convex relations when probabilistic γ is satisfied. One more unanswered question is whether other characterizations can be reproduced in the probabilistic setting. In first naive attempts, finding proofs for characterizing γ and Aizerman, as well as β^+, seemed to be complicated. It might not be possible at all.
We characterized transitive downwards rationalizability using completeness and acyclicity of the competing relation in Section 6. While completeness is equivalent to WWARP, acyclicity of the competing relation seems to be of abstract nature. Can we characterize acyclicity of the competing relation using more intuitive properties?

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