Loyalty in Cardinal Hedonic Games

Martin Bullinger and **Stefan Kober**
Technical University of Munich
bullinge@in.tum.de, stefan.kober@tum.de

Abstract

A common theme of decision making in multi-agent systems is to assign utilities to alternatives, which individuals seek to maximize. This rationale is questionable in coalition formation where agents are affected by other members of their coalition. Based on the assumption that agents are benevolent towards other agents they like to form coalitions with, we propose loyalty in hedonic games, a binary relation dependent on agents' utilities. Given a hedonic game, we define a loyal variant where agents' utilities are defined by taking the minimum of their utility and the utilities of agents towards which they are loyal. This process can be iterated to obtain various degrees of loyalty, terminating in a locally egalitarian variant of the original game.

We investigate axioms of group stability and efficiency for different degrees of loyalty. Specifically, we consider the problem of finding coalition structures in the core and of computing best coalitions, obtaining both positive and intractability results. In particular, the limit game possesses Pareto optimal coalition structures in the core.

1 Introduction

Decision making in multi-agent systems is highly driven by the idea of the *homo economicus*, a rational decision taker that seeks to maximize her individual well-being. Following the classical *Theory of Games and Economic Behavior* by von Neumann and Morgenstern, agents assign utilities to alternatives and aim for an outcome that maximizes individual utility. Such behavior entails many delicate situations in non-cooperative game theory such as the prisoner's dilemma or the tragedy of the commons [Hardin, 1968], where agents take decisions in their individual interest without regarding other agents. This leads to outcomes that are bad for the society as a whole and often, as it is the case in the prisoner's dilemma, agents have an incentive to coordinate to improve their respective situation.

From the theoretical point of view, one can either accept the existence of such dilemmata and study their social impact, for instance, by means of the price of anarchy [Koutsoupias and Papadimitriou, 1999], or one can ask for the degree of individual dependency on the social outcome necessary to escape a situation of inferior welfare. The latter idea is implemented by adapting the utility function of players as a weighted sum of individual and joint utility, an idea repeatedly developed in network design [Elias *et al.*, 2010], artificial intelligence [Apt and Schäfer, 2014], or public choice [Mueller, 1986]. Specifically, the *selfishness level* by Apt and Schäfer is the lowest weight on the joint utility such that a Nash equilibrium becomes a social optimum.

On the other hand, empirical evidence does not only question whether agents behave according to the utility model by von Neumann and Morgenstern [Kahneman and Tversky, 1979], but even supports the hypothesis that human behavior is steered by the well-being of the whole society [Colman *et al.*, 2008]. However, in scenarios of high competition, agents might also act spiteful towards other agents, i.e., there is an incentive to harm other agents [Levine, 1998].

In cooperative game theory, it seems to be an even more reasonable assumption to include other agents into the own valuation. We follow this line of thought in the setting of coalition formation, where we propose loyalty, a possibility to modify utilities by taking into account other agents' utilities towards which loyalty is perceived. Loyalty is a binary relation directly extracted from the agents' utilities over partnership, i.e., coalitions of size 2. Loyalty is sensed towards the agents within the own coalition that yield positive utility in a partnership. Following the paradigm of a chain that is only as strong as its weakest link, loyal utilities are obtained by taking the minimum of the own utility and the utilities of agents receiving our loyalty. As such, we obtain a loyal variant of the original game, which is itself a coalition formation game, and we can iterate towards various degrees of loyalty. As we will see, this process terminates in a game which satisfies a high degree of egalitarianism. We consider common solution concepts concerning group stability and efficiency for different degrees of loyalty and the limit game, and provide both existential and computational results.

We study coalition formation in the framework of hedonic games [Drèze and Greenberg, 1980; Banerjee *et al.*, 2001; Bogomolnaia and Jackson, 2002]. Our contribution lies in studying aspects of empathy in hedonic games [Brânzei and Larson, 2011; Monaco *et al.*, 2018; Nguyen *et al.*, 2016]. Previous work considers empathy between agents through various alternative utility functions based on friendship relations among the agents extracted from utility functions or a social network. Closest to our work are altruistic hedonic games introduced by Nguyen *et al.* [2016] and subsequently studied by Wiechers and Rothe [2020], Kerkmann and Rothe [2020], and Schlueter and Goldsmith [2020]. Our first degree of loyalty in symmetric friend-oriented hedonic games coincides with minimum-equal-treatment altruistic hedonic games as defined by Wiechers and Rothe [2020]. We significantly extend their model, but since most of our hardness results work for the restricted class of symmetric friend-oriented hedonic games, they have immediate consequences for this type of altruistic hedonic games. Also, loyal variants of hedonic games fit into the framework of super altruistic hedonic games by Schlueter and Goldsmith [2020] if their aggregation is modified by taking the average instead of the minimum of other agents' utilities.

2 Preliminaries and Model

We start with some notation. Define $[i] = \{1, \dots, i\}$ and $[i, j] = \{i, \dots, j\}$ for $i, j \in \mathbb{Z}$, $i \leq j$.

Also, we use standard notions from graph theory. Let $G = (V, E)$ be an undirected graph. For a subset of agents $W \subseteq V$, denote by $G[W]$ the subgraph of G *induced by* W. Given two vertices $v, w \in V$, we denote by $d_G(v, w)$ their *distance* in G, i.e., the length of a shortest path connecting them. The graph G is called *regular* if there exists a non-negative integer r such that every vertex of G has degree r.

In the following subsections, we introduce hedonic games, our concept of loyalty, and desirable properties of coalition structures.

2.1 Cardinal Hedonic Games

Let $N = \{1, \dots, n\}$ be a finite set of agents. A *coalition* is a non-empty subset of N. By $\mathcal{N}_i$ we denote the set of coalitions agent i belongs to, i.e., $\mathcal{N}_i = \{S \subseteq N : i \in S\}$. A *coalition structure*, or simply a *partition*, is a partition π of the agents N into disjoint coalitions, where $\pi(i)$ denotes the coalition agent i belongs to. A *hedonic game* is a pair $(N, \succsim)$, where $\succsim = (\succsim_i)_{i \in N}$ is a preference profile specifying the preferences of each agent i as a complete and transitive preference relation $\succsim_i$ over $\mathcal{N}_i$. In hedonic games, agents are only concerned about their own coalition. Accordingly, preferences over coalitions naturally extend to preferences over partitions as follows: $\pi \succsim_i \pi'$ if and only if $\pi(i) \succsim_i \pi'(i)$.

Throughout the paper, we assume that rankings over the coalitions in $\mathcal{N}_i$ are given by utility functions $u_i : \mathcal{N}_i \to \mathbb{R}$, which are extended to evaluate partitions in the hedonic way by setting $u_i(\pi) = u_i(\pi(i))$. A hedonic game together with a representation by utility functions is called *cardinal hedonic game*. Because the sets $\mathcal{N}_i$ are finite, preferences could in principle always be represented by cardinal values. This is impractical due to two reasons. First, such utility functions require exponential space to represent. Therefore it would be desirable to consider classes of hedonic games with succinct representations. Second, we would like to compare different agents' utility functions such that a certain cardinal value expresses the same intensity of a preference for all agents. This cannot be guaranteed by arbitrary utility representations of ordinal preferences. Our model of loyalty is therefore particularly meaningful in succinctly representable classes of cardinal hedonic games. These include the following classes of hedonic games, which aggregate utility functions over single agents of the form $u_i : N \to \mathbb{R}$ where $u_i(i) = 0$, which can be represented by a complete weighted digraph.

- *Additively separable hedonic games (ASHGs)* [Bogomolnaia and Jackson, 2002]: utilities are aggregated by taking the sum of single utilities, i.e., $u_i(\pi) = \sum_{j \in \pi(i)} u_i(j)$.
- *Friend-oriented hedonic games (FOHGs)* [Dimitrov *et al.*, 2006]: the restriction of ASHGs where utilities for other agents are either n (the agent is a *friend*) or -1 (the agent is an *enemy*), i.e., for all $i, j \in N$ with $i \neq j$, $u_i(j) \in \{n, -1\}$. Given an FOHG, the set $F_i = \{j \in N : u_i(j) = n\}$ is called *friend set* of agent i. The unweighted digraph $G_F = (N, A)$ where $(i, j) \in A$ if and only if $j \in F_i$ is called *friendship graph*. An FOHG can be represented by specifying the friend set for every agent or by its friendship graph.
- *Modified fractional hedonic games (MFHGs)* [Olsen, 2012]: utilities are aggregated by dividing the sum of single utilities by the size of the coalition minus 1, i.e., $u_i(\pi) = 0$ if $\pi(i) = \{i\}$, and $u_i(\pi) = \frac{\sum_{j \in \pi(i)} u_i(j)}{|\pi(i)| - 1}$, otherwise. In other words, the utility of a coalition structure is the expected utility achieved through another agent in the own coalition selected uniformly at random.

A cardinal hedonic game is called *mutual* if, for all pairs of agents $i, j \in N$, $u_i(j) > 0$ implies $u_j(i) > 0$. It is called *symmetric* if, for all pairs of agents $i, j \in N$, $u_i(j) = u_j(i)$. Clearly, symmetric games are mutual. Throughout most of the paper, we will consider at least mutual variants of the classes of hedonic games, which we just introduced.

2.2 Loyalty in Hedonic Games

We are ready to define our concept of loyalty. Given a cardinal hedonic game, its loyal variant needs to specify two key features. First, for every agent, we need to identify a loyalty set, which contains the agents towards which loyalty is expressed. Second, we need to specify how loyalty is expressed, i.e., how to obtain new, loyal utility functions.

Formally, given a cardinal hedonic game and an agent $i \in N$, we define her *loyalty set* as $L_i = \{j \in N \setminus \{i\} : u_i(\{i, j\}) > 0\}$. In other words, agents are affected by agents that influence them positively when being in a joint coalition. Note that for all hedonic games considered in this paper, the loyalty set is equivalently given by $L_i = \{j \in N \setminus \{i\} : u_i(\{i, j\}) > u_i(i)\}$, i.e., it contains the agents with which i would rather form a coalition of size 2 than staying on her own. The *loyalty graph* is the directed graph $G_L = (N, A)$ where $(i, j) \in A$ if and only if $j \in L_i$.

It remains to specify how agents aggregate utilities in a loyal way. Given a cardinal hedonic game, its *loyal variant* is defined on agent set N by the utility function $u_i^L(\pi) = \min_{j \in \pi(i) \cap (L_i \cup \{i\})} u_j(\pi(i))$. Interestingly, the loyal variant is itself a hedonic game, and we can consider its own loyal variant. Following this reasoning, we recursively define the k*-fold loyal variant* by setting the 1-fold loyal variant to the loyal variant and the $(k+1)$-fold loyal variant to the loyal variant of the k-fold loyal variant. Also, we denote by u_i^k and L_i^k the utility function and the loyalty set of an agent i, and by G_L^k the loyalty graph of the k-fold loyal variant.

In fact, we will see that this process terminates after at most n steps in a limit game that satisfies egalitarianism at the level of coalitions. For simplicity, we restrict attention to mutual cardinal hedonic games, where the loyalty sets defines a symmetric binary relation and the loyalty graph can be represented by an undirected graph.[1] For an agent $i \in N$, let $G_L^\pi(i)$ be the agents in the connected component of the subgraph of G_L induced by $\pi(i)$ containing i. Now, define the *locally egalitarian variant* of a cardinal hedonic game as the game on agent set N with utilities given by $u_i^E(\pi) = \min_{j\in G_L^\pi(i)} u_j(\pi)$. In other words, an agent receives the minimum utility among all agents reachable within her coalition in the loyalty graph.

Finally, we introduce a technical assumption. A mutual cardinal hedonic game is called *loyalty-connected* if, for all agents $i \in N$ and coalition structures π, $u_i(G_L^\pi(i)) \geq u_i(\pi)$. This property precludes negative influence through agents outside the reach of loyalty, and is satisfied by reasonable classes of cardinal hedonic games like ASHGs, MFHGs, or fractional hedonic games [Aziz *et al.*, 2019].

2.3 Solution Concepts

We evaluate the quality of coalition structures by measures of stability and efficiency.

A common concept of group stability is the core. Given a coalition structure π, a coalition $C \subseteq N$ is *blocking* π (respectively, *weakly blocking* π) if for all agents $i \in C$, $u_i(C) > u_i(\pi)$ (respectively, for all agents $i \in C$, $u_i(C) \geq u_i(\pi)$, where the inequality is strict for some agent in C). A coalition structure π is in the *core* (respectively, *strict core*) if there exists no non-empty coalition blocking (respectively, weakly blocking) π.

A fundamental concept of efficiency is Pareto optimality. A coalition structure π' *Pareto dominates* a coalition structure π if, for all $i \in N$, $u_i(\pi'(i)) \geq u_i(\pi(i))$, where the inequality is strict for some agent in N. A coalition structure π is called *Pareto optimal* if it is not Pareto dominated by another coalition structure. In other words, given a Pareto optimal coalition structure, every other coalition structure that is better for some agent, is also worse for another agent.

Another concept of efficiency concerns the welfare of a coalition structure. There are many notions of welfare dependent on how to aggregate single agents' utilities for a social evaluation. In the context of loyalty, egalitarianism seems to be especially appropriate. It aims to maximize the well-being of the agent that is worst off. Formally, the *egalitarian welfare* of a partition π is defined as $\mathcal{E}(\pi) = \min_{i\in N} u_i(\pi(i))$. Also, let $\mathcal{E}^k(\pi)$ denote the egalitarian welfare of the k-fold loyal variant. Following this definition, coalition structures maximizing egalitarian welfare are not necessarily Pareto optimal. However, there exists always a Pareto optimal coalition structure maximizing egalitarian welfare. Specifically, a coalition structure maximizes *leximin welfare* if its utility vector, sorted in non-decreasing order, is lexicographically largest. A coalition structure maximizing leximin welfare is Pareto optimal and maximizes egalitarian welfare.

Apart from finding efficient coalition structures, an individual goal of an agent i is to be in a *best coalition*, i.e., in a coalition in $\mathcal{N}_i$ maximizing her utility. Formally, the problem of, given a cardinal hedonic game, an agent $i^* \in N$, and a rational number $q \in \mathbb{Q}$, deciding if there exists a subset $C \subseteq N$ with $i^* \in C$ and $u_{i^*}(C) \geq q$, is called `BestCoalition`.

3 Loyalty Propagation and Best Coalitions

Our first proposition collects some initial observations. It states, how loyalty propagates through the loyalty graph for higher degree loyal variants, terminating with the locally egalitarian variant, and considers egalitarian welfare.

Proposition 1. *Let a mutual cardinal hedonic game on agent set N with $|N| = n$ be given. Let $k \geq 1$, $i \in N$, and π a coalition structure. Then, the following statements hold.*

1. *The loyalty graph and loyalty sets are the same for all loyal variants, i.e., $G_L^k = G_L^1$ and $L_i^k = L_i^1$.*
2. *Loyalty extends to agents at distance k, i.e., $u_i^k(\pi) = \min\{u_j(\pi)\colon j \in \pi(i) \text{ with } d_{G_L[\pi(i)]}(i,j) \leq k\}$.*
3. *Utilities converge to the utilities of the locally egalitarian variant, i.e., $u_i^l = u_i^E$ for all $l \geq n$.*
4. *Egalitarian welfare is preserved, i.e., $\mathcal{E}^k(\pi) = \mathcal{E}(\pi)$.*

Proof. The first statements follow immediately from mutuality. We prove the second statement by induction over k. For $k = 1$, the assertion follows directly from the definition of the loyal variant.

Now, let $k \geq 2$ be an integer. Let $C = \pi(i)$, $C_L = \pi(i) \cap (L_i \cup \{i\})$, $H = G_L[\pi(i)]$, and for $p \geq 1$, let $C_p(j) = \{m \in C \text{ with } d_H(j,m) \leq p\}$. Then,

$$\begin{aligned}
u_i^k(\pi) &= \min_{j\in C_L} u_j^{k-1}(C)\\
&= \min_{j\in C_L} \min\{u_m(C)\colon m \in C_{k-1}(j)\}\\
&= \min_{j\in C\colon d_H(i,j)\leq 1} \min\{u_m(C)\colon m \in C_{k-1}(j)\}\\
&= \min\{u_j(C)\colon j \in C \text{ with } d_H(i,j) \leq k\}.
\end{aligned}$$

There, the second equality follows by induction, the third equality by definition of the loyalty graph, and the last equality by observing that the vertices with a distance of at most k from i are precisely the vertices with a distance of at most $k-1$ from an arbitrary neighbor.

The third statement follows from the second one, and the final statement follows from the observation that the minimum utility among agents in a coalition structure is preserved when transitioning to a loyal variant. □

Example 1. *We provide an example showing that part 4 of Proposition 1 does not extend to leximin welfare.*

[1] This restriction is in accordance with our results, but it can be lifted with some technical effort.

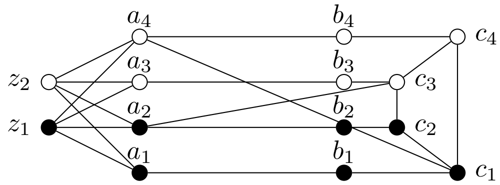


Figure 1: Friendship graph of Example 1. The black and white coalitions constitute a coalition structure minimizing leximin welfare for the 2-fold loyal variant, which is not Pareto optimal under the original utilities.

Consider a symmetric FOHG with agent set $N = \{a_i, b_i, c_i : 1 \leq i \leq 4\} \cup \{z_1, z_2\}$*, and the friendship graph in Figure 1. It can be shown that the coalition structure* $\pi = \{\{z_i, a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i}, c_{2i-1}, c_{2i}\} : i = 1, 2\}$ *maximizes leximin welfare for its 2-fold loyal variant (consider agents of type* b_i*). However,* π *is not even Pareto optimal under the original utilities. Indeed,* $\pi' = \{\{z_1, a_1, a_4, b_1, b_4, c_1, c_4\}, \{z_2, a_2, a_3, b_2, b_3, c_2, c_3\}\}$ *is a Pareto improvement. All agents receive at least the same utility, and* a_2, a_4, c_1*, and* c_3 *are better off.*

Our next goal is to reason about finding best coalitions for an agent. Note that this problem can usually be solved in polynomial time. For instance, in ASHGs, given an agent i, every coalition that contains i together with all agents that give positive utility to i and no agent that gives negative utility to i is a best coalition for i. By contrast, we obtain hardness results for loyalty even in symmetric FOHGs. While it is possible to determine the number of friends of the unhappiest friend in a best coalition in polynomial time [Wiechers and Rothe, 2020], the problem becomes hard if the number of enemies is to be minimized at the same time. We defer some proof details and proofs to the appendix.

Theorem 2. *Let* $k \geq 1$*. Then,* `BestCoalition` *is NP-complete for the* k*-fold loyal variant of symmetric FOHGs.*

Proof sketch. Membership in NP is clear. For hardness, we provide a reduction from the NP-complete problem `SetCover` [Karp, 1972]. An instance of `SetCover` consists of a triple (A, S, κ), where A is some finite ground set, $S \subseteq 2^A$ is a set of subsets of A, and κ is an integer. The instance (A, S, κ) is a Yes-instance if there exists $S' \subseteq S$ with $\bigcup_{B \in S'} B = A$ and $|S'| \leq \kappa$. The reduction is illustrated in Figure 2.

Let $k \in \mathbb{N}$. Define $M = \left\lfloor \frac{k-1}{2} \right\rfloor$. Given an instance (A, S, κ) of `SetCover`, define $a = |A|$. We define an instance $((N, (F_i)_{i \in N}), i^*, q)$ of `BestCoalition` based on an FOHG $(N, (F_i)_{i \in N})$ represented via friend sets by specifying each individual component. The agent set is defined as $N = \{w_i : i \in [0, a+2]\} \cup \{v_i : i \in [0, a-1]\} \cup \{\alpha_i^j, \beta_i^j : i \in [a], j \in [M]\} \cup A \cup S$, and consists of representatives of the elements of A and S, and auxiliary agents. If k is even, set $i^* = w_0$ and if k is odd, $i^* = v_0$. The friend sets are given as

- $F_{w_0} = \{w_1, v_0, \ldots, v_{a-1}\}$,
- $F_{w_1} = \{w_0, w_2, w_3, \ldots, w_{a+2}\}$,
- $F_{w_i} = \{w_j : j \in [a+2], j \neq i\}$ for $i \in [2, a+2]$,

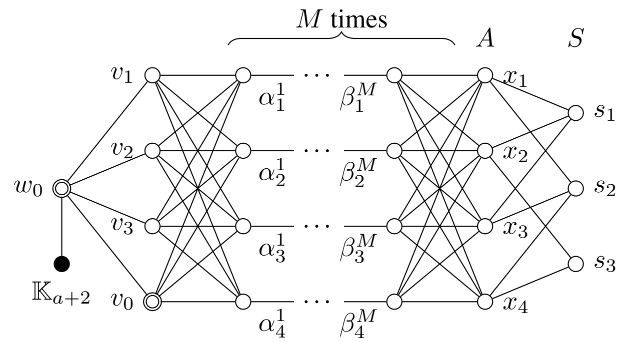


Figure 2: Schematic of the hardness reduction in Theorem 2 for $k \geq 2$. The figure shows the friendship graph for the instance of `SetCover` given by $A = \{x_1, x_2, x_3, x_4\}$ and $S = \{s_1 = \{x_1, x_2, x_3\}, s_2 = \{x_1, x_3, x_4\}, s_3 = \{x_2, x_4\}\}$. The black vertex indicates a complete subgraph on $a + 2$ vertices. We ask `BestCoalition` for the agents v_0 and w_0, respectively, indicated by double circles.

- $F_{v_i} = \{w_0, \alpha_1^1, \ldots, \alpha_a^1\}$ for $i \in [0, a-1]$ if $k > 2$,
- $F_{v_i} = \{w_0\} \cup A$ for $i \in [0, a-1]$ if $k \leq 2$,
- $F_{\alpha_i^1} = \{v_0, \ldots, v_{a-1}, \beta_i^1\}$ for $i \in [a]$,
- $F_{\alpha_i^j} = \{\beta_1^{j-1}, \ldots, \beta_a^{j-1}, \beta_i^j\}$ for $i \in [a], j \in [2, M]$,
- $F_{\beta_i^j} = \{\alpha_i^j, \alpha_1^{j+1}, \ldots, \alpha_a^{j+1}\}$ for $i \in [a], j \in [M-1]$,
- $F_{\beta_i^M} = \{\alpha_i^M\} \cup A$ for $i \in [a]$,
- $F_x = \{\beta_i^M : i \in [a]\} \cup \{s \in S : x \in s\}$ for $x \in A$ if $k > 2$,
- $F_x = \{v_0, \ldots, v_{a-1}\} \cup \{s \in S : x \in s\}$ for $x \in A$ if $k \leq 2$, and
- $F_s = \{x \in A : x \in s\}$ for $s \in S$ (in other words, $F_s = s$).

Finally, with $n = |N|$, specify the threshold utility $q = n(a+1) - (a+\kappa)$ for $k = 1$ and $q = n(a+1) - (1 + \kappa + 2(M+1)a)$, otherwise. Note that the distance between i^* and the x_i in the loyalty graph is exactly k.

If (A, S, κ) is a Yes-instance, let $S' \subseteq S$ be a set cover of A with at most κ sets. For $k = 1$, consider the coalition $C = A \cup S' \cup \{v_0, \ldots, v_{a-1}, w_0, w_1\}$. For $k \geq 2$, consider the coalition $C = (N \setminus S) \cup S'$. It is quickly checked that in each case $u_{v_0}^k(C) \geq q$.

Conversely, assume that C is a coalition with $i^* \in C$ and $u_{i^*}^k(C) \geq q$. Then, all agents that have a distance of at most k in the loyalty graph have to be included due to the degrees of vertices at a distance of at most k. In particular, $A \subseteq C$ for any k. Let $S' = C \cap S$.

First, consider the case $k = 1$. Then, $u_{v_0}(C) = n(a + 1) - a - |S'|$. Hence $u_{v_0}^1(C) \geq q$ implies that $|S'| \leq \kappa$. In addition, every agent $x \in A$ must have at least $a + 1$ friends present in C. In other words, for every $x \in A$ there exists $s \in S'$ with $x \in s$. Hence, S' is a cover of A with at most κ elements.

For arbitrary $k \geq 2$, it holds that $u_{i^*}(C) = n(a+1) - 1 - |S'| - (M+2)a$. Hence $u^1_{v_0}(C) \geq q$ implies that $|S'| \leq \kappa$. The remainder follows analogous to the case $k = 1$. □

Since the instances in the previous reduction contain agents with an arbitrarily large distance (parametrized by k), we cannot deduce direct consequences for the locally egalitarian variant. However, it is possible to bound the diameter in the reduced instances globally to obtain a similar result.

Theorem 3. *`BestCoalition` is NP-complete for the locally egalitarian variant of symmetric FOHGs.*

If we change the underlying class of hedonic games, we can circumvent the hardness results of the last two theorems.

Theorem 4. *Let $k \geq 1$. Then, `BestCoalition` can be solved in polynomial time for the k-fold loyal variant and the locally egalitarian variant of symmetric MFHGs.*

4 Coalition Structures in the Core

In this section we consider group stability in the locally egalitarian variant and the loyal variants.

4.1 Core in the Locally Egalitarian Variant

We start with a general lemma yielding a sufficient condition for existence of Pareto optimal coalition structures in the core.

Lemma 5. *Consider a class of hedonic games with the following two properties:*

1. *Restrictions of the game to subsets of agents are in the class.*
2. *For every coalition in any game of the class, the value of the coalition is the same for every player in the coalition.*

Then, for every game in the class, there exists a coalition structure in the core which is Pareto optimal.

Weakening the second condition of the lemma to the existence of some coalition that is best for all of its members is sufficient to find a coalition structure in the core. We discuss this in the appendix (Lemma 16, Theorem 17). Interestingly, the lemma can be applied to the locally egalitarian variant of cardinal hedonic games under fairly weak assumptions.

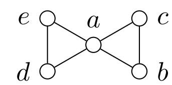
Theorem 6. *Let a loyalty-connected, mutual cardinal hedonic game be given. Then, there exists a Pareto optimal coalition structure in the core of its locally egalitarian variant.*

Proof. Let a loyalty-connected, mutual cardinal hedonic game be given and consider its locally egalitarian variant. We modify the utility functions such that $u^E_i(C)$ stays the same if C is connected in the loyalty graph, and set it to 0, otherwise. It suffices to find a Pareto optimal member of the core under this modification, because, by loyalty-connectivity, splitting coalitions into their connected components in the loyalty graph is weakly better for every agent, even under u^E. Consider the class of hedonic games given by this modified n-fold loyal variant together with all of its restrictions, in which we apply the same modifications towards the utility values for non-connected coalitions.

By Proposition 1, the utility for a coalition is the same for every player in the coalition. Hence, all requirements of Lemma 5 are satisfied and we find the desired coalition structure. □

Example 3 in the appendix shows the necessity of loyalty-connectivity in the previous theorem.

Example 2. *We extend an example by Wiechers and Rothe [2020] that shows that the previous result cannot be strengthened to find a coalition structure in the strict core. Consider the symmetric FOHG on agent set $\{a, b, c, d, e\}$ with loyalty graph depicted below.*



Consider its locally egalitarian variant. Then, $\{a, b, c\}$ is the unique best coalition for agents b and c and among the best coalitions for agent a. Hence, it has to be contained in every coalition structure in the strict core. Similarly, $\{a, d, e\}$ has to be a coalition in the strict core. As these conditions cannot be satisfied simultaneously, the strict core is empty.

Note that both the coalition structure $\{\{a, b, c\}, \{d, e\}\}$ and $\{\{a, d, e\}, \{b, c\}\}$ are in the core and Pareto optimal.

The construction in Lemma 5 gives rise to a simple recursive algorithm that computes Pareto optimal coalition structures in the core. Still, the computational complexity highly depends on the underlying cardinal hedonic game. While a modified version of the algorithm by Bullinger [2020] for computing Pareto optimal coalition structures in symmetric MFHGs finds a coalition structure in the core of their locally egalitarian variants, a version of our reduction on best coalitions shows an intractability for FOHGs.

Theorem 7. *The following statements hold.*

1. *Computing a coalition structure in the core can be done in polynomial time for the locally egalitarian variant of symmetric MFHGs.*
2. *Computing a coalition structure in the core is NP-hard for the locally egalitarian variant, even in the class of symmetric FOHGs with non-empty core.*

4.2 Core in the Loyal Variants

In contrast to the locally egalitarian variant, the k-fold loyal variant may have an empty core for arbitrary k. This is even true in a rather restricted class of symmetric ASHGs with individual values restricted to $\{n, n+1, -1\}$.

Proposition 8. *For every $k \geq 1$, there exists a symmetric ASHG with $\mathcal{O}(k)$ agents such that the core of its k-fold loyal variant is empty.*

Proof sketch. We only describe the instance. Let $k \in \mathbb{N}$. We define an ASHG $(N, (u_i)_{i \in N})$. Set $m = k$ if k is an even number and $m = k + 1$ if k is odd. Let $A_i = \{a_i, b^i_1, \ldots, b^i_m, c^i_1, \ldots, c^i_m\}$ for $i \in [3]$. Define $N = \bigcup_{i=1}^{3} A_i$ as the set of agents and let $n = |N|$. Reading indices i modulo 3, we define symmetric utilities according to

- $u(a_i, b^i_1) = u(a_i, c^i_1) = n + 1$ for $i \in [3]$,
- $u(b^i_m, a_{i+1}) = u(c^i_m, a_{i+1}) = n$ for $i \in [3]$,

- $u(b_j^i, b_{j+1}^i) = u(c_j^i, c_{j+1}^i) = n + 1$ for $i \in [3], j \in [m-1]$, and
- $u(v, w) = -1$ for all other utilities.

Note that $|N| = 3(2m + 1) = \mathcal{O}(k)$. □

We can use the previous counterexample as a gadget in a sophisticated reduction to obtain computational hardness.

Theorem 9. *Let $k \geq 1$. Deciding whether the core is non-empty is NP-hard for the k-fold loyal variant of symmetric ASHGs.*

Naturally, the question arises whether the core is always non-empty for loyal variants of FOHGs. While we leave the ultimate answer to this question as an open problem, we give evidence into both directions. First, we determine certain graph topologies that allow for coalition structures in the core. By contrast, we provide an intractability result for the computation of coalition structures in the core, and in Proposition 18 in the appendix we show that the dynamics related to blocking coalitions can cycle.

Proposition 10. *Let a symmetric FOHG with connected, regular friendship graph be given. Then the coalition structure consisting of the grand coalition is in the strict core for the k-fold loyal variant for every $k \geq 1$.*

Proof. Assume that the friendship graph is regular with every vertex having degree r. Singleton coalitions are clearly not weakly blocking, so we may assume that $r \geq 2$. In addition, we may assume that a weakly blocking coalition induces a connected subgraph of G. In a weakly blocking coalition $C \subsetneq N$, some agent would have less than r friends, strictly decreasing her utility. Hence, the grand coalition is in the strict core. □

Albeit the previous proposition may look rather innocent, regular substructures in the loyalty graph have been very useful in dealing with core (non-)existence (see, e.g., the many cycles in the games of Proposition 8 and Theorem 9).

For symmetric FOHGs with a tree as loyalty graph, it is easy to see that a coalition structure is in the core if and only if its coalitions form an inclusion-maximal matching. In the case of ASHGs, we can apply a greedy matching algorithm to compute coalition structures in the core.

Proposition 11. *Let $k \geq 1$. A coalition structure in the core of the k-fold loyal variant can be computed in polynomial time for symmetric ASHGs with a tree as loyalty graph.*

On the negative side, even under the existence of core partitions, it may be hard to compute them. Interestingly, the next theorem does not cover the case $k = 1$.

Theorem 12. *Let $k \geq 2$. Computing a coalition structure in the core is NP-hard for the k-fold loyal variant of symmetric FOHGs with non-empty core.*

On the other hand, if the games originate from symmetric MFHGs, we obtain a polynomial-time algorithm by a modification of the algorithm in Theorem 7.

Theorem 13. *Let $k \geq 1$. Computing a coalition structure in the core can be done in polynomial time for the k-fold loyal variant of symmetric MFHGs.*

Symmetric k-fold loyal variant		Best Coalition	Core Solution
FOHGs	orig.	poly.	poly.⊕ [Dimitrov *et al.*, 2006]
	$k = 1$	NP-h.[Thm. 2]	open(?)
	$k \geq 2$	NP-h.[Thm. 2]	NP-h.(?) [Thm. 12]
	limit	NP-h.[Thm. 3]	NP-h.⊕ [Thm. 7]
ASHGs	orig.	poly.	NP-h.⊖ [Aziz *et al.*, 2013]
	$k \geq 1$	NP-h.[Thm. 2]	NP-h.⊖ [Thm. 9]
	limit	NP-h.[Thm. 3]	NP-h.⊕ [Thm. 7]
MFHGs	all	poly.[Thm. 4]	poly.⊕ [Thms. 7,13]

Table 1: Computational complexity of computing best coalitions and core partitions. The circled $+$, $-$, and ? indicate whether elements in the core always exist, may not exist, or whether this is unknown.

5 Conclusion and Open Problems

We have introduced loyalty in hedonic games as a possible way to integrate relationships of players in a coalition into the coalition formation process. Given a hedonic game, players can modify their utilities to obtain a new hedonic game which regards loyalty among coalition partners. Applying loyalty multiple times yields a sequence of hedonic games with increasing loyalty, eventually terminating in a hedonic game with utilities that represent a local form of egalitarianism. The limit game usually contains Pareto optimal coalition structures in the core, but their efficient computability is dependent on the initial input game. We show that computing best coalitions is hard if the input is an FOHG, a reduction that can also be applied to the computation of coalition structures in the core, revealing a close relationship of the two problems. An overview of our results is given in Table 1.

Our work offers plenty directions for further investigation. First, similarly to altruistic hedonic games, one can make the aggregation mechanism for loyal utilities dependent on a priority amongst the agents, or take averages instead of sums. This yields new notions of loyalty that are worth to investigate and compare. Second, it would be interesting to approach loyalty for other underlying classes of hedonic games such as fractional hedonic games. This includes also to find a reasonable way to define loyalty for purely ordinal input. Note that our (equivalent) definition of the loyalty set is also applicable in this case. Finally, an intriguing open problem concerns the existence of coalition structures in the core for loyal variants of FOHGs, in particular for the 1-fold variant, where we could not show hardness of the computational problem.

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A Missing Proofs

In this section, we provide proofs missing in the body of the paper. We use the notation of the degree of a vertex. Formally, given an undirected graph $G = (V, E)$, denote by $\deg_G(v)$ the *degree* of v in G.

Theorem 2. *Let $k \geq 1$. Then,* `BestCoalition` *is NP-complete for the k-fold loyal variant of symmetric FOHGs.*

Proof details. We provide details for the computations regarding the correctness of the reduction.

We start with the case when (A, S, κ) is a Yes-instance, where we considered a set cover $S' \subseteq S$ of A with at most κ sets. For $x \in A$, let $\kappa_x = |\{s \in S' : x \in s\}|$. Note that for all $x \in A$, $\kappa_x \geq 1$, because S' covers every element in the set A at least once.

Recall that we considered the coalition $C = A \cup S' \cup \{v_0, \dots, v_{a-1}, w_0, w_1\}$ for $k = 1$. We need to compute the minimum utility of the agents $\{v_0, w_0\} \cup A$. It holds that

- $u_{v_0}(C) = u_{w_0}(C) \geq n(a+1) - (a+\kappa)$ and
- $u_x(C) \geq n(a+\kappa_x) - (a+1+\kappa-\kappa_x) \geq u_{v_0}(C)$ for $x \in A$.

Hence, $u^1_{v_0}(C) \geq q$.

Now recall the coalition $C = (N \setminus S) \cup S'$ for $k \geq 2$. By part 2 of Proposition 1, we need to compute the minimum utility of all agents with a distance of at most k from i^*, i.e., of the set $N \setminus S$. It holds that

- $u_{v_0}(C) = u_{v_i}(C) = u_{w_0}(C) = u_{w_j}(C) = u_{\alpha_p^m}(C) = u_{\beta_p^m}(C) \geq n(a+1) - (1+\kappa+2(M+1)a)$ for $i \in [a-1]$, $j \in [2, a+2]$, $m \in [M]$, and $p \in [a]$,
- $u_{w_1}(C) \geq n(a+2) - (\kappa + 2(M+1)a) > u_{v_0}(C)$, and
- $u_x(C) \geq n(a+\kappa_x) - (2+\kappa-\kappa_x+2(M+1)a) \geq u_{v_0}(C)$ for $x \in A$.

Together, $u^k_{v_0}(C) \geq q$.

For the reverse direction, we assumed that C is a coalition with $i^* \in C$ and $u^k_{i^*}(C) \geq q$ and claimed that all agents that have a distance of at most k in the loyalty graph have to be included in C.

First, all friends of i^* must be present in C. Hence, $\{v_0, w_0\} \subseteq C$. But then, also all friends of w_0 must be present, which implies that $\{v_1, \dots, v_{a-1}, w_1\} \subseteq C$. It follows that all agents at a distance of at most k in the loyalty graph have to be included due to the degrees of vertices at a distance of at most k. Consequently, A must be included for any k. For $k \geq 2$, also the friends of w_1 must be included, i.e., w_i for $i \in [2, a+2]$. For $k \geq 3$, the agents α_i^j and β_i^j must be present in C for all $i \in [a]$ and all $j \in [M]$. The remaining proof is contained in the main text. ☐

Theorem 3. `BestCoalition` *is NP-complete for the locally egalitarian variant of symmetric FOHGs.*

Proof. Membership in NP is clear. For hardness, we provide a reduction from `SetCover`. The reduction is illustrated in Figure 3.

Given an instance (A, S, κ) of `SetCover`, define $a = |A|$. We define an instance $((N, (F_i)_{i \in N}), i^*, q)$ of `BestCoalition` based on an FOHG $(N, (F_i)_{i \in N})$ represented via friend sets by specifying each individual component.

First, the set of players is $N = \{w_i, \alpha_i, \beta_i : i \in [a]\} \cup \{v_0, v_1\} \cup A \cup S$. Further, let $i^* = v_0$. Formally, we define the friend sets as

- $F_{v_0} = \{v_1, w_1, \dots, w_a\}$,
- $F_{v_1} = \{v_0, \beta_1, \dots, \beta_a\}$,
- $F_{w_i} = \{v_0\} \cup A$ for $i \in [a]$,
- $F_{\beta_i} = \{v_1, \alpha_1, \dots, \alpha_a\}$ for $i \in [a]$,
- $F_{\alpha_i} = \{\beta_1, \dots, \beta_a\} \cup S$ for $i \in [a]$,
- $F_x = \{w_1, \dots, w_a\} \cup \{s \in S : x \in s\}$ for $x \in A$, and
- $F_s = \{a \in A : a \in s\} \cup \{\alpha_1, \dots, \alpha_a\}$ for $s \in S$.

Finally, with $n = |N|$, specify the threshold utility $q = n(a+1) - (\kappa + 3a)$.

If (A, S, κ) is a Yes-instance, let $S' \subseteq S$ be a set cover of A with at most κ sets. For $x \in A$, define $\kappa_x = |\{s \in S' : x \in s\}|$. Note that for all $x \in A$, $\kappa_x \geq 1$, because S' covers every element in the set A at least once.

Now consider the coalition $C = (N \setminus S) \cup S'$. By part 3 Proposition 1, we need to compute the minimum utility of all agents in C. It holds that

- $u_{v_0}(C) = u_{v_1}(C) = u_{w_i}(C) = u_{\beta_i}(C) = n(a+1) - (3a+\kappa)$ for $i \in [a]$,
- $u_{\alpha_i}(C) = n(a+\kappa) - (3a+1) \geq u_{v_0}(C)$ for $i \in [a]$,
- $u_x(C) = n(a+\kappa_x) - (a+1+\kappa-\kappa_x)) \geq u_{v_0}(C)$ for $x \in A$, and
- $u_s(C) = n(2a) - (2a+1+\kappa) \geq u_{v_0}(C)$ for $s \in S \cap C$.

Hence, $u^E_{v_0}(C) \geq q$.

Conversely, assume that C is a coalition with $v_0 \in C$ and $u^E_{v_0}(C) \geq q$. Then, all friends of v_0 must be present in C. Hence, $\{v_1, w_1, \dots, w_a\} \subseteq C$. Following a similar line of argumentation, we can also deduce that A, α_i for $i \in [a]$ and β_i for $i \in [a]$ must be present in C. Let $S' = C \cap S$. Then, $u_{v_0}(C) = n(a+1) - 3a - |S'|$. Hence $u^E_{v_0}(C) \geq q$ implies that $|S'| \leq \kappa$. In addition, every agent $x \in A$ must have at least $a+1$ friends present in C. In other words, for every $x \in A$ there exists $s \in S'$ with $x \in s$. Hence, S' is a cover of A with at most κ elements. ☐

Theorem 4. *Let $k \geq 1$. Then,* `BestCoalition` *can be solved in polynomial time for the k-fold loyal variant and the locally egalitarian variant of symmetric MFHGs.*

Proof. Let $k \geq 1$. Let $(N, (u_i)_{i \in N})$ be an MFHG and let $i \in N$ be some fixed agent. Let $j \in \arg\max_{j \in N} u_i(j)$. Clearly, $C = \{i, j\}$ is a best coalition for the original MFHG. Note that $u^E_i(C) = u^k_i(C) = u_i(C)$. Also note that the aggregation by minimization and Proposition 1.2 imply that, for every coalition $D \subseteq N$ with $i \in D$, $u_i(D) \geq u^k_i(D) \geq u^E_i(D)$, i.e., the utility for a coalition can only decrease in the loyal variants. Hence, C remains a best coalition for i in the k-fold loyal variant and the locally egalitarian variant. ☐

Lemma 5. *Consider a class of hedonic games with the following two properties:*

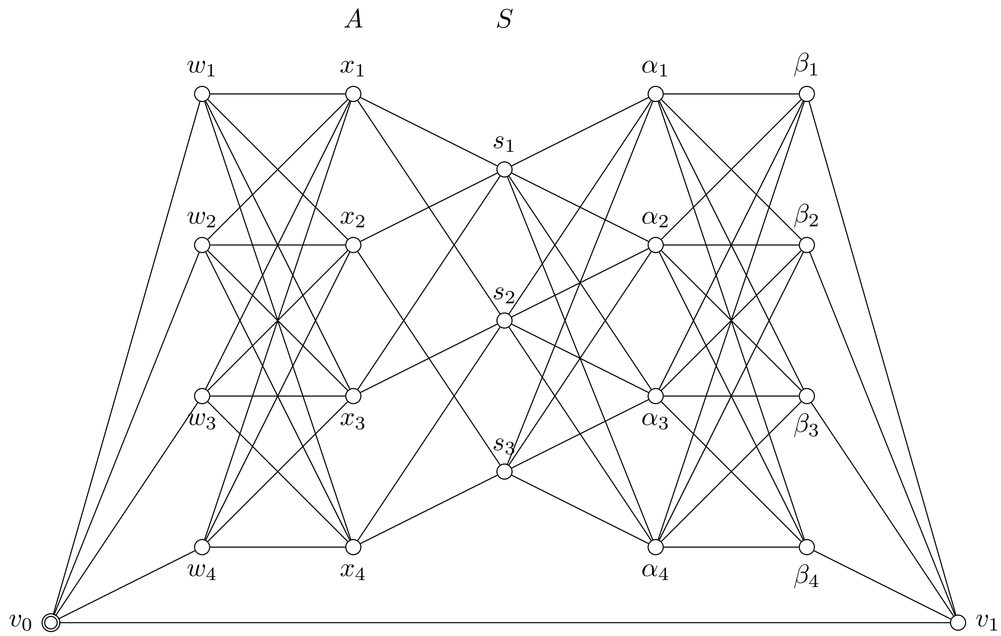

Figure 3: Schematic of the reduction of the hardness construction in Theorem 3 and Theorem 7. We depict the friendship graph of the reduced instance for the source instance of `SetCover` given by $A = \{x_1, x_2, x_3, x_4\}$ and $S = \{s_1 = \{x_1, x_2, x_3\}, s_2 = \{x_1, x_3, x_4\}, s_3 = \{x_2, x_4\}\}$.

1. *Restrictions of the game to subsets of agents are in the class.*
2. *For every coalition in any game of the class, the value of the coalition is the same for every player in the coalition.*

Then, for every game in the class, there exists a coalition structure in the core which is Pareto optimal.

Proof. We prove the statement by induction over the number of agents. By the first property, games which contain only a single agent are in the class and they have a unique Pareto optimal coalition structure in the core.

Now, assume that there are at least two agents in the agent set N. Since the value of a coalition is the same for all agents in the coalition, we define $u(C) = u_i(C)$, for all non-empty $C \subseteq N$ where $i \in C$ is chosen arbitrarily. Let v be the value of a coalition with highest overall value, i.e., $v = \max\{u(C) : C \subseteq N\}$. Given an arbitrary coalition structure σ, let $h(\sigma) = |\{i \in N : u_i(\sigma) = v\}|$ be the number of agents that attain utility v. Let π' be a coalition structure that maximizes this quantity, i.e., $\pi' \in \arg\max\{h(\sigma) : \sigma \text{ partitions } N\}$. The idea is to have the agents with utility v in their coalitions of π' and apply induction for the remaining set of agents. Therefore, let $B = \{i \in N : u_i(\pi') = v\}$.

By induction, we find a partition π'' in the core of the game induced by the agents in $N \setminus B$, which is also Pareto optimal. Consider the partition $\pi = \pi'' \cup \{\pi'(i) : i \in B\}$. Note that π still maximizes h. Clearly, π is in the core. No agent in B can be part of a blocking coalition, because they are in one of their best coalitions. Since π'' is in the core, no subset of agents of $N \setminus B$ can form a blocking coalition.

Now, assume for contradiction that π is Pareto dominated by a partition σ. Since all agents in B are in a best coalition, $u_i(\pi) = u_i(\sigma)$ for all $i \in B$. Moreover, since π maximizes the number of agents with utility v, no agent outside B can have a utility of v. Hence, because the value of a coalition is the same for all of its agents, it follows that $\sigma(i) \subseteq B$ for all $i \in B$. Consequently, $\sigma'' = \{\sigma(i) : i \in N \setminus B\}$ is a Pareto improvement of π'' (on the game restricted to $N \setminus B$). This contradicts the assumption that π'' was Pareto optimal. Hence, π is Pareto optimal, which completes the proof. □

Theorem 7. *The following statements hold.*

1. *Computing a coalition structure in the core can be done in polynomial time for the locally egalitarian variant of symmetric MFHGs.*
2. *Computing a coalition structure in the core is NP-hard for the locally egalitarian variant, even in the class of symmetric FOHGs with non-empty core.*

We split the theorem into two lemmas with separate proofs. Together, they prove Theorem 7.

Lemma 14. *Computing a coalition structure in the core can be done in polynomial time for the locally egalitarian variant of symmetric MFHGs.*

Proof. Let $(N, (u_i)_{i \in N})$ be an MFHG and consider its underlying complete weighted graph $G = (N, E, w)$ with $E = \{\{i, j\} : i, j \in N\}$ and edge weights $w(i, j) = u_i^E(\{i, j\}) = u_i(j)$ for $i, j \in N$. Given a subset of edges $F \subseteq E$, we denote by $G[F]$ the subgraph of G with edge set F. Also, given a graph G, denote its edge set by $E(G)$. We consider Algorithm 1, which is a variant of the algorithm by

Bullinger [2020] for computing a Pareto-optimal and individually rational coalition structure in symmetric MFHGs. It uses the subroutine `MaxMatching` which finds a maximum matching in a graph.

Algorithm 1 Partition in the core of the locally egalitarian variant of a symmetric MFHG

Input: Symmetric MFHG given by graph $G = (N, E, w)$
Output: Coalition structure π in the core of its locally egalitarian variant

$\pi \leftarrow \emptyset, A \leftarrow N, G' \leftarrow G[\{e \in E : w(e) > 0\}]$
while $E(G') \neq \emptyset$ **do**
  $w_{\max} \leftarrow \max\{w(e) : e \in E, w(e) > 0\}$
  $E_H \leftarrow \{e \in E : w(e) = w_{\max}\}$
  $H \leftarrow G'[E_H]$
  $\mathcal{C} \leftarrow \texttt{MaxMatching}(H)$
  $\pi \leftarrow \pi \cup \mathcal{C}$
  $A \leftarrow \{a \in A : a$ not covered by $\mathcal{C}\}$
  $G' \leftarrow G'[A]$
end while
return $\pi \cup \{\{a\} : a \in A\}$

Since the subroutine `MaxMatching` runs in polynomial time [Edmonds, 1965], the whole algorithm runs in polynomial time. It remains to show that the partition π returned by the algorithm is in the core of the locally egalitarian variant of the MFHG. The proof is similar to the proof of Pareto optimality of the related algorithm by [Bullinger, 2020].

Assume that the while loop took m iterations and subdivide $\pi = \mathcal{S} \cup \bigcup_{k=1}^{m} \mathcal{C}_k$, where $\mathcal{C}_k$ is the matching in iteration k, and $\mathcal{S}$ contains the singleton coalitions added to π after the while loop. We will show by induction over m that if the algorithm uses m iterations of the while loop, then it returns a coalition structure in the core. If $m = 0$, then all utilities in the locally egalitarian variant are non-positive, and therefore the coalition structure consisting of singleton coalitions is in the core.

For the induction step, let $m \geq 1$. Assume for contradiction that C is a coalition blocking π, i.e., for all agents $i \in C$, $u_i^E(\pi) < u_i^E(C)$. Let H be the auxiliary graph of the first while loop. Note that within π, agents in $\mathcal{C}_1$ are in a best partition. In particular, they cannot be better off and therefore $C \cap \bigcup_{D \in \mathcal{C}_1} D = \emptyset$. Define $W = \{i \in N : i$ not covered by $\mathcal{C}_1\}$ and consider $\hat{G} = G[W]$. By assumption, C is also a blocking coalition to the partition $\hat{\pi} = \mathcal{S} \cup \bigcup_{k=1}^{m-1} \mathcal{C}_{k+1}$ of the restriction of the locally egalitarian variant of the MFHG induced by $\hat{G}$. Note that $\hat{\pi}$ is a possible outcome of Algorithm 1 for input $\hat{G}$. Hence, by induction, $\hat{\pi}$ is in the core of the restricted locally egalitarian variant, a contradiction. □

Lemma 15. *Computing a coalition structure in the core is NP-hard for the locally egalitarian variant, even in the class of symmetric FOHGs with non-empty core.*

Proof. It remains a proof for the second statement.

We provide a Turing reduction from the optimization variant of `SetCover`, i.e., the problem of finding a set cover of minimum size. We denote this problem by `MinSetCover`. By restricting the problem to instances that evolve from the NP-hard vertex cover problem [Karp, 1972], we may assume that in every instance (A, S), every element $a \in A$ is contained in at most two sets of S. Additionally, we may assume that $|A| \geq 2$.

We will show, that if we can find a coalition structure in the core of the locally egalitarian variant of certain FOHGs, then we can find a minimum set cover for each set cover problem under the given restriction.

Now, let such an instance (A, S) of `MinSetCover` be given and define $a = |A|$. We consider the FOHG $(N, \{F_i\}_{i \in N})$ described in the proof of Theorem 3 and depicted in Figure 3.

By Proposition 1, any agent in a coalition C that induces a connected subgraph of the loyalty graph has the utility in the locally egalitarian variant, i.e., of this coalition, i.e. $u_i^E(C) = u_j^E(C)$ for all $i, j \in C$. Therefore, it suffices to know the utility of any agent in a connected coalition in the locally egalitarian variant, or the agent with the minimum utility in the original FOHG.

We claim, that the core of the locally egalitarian variant of $(N, \{F_i\}_{i \in N})$ consists exactly of the coalition structures of the form $\{(N \setminus S) \cup S'\} \cup \{\{s\} : s \in S \setminus S'\}$ for some minimum set cover $S' \subseteq S$. Note that $u_{v_0}^E((N \setminus S) \cup S') = n(a+1) - 3a - |S'|$.

First, let $S' \subseteq S$ be a minimum set cover. We will show that $\pi = \{(N \setminus S) \cup S'\} \cup \{\{s\} : s \in S \setminus S'\}$ is in the core.

Assume for contradiction that there is a coalition C blocking π. If $C \cap \{v_0, v_1, w_1, \dots, w_a, \beta_1, \dots, \beta_a\} \neq \emptyset$, it is quickly checked that $N \setminus S \subseteq C$, because each of these agents has exactly $a + 1$ friends and they together with their friends need to be added subsequently to C. Furthermore, for each agent $x \in A$ to have enough friends, an agent $s \in S$ with $a \in S$ needs to be present. Hence, $C = (N \setminus S) \cup \hat{S}$ for some set cover $\hat{S} \subseteq S$. But then $u_{v_0}^E(C) \leq u_{v_0}(C) = n(a+1) - 3a - |\hat{S}| \leq n(a+1) - 3a - |S'| = u_{v_0}^E(\pi)$, and C is not blocking. Hence, $C \subseteq \{\alpha_1, \dots, \alpha_a\} \cup S \cup A$. Elements from A cannot be in C, because they could have at most $2 < a + 1$ friends. The coalition C can only be blocking if it contains an agent of positive utility (because even the agents in $S \setminus S'$ obtain utility 0 in π). Hence, we may assume that there is an $s \in S \cap C$ and an agent $\alpha \in \{\alpha_1, \dots, \alpha_a\} \cap C$. But then $u_\alpha^E(C) \leq u_s(C) \leq na < u_\alpha^E(\pi)$, a contradiction. Hence, the coalition structure π is in the core.

Now, assume that π is a coalition structure in the core. Define $D = \{v_0, v_1, x_1, \dots, x_a, w_1, \dots, w_a, \beta_1, \dots, \beta_a\}$ and consider an agent in $z \in D$. If $u_z^E(\pi) > na$, then $D \subseteq \pi(z)$ and each of these agents has a utility of more than na. This implies that if any agent in D has less than $a+1$ friends, then all agents in D have utility of at most na. Assume now that some agent in D has at most a friends. From the low utility of agents in D, we can infer that also agents in $S \cup \{\alpha_1, \dots, \alpha_a\}$ have at most utility na (because they would need a friend in D for more utility). But it holds that $u_y(N) > na$ for all $y \in N$, and therefore N would be blocking.

We can conclude that every agent in D has at least $a+1$ friends in her coalition in π. In particular, $\pi(v_0) = (N \setminus S) \cup S'$ for some set cover $S' \subseteq S$. It is easily checked that $(N \setminus S) \cup \hat{S}$ would be blocking for a set cover $\hat{S} \subseteq S$ with $|\hat{S}| < |S'|$. Hence, S' is a minimum set cover.

It is also easily seen that the best coalition for each agent in $S \setminus S'$ within this set is a singleton coalition (the only coalition with utility 0). Hence, π is of the desired form.

Now, we can solve `MinSetCover` by applying the reduction, finding a coalition structure π in the core of the reduced instance, and returning $S \cap \pi(v_0)$.

□

Proposition 8. *For every $k \geq 1$, there exists a symmetric ASHG with $\mathcal{O}(k)$ agents such that the core of its k-fold loyal variant is empty.*

Full proof. Let $k \in \mathbb{N}$. We define an ASHG $(N, (u_i)_{i \in N})$. Set $m = k$ if k is an even number and $m = k+1$ if k is odd.[2] Let $A_i = \{a_i, b_1^i, \dots, b_m^i, c_1^i, \dots, c_m^i\}$ for $i \in [3]$. Consider the set of agents $N = \bigcup_{i=1}^{3} A_i$. Let $n = |N|$ be the number of agents. We define symmetric utilities according to

- $u(a_i, b_1^i) = u(a_i, c_1^i) = n+1$ for $i \in [3]$,
- $u(b_m^i, a_{i+1}) = u(c_m^i, a_{i+1}) = n$ for $i \in [3]$,
- $u(b_j^i, b_{j+1}^i) = u(c_j^i, c_{j+1}^i) = n+1$ for $i \in [3], j \in [m-1]$, and
- $u(v,w) = -1$ for all utilities which are not defined, yet.

Indices i are to be read modulo 3. Note that $|N| = 3(2m+1) = \mathcal{O}(k)$. The game for $k=1$ or $k=2$ is depicted in Figure 4.

We show now that the core of its k-fold loyal variant is empty. Let π be an arbitrary coalition structure of N, and assume for contradiction that π is in the core of the k-fold loyal variant.

Let $H = (N, E)$ be the simple graph with edge set $E = \{\{v,w\} \subseteq N \colon u(v,w) > 0\}$. Let $C \in \pi$ be a coalition. Consider $L = H[C]$, i.e., the subgraph of H induced by C. Then, L cannot contain a vertex of degree 1 unless $|C| = 2$. Indeed, if $v \in C$ has degree 1 in L with neighbor w, then $\{v,w\}$ is blocking if $|C| \geq 3$.

Consequently, every coalition of π with more than 2 agents contains no agent with just one neighbor in her loyalty set. First assume that there is no coalition of π with more than 2 agents. Then, $A_1 \cup \{a_2\}$ is a blocking coalition. Hence, there is a coalition $C \subseteq \pi$ with $|C| \geq 3$. Let $A_C = C \cap \{a_1, a_2, a_3\}$. It is easily seen that $|A_C| \geq 2$. If $|A_C| = 3$, then $C = A_i \cup A_j \cup \{a_1, a_2, a_3\}$ for some $i, j \in \{1,2,3\}$, or $C = N$, and $A_1 \cup \{a_2\}$ is blocking. Otherwise, without loss of generality, $C = A_1 \cup \{a_2\}$, and $A_2 \cup \{a_3\}$ is blocking. □

Theorem 9. *Let $k \geq 1$. Deciding whether the core is non-empty is NP-hard for the k-fold loyal variant of symmetric ASHGs.*

[2]The example would also work for $m = k$, but it is convenient to have this slight modification, because then the example can be directly used in the reduction of Theorem 9.

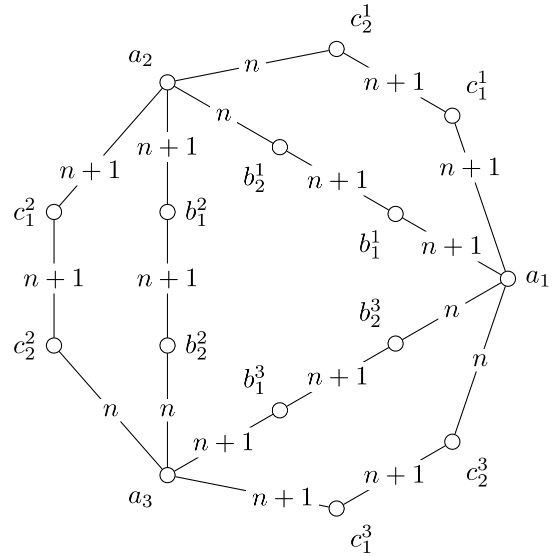

Figure 4: ASHG without a core partition for the 1-fold and 2-fold loyal variant.

Proof. Fix $k \in \mathbb{N}$. We provide a reduction from the NP-complete problem `Exact 3-Cover`. An instance of `Exact 3-Cover` consists of a tuple (R,S), where R is a ground set together with a set S of 3-element subsets of R. A Yes-instance is an instance such that there exists a subset $S' \subseteq S$ that partitions R.

The reduction is illustrated in Figure 5. Let (R,S) be an instance of `Exact 3-Cover`. Let $l = |R|/3$ be the size of a potential 3-partition of the elements in R. Without loss of generality, we may assume that $|S| > l$. We define a symmetric ASHG as follows.

The agent set consists of copies of a close variant of the game from Proposition 8 for every element of R, copies of the elements of the 3-elementary sets in S, and some auxiliary agents who aim at achieving 3-covers of the right size. Formally, set $m = k$ if k is an even number and $m = k+1$ if k is odd. For $r \in R$ and $i \in [3]$, let $A_i^r = \{a_i^r, b_{i,1}^r, \dots, b_{i,m}^r, c_{i,1}^r, \dots, c_{i,m}^r\}$, and let $A^r = A_1^r \cup A_2^r \cup A_3^r$. For $s \in S$, let $T^s = \{t^s\} \cup \{r_i^s \colon i \in [3], r \in s\}$. For $i \in [l]$, let $Z^i = \{z_1^i, z_2^i\}$. Let the agent set be given by $N = \bigcup_{r \in R} A^r \cup \bigcup_{s \in S} T^s \cup \bigcup_{i=1}^{l} Z^i$.

Let $n = |N|$ be the number of agents. In the definition of the utilities, the elements in each set $s \in S$ have to be ordered in an arbitrary direction to assign 'circular' utilities to some of the agents in T^s. Therefore, fix for every $s \in S$ a bijection $\rho_s : [3] \to s$. Formally, the utilities are given as follows.

- $u(a_i^r, b_{i,1}^r) = u(a_i^r, c_{i,1}^r) = n+1$ for $i \in [3], r \in R$,
- $u(b_{i,m}^r, a_{i+1}^r) = u(c_{i,m}^r, a_{i+1}^r) = n$ for $i \in [3], r \in R$,
- $u(b_{i,j}^r, b_{i,j+1}^r) = u(c_{i,j}^r, c_{i,j+1}^r) = n+1$ for $i \in [3], j \in [m-1], r \in R$,
- $u(r_1^s, r_2^s) = u(r_1^s, r_3^s) = u(r_2^s, a_1^r) = u(r_3^s, a_1^r) = n$, for $s \in S, r \in s$,

- $u(t^s, \rho_s(1)^s_1) = u(\rho_s(1)^s_1, \rho_s(2)^s_1) = u(\rho_s(2)^s_1, \rho_s(3)^s_1) = u(\rho_s(3)^s_1, t^s) = n+1$, for $s \in S$,
- $u(t^s, z^i_1) = u(t^s, z^i_2) = u(z^i_1, z^i_2) = n+1$, for $i \in [l], s \in S$,
- $u(v,w) = -(n+1)^2$, for $v \in Z^i, w \in Z^j$ with $j \in [2,l]$ and $i \in [j-1]$, and
- $u(v,w) = -1$ for all utilities which are not defined, yet.

Assume first that (R,S) is a Yes-instance and let $S' \subseteq S$ be a partition of R, say $S' = \{s_1, \dots, s_l\}$. Define the coalition structure $\pi = \{\{t^{s_i}, z^i_1, z^i_2\} \colon 1 \le i \le l\} \cup \{\{t^s, \alpha^s_1, \beta^s_1, \gamma^s_1\} \colon s \in S \setminus S', s = \{\alpha,\beta,\gamma\}\} \cup \bigcup_{s\in S'} \bigcup_{r\in s} \{\{r^s_1, r^s_2, r^s_3, a^r_1\}\} \cup \bigcup_{r \in R}(\{\{a^r_3\} \cup A^r_2\} \cup \{\{b^r_{i,2j-1}, b^r_{i,2j}\}, \{c^r_{i,2j-1}, c^r_{i,2j}\} \colon i \in \{1,3\}, j \in [m/2]\} \cup \bigcup_{s \in S\setminus S'} \bigcup_{r\in s} \{\{r^s_i\} \colon i = 2,3\}$.

We claim that π is in the core. Define the undirected graph $H = (N,E)$ with edge set $E = \{\{v,w\} \subseteq N \colon u(v,w) > 0\}$. Assume for contradiction that $C \subseteq N$ is a blocking coalition and let $L = H[C]$ be the subgraph of H induced by the agents of C. A key observation is that L cannot contain vertices of degree 1. This is immediate for all agents v with $\deg_v(H[\pi(v)]) \ge 2$, because their utility would be strictly worse. On the other hand, agents w of the type r^s_2 or r^s_3 with $u^k_w(\pi) = 0$ would make their neighbor in L strictly worse off. In particular, it suffices to prove that agents which have a degree of at least 3 in H cannot be part of a blocking coalition. Indeed, by excluding vertices of degree 1, all the neighbors of vertices of degree 2 need to be in the blocking coalition, and we could traverse C until we find a vertex of degree at least 3.

Consider first agents v of the type a^r_2. If $a^r_2 \in C$, then at least 3 neighbors of v are in C, which means that $|C| \ge 3m+3 > 2m+2$. Hence, a neighbor of v in A^r_2 would be worse off.

Next, let v be an agent of the type a^r_3 and assume that $v \in C$. If $C = A^r_3 \cup \{a^r_1\}$, then a^r_1 is worse off (because of having more enemies in her coalition). Otherwise, $C \supsetneq A^r_3 \cup \{a^r_1\}$ and C needs to contain at least 3 agents outside A^r. Thus, v would be worse off.

Now, let v be an agent of the type a^r_1 and assume that $v \in C$. Then, at least 3 neighbors of v in H have to be present in C which can—by the previous arguments—not be part of A^r. In particular, $|C| \ge 6$. Let $z \in C$ be one of v's neighbors of type r^s_2 or r^s_3. Then, $u^k_v(C) \le u_z(C) \le 2n-4 < u^k_v(\pi)$, and v would not be improving.

So far, we can conclude that $C \subseteq \bigcup_{s\in S} T^s \cup \bigcup_{i=1}^{l} Z^i$. Thus, agents of the type r^s_i for $i = 2,3$ cannot be in C, because then a^r_1 would also be in C. Consider now an agent v of the type r^s_1. She could only form a blocking coalition together with t^s. But t^s can only improve by having at least two other agents outside T^s in the blocking coalition. Hence, v would not be better off.

Finally, if an agent v of the type z^i_j is in C, then multiple agents of type t^s need to be in C, but due to the large weights no agent in $Z^{i'}$ for $i' \neq i$. Hence, the agents t^s would not improve their utility. Since agents of the type t^s cannot form a blocking coalition on their own, we conclude that there is no blocking coalition. Hence, π is in the core.

Conversely, assume that there exists a coalition structure π which is in the core. Note that for all coalitions $C \in \pi$, $H[C]$ has no vertices of degree 1 unless $|C| = 2$. We first claim that for each $i \in [l]$, there exists $s \in S$ with $\{t^s, z^i_1, z^i_2\} \in \pi$. Let therefore $i \in [l]$ and consider $C = \pi(z^i_1)$ and set $L = H[C]$. Note that for all $i' \in [l]$ with $i' \neq i$, it holds that $Z^{i'} \cap C = \emptyset$.

We start with the assumption that $|C| \ge 4$. If there is a unique agent of the type t^s in C, then $\{t^s\} \cup Z^i$ would be blocking. Hence, we can assume that there are at least two agents of this type present. Since a coalition $\{t^s\} \cup Z^i$ would be blocking if $\deg_L(t^s) = 2$, we may assume that for some $r \in s$, $r^s_1 \in C$. Note that it cannot happen that $r^s_2 \in C$ or $r^s_3 \in C$ (because $\{r^s_1, r^s_2, r^s_2, a^r_1\}$ would be block). Hence, $B = \{t^s, \alpha^s_1, \beta^s_1, \gamma^s_1\}$ for $s = \{\alpha,\beta,\gamma\}$ fulfills $B \subsetneq C$ and would be blocking. Together, we deduce that $|C| \le 3$. In particular, $z^i_2 \in C$ (otherwise, there would be a vertex of degree 1). Finally, if $C = Z^i$, then we find $s \in S$ such that $Z^j \cap \pi(t^s) = \emptyset$ for all $j \in [l]$. But then, $\{t^s\} \cup Z^i$ would be blocking. Hence, there is a unique $s \in S$ such that $C = \{t^s\} \cup Z^i$.

Now, set $S' = \{s \in S \colon \pi(t^s) \cap Z^i \neq \emptyset \text{ for some } i \in [l]\}$. Note that $|S'| = l$. We conclude the proof by showing that S' covers R. Therefore, let $r \in R$ and set $C = \pi(a^r_1)$. If $C \subseteq A^r$, the partition restricted to A^r would be in the core, contradicting Proposition 8. Hence, there exists $s \in S$ with $r \in s$ such that $C \cap T^s \neq \emptyset$. More precisely, $r^s_1 \in C$. Assume that $s = \{r,p,q\}$. It is easy to see that $s \in S'$ or $\{t^s\} \cup \{r^s_1, p^s_1, q^s_1\}$ is blocking. Hence, r is covered by S'. □

Proposition 11. *Let $k \ge 1$. A coalition structure in the core of the k-fold loyal variant can be computed in polynomial time for symmetric ASHGs with a tree as loyalty graph.*

Proof. Consider some symmetric ASHG $(N, (u_i)_{i\in N})$ such that its loyalty graph is a tree. Consider some coalition structure π and a coalition $C \subseteq N$ blocking π with $|C| \ge 3$. Clearly, there exists an agent $i \in C$ with $|L_i \cap C| = 1$, say $L_i \cap C = \{j\}$. It holds that $u^k_i(\{i,j\}) = u^k_j(\{i,j\}) \ge \max(u^k_i(C), u^k_j(C))$, i.e., i and j weakly prefer $\{i,j\}$ over C. Therefore, whenever there exists a blocking coalition, then there is a blocking coalition of size 2.

Now consider the following greedy matching algorithm iteratively adding pairs of highest mutual utility. More formally, we start with an empty coalition structure π. From the set of agents, choose a pair with maximum utility, i.e., $\{i^*, j^*\} \in \arg\max_{i,j\subseteq N}(u^k_i(\{i,j\}))$, and add it to π. Delete these agents from the set and continue this process, until no pair with positive utilities is left.

We claim that the obtained coalition structure is in the core. Assume that there is some pair $\{i,j\} \subseteq N$ blocking a coalition structure π calculated by the algorithm. This means, that $u^k_i(\{i,j\}) = u^k_j(\{i,j\}) > \max(u^k_i(\pi(i)), u^k_j(\pi(j)))$, which is in contrast to the greedy choice, as $\{i,j\}$ would have been chosen before $\pi(i)$ and $\pi(j)$. Hence, there is no blocking coalition of size 2. Due to the first part of the proof, this means that π cannot be blocked by any coalition and consequently lies in the core.

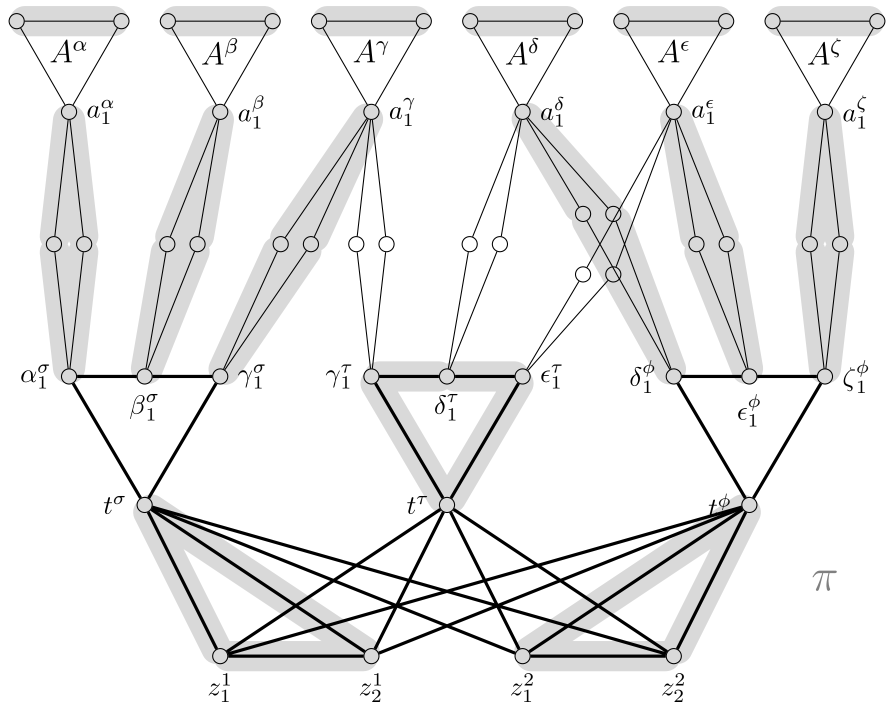

Figure 5: Schematic of the reduction of the hardness construction in Theorem 9 for the instance of `Exact 3-Cover` given by $R = \{\alpha, \beta, \gamma, \delta, \epsilon, \zeta\}$ and $S = \{\sigma = \{\alpha, \beta, \gamma\}, \tau = \{\gamma, \delta, \epsilon\}, \phi = \{\delta, \epsilon, \zeta\}\}$. A normal edge between agents x and y means that $u(x, y) = n$. A bold edge between agents x and y indicates that $u(x, y) = n+1$. The gadgets of the game from Proposition 8 are indicated by the triangles at the top with agent sets A^ξ for $\xi \in R$. Edges with negative weight, including the edge with large negative weight between agents of different sets Z^i are omitted. The gray coalition structure indicates a coalition structure π in the core for the set exact 3-cover given by $\{\sigma, \phi\}$.

The described algorithm clearly runs in polynomial time. □

Theorem 12. *Let $k \geq 2$. Computing a coalition structure in the core is NP-hard for the k-fold loyal variant of symmetric FOHGs with non-empty core.*

Proof. We consider again the reduction from Theorem 7. Apart from the properties already defined in the proof of Theorem 7, we may assume without loss of generality that for every instance (A, S) of `MinSetCover`, $a \geq 3$ and $|S| \leq |A|$.

Fix some $k \geq 2$ and let (A, S) be an instance of `MinSetCover` under the described restrictions. Consider the reduced FOHG $(N, \{F_i\}_{i \in N})$ from the proof of Theorem 7. We will prove that the core of the k-fold loyal variant coincides with the core of the locally egalitarian variant, i.e., the coalition structures in the core are precisely of the form $\pi = \{(N \setminus S) \cup S'\} \cup \{\{s\} \colon s \in S \setminus S'\}$ for some minimum set cover $S' \subseteq S$. The proof that each of these coalition structure is contained in the core is identical to the respective part of the proof of Theorem 7.

It remains to show that no coalition structure of a different form is in the core.

First, consider some coalition structure π that contains a coalition of the form $(N \setminus S) \cup \hat{S}$ for some set covering $\hat{S}$. Then, it is easy to see that $\hat{S}$ is a minimum set cover. Otherwise, π can be blocked by $(N \setminus S) \cup S'$ for some minimum set cover S'. Additionally, all other agents must be contained in singleton coalitions.

Now, assume for contradiction that there is a coalition structure π in the core, which does not contain a coalition of type $(N \setminus S) \cup S'$ for some set cover S'. First, there must be an agent $i \in N$ with $u_i^k(\pi(i)) \geq n(a+1) - 3a - |S| := M$ (otherwise, N would block π). Let $C = \pi(i)$. We will perform an exhaustive case distinction over the type of the agent i to show that then either $A \cup \{w_1, \dots, w_a\}$ or $\{\alpha_1, \dots, \alpha_a, \beta_1, \dots, \beta_a\}$ is blocking.

First, assume $i = v_0$. Then, clearly $\{v_1, w_1, \dots, w_a\} \subseteq C$ as the friends of v_0 have to be in C. By propagating into both directions, we can see that also $\{x_1, \dots, x_a, \beta_1, \dots, \beta_a, \alpha_1, \dots, \alpha_a\} \subseteq C$, and in addition, some cover of the $x_j, j \in [a]$ has to be in C. But this contradicts the assumption that C does not contain $(N \setminus S) \cup S'$ for some set cover S'.

Next, assume $i = v_1$. By propagating into both directions, we can see that $\{v_0, w_1, \dots, w_a, x_1, \dots, x_a, \beta_1, \dots, \beta_a, \alpha_1, \dots, \alpha_a\} \subseteq C$. In order for all $\alpha_j, j \in [a]$ to have $a+1$ friends, there must also be some $s \in S \cap C$. Now, if C does not contain a cover of the $x_j, j \in [a]$, then some of the x_j only have a friends in C. Therefore, $u_m^k(C) \leq na - (3a+2)$ for all $m \in \{w_1, \dots, w_a\} \cup A$. But $u_m^k(C') = na - (a-1)$ for $C' = \{w_1, \dots, w_a\} \cup A$ and $m \in C'$. Hence, C' blocks π. Otherwise, if C does contain a cover of the $x_j, j \in [a]$, then

C contradicts again our assumption on its structure.

Next, assume $i = \beta_j$ for some $j \in [a]$. By propagating into both directions, we can see that $\{v_0, v_1, w_1, \dots, w_a, \alpha_1, \dots, \alpha_a, \beta_1, \dots, \beta_a\} \subseteq C$. In order for all $\alpha_j, j \in [a]$ to have $a+1$ friends, there must also be some $s \in S \cap C$. In order for s to have $a+1$ friends, there must also be some $x \in A \cap C$. Now, if $A \nsubseteq C$, then $u_m^k(C) \leq na - (3a+1)$ for all $m \in (\{w_1, \dots, w_a\} \cup A) \cap C$, because the $w_j, j \in [a]$ have at most a friends. All $x \in A \setminus C$ have at most 2 friends in S. Therefore, it holds that $u_x^k(\pi(x)) \leq 2n < na - (a-1)$ for each $x \in A \setminus C$. The last inequality holds, since $a \geq 3$ and $a - 1 < n$. Thus, $A \subseteq C$, because otherwise $C' = \{w_1, \dots, w_a\} \cup A$ would block π. Similar to the case of $i = v_1$, we can deduce that C must contain a cover of the $x_j, j \in [a]$, which leads to a contradiction.

Next, assume $i = \alpha_j$ for some $j \in [a]$. By propagating into both directions, we can see that $\{v_0, v_1, \alpha_1, \dots, \alpha_a, \beta_1, \dots, \beta_a\} \subseteq C$. In order for all $\alpha_j, j \in [a]$ to have $a+1$ friends, there must also be some $s \in S \cap C$. In order for s to have $a+1$ friends, there must also be some $x \in A \cap C$. In order for x to have $a+1$ friends, at least $a-1$ agents from $\{w_1, \dots, w_a\}$ need to be in C (since x has at most 2 friends in S). Now, if $\{w_1, \dots, w_a\} \nsubseteq C$, then fix the unique $w \in \{w_1, \dots, w_a\}$ with $w \notin C$.

In the next step, we calculate the utilities of all agents in $\{w_1, \dots, w_a\} \cup A$ in order to show that it must hold that $w \in C$. First, $u_{w'}^k(C) \leq na - (2a+3)$ for all $w' \in \{w_1, \dots, w_a\} \setminus \{w\}$ because v_0 only has a friends. Further, $u_w^k(\pi(w)) \leq n(a-1)$, as at most $a-1$ elements of A are not in C. For the $x \in A \cap C$, we see that $u_x^k(C) \leq na - (2a+3)$ because of v_0. For the $x \in A \cap \pi(w)$, we see that $u_x^k(C) \leq n(a-1)$ because of w. Finally, it holds that $u_x^k(\pi(x)) \leq 2n < na - (a-1)$ for each $x \in A \setminus (C \cup \pi(w))$, since these agents have at most 2 friends in S. Consequently, we see that $w \in C$. We obtain a contradiction as in the case $i = \beta_j$.

Next, assume $i = w_j$ for some $j \in [a]$. By propagating into both directions, we can see that $\{v_0, v_1, w_1, \dots, w_a, x_1, \dots, x_a, \beta_1, \dots, \beta_a\} \subseteq C$. In order for the $x \in A$ to have $a+1$ friends, C must contain some cover of A. In order for the $s \in S \cap C$ to have $a+1$ friends, there must be some $\alpha_j, j \in [a]$ with $\alpha_j \in C$. In order to show that $\{\alpha_1, \dots, \alpha_a\}$ must be contained in C, we can follow a similar line of arguments as in the case of $i = \beta_j$. Define $C' := \{\alpha_1, \dots, \alpha_a, \beta_1, \dots, \beta_a\}$. Assume there is at least one α_j,$j \in [a]$ with $\alpha_j \notin C$. Then, $u_m^k(C) \leq na - 3a - |S'|$, for $m \in C' \cap C$, since the $\beta_j, j \in [a]$ have at most a friends in C. On the other hand, $u_m^k(C) \leq n(a-1)$, for $m \in C' \setminus C$, since m can only have friends in S, $|S| \leq a$ and C contains at least one agent in S. Hence, C' blocks π and we again have the desired contradiction.

Next, assume $i = x$ for some $x \in A$. By propagating into both directions, we can see that $\{v_0, v_1, w_1, \dots, w_a, x_1, \dots, x_a\} \subseteq C$. In order for the $x \in A$ to have $a+1$ friends, C must contain some cover of A. In order for the $s \in S$ with $s \in C$ to have $a+1$ friends, there must be some α_j, $j \in [a]$ with $\alpha_j \in C$. In order for α_j to have $a+1$ friends, there must be some β_m, $m \in [a]$ with $\beta_m \in C$. Assume that $\{\beta_1, \dots, \beta_a\} \nsubseteq C$. Then, $u_m^k(C) \leq na - 3a - |S'|$, for $m \in C' \cap C$, since v_1 has at most a friends in C. Further, $u_m^k(C) \leq n(a-1)$, for $m \in C' \setminus C$. This is clear for $m = \beta_j$, since these agents have at most $a-1$ friends outside of C. For $m = \alpha_j$, we can see that m needs to be in a coalition with some $s \in S$ or some β_j. Both s and β_j only have at most $a-1$ friends outside of C. Therefore, either $\{\beta_1, \dots, \beta_a\} \subseteq C$, or C' blocks π. In order to see that $\{\alpha_1, \dots, \alpha_a\} \subseteq C$, we follow the steps of the case $i = w_j$. Therefore, we obtain the contradiction as in the previous case.

Finally, assume $i = s$ for some $s \in S$. By propagating into both directions, we can see that $\{v_0, v_1, x_1, \dots, x_a, \alpha_1, \dots, \alpha_a\} \subseteq C$. In order for the $x \in A$ to have $a+1$ friends, C must contain at least $a-1$ of $\{w_1, \dots, w_a\}$. In order for the α_j, $j \in [a]$ to have $a+1$ friends, C must contain at least one of $\{\beta_1, \dots, \beta_a\}$. It is easy to see that $\{w_1, \dots, w_a\} \in C$, since otherwise $\{w_1, \dots, w_a\} \cup A$ would be blocking, because v_0 has at most a friends. A similar argument holds for the β_j in conjunction with v_1. Lastly, we see that C also has to contain a set cover in order for $\{w_1, \dots, w_a\} \cup A$ not to be blocking by the argument illustrated for the case $i = v_1$. Therefore, $C \supseteq (N \setminus S) \setminus S'$ for some set cover S', resulting in the final contradiction.

Hence, the core of the k-fold loyal variant is equal to the locally egalitarian variant of the FOHG, and the hardness result can be transferred. □

Note that this reduction cannot easily be extended to the 1-fold loyal variant, because then, for some fixed $s^* \in S$ with $|s^*| \geq 2$, $\pi = \{(N \setminus S) \cup \{s^*\} \cup \{\{s\} \colon s \in S \setminus \{s^*\}\}$ is another coalition structure in the core of the reduced instance. Indeed, assume for contradiction that C is a coalition blocking π.

Then, $v_0 \in C$ or $v_1 \in C$ can only happen if their number of enemies is reduced and all agents at distance at most 2 are still present. In other words, $\{v_0, v_1, w_1, \dots, w_a, \beta_1, \dots, \beta_a\} \subseteq C$. If $S \cap C = \emptyset$ or $\{\alpha_1, \dots, \alpha_a\} \nsubseteq C$, then agents β_i would not be improving. If there is a single agent in $A \setminus C$, then some of the remaining agents in A contained in s^* would be worse off. But if $|A \setminus C| \geq 2$, then the agents in $\{w_1, \dots, w_a\}$ would be worse off.

If $w_i \in C$ for some $i \in [a]$, then $A \subseteq C$, but agents contained in s^* would be worse off (because w_i can have at most a friends). Then, also agents in A could only have $2 < a$ friends in a blocking coalition which is insufficient.

If $\beta_i \in C$ for some $i \in [a]$, then $\{\alpha_1, \dots, \alpha_a\} \subseteq C$, but all of these agents would be worse off (having less than a friends).

Note that agents $u_{\alpha_i}^1(\pi) > na \in C$ for all $i \in [a]$. Since we already excluded the β_i from C, they can also not be in C, because $|S| \leq |A|$.

Finally, the agents in S cannot form a blocking coalition of their own. Together, we have shown that a blocking coalition cannot exist and π is in the core.

Theorem 13. *Let $k \geq 1$. Computing a coalition structure in the core can be done in polynomial time for the k-fold loyal variant of symmetric MFHGs.*

Proof. Let $k \geq 1$. The algorithm in Lemma 14 also works in this case. Note that $w(i,j) = u_i^k(\{i,j\}) = u_i(j)$. □

B Additional Results

In this section, we will provide some additional results that help to broaden the picture about loyalty in hedonic games.

First, we provide a weakening of Lemma 5, which is still sufficient for the computation of coalition structure in the core and which might by applicable more widely. It is particularly useful, because it can also be applied directly to hedonic games with ordinal preferences.

Lemma 16. *Consider a class of hedonic games with the following two properties:*

1. *Restrictions of the game to subsets of agents are in the class.*
2. *Every game in the class has a coalition which is best for all of its participants.*

Then, for every game in the class, the core is non-empty.

Proof. We prove the statement by induction over the number of agents. By the first property, games which contain only a single agent are in the class and they clearly have a non-empty core. So assume that the set of agents N satisfies $|N| \geq 2$. Let $C \subseteq N$ be a coalition that is best for all agents in C. By induction, we find a partition π' in the core of the restriction of the game induced by the agents $N \setminus C$. We claim that the partition $\pi = \pi' \cup \{C\}$ is in the core. Clearly, the agents in C cannot be in a blocking coalition. Hence, the statement follows, because π' is assumed to be in the core on the game restricted to the agent set $N \setminus C$. □

It is possible to apply Lemma 16 directly to the locally egalitarian variant to obtain a coalition structure in the core.

Theorem 17. *Given a loyalty-connected, mutual cardinal hedonic game, the core of its locally egalitarian variant is always non-empty.*

Proof. Let a cardinal hedonic game be given and consider the class of games which consists of its locally egalitarian variant, and the subgames of the locally egalitarian variant induced by an arbitrary subset of agents. We claim that every game in the class has a coalition that is best for all of its agents. By loyalty-connectivity, we can restrict our search to connected components of the loyalty graph. Now, we can apply part 3 of Proposition 1. The value of a coalition is the same for all of its players. Hence the coalition with the highest value for any player satisfies the desired property. By Lemma 16, the core is non-empty. □

Next, we provide an example showing that loyalty-connectivity is a necessary condition for Theorem 6 and Theorem 17. In fact, if it is not satisfied, there need not exist a coalition structure in the core.

Example 3. *Consider the following cardinal hedonic game. The agent set is $N = \{a_i, b_i : i \in [3]\}$. The utilities are based on an ASHG with utilities $(u_i^A)_{i\in N}$ induced by the symmetric utilities over single agents*

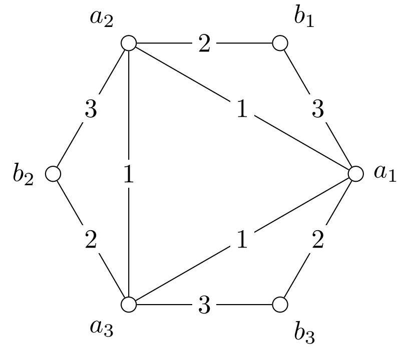

Figure 6: The preferences of Example 3 are based on additively separable utilities given by the depicted weighted graph. Negative weights are omitted from the figure.

- $u^A(a_i, b_i) = 3$ *for* $i \in [3]$,
- $u^A(b_i, a_{i+1}) = 2$ *for* $i \in [3]$,
- $u^A(a_i, a_{i+1}) = 1$ *for* $i \in [3]$, *and*
- $u^A(v, w) = -8$ *for all utilities not defined, yet.*

We read all indices modulo 3. *The underlying weighted graph is depicted in Figure 6. Note that the large negative weight is chosen such that no two agents with mutual negative utility can be together in a coalition structure in the core. Now, we modify the additively separable utilities to obtain utilities $(u_i)_{i\in N}$ as*

$$u_i(C) = \begin{cases} 0 & \text{if } |C| \leq 2, \\ u_i^A(C) & \text{if } |C| \geq 3. \end{cases}$$

Clearly, the loyalty graph of the game $(N, (u_i)_{i\in N})$ contains no edges. Hence, all its loyal variants including its locally egalitarian variant are identical to the ground game. Next, we will show that this game has an empty core.

Assume for contradiction that π is a coalition structure in its core. Then, there exists a coalition $C \in \pi$ with $|C| \geq 3$. Otherwise, the coalition $\{a_1, a_2, b_1\}$ is blocking. Assume first that there exists $i \in [3]$ with $b_i \in C$, say $i = 1$. Due to the large negative utilities, this implies that $C \subseteq \{a_1, a_2, b_1\}$ and therefore $C = \{a_1, a_2, b_1\}$. For the same reason, b_2 and b_3 cannot be in a joint coalition. Hence, the coalition sizes of the coalitions of b_2 and a_3 are each at most 2, which yields $u_{b_2}(\pi) = u_{a_3}(\pi) = 0$. Together, $\{a_2, b_2, a_3\}$ is a blocking coalition. This leaves the case $C = \{a_i : i \in [3]\}$. But then $\{a_1, a_2, b_1\}$ is blocking. Together, we have derived a contradiction and there is no coalition structure in the core.

Finally, we will provide more evidence against the existence of coalition structure in the core for FOHGs. A necessary condition for the existence of games with empty core is cycling of the natural dynamics induced by blocking coalitions. The *core dynamics* is the process of moving among coalition structure by group deviations of blocking coalitions.

Proposition 18. *The core dynamics can cycle in the loyal variant of symmetric FOHGs.*

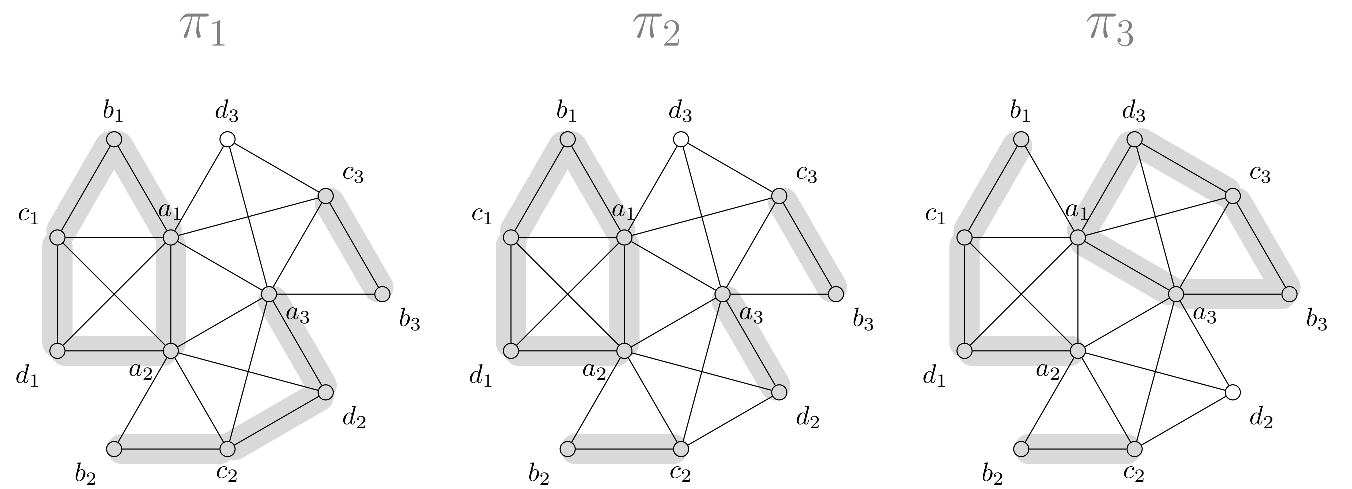

Figure 7: Cycling of the core dynamics in Proposition 18. The cycle consists essentially of the two deviations from π_1 to π_2 and from π_2 to π_3 between the depicted three coalition structures. The third coalition structure is a rotated version of the first one.

Proof. Consider the FOHG given by agent set $N = \{a_i, b_i, c_i, d_i : i \in [3]\}$ and symmetric friend sets as $F_{a_i} = \{b_i, c_i, d_i, a_{i+1}, a_{i+2}, c_{i+2}, d_{i+2}\}$, $F_{b_i} = \{a_i, c_i\}$, $F_{c_i} = \{a_i, b_i, d_i, a_{i+1}\}$, and $F_{d_i} = \{a_i, c_i, a_{i+1}\}$.

Consider the partitions $\pi_1 = \{\{a_1, b_1, c_1, d_1, a_2\}, \{b_2, c_2, d_2, a_3\}, \{b_3, c_3\}, \{d_3\}\}$, $\pi_2 = \{\{a_1, b_1, c_1, d_1, a_2\}, \{b_2, c_2\}, \{d_2, a_3\}, \{b_3, c_3\}, \{d_3\}\}$, and $\pi_3 = \{\{b_1, c_1, d_1, a_2\}, \{b_2, c_2\}, \{d_2\}, \{a_3, b_3, c_3, d_3, a_1\}\}$. Then, π_2 evolves from π_1 by a group deviation of the blocking coalition $\{b_2, c_2\}$, and π_3 evolves from π_2 by a group deviation of the blocking coalition $\{a_3, b_3, c_3, d_3, a_1\}$. Also, π_1 and π_3 differ only by a circular renaming of the agents. Continuing with the same deviations while renaming the agents, we obtain a cycling core dynamics. A visualization of the dynamics is given in Figure 7. $\square$