Selecting Interlacing Committees

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ABSTRACT

Polarization is a major concern for a well-functioning society. Often, mass polarization of a society is caused by a polarizing political representation, even though it could be easily prevented on this level. This is an effect inherent to the current theory on committee selection from the view of computational social choice. We enhance the standard model of committee selection by defining two quantitative values that measure how much a given committee interlaces voters. Maximizing these values aims at avoiding polarizing committees. While the corresponding maximization problem is NP-complete in general, we obtain an efficient algorithm for profiles in the voter-candidate interval domain. Moreover, we analyze the compatibility of our goals with other objectives of excellent, diverse, and proportional representation. We identify trade-offs between approximation guarantees and describe algorithms that achieve simultaneous constant-factor approximations.

KEYWORDS

Computational Social Choice, Approval-Based Committee Voting, Polarization

1 INTRODUCTION

In recent years, the emergent phenomenon of polarization has been a major concern, discussed not just by social scientists, but by society at large, and accompanied by extensive media coverage [19, 26]. Polarization is commonly defined as the division of a group into clusters of completely different opinions or ideologies. It is a major concern for the modern society, which has to work towards a consensus when resolving global challenges, such as fighting poverty, climate change, or pandemics (see [27] and the references therein).

Importantly, polarization can occur as a phenomenon concerning a whole society or only at the level of political representation, e.g., when considering the distribution of opinions among the delegates in a parliament. The former is often referred to as *mass polarization*, while the latter is known as *elite polarization* [see, e.g., 1].

The academic opinion broadly agrees that the phenomenon of elite polarization is on the rise. For example, when depicting the members of the US Congress in terms of their ideology on a scale ranging from the most liberal to the most conservative, one can observe a significant shift when comparing the 87th Congress in the 1960s and the 111th Congress around 2010, see Figure 2.1 in the book by Fiorina [18]. However, whether polarization concerns the opinions of a society as a whole is in huge debate. Fiorina et al. [19] argue that there is no conclusive evidence for mass polarization, even when considering highly sensitive topics such as abortion. For instance, they provide evidence that the elite polarization among delegates is already much higher than the polarization among party identifiers [19, Table 2.1]. By contrast, they ascribe an important role in creating an inaccurate picture of mass polarization to the media [19]. In fact, the media can have a huge effect on the perception of and conclusions drawn from elite polarization [26].

This view is opposed by Abramowitz and Saunders [1] when analyzing data from the American National Election Studies. They provide extensive evidence that mass polarization has increased signficiantly since the 1970s. Moreover, their results suggest mass polarization based on geography (i.e., different ideologies across US states) or religious beliefs.

Against this background, we aim to offer a novel perspective on the intertwined phenomena of mass polarization at the broad level of a society as a whole and elite polarization at the level of the society's political, parliamentary representation. We highlight how an election can lead to a parliament that is far more polarized than the society it represents, and we propose quantitative measures that evaluate a set of representatives according to how well it interlaces the electorate. We believe that our ideas can be developed to prevent societies with broadly moderate opinions being represented by unnecessarily polarized parliaments.

We approach polarized democratic representation through the lens of social choice theory. In this line of research, parliamentary elections have been conceptualized as so-called multiwinner voting rules. Their formal study, especially in an approval-based setting, in which voters' ballots specify a set of approved candidates, has received extensive attention in recent years [17, 25].

Example 1.1. As a motivating example, consider the voting scenario illustrated in Figure 1. There are four voters, indicated by the gray circles, as well as six candidates. Each candidate is represented by an ellipse that encompasses the voters approving the candidate. For instance, candidate b_1 is approved by voters v_1 and v_2, whereas candidates c_1 and c_3 are both approved by the same set of voters, namely v_1 and v_3. In practice, this is likely to happen when c_1 and c_3 represent very similar ideologies.

Assume that we want to select a committee consisting of 4 candidates. Two reasonable choices would be to select $W = \{c_1, c_2, c_3, c_4\}$ or $W' = \{c_1, c_2, b_1, b_2\}$. Both selections lead to committees in which each voter approves exactly two selected candidates. Moreover, multiwinner voting rules typically considered in the literature, such as Thiele rules and their sequential variants [32], Phragmén's rule [31], or the more recently introduced method of equal shares [30], do not distinguish between these two choices. There is, however, a difference. While W divides the electorate into two perfectly separated

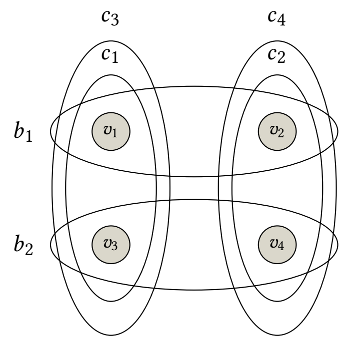

Figure 1: A preference profile with four voters $v_1, \ldots, v_4$ is depicted as hypergraph, where the voters are nodes and the candidates b_i, c_j are hyperedges connecting the voters approving them. In this profile, typical multiwinner voting rules do not distinguish between selecting $\{c_1, c_2, c_3, c_4\}$ and $\{c_1, c_2, b_1, b_2\}$.

subsets of voters, W' connects all voters. From the perspective of polarization, W looks polarizing while W' bridges all voters. Thus, we need novel voting rules that can tease out this distinction. In our paper, we aim to provide a principled approach that favors committees resembling W'.[1] ◁

We define two simple objectives that aim to interlace voters by means of committees. First, we consider maximizing the number of *pairs* of voters approving a common candidate (the Pairs objective). While this leads to the selection of W' in Example 1.1, it can still cause clusters of voters disconnected in terms of their representation (cf. Example 3.1). The reason is that Pairs only counts direct, but not indirect links. Hence, as a second objective we count the number of pairs of voters that are *connected* by a sequence of candidates (the Cons objective).

While both objectives immediately give rise to voting rules—select a committee that maximizes Pairs or Cons—we primarily view them as measures of polarization. Whenever they are high, polarization in the selected committee is low. Thus, we investigate the feasibility of maximizing our objectives, both on their own and in combination with the goals of diversity and proportionality.

We first consider the computational problem of maximizing Pairs or Cons in isolation (Section 4). Unfortunately, for unrestricted preferences this problem is NP-hard. However, we obtain a polynomial-time algorithm for the structured domain of voter-candidate interval (VCI) preferences [20], where voters and candidates are represented by intervals on the real line and a voter approves a candidate if and only if their intervals intersect. Such preferences are reasonable in parliamentary elections where candidates can often be ordered on a left-right spectrum and voters approve candidates that are close to them on this spectrum.

In Section 5, we investigate whether one can select interlacing committees while achieving other desiderata. We first consider *excellence*, as measured by the *approval voting* (AV) score, i.e., the total number of approvals received by committee members. There is a straightforward way to obtain what is essentially an α-approximation of the Pairs objective together with an $(1-\alpha)$-approximation of the AV score: one can simply use an α-fraction of the committee for the former and an $(1-\alpha)$-fraction for the latter. Unfortunately, it turns out that this is as good as it gets: We prove that if a voting rule provides an α-approximation of the Pairs objective and a β-approximation of the AV score, then necessarily $\alpha + \beta \leq 1$. Next, we look at *diversity*, as captured by the *Chamberlin–Courant* (CC) score, which is the number of voters who approve at least one candidate in the committee. The CC score is closely related to the Pairs objective: both measure the coverage of voters and pairs of voters, respectively. Hence, it is quite surprising that the trade-off we get here matches the one for Pairs and AV. Further, we study the compatibility with *proportionality*, as captured by the extended justified representation axiom (EJR). Again, we show the same tight trade-off: If a voting rule provides an α-approximation of the Pairs and β-approximate EJR, then $\alpha + \beta \leq 1$.

It is more challenging to combine the Cons objective with AV, CC, EJR or even Pairs. This is due to an interesting qualitative difference between Pairs and Cons. While a constant fraction of the best candidates achieves a constant approximation of Pairs, for Cons this is not the case. Hence, we obtain worse trade-offs: If a voting rule provides an α^2-approximation of Cons and a β-approximation of AV, CC, EJR, or Pairs, then $\alpha + \beta \leq 1$. Note that since $\alpha < 1$, it holds that $\alpha^2 < \alpha$. Hence, for instance, $\alpha^2 = \frac{1}{3}$ and $\beta = \frac{1}{2}$ is already impossible. Moreover, for Cons and AV specifically, the trade-off that we obtain is even more subtle, which suggests that finding a matching lower bound might be challenging. Nevertheless, we make first steps towards this goal, by showing that under suitable domain restrictions there always exists a committee that achieves a $\frac{1}{4}$-approximation of Cons and a $\frac{1}{2}$-approximation of AV, CC, EJR, or Pairs, which matches our upper bound.

2 RELATED WORK

In the existing literature, multiwinner voting rules usually aim to guarantee the selection of the best candidates based on their individual quality [3, 13], representation of diverse opinions [7, 14], or proportional treatment of different groups of interests [28, 30–32]. An overview of the most common approval-based multiwinner voting rules is given in the book by Lackner and Skowron [25]. To the best of our knowledge, no rules were proposed so far with the explicit goal of reducing polarization or connecting voters.

A line of research in multiwinner voting looks at the possibility of combining various objectives as well as their inherent trade-offs, similar to our study in Section 5. Lackner and Skowron [24] provide worst-case bounds on how several established rules approximate AV and CC scores. For ordinal preferences, Kocot et al. [22] analyze the complexity of finding committees giving an optimal combination of approximations of two objectives. Moreover, a series of works look at achievable AV and CC guarantees when we require that a committee satisfies a certain proportionality axiom [6, 12, 16].

A number of authors study the relationship between an electoral system (or, more narrowly, a voting rule) and the way the candidates choose to strategically place themselves on the political spectrum [5, 9, 23, 29]. Such an analysis can indicate whether a rule prevents,

[1] Of course, while we try to highlight the phenomenon at hand with a simple example, it is easy to extend this to elections with large sets of voters or candidates, e.g., each voter in the example might represent a quarter of a large electorate.

or reinforces, polarization. Our approach differs in that we analyze the direct effect of a voting rule on the polarization caused by a chosen committee, while the aforementioned works analyze how preferences evolve based on a given rule.

Delemazure et al. [10] pursue a goal that can be seen as opposite to ours: selecting a most polarizing committee of size 2; they focus on ordinal preferences. In a similar vein, Colley et al. [8] proposed measures of how *divisive*, or polarizing, a single candidate is.

3 MODEL

We start by introducing key notation and proposing two ways of measuring how well a committee interconnects the voters. For a positive integer $k \in \mathbb{N}$, define $[k] := \{1, \ldots, k\}$.

3.1 Approval-Based Multiwinner Voting

We consider the standard setting of approval-based multiwinner voting [25]. Given a set of m candidates C, an *election instance* $\mathcal{E} = (V, A, k)$ consists of a set of n voters V, an approval profile $A = (A_v)_{v \in V}$ with $A_v \subseteq C$ for all $v \in V$, and a target committee size $k \in [m]$. Intuitively, a voter $v \in V$ approves precisely the candidates in A_v. Throughout the paper, we view a profile A as a hypergraph with vertex set V, and, for each $c \in C$, a hyperedge $V_c = \{v \in V : c \in A_v\}$. Throughout the remainder of this section, we consider an election instance $\mathcal{E} = (V, A, k)$ over a candidate set C.

Besides the general setting, we also consider structured domains of spatial one-dimensional preferences. An election belongs to the voter-candidate interval domain if each voter and candidate can be represented as an interval on the real line and a voter approves a candidate if and only if their respective intervals intersect. Formally, following Godziszewski et al. [20], we say that an election (V, A, k) belongs to the *voter-candidate interval (VCI) domain* if there exist a collection of positions $\{x_c\}_{c \in C} \cup \{x_v\}_{v \in V} \subseteq \mathbb{R}$ and a collection of nonnegative radii $\{r_c\}_{c \in C} \cup \{r_v\}_{v \in V} \subseteq \mathbb{R}^+ \cup \{0\}$ such that for all $v \in V$, $c \in C$ it holds that $c \in A_v$ if and only if $|x_c - x_v| \leq r_c + r_v$.

The VCI domain is the most general domain of one-dimensional approval preferences considered in the literature. In particular, it generalizes the voter interval (VI) and candidate interval (CI) domains, defined as follows [15]. An election belongs to the *voter interval (VI) domain* if there is an ordering of the voters $v_1, \ldots, v_n$ such that each candidate is approved by some interval of this ordering, i.e., for each $c \in C$ there exist $i, j \in [n]$ such that $V_c = \{v_i, \ldots, v_j\}$. Similarly, an election belongs to the *candidate interval (CI) domain* if there is an ordering of the candidates $c_1, \ldots, c_m$ such that each voter's approval set forms an interval of this ordering, i.e., for each $v \in V$ there exist $i, j \in [m]$ such that $A_v = \{c_i, \ldots, c_j\}$. It is easy to see that the VI domain and the CI domain are contained in the VCI domain.[2]

A *feasible committee* for an instance (V, A, k) is a subset $W \subseteq C$ with $|W| = k$. A *(multiwinner) voting rule* f takes as input an instance (V, A, k) and outputs a feasible committee $f(V, A, k)$.

[2]For instance, given an election $\mathcal{E} = (V, A, k)$ in VI, as witnessed by voter ordering $v_1, \ldots, v_n$, we can set $x_{v_i} = i$ and $r_{v_i} = 0$ for each $i \in [n]$. To position the candidates, for each $c \in C$ we compute $c^- = \min\{i : c \in A_{v_i}\}$ and $c^+ = \max\{i : c \in A_{v_i}\}$ and set $x_c = (c^- + c^+)/2$, $r_c = (c^+ - c^-)/2$. Clearly, these positions and radii certify that $\mathcal{E}$ belongs to the VCI domain. For CI, the construction is analogous.

3.2 Classic Committee Selection

A popular classification of multiwinner voting rules is in terms of the main objective in electing the committee, with three most commonly studied objectives being *excellence*, *diversity*, and *proportionality* [17].

Both excellence and diversity are defined quantitively: each of these objectives is associated with a function that assigns a numerical score to each feasible committee, with higher score associated with better performance. Formally, given an instance $\mathcal{E} = (V, A, k)$ and a feasible committee W, we define

$$\mathrm{AV}(W, \mathcal{E}) := \sum_{v \in V} |A_v \cap W|,$$
$$\mathrm{CC}(W, \mathcal{E}) := |\{v \in V : A_v \cap W \neq \varnothing\}|.$$

For both objectives (as well as the two novel objectives defined in Section 3.3) we omit $\mathcal{E}$ from the notation when it is clear from the context. The quantities AV and CC are referred to as, resp., the *approval score* and the *Chamberlin–Courant score* of committee W in election $\mathcal{E}$. Intuitively, AV counts the number of approvals received by the members of W and is viewed as a measure of excellence, while CC counts the number of voters represented by W, i.e., voters who approve at least one member of W, and is viewed as a measure of diversity. The voting rule that outputs a committee maximizing AV (resp., CC) is known as the *approval voting rule* (resp., the *Chamberlin–Courant rule*).

Consider any function S that assigns scores to feasible committees (e.g., S = AV or S = CC). Given $\alpha \in [0, 1]$, we say that a committee W^* *satisfies α-S* for an election $\mathcal{E} = (V, A, k)$ if it holds that

$$S(W^*, \mathcal{E}) \geq \alpha \cdot \max_{\substack{W \subseteq C, \\ |W| = k}} S(W, \mathcal{E}).$$

Moreover, we say that a voting rule f *satisfies α-S* if, for every election $\mathcal{E}$, it holds that $f(\mathcal{E})$ satisfies α-S for $\mathcal{E}$. For instance, the Chamberlin–Courant rule satisfies 1-CC.

In contrast, proportionality is typically captured by representation axioms. A prominent axiom of this type is *extended justified representation* (EJR) [2]; intuitively, it states that sufficiently large groups of voters with similar preferences should be appropriately represented in the selected committee. We will now define what it means for a committee to satisfy approximate EJR.

Given an election (V, A, k) over C and $\alpha \in (0, 1]$, a committee $W \subseteq C$ is said to satisfy α-EJR if for every $\ell \in [k]$ and every subset $S \subseteq V$ such that $\alpha \cdot |S| \geq \frac{\ell}{k} \cdot |V|$ and $|\bigcap_{i \in S} A_i| \geq \ell$, there exists at least one voter $i \in S$ such that $|W \cap A_i| \geq \ell$. We say that a rule f satisfies α-EJR, if for every election $\mathcal{E}$ it holds that $f(\mathcal{E})$ satisfies α-EJR. Setting α to 1 gives the standard EJR axiom.

3.3 Interlacing Committee Selection

We now define two new objectives, which assess committees based on how well they interlace voters.

Our first objective is the number of *pairs* of voters that jointly approve a selected candidate. Given an election $\mathcal{E} = (V, A, k)$, let $V^{(2)} := \{\{u, v\} \subseteq V : u \neq v\}$ be the set of all voter pairs. We set

$$\textsc{Pairs}(W, \mathcal{E}) := |\{\{u, v\} \in V^{(2)} : A_u \cap A_v \cap W \neq \varnothing\}|.$$

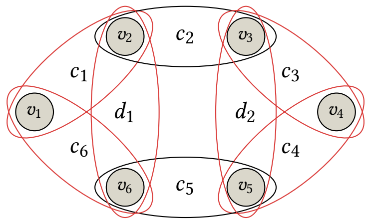

Figure 2: Illustration of Example 3.1.

Note that for every instance $\mathcal{E} = (V, A, k)$ one can define an *associated pair instance* $\mathcal{E}^{(2)} = (V^{(2)}, A^{(2)}, k)$, where $A^{(2)}_{\{u,v\}} = A_u \cap A_v$ for every $\{u, v\} \in V^{(2)}$. For each instance $\mathcal{E}$ and committee $W \subseteq C$ we have $\text{Pairs}(W, \mathcal{E}) = \text{CC}(W, \mathcal{E}^{(2)})$.

While the Pairs objective only considers direct links between voters, our second objective takes into account indirect connections as well. Given an instance $\mathcal{E} = (A, V, k)$ and a subset of candidates $W \subseteq C$, we say that two voters $u, v \in V$ are *connected by* W (and write $u \sim_W v$) if there is a sequence of voters $u = v_0, v_1, \ldots, v_s = v$ with $A_{v_{i-1}} \cap A_{v_i} \cap W \neq \varnothing$ for every $i \in [s]$. To evaluate a committee W, we count pairs of voters connected by W. Formally,

$$\text{Cons}(W, \mathcal{E}) := \left|\{\{u, v\} \in V^{(2)} : u \sim_W v\}\right|.$$

Since both Pairs and Cons assign scores to committees, we also consider their approximate versions, as captured by α-Pairs and α-Cons.

Our interest in Cons is motivated by the following example.

Example 3.1. Consider a profile with six voters $v_1, \ldots, v_6$, six *cycle* candidates $c_1, \ldots, c_6$, and two *diagonal* candidates d_1 and d_2, whose hypergraph is depicted in Figure 2. Each cycle candidate is approved by two consecutive voters: for $i = 1 \ldots, 5$ candidate c_i is approved by v_i and v_{i+1}, while c_6 is approved by v_1 and v_6. Also, d_1 is approved by v_2 and v_6 and d_2 by v_3 and v_5. Let $k = 6$.

Consider two committees: $W = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ contains all cycle candidates, whereas in $W' = \{c_1, c_3, c_4, c_6, d_1, d_6\}$ two cycle candidates are exchanged for the diagonal candidates (W' is shown in red in Figure 2). Common voting rules, including the approval rule and the Chamberlin–Courant rule, do not distinguish between W and W', as each voter approves exactly two candidates in either committee. Moreover, the rule that maximizes Pairs is also unable to distinguish them, as both W and W' cover exactly 6 pairs of voters. However, intuitively, W' seems more polarizing: under W', there are two disconnected groups of voters, each supporting (though not fully) their own set of candidates.

In contrast, a rule that maximizes Cons is sensitive to the differences between the two committees. Under W, all 15 pairs of voters are connected, while W' only achieves 6 connections. ◁

4 COMPUTATION OF THE NEW OBJECTIVES

In this section, we show that maximizing Pairs and Cons is NP-hard in general, but tractable on well-structured domains. All proofs missing from this section can be found in Appendix A of the supplementary material.

4.1 General Preferences

Both hardness proofs in this section are based on the NP-complete problem Exact Cover by 3-Sets (X3C) [21]. An instance of X3C is a pair $(R, \mathcal{S})$, where R is a ground set of size 3ρ and $\mathcal{S}$ is a collection of 3-element subsets of R; it is a Yes-instance if and only if there exists a subset $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = \rho$ that covers R.

We start by considering the decision problem associated with maximizing the Pairs objective.

Theorem 4.1. *It is* NP*-complete to decide whether, given an election $\mathcal{E} = (V, A, k)$ and a threshold $q \in \mathbb{N}$, there exists a committee W of size at most k such that $\text{Pairs}(W, \mathcal{E}) \geq q$.*

Proof. Membership in NP is immediate: for a given committee, its size and the number of pairs of voters approving a common candidate can be checked in polynomial time.

To show NP-hardness, we present a reduction from X3C. Given an instance $(R, \mathcal{S})$ of X3C with $|R| = 3\rho$, we construct an instance of our problem as follows. We create one candidate for each set in $\mathcal{S}$ and two voters for each element of the ground set, i.e., we set $C = \{c_S : S \in \mathcal{S}\}$ and $V = \{v_r, v'_r : r \in R\}$. For each $S \in \mathcal{S}$, candidate c_S is approved by voters $\{v_r, v'_r : r \in S\}$. We set the target committee size k to ρ and the threshold q to 15ρ. We will show that we can cover q pairs of voters if and only if the source instance is a Yes-instance of X3C.

Suppose first there exists a feasible committee W that covers q pairs of voters. Each $c \in W$ is approved by exactly 6 voters, so it can cover at most $\binom{6}{2} = 15$ pairs of voters. Moreover, the candidates' support sets are either disjoint or overlap in at least two voters. As $q = 15k$, this means that candidates in W have pairwise disjoint support sets. Since $|W| = k$, it follows that $\{S \in \mathcal{S} : c_S \in W\}$ forms a cover of R, i.e., our instance of X3C is a Yes-instance.

Conversely, assume that there exists a subset $\mathcal{S}' \subseteq \mathcal{S}$ of size k that covers R. Consider the committee $W = \{c_S : S \in \mathcal{S}'\}$. Then, $|W| = |\mathcal{S}'| = \rho = k$. Moreover, since, all of the sets in $\mathcal{S}'$ are pairwise disjoint, the support sets of the candidates in W are pairwise disjoint and contains exactly 6 voters each. Hence, there are $k \cdot \binom{6}{2} = 15\rho = q$ pairs of voters who approve a common candidate. □

A similar hardness result holds for Cons. The proof idea is to introduce an auxiliary voter that is the focal point in connecting all voters.

Theorem 4.2. *It is* NP*-complete to decide whether, given an election $\mathcal{E} = (V, A, k)$ and a threshold $q \in \mathbb{N}$, there exists a committee W of size k such that $\text{Cons}(W, \mathcal{E}) \geq q$. The hardness result holds even if $q = \binom{n}{2}$, i.e., if the goal is to connect all n voters.*

4.2 One-dimensional Preferences

In Section 4.1, we have shown that the computational problems associated with selecting interlacing committees are NP-hard. In contrast, we will now show that these problems can be solved in polynomial time on the VCI domain.

We start by observing that, for the objectives we consider, a VCI instance can be transformed into a CI instance without changing the value of these objectives. To this end, we define a notion of dominance among candidates and prove that, in the absence of dominated candidates, every VCI instance is a CI instance.

4.2.1 *Relationship between VCI and CI.* Given an election $\mathcal{E} = (V, A, k)$ over a candidate set C, we say that candidate $c' \in C$ is *dominated* by a candidate $c \in C$ if every voter approving c' also approves c, and some voter approves c but not c', i.e., $V_{c'}$ is a proper subset of V_c.

Our next result shows that if an election in the VCI domain contains no dominated candidates, it belongs to the (much simpler to analyze) CI domain. This result is very useful for our purposes: Indeed, removal of dominated candidates from a winning committee does not affect the Pairs and Cons objectives, so we can simply remove all dominated candidates from the input instance. It is also of independent interest, as it points out a surprising relationship between the two domains.

Proposition 4.3. *Let $\mathcal{E}$ be an instance in the VCI domain. If $\mathcal{E}$ contains no dominated candidates, then it belongs to the CI domain.*

In what follows, we state our results for the VCI domain, but assume that the input election belongs to the CI domain, and we are explicitly given the respective candidate order. It will also be convenient to assume that this order is $c_1, \dots, c_m$. This requires two preprocessing steps: first, we eliminate all dominated candidates (which, by Proposition 4.3, results in a CI election), and second, we compute an ordering of the candidates witnessing that our instance belongs to the CI domain. Both steps can be implemented in polynomial time (for the second step, see, e.g., [15]).

4.2.2 *Efficient Algorithms.* We are ready to present polynomial-time algorithms for Pairs and Cons. Since Pairs is identical to CC on the associated pair instance, we can compute Pairs by leveraging an existing algorithm for CC in the CI domain [4, 15].

Proposition 4.4. *In the VCI domain, a committee that maximizes Pairs can be computed in polynomial time.*

In the VCI domain, we can also compute a committee that maximizes Cons in polynomial time; however, the argument is significantly more complicated. Again, we assume that the input profile belongs to the CI domain, as witnessed by the candidate ordering $c_1, \dots, c_m$. A natural idea, then, is to use dynamic programming to compute, for each $b \in [k]$ and $i \in [m]$, an optimal subcommittee of size b with rightmost candidate c_i. For $b = 1$, the computation is straightforward, and for $b = k$, one of the resulting m committees globally maximizes Cons. However, computing the value of adding c_i to a committee of size $b - 1$ that has c_j as its rightmost candidate is a challenging task: this is because the number of connections that c_i adds depends on the size of the connected component associated with c_j. To handle this, we add a third dimension to the dynamic program: the number of voters $x \in [n]$ in the connected component of the last selected candidate. The resulting dynamic program has $\mathcal{O}(mnk)$ cells, and each cell can be filled in polynomial time given the values of the already-filled out cells.

Theorem 4.5. *In the VCI domain, a committee that maximizes Cons can be computed in polynomial time.*

5 COMBINING OBJECTIVES

While interlacing objectives can be viewed in isolation, in many cases, standard objectives of excellence, diversity, or proportionality continue to be important for the election of a committee. In this section, we investigate to what extent we can select committees that simultaneously perform well with respect to both interlacing and standard objectives. Missing proofs can be found in Appendix B.

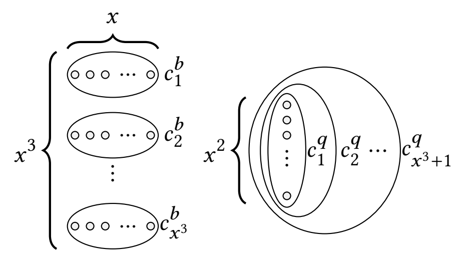

Figure 3: Illustration of the profile constructed in the proof of Proposition 5.3. Block voters are on the left, central voters in the middle, and arm voters on the right. Block candidates are each approved by x block voters whereas arm candidates are each approved by all central voters and one arm voter.

5.1 Pairs Objective

First, we consider combining the Pairs objective with individual excellence of the chosen candidates, as measured by AV. For every $\alpha \in [0, 1]$ and every election $\mathcal{E} = (V, A, k)$, there is a simple way to obtain a simultaneous $\lceil \alpha k \rceil / k$-approximation of Pairs and $\lfloor (1 - \alpha)k \rfloor / k$-approximation of AV. Indeed, we can split the k positions on the committee into two parts of size $k_1 = \lceil \alpha k \rceil$ and $k_2 \lfloor (1 - \alpha)k \rfloor$, respectively, and then select k_1 candidates so as to maximize Pairs and k_2 candidates so as to maximize AV (if some candidate is selected both times, we replace their second copy by an arbitrary unselected candidate). Since the marginal gain for Pairs and AV objectives from each additional candidate is non-increasing, this procedure obtains the desired guarantees. Note that Lackner and Skowron [24] propose a similar method for combining AV and CC.

Proposition 5.1. *For every $\alpha \in [0, 1]$ and election $\mathcal{E}$, there exists a committee that satisfies $\lceil \alpha k \rceil / k$-Pairs and $\lfloor (1 - \alpha)k \rfloor / k$-AV.*

We can use the same technique to combine Pairs with the goal of diverse representation, as measured by CC.

Proposition 5.2. *For every $\alpha \in [0, 1]$ and election $\mathcal{E}$, there exists a committee that satisfies $\lceil \alpha k \rceil / k$-Pairs and $\lfloor (1 - \alpha)k \rfloor / k$-CC.*

It turns out that, for both combinations, this is the best we can hope for.

Proposition 5.3. *For every $\alpha, \beta \in [0, 1]$ if a voting rule satisfies α-Pairs and β-AV, then $\alpha + \beta \leq 1$.*

Proof. Assume for the sake of contradiction that some voting rule satisfies α-Pairs and β-AV with $\alpha + \beta = 1 + \varepsilon$ for some $\varepsilon > 0$. For a given constant $x \in \mathbb{N}$, consider the election $\mathcal{E} = (V, A, k)$, defined as follows (see Figure 3 for an illustration). The set V consists of x^4 *block* voters $(v^b_{i,j})_{i \in [x], j \in [x^3]}$ and x^2 *central* voters $(v^q_i)_{i \in [x^2]}$. Also, the set C contains x^3 *block* candidates $(c^b_i)_{i \in [x^3]}$, and $x^3 + 1$ *central* candidates $(c^q_i)_{i \in [x^3+1]}$. For every $i \in [x]$ and $j \in [x^3]$, the block voter $v^b_{i,j}$ only approves the block candidate c^b_j, but all central voters approve all central candidates, i.e., for every $i \in [x^2]$ and

$j \in [x^3+1]$, voter v_i^q approves candidate c_j^q. The target committee size is set to $k = x^3 + 1$.

By symmetry, without loss of generality, this means that for some $\gamma \in \{0, 1/x^3, 2/x^3, \dots, 1\}$ we select a committee $W_\gamma \subseteq C$ with central candidates $c_1^q, c_2^q, \dots, c_{\gamma x^3+1}^q$ and block candidates $c_1^b, c_2^b, \dots, c_{(1-\gamma)x^3}^b$. Observe that every selected block candidate is approved by x voters and covers $x(x-1)/2$ pairs of voters. In turn, every selected central candidate is approved by x^2 voters, but all $(x^2-1)x^2/2$ pairs of central voters are just covered once, no matter the value of γ. Thus, when we select $\gamma x^3 + 1$ central candidates and $(1-\gamma)x^3$ block candidates, we get the following AV and Pairs scores.

$$\mathrm{AV}(W_\gamma, \mathcal{E}) = \gamma x^5 + (1-\gamma)x^4 \le \gamma x^5 + \mathcal{O}(x^4), \text{ and}$$

$$\textsc{Pairs}(W_\gamma, \mathcal{E}) = (1-\gamma)\frac{x^5 - x^4}{2} + \frac{x^4 - x^2}{2} \le (1-\gamma)x^5 + \mathcal{O}(x^4),$$

where the $\mathcal{O}(\cdot)$ terms are independent of γ. Observe that the maximum AV score is obtained when we take $\gamma = 1$, and the maximum Pairs score is obtained when $\gamma = 0$. Also,

$$\frac{\mathrm{AV}(W_\gamma, \mathcal{E})}{\mathrm{AV}(W_1, \mathcal{E})} + \frac{\textsc{Pairs}(W_\gamma, \mathcal{E})}{\textsc{Pairs}(W_0, \mathcal{E})} \le 1 + \mathcal{O}(1/x).$$

Therefore, for x large enough, regardless of the value of γ, the sum of approximation ratios for AV and Pairs is less than $1+\varepsilon$, a contradiction. □

It may seem that the Pairs and CC objectives are more aligned than Pairs and AV. Indeed, both CC and Pairs only demand that a voter (resp., a pair of voters) has at least one candidate in the selected committee that they (jointly) approve. However, surprisingly, the worst-case trade-off for this pair of objectives is the same as for Pairs and AV.

Proposition 5.4. *For every $\alpha, \beta \in [0,1]$, if a voting rule satisfies α-Pairs and β-CC, then $\alpha + \beta \le 1$.*

Proof sketch. The proof is similar to the one of Proposition 5.3. This time, the idea is that the block candidates are required to achieve a large CC-score, while the central candidates are required to achieve a large Pairs-score, see Figure 4.

We now increase the number of blocks from x^3 to x^4, and the number of central voters from x^2 to x^3. Accordingly, we increase the number of block candidates to x^4 so that each block still approves precisely one block candidate, and the number of central candidates to x^4+1, where still all central voters approve all central candidates. Further, we add x^4+1 *arm voters* $(v_i^a)_{i \in x^4+1}$, where each v_i^a further approves the central candidate i. Finally, we set the target committee size to $k = x^4 + 1$.

Just as in the proof of Proposition 5.3, the choice of the committee boils down to choosing γx^4 central candidates and $(1-\gamma)x^4$ block candidates for some $\gamma \in \{0, 1/x^4, \dots, 1\}$. Then, by a similar analysis, we get that

$$\frac{\mathrm{CC}(W_\gamma, \mathcal{E})}{\mathrm{CC}(W_1, \mathcal{E})} + \frac{\textsc{Pairs}(W_\gamma, \mathcal{E})}{\textsc{Pairs}(W_0, \mathcal{E})} \le 1 + \mathcal{O}(1/x),$$

which concludes the proof. □

Finally, we investigate how we can combine the Pairs objective with proportional representation, as captured by the EJR axiom.

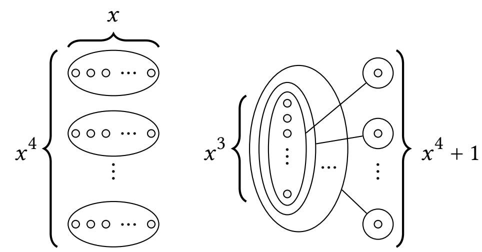

Figure 4: An illustration of the profile constructed in the proof of Proposition 5.4. Block voters are on the left, central voters in the middle, and arm voters on the right. Block candidates are approved by x block voters each whereas arm candidates are approved by all central voters and one arm voter each.

Again, we can use the committee-splitting technique to show that for every election $\mathcal{E} = (V, A, k)$ there is a committee that satisfies $\lceil \alpha k \rceil / k$-Pairs and $(1-\alpha)$-EJR. For this, we first need to show that we can guarantee $(1-\alpha)$-EJR with a $(1-\alpha)$-fraction of the committee seats. To obtain this, we utilize a variant of the *method of equal shares* (MES) [30]. Roughly speaking, this rule gives each voter k/n units of money and then sequentially selects candidates that are best for voters that still have money, and subtracts money from the supporters of the selected candidate (the formal definition is in Appendix B). Mimicking the proof that MES satisfies EJR by Peters and Skowron [30], we show that a variant of MES in which we scale money allocated to voters by α provides α-EJR for the original instance, which can be of independent interest.[3]

Lemma 5.5. *Let $\alpha \le 1$ be given. For every election $\mathcal{E} = (V, A, k)$, executing MES on $(V, A, \alpha k)$ returns a committee of size $\lfloor \alpha k \rfloor$ satisfying α-EJR in polynomial time.*

Using this lemma, we now easily obtain the desired guarantees.

Proposition 5.6. *For every $\alpha \in [0,1]$ and election $\mathcal{E}$, there exists a committee that satisfies α-Pairs and $(1-\alpha)$-EJR.*

Proof. Consider an election $\mathcal{E}$. By Lemma 5.5, we can satisfy $(1-\alpha)$-EJR using $\lfloor (1-\alpha)k \rfloor$ candidates. With the remaining $k - \lfloor (1-\alpha)k \rfloor = \lceil \alpha k \rceil$ candidates, we can guarantee α-CC on the associated pair instance $\mathcal{E}^{(2)}$. This is equivalent to satisfying α-Pairs on $\mathcal{E}$, concluding the proof. □

As before, we provide the matching upper bound. We note that our proof even works if, instead of EJR, we considere the much weaker axiom of *justified representation* (JR) [2].

Proposition 5.7. *For every $\alpha, \beta \in [0,1]$, if a voting rule satisfies α-Pairs and β-EJR, then $\alpha + \beta \le 1$.*

To conclude this section, we note that the algorithms for obtaining the guarantees of Propositions 5.1, 5.2, and 5.6 can be made polynomial-time using greedy approximations of CC and Pairs, but this will result in the respective guarantees being multiplied by $(1-1/e)$ [24]. Indeed, a $(1-1/e)$-approximation of Pairs can

[3] A similar observation was made by Dong and Peters [11], but requires $\lceil (1-\alpha)k \rceil$ seats, which in our case would allow only for a rounded-down Pairs guarantee.

be computed in polynomial time using the sequential Chamberlin–Courant rule on the associated pair instance.

5.2 Cons Objective

An important reason why we obtained good approximations of Pairs, AV, and CC was that these objectives are subadditive, i.e., for every two committees W and W', the value for committee $W \cup W'$ is never larger than the sum of the values for W and W'. As a consequence, these objectives are sublinear with respect to the committee size, in the sense that if we only use an α-fraction of the k committee seats, we can obtain at least an α-fraction of the original value for a committee of size k (up to rounding).

In contrast, the Cons objective is not subadditive, so we cannot use the same technique. In fact, the following result shows that the trade-off between Cons and any of AV, CC, or Pairs is strictly worse (on the side of the Cons) than the trade-offs we have established in Section 5.1. Notably, our upper bound applies even to instances that belong to the VI domain.

Proposition 5.8. *For every $\alpha, \beta \in [0, 1]$, if a voting rule satisfies α^2-Cons and β-AV, β-CC, or β-Pairs, then $\alpha + \beta \leq 1$. This already holds in the VI domain.*

Proof sketch. For the proof of all three statements, consider an instance with x^3 blocks, with each block consisting of x voters approving the corresponding block candidate. Further, we have $x^3 + 1$ central voters ordered on a line, with each pair of adjacent central voters approving a designated central candidate.

This instance belongs to the VI domain, as we can first enumerate each block and then the central voters on the line. The remainder of the proof consists of two parts. In part one, we show that, to satisfy β-Pairs, β-CC, or β-AV, we require at least $\beta x^3 - \mathcal{O}(x^2)$ block candidates. In part two, we show that with the remaining candidates, we can obtain at most a $(1 - \beta)^2$-approximation of Cons. □

We obtain an analogous result for EJR by reducing the number of blocks from x^3 to slightly less than βx^3.

Proposition 5.9. *For every $\alpha, \beta \in [0, 1]$, if a voting rule satisfies α^2-Cons and β-EJR, then $\alpha + \beta \leq 1$. This already holds in the VI domain.*

However, in some cases the trade-off is even worse than the one presented in the above results. Consider a stepwise function $s\colon [0, 1] \to [0, 1]$ given by $s(\alpha) = 1/(\lceil 2/\alpha \rceil - 1)$, as illustrated in Figure 6. Intuitively, it finds the smallest $p \in \mathbb{N}$ such that $\alpha \geq 2/p$ and returns $1/(p-1)$. We then have the following trade-off between Cons and AV.

Proposition 5.10. *For every $\alpha, \beta \in [0, 1]$, if a voting rule satisfies $s(\alpha)$-Cons and β-AV, then $\alpha + \beta \leq 1$.*

Proof. Let $y = 1/s(1 - \beta)$. For an arbitrary constant $x \in \mathbb{N}$, consider the election $\mathcal{E} = (V, A, k)$, defined as follows (see Figure 5 for an illustration). Let V consist of x^2 *block* voters $(v_i^b)_{i \in [x^2]}$, as well as yx^3 *arm* voters split into y arms $(v_{i,j}^a)_{i \in [x^3], j \in [y]}$, $y(x^2 - 1)$ *chain* voters also split into y arms, $(v_{i,j}^c)_{i \in [x^2-1], j \in [y]}$, and one *central* voter v^q. Moreover, let C contain $yx^2 + 1$ *block* candidates

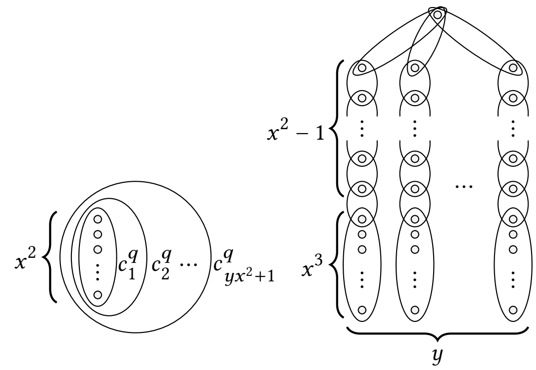

Figure 5: An illustration of the profile constructed in the proof of Proposition 5.10.

$(c_i^b)_{i \in [yx^2+1]}$, y *arm* candidates $(c_i^a)_{i \in [y]}$, and yx^2 *chain* candidates $(c_{i,j}^c)_{i \in [x^2], j \in [y]}$.

All block voters approve all block candidates, i.e., for every $i \in [x^2]$ and $j \in [yx^2 + 1]$ voter v_i^b approves c_j^b. Next, arm voters in each arm approve the respective arm candidate, i.e., for every $i \in [x^3]$ and $j \in [y]$, voter $v_{i,j}^a$ approves c_j^a. Then, every chain voter approves the chain candidate corresponding to their index and the next one, i.e., for every $i \in [x^2 - 1]$ and $j \in [y]$ voter $v_{i,j}^c$ approves candidates $c_{i,j}^c$ and $c_{i+1,j}^c$. Finally, the first arm voter in each arm approves the last chain candidate in each arm, i.e., for every $j \in y$ voter $v_{1,j}^a$ approves $c_{x^2,j}^c$ and the central voter v^q approves the first chain candidates in each arm, i.e., $c_{1,j}^c$ for every $j \in [y]$.

Now, assume that we want to select a committee of size $k = y(x^2 + 1) + 1$. The high-level idea for the proof is that for Cons it is important to connect the arm voters through the selection of chain candidates. However, if we select a β-fraction of block candidates in order to guarantee β-AV, then we cannot connect any two such groups of voters.

Observe that at least one block candidate will always be chosen, as there are not enough other candidates. Moreover, if we want to maximize either AV or Cons, it is always better to select an arm candidate than any other candidate, thus we can assume that we select all y of them. Then, from the remaining yx^2 slots in the committee, we can select any number of $z \leq yx^2$ block candidates and $yx^2 - z$ chain candidates. Let us denote an arbitrary committee with such a selection of candidates by $W_z \subseteq C$.

For AV, we get score x^2 for every selected block candidate, x^3 for every selected arm candidate, and 2 for every selected chain candidate. Thus, we obtain

$$\mathrm{AV}(W_z, \mathcal{E}) = zx^2 + yx^3 + x^2 + 2(yx^2 - z).$$

Observe that we maximize AV when $z = yx^2$, thus, in order to obtain β-AV it has to be the case that $z \geq \beta yx^2 - \mathcal{O}(x)$.

Now, for Cons, we claim that for large enough x, with the remaining $yx^2 - z$ chain candidates, we cannot connect arm voters from any two arms. To this end, recall that $y = 1/s(1 - \beta)$, and let p be smallest integer such that $1 - \beta \geq 2/p$. Thus, $1 - \beta < 2/(p - 1)$. Also, by definition $y = p - 1$. Thus, we get $(1 - \beta)y < 2(p - 1)/(p - 1) = 2$.

In other words, there is an $\varepsilon > 0$ such that $(1-\beta)y = 2-\varepsilon$. Since $z \geq \beta y x^2 - \mathcal{O}(x)$, we choose at most $x^2 y - z \leq (1-\beta)yx^2 + \mathcal{O}(x) < 2x^2 - \varepsilon x^2 + \mathcal{O}(x)$ arm candidates. Thus, for large enough x we have strictly fewer than $2x^2$ chain candidates, which proves the claim.

Let us now calculate the value of the Cons objective. For block voters we get $\binom{x^2}{2}$, no matter how many block candidates we choose. For connections of chain voters to themselves we get at most $\binom{yx^2}{2}$. For connections between chain voters and arm voters we get at most yx^5. And finally, for connections of arm voters to themselves, since we did not connected any two arms with each other, we get $y\binom{x^3}{2}$. In total

$$\text{Cons}(W_z, \mathcal{E}) = \frac{y}{2}x^6 + \mathcal{O}(x^5).$$

The maximum value of Cons is obtained when we select all yx^2 chain voters, which yields

$$\binom{yx^3 + x^2 y + 1}{2} = \frac{y^2}{2}x^6 + \mathcal{O}(x^5).$$

Thus, the fraction of Cons we can obtain while satisfying β-AV converges to $\frac{1}{y}$ for $x \to \infty$. As we assumed, $\frac{1}{y} = s(1-\beta)$, which concludes the proof. □

Consider the two upper bounds that we have obtained for α-approximation of Cons given that a voting rule satisfies β-AV in Propositions 5.8 and 5.10 (their plots are presented in Figure 6). Since the upper bounds intersect several times and they are of different nature (stepwise vs. continuous), it seems that establishing tight trade-offs might be a challenging and interesting problem. Similarly, because Cons is not subadditive, finding a general lower bound for these trade-offs seems highly non-trivial as well. Nevertheless, we conclude this section with a positive result on guarantees that we can obtain for a combination of Pairs and Cons objectives in the VI domain, which matches our upper bound.

Proposition 5.11. *For instances (V, A, k) in the VI domain with even k, there exists a committee satisfying $\frac{1}{2}$-AV, $\frac{1}{2}$-CC, $\frac{1}{2}$-EJR, or $\frac{1}{2}$-Pairs and $\frac{1}{4}$-Cons.*

Proof sketch. Let $\mathcal{E}$ be an election with even k that belongs to the VI domain. Take an optimal committee W with respect to Cons. In the corresponding hypergraph, W consists of one or more connected components.

For components in which we have an even number of candidates, we can show that half of these candidates connect more than half of the voters covered by the whole component. This gives us at least $\frac{1}{4}$ of connections inside the component.

For components with an odd number of candidates, we observe that since k is even, there is an even number of them. We arbitrarily group them into pairs. Then, for every pair, we show that it is possible to select half of the candidates rounded up in one of them and half of the candidates rounded down in the other so that in total we cover $\frac{1}{4}$ connections from both. □

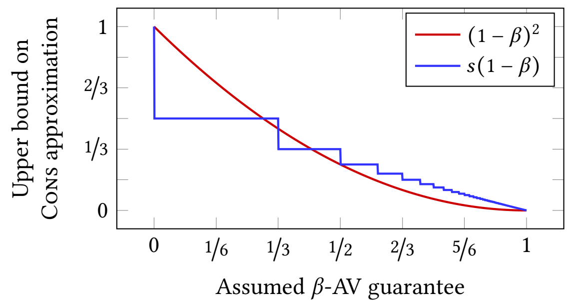

Figure 6: Two different upper bounds on the possible α-approximation of Cons for rules that satisfy β-AV. The $(1-\beta)^2$ upper bound is the result of Proposition 5.8 and $s(1-\beta)$ is implied by Proposition 5.10.

6 CONCLUSION

Our paper sheds new light on the interdependency of mass and elite polarization. We observe that the selection of a representative committee can significantly influence elite polarization independently of mass polarization. With the aim of avoiding polarization at the level of the representation, we have introduced Pairs and Cons, two numerical objectives that measure how well a committee interlaces the electorate.

We show that, while maximizing both objectives is NP-hard, a committee maximizing either of them can be computed in polynomial time on the voter-candidate interval domain. Also, we study the compatibility of our objectives with measures of excellence, diversity, and proportionality. We find approximation trade-offs suggesting that there is nothing better than dividing the committee seats among different objectives and trying to maximize each objective with their designated share of the committee: in the worst case, the synergies are negligible. While a subcommittee yields the approximation of an objective proportional to its size, the dependency for Cons is quadratic (or even worse0, leading to inferior guarantees.

We believe that our work offers an important perspective that has been missing from the social choice literature on multiwinner voting. As such, it calls for further research; in what follows, we suggest some promising directions.

An immediate open question is to determine the exact trade-off between Cons and other objectives. While we have a bound for α^2-Cons and β-approximations of other objectives, Proposition 5.10 shows that the picture is more nuanced.

Going beyond our base model, another direction is to consider our objectives in the broader context of participatory budgeting (PB), where each candidate has a cost, and the committee needs to stay within a given budget. In this setting, candidates are usually projects, such as a playground, a community garden, or a cycling path. Interlacing voters by projects in PB has an additional interpretation apart from linking similar opinions: the funded projects may lead to interaction among the agents who use them (e.g., working together in a community garden). This seems quite desirable in the context of PB, where one of the goals is community building.

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APPENDIX FOR SUBMISSION 933

In the supplementary material, we present proofs missing from the main part of the paper.

A MISSING PROOFS FROM SECTION 4

In this section, we consider the missing proof about the computation of Pairs and Cons. We start with the hardness result for Cons.

Theorem 4.2. *It is* NP*-complete to decide whether, given an election $\mathcal{E} = (V, A, k)$ and a threshold $q \in \mathbb{N}$, there exists a committee W of size k such that* $\text{Cons}(W, \mathcal{E}) \geq q$. *The hardness result holds even if $q = \binom{n}{2}$, i.e., if the goal is to connect all n voters.*

Proof. Membership in NP is clear: checking if a committee connects q pairs of voters reduces to finding connected components of the associated graph (where there is an edge from v to v' if v and v' approve the same committee member).

To show NP-hardness, we present a reduction from X3C. Given an instance $(R, \mathcal{S})$ of X3C with $|R| = 3\rho$, we construct an instance of our problem as follows. We create a candidate for each set in $\mathcal{S}$ and a voter for each element of the ground set R, as well as one additional voter, i.e., we set $C = \{c_S : S \in \mathcal{S}\}$, $V = \{v\} \cup \{v_r : r \in R\}$. For each $S \in \mathcal{S}$, c_S is approved by $\{v\} \cup \{v_r : r \in S\}$. We want to select a committee $W \subseteq C$ of size $k = \rho$ and set the threshold q to $\binom{n}{2}$, where $n = |V|$.

Consider a collection $\mathcal{S}' \subseteq \mathcal{S}$ of size k and the respective committee $W = \{c_S : S \in \mathcal{S}'\}$. If $\mathcal{S}$ covers R, each voter in V approves a candidate in W, and v approves all candidates, so all voters are connected via v. Conversely, if all pairs of voters are connected, then each voter must approve some candidate in W and hence $\mathcal{S}'$ covers R. This completes the proof. □

Next, we provide the proof that the VCI domain is equal to the CI domain in the absence of dominated candidates.

Proposition 4.3. *Let $\mathcal{E}$ be an instance in the VCI domain. If $\mathcal{E}$ contains no dominated candidates, then it belongs to the CI domain.*

Proof. Consider an election $\mathcal{E} = (V, A, k)$ over the candidate set C that belongs to the VCI domain, as witnessed by positions $\{x_c\}_{c\in C} \cup \{x_v\}_{v\in V}$ and radii $\{r_c\}_{c\in C} \cup \{r_v\}_{v\in V}$. Renumber the candidates so that $x_{c_1} \leq x_{c_2} \leq \cdots \leq x_{c_m}$.

Suppose for the sake of contradiction that this ordering of the candidates does not witness that $\mathcal{E}$ belongs to CI. Then, there exists a voter $v \in V$ and $h < i < j$ such that v approves c_h and c_j, but not c_i. For readability, we will refer to the positions and radii of c_h, c_i and c_j as x_h, x_i, x_j and r_h, r_i, r_j, respectively. Since v does not approve c_i, we have $x_v \neq x_i$; we can then assume without loss of generality that $x_v < x_i \leq x_j$. To obtain a contradiction, we will show that c_i is dominated by c_j.

We will first argue that $[x_i - r_i, x_i + r_i] \subseteq [x_j - r_j, x_j + r_j]$. Indeed, $c_i \notin A_v$ implies $x_i - r_i > x_v + r_v$ whereas $c_j \in A_v$ implies $x_j - r_j \leq x_v + r_v$. Rearranging the terms, we obtain $r_i < x_i - x_v - r_v \leq x_j - x_v - r_v \leq r_j$. Thus, the right endpoint of c_j's interval is at $x_j + r_j \geq x_i + r_i$. For the left endpoints of both intervals, in a similar manner we obtain $x_i - r_i > x_v + r_v \geq x_j - r_j$. Thus, the interval of c_i is subsumed by that of c_j, and hence every voter who approves x_i also approves x_j. Moreover, v approves c_j, but not c_i. We have shown that c_i is dominated, concluding the proof. □

Finally, we provide the details for the polynomial time computation of our polarization-preventing objectives in the VCI domain. We start with the simple proof for Pairs.

Proposition 4.4. *In the VCI domain, a committee that maximizes* Pairs *can be computed in polynomial time.*

Proof. Fix an election instance $\mathcal{E}$. As argued earlier, we can assume that $\mathcal{E}$ is in the CI domain with respect to candidate ordering $c_1, \ldots, c_m$. Recall that $\text{Pairs}(W, \mathcal{E}) = \text{CC}(W, \mathcal{E}^{(2)})$. Now, note that for all $\{u, v\} \in V^{(2)}$, it holds by definition that $A_{\{u,v\}} = A_u \cap A_v$ is the intersection of two intervals of that ordering and hence itself an interval. Thus, $\mathcal{E}^{(2)}$ is in the CI domain with respect to the same candidate ordering. For instances in the CI domain, CC can be maximized in polynomial time [4, 15]. This concludes the proof. □

We conclude the section by providing the proof concerning the dynamic program that computes a committee maximizing Cons on the VCI domain.

Theorem 4.5. *In the VCI domain, a committee that maximizes* Cons *can be computed in polynomial time.*

Proof. Consider an election $\mathcal{E} = (V, A, k)$; again, we assume that $\mathcal{E}$ belongs to the CI domain, as witnessed by the candidate order $c_1, \ldots, c_m$. For each voter $v \in V$, let $\ell(v) := \min\{i : c_i \in A_v\}$ and $r(v) := \max\{i : c_i \in A_v\}$ be the leftmost and rightmost approved candidates of voter v, respectively. Given candidate indices $1 \leq j < i \leq m$, let $V(\neg j, i)$ be the set of voters that approve c_i, but not c_j, i.e., $V(\neg j, i) := \{v \in V : j < \ell(v) \leq i \leq r(v)\}$. Further, we introduce an indicator variable $\mathbb{1}(j \wedge i)$ defined by

$$\mathbb{1}(j \wedge i) := \begin{cases} \text{true, if } V_{c_j} \cap V_{c_i} \neq \varnothing \\ \text{false, otherwise.} \end{cases}$$

Hence, $\mathbb{1}(j \wedge i)$ is true if and only if there is a voter that approves both c_j and c_i.

Calculating connected pairs after the addition of a candidate. Consider adding c_i to a committee $W \subseteq \{c_1, \ldots, c_{i-1}\}$. Let j^* be the index of the rightmost candidate in W. We will now show how to update $\text{Cons}(W \cup \{c_i\})$ dependent on whether there is some voter approving c_{j^*} and c_i, i.e., dependent on the value of $\mathbb{1}(j^* \wedge i)$.

First, if $\mathbb{1}(j^* \wedge i)$ is false (or if $W = \varnothing$, i.e., c_i is the only candidate), then no voter in V_{c_i} approves any candidate in W. This is because, in the CI domain, if $c_j, c_i \in A_v$ for some $j < j^*$ and $v \in V$, then also $c_{j^*} \in A_v$. Thus, adding c_i only connects the voters approving c_i and no members of W, i.e., $|V(\neg j^*, i)|$ additional voters. Thus, after we add c_i to W, we obtain

$$\text{Cons}(W \cup \{c_i\}) = \text{Cons}(W) + \binom{|V(\neg j^*, i)|}{2}$$

connected pairs.

Otherwise, $\mathbb{1}(j^* \wedge i)$ is true, and, by CI, if a voter approves c_i and some $c_j \in W$, they also approve c_{j^*}. The update now depends on the connected component containing c_{j^*} on the hypergraph induced by $W \cup \{c_i\}$. Consider a candidate $c \in C$ and a voter $v \in V_c$. We define $K_W(c) := \{u \in V : u \sim_W v\}$, i.e., $K_W(c)$ is the set of voters contained in the connected component containing the hyperedge c.

Note that $K_W(c)$ is well defined because $\sim_W$ is an equivalence relation and $u \sim_W u'$ for all $u, u' \in V_c$.

We have that

$$K_{W\cup\{c_i\}}(c_i) = V(\neg j^*, i) \cup K_W(c_{j^*}),$$

where the union is disjoint. Since $K_W(c_{j^*})$ is already connected, adding c_i creates two types of connections: those within the newly connected voters in $V(\neg j^*, i)$ and those between $V(\neg j^*, i)$ and the voters in the connected component to which they connect, i.e., $K_W(c_{j^*})$. This leads to

$$\textsc{Cons}(W\cup\{c_i\}) = \textsc{Cons}(W)+\binom{|V(\neg j^*, i)|}{2}+|V(\neg j^*, i)||K_W(c_{j^*})|$$

connected pairs after the addition of c_i.

Defining the dynamic program. Let $\mathbf{opt}[i, x, b]$ denote the maximum number of connected pairs that can be achieved by a committee of size at most b that has c_i as its rightmost candidate, while c_i is in a connected component of size x. We use the convention that $\mathbf{opt}[i, x, b] = -1$ if there is no such committee. We will define functions $\mathbf{dp}[i, x, b]$ and $W[i, x, b]$ and argue that for all i, x, b it holds that $\mathbf{dp}[i, x, b] = \mathbf{opt}[i, x, b]$ and, moreover, if this value is nonnegative, $W[i, x, b]$ is a committee of size at most b with rightmost candidate c_i having a connected component of size x satisfying $\textsc{Cons}(W[i, x, b], \mathcal{E}) = \mathbf{opt}[i, x, b]$.

For the initialization, we consider $b = 1$. We further take care of the trivial solution when $i = 1$ to avoid a case distinction later.

- For $i \in [m]$, we initialize $\mathbf{dp}[i, |V_{c_i}|, 1] = \textsc{Cons}(\{c_i\}, \mathcal{E})$. This is the number of pairs connected by $\{c_i\}$. Moreover, set $W(i, |V_{c_1}|, 1) = \{c_i\}$, and $\mathbf{dp}[i, x, 1] = -1$ for all other x.
- For $b \in [k]$, we initialize $\mathbf{dp}[1, |V_{c_1}|, b] = \textsc{Cons}(\{c_i\}, \mathcal{E})$. Again, this is the number of pairs connected by $\{c_1\}$. Also, set $W(1, |V_{c_1}|, b) = \{c_1\}$, and $\mathbf{dp}[1, x, b] = -1$ for all other x.

Clearly, for $b = 1$ (resp., $i = 1$) the claim is correct, i.e., it holds that $\mathbf{dp}[i, x, p] = \mathbf{opt}[i, x, p]$ for all $x, i \leq m$ (resp., $x \leq m$, $b \leq k$) and if the value is nonnegative, then $W[i, x, b]$ is a committee achieving this value.

For the induction step, let i, x, and b be given with $i, b \geq 2$. Any committee of size at most b with rightmost candidate c_i whose connected component is of size x is induced by a committee of size at most $b - 1$ with rightmost candidate c_j, where $j < i$, inducing a connected component of size $y \leq m$. The sizes of the connected components need to align, i.e., if $\mathbb{1}(j \wedge i)$ is false, the only feasible size is $x = |V(\neg j, i)|$, while only $x = |V(\neg j, i)| + y$ is possible when $\mathbb{1}(j \wedge i)$ is true. To choose the committee $W[i, x, b]$, we calculate the numbers of pairs induced when adding c_i to $W[j, y, b - 1]$ for each feasible j, y, then go with the best extension. For this, define $\mathbf{score}(i, x, b, j, y)$ for each $j < i$, $y \leq m$ as follows:

- If $\mathbb{1}(j \wedge i)$ is true, $y + |V(\neg j, i)| = x$, and $\mathbf{dp}[j, y, b - 1] \geq 0$, set $\mathbf{score}(i, x, b, j, y) = \mathbf{dp}(j, y, b - 1) + \binom{x-y}{2} + (x - y)y$.
- If $\mathbb{1}(j \wedge i)$ is false, $|V(\neg j, i)| = x$, and $\mathbf{dp}(j, y, b - 1) \geq 0$, set $\mathbf{score}(i, x, b, j, y) = \mathbf{dp}(j, y, b - 1) + \binom{x}{2}$.
- Else, set $\mathbf{score}(i, x, b, j, y) = -1$.

As, intuitively, $\mathbf{score}(i, x, b, j, y)$ calculates the number of pairs that can be obtained when (successfully) extending the committee $W[j, y, b - 1]$ by adding c_i, we define $\mathbf{dp}[i, x, b]$ and $W[i, x, b]$ to maximize this score.

- $\mathbf{dp}[i, x, b] = \max_{(j,y)} \mathbf{score}(i, x, b, j, y)$.
- $W[i, x, b] = W(j^*, y^*, b - 1) \cup \{c_i\}$ for some maximizers j^*, y^* of $\mathbf{score}(x, i, b, j, y)$, if $\mathbf{dp}[i, x, b] \geq 0$.

Correctness of the dynamic program. By construction, it holds that if $\mathbf{dp}[i, x, b]$ is nonnegative, then $\textsc{Cons}(W[i, x, b], \mathcal{E}) = \mathbf{dp}[i, x, b]$.

It remains to prove correctness of the update formulas for the dynamic program, i.e., $\mathbf{dp}[i, x, b] = \mathbf{opt}[i, x, b]$. First, note that, by definition, $\mathbf{dp}[i, x, b] = \mathbf{opt}[i, x, b] = -1$ if there is no committee satisfying the constraints. To consider satisfiable (i, x, b), we split the proof into two inequalities.

For "≤", let $W[i, x, b]$ be induced by the score maximizer $W' = W(j^*, y^*, b - 1)$. Then, W' is of size at most $b - 1$, with rightmost candidate index $j^* < i$ having a connected component of size y^* and $W = W' \cup \{c_i\}$. By our previous observations, if $\mathbb{1}(j^* \wedge i)$ is true, the number of connected pairs $W[i, x, b]$ induces is equal to

$$\begin{aligned}\textsc{Cons}(W[i, x, b]) &= \textsc{Cons}(W') + \binom{x - y^*}{2} + (x - y^*)y^* \\ &= \mathbf{dp}(j^*, y^*, b - 1) + \binom{x - y^*}{2} + (x - y^*)y^* \\ &= \mathbf{score}[i, x, b, j^*, y^*] \\ &= \mathbf{dp}[i, x, b],\end{aligned}$$

and if $\mathbb{1}(j^* \wedge i)$ is false, the number of pairs $W(i, x, b)$ induces is equal to

$$\begin{aligned}\textsc{Cons}(W[i, x, b]) &= \textsc{Cons}(W') + \binom{x}{2} \\ &= \mathbf{dp}(j^*, y^*, b - 1) + \binom{x}{2} \\ &= \mathbf{score}[i, x, b, j^*, y^*] \\ &= \mathbf{dp}[i, x, b].\end{aligned}$$

As $W[i, x, b]$ is a feasible committee, we conclude that $\mathbf{dp}[i, x, b] \leq \mathbf{opt}[i, x, b]$.

For "≥", consider the committee W^* that achieves $\mathbf{opt}[i, x, b]$. Since $b \geq 2$ and $i \geq 2$, we may assume without loss of generality that $|W| \geq 2$. We set $W' = W^* \setminus \{c_i\}$, j as the candidate with rightmost index in W', and y as the size of its connected component under W'. If $\mathbb{1}(j^* \wedge i)$ is true, we have

$$\begin{aligned}\mathbf{dp}[j, y, b - 1] &= \mathbf{opt}[j, y, b - 1] \\ &\geq \textsc{Cons}(W') \\ &= \textsc{Cons}(W) - \binom{x - y}{2} - (x - y)y \\ &= \mathbf{opt}[i, x, b] - \binom{x - y}{2} - (x - y)y,\end{aligned}$$

and if $\mathbb{1}(j^* \wedge i)$ is false, we have

$$\begin{aligned}\mathbf{dp}[j, y, b-1] &= \mathbf{opt}[j, y, b-1] \\ &\geq \textsc{Cons}(W') \\ &= \textsc{Cons}(W) - \binom{x}{2} \\ &= \mathbf{opt}[i, x, b] - \binom{x}{2}.\end{aligned}$$

In the first case, we have $\mathbf{dp}[i, x, b] \geq \mathbf{score}[i, x, b, j, y] = \mathbf{dp}[j, y, b-1] + \binom{x-y}{2} + (x-y)y \geq \mathbf{opt}[i, x, b]$. Similarly, in the second case, we have $\mathbf{dp}[i, x, b] \geq \mathbf{score}[i, x, b, j, y] = \mathbf{dp}[j, y, b-1] + \binom{x}{2} \geq \mathbf{opt}[i, x, b]$.

Together, we obtain $\mathbf{dp}[i, x, b] = \mathbf{opt}[i, x, b]$. Finally, to compute a feasible committee that maximizes Cons, we output any committee $W \in \arg\max_{i \in [m], x \in [n]} \textsc{Cons}(W[i, x, b])$.

Note that the dynamic program has $\mathcal{O}(mnk)$ cells as $i \in [m]$, $x \in [n]$, and $b \in [k]$. Moreover, every cell can be computed in polynomial time given the values of previously computed cells. Hence, we have obtained a polynomial-time algorithm to compute a committee maximizing Cons. □

B MISSING PROOFS FROM SECTION 5

In this section, we provide the missing proofs for all results on tradeoffs between classic and interlacing committee selection objectives.

B.1 Proofs from Section 5.1

We start with trade-offs that concern the Pairs objective.

Proposition 5.4. *For every $\alpha, \beta \in [0, 1]$, if a voting rule satisfies α-Pairs and β-CC, then $\alpha + \beta \leq 1$.*

Proof. For a given constant $x \in \mathbb{N}$, consider the election instance $\mathcal{E} = (V, A, k)$, defined as follows. (see Figure 4 in the main paper for an illustration). Let there be x^5 *block* voters $(v^b_{i,j})_{i \in [x], j \in [x^4]}$, x^4+1 *arm* voters $(v^a_i)_{i \in [x^4+1]}$, and x^3 *central* voters $(v^q_i)_{i \in [x^3]}$. In total, there are $x^5+x^4+x^3+1$ voters in V. Additionally, we will have x^4 *block* candidates $(c^b_i)_{i \in [x^4]}$ and x^4+1 *arm* candidates $(c^a_i)_{i \in [x^4+1]}$. In total, there are $2x^4+1$ candidates. We define that for every $i \in [x]$ and $j \in [x^4]$, block voter $v^b_{i,j}$ approves only block candidate c^b_j and, similarly, for every $i \in [x^4+1]$, arm voter c^a_i approves only arm candidate c^a_i. On the other hand, for every $i \in [x^3]$, central voter v^q_i approves all arm candidates and no further candidates.

Now, assume that we want to select a committee of size $k = x^4 + 1$. By symmetry, without loss of generality, this means that for some $\alpha \in \{0, 1/x^4, 2/x^4, \ldots, 1\}$ we select committee $W_\alpha \subseteq C$ containing arm candidates $c^a_1, c^a_2, \ldots, c^a_{\alpha x^4+1}$ and block candidates $c^b_1, c^b_2, \ldots, c^b_{(1-\alpha)x^4}$. Observe that every selected block candidate covers x voters and $x(x-1)/2$ pairs of voters. In turn, the first selected arm candidate covers x^3+1 voters and $(x^3+1)x^3/2$ pairs of voters and every following arm candidate covers 1 voter and x^3 pairs of voters. Thus, when we select $\alpha x^4 + 1$ arm candidates and $(1-\alpha)x^4$ block candidates, we get the following CC and Pairs scores:

$$\text{CC}(W_\alpha, \mathcal{E}) = (1-\alpha)x^5 + \alpha x^4 + x^3 + 1, \text{ and}$$
$$\textsc{Pairs}(W_\alpha, \mathcal{E}) = \alpha x^7 + \frac{2-\alpha}{2}x^6 - \frac{1-\alpha}{2}x^5 + \frac{1}{2}x^3.$$

Observe that the maximum number of covered voters is obtained when we take $\alpha = 0$, and the maximum number of covered pairs of voters is obtained when $\alpha = 1$. Also we have,

$$\frac{\text{CC}(W_\alpha, \mathcal{E})}{\text{CC}(W_0, \mathcal{E})} = 1 - \alpha + \mathcal{O}(1/x) \quad \text{and} \quad \frac{\textsc{Pairs}(W_\alpha, \mathcal{E})}{\textsc{Pairs}(W_1, \mathcal{E})} = \alpha + \mathcal{O}(1/x).$$

Therefore, when x goes to infinity, every possible solution becomes at most an α-approximation of Pairs and an $(1-\alpha)$-approximation of CC for some $\alpha \in [0, 1]$. □

We now define MES formally: At the start, every voter $v \in V$ is assigned a budget $\text{bud}(v) = \alpha \frac{k}{n}$, the committee W is initialized as empty set and every candidate has a cost of 1. In each step, we consider the candidates in $C \setminus W$ that can be bought by the voters approving it. We then choose a candidate minimizing the maximum amount of budget a voter approving it has to spend to buy it into the committee. More formally, for each $c \in C \setminus W$ such that $\sum_{v : c \in A_v} \text{bud}(v) \geq 1$, we set $\rho(c)$ as the minimal value $\rho \geq 0$ such that $\sum_{v : c \in A_v} \min(\text{bud}(v), \rho) \geq 1$. If there is no such c, the algorithm terminates and returns W. Else, we add c^* to W with c^* minimizing $\rho(c)$. We further update the voter budgets $\text{bud}(v)$ ad $\text{bud}(v) - \min(\text{bud}(v), \rho(c^*))$ for all voters v with $c^* \in A_v$.

Lemma 5.5. *Let $\alpha \leq 1$ be given. For every election $\mathcal{E} = (V, A, k)$, executing MES on $(V, A, \alpha k)$ returns a committee of size $\lfloor \alpha k \rfloor$ satisfying α-EJR in polynomial time.*

Proof. For brevity, set $n = |V|$. MES terminates after at most $\lfloor \alpha k \rfloor$ rounds, as in each round the total budget is reduced by 1 and the total budget is αk. Hence, the returned committee W is also of size at most $\lfloor \alpha k \rfloor$.

Assume for contradiction that W violates α-EJR, i.e., there is $S \subseteq V$ and $\ell \leq k$ with $|S| \geq \frac{l}{\alpha k} n$ and $|\bigcap_{v \in S} A_v| \geq l$, but $|A_v \cap W| < \ell$ for all $v \in S$.

From this, we will infer the following: For any purchase done during the run of the algorithm, each voter in S spends at most $\alpha \frac{k}{n\ell}$. If this claim was false, consider for contradiction the first time t at which a voter $v \in S$ spends strictly more than $\alpha \frac{k}{n\ell}$ to buy the candidate c_t into $W = \{c_1, \ldots, c_t, \ldots, c_r\}$. By definition of the rule, then $\rho(c_t) > \alpha \frac{k}{n\ell}$ in round t. Before this purchase, all voters spent at most $\alpha \frac{k}{n\ell}$ per candidate c_i, $i < t$. By assumption, they approve at most $\ell - 1$ candidates from W. Thus, before the purchase of c_t, each voter has a remaining budget of at least $\alpha \frac{k}{n} - (\ell - 1)\alpha \frac{k}{n\ell} = \alpha \frac{k}{n\ell}$. Together, they thus have a budget of $\alpha \frac{k}{n\ell}|S| \geq \alpha \frac{k}{n\ell} \frac{\ell}{\alpha k} n = 1$. This means that for all candidates $c^* \in \bigcap_{v \in S} A_v$ with $c^* \neq c_1, \ldots c_{t-1}$, we have $\rho(c^*) \leq \alpha \frac{k}{n\ell}$ in round t. This is the desired contradiction, as c_t was not a minimizer of ρ in round t.

Following from the claim, after the rule has terminated, each voter in S still has a remaining budget of at least $\alpha \frac{k}{n\ell}$. Again, this yields a total budget for S of at least 1, a contradiction to the fact that our modification of the method of equal shares terminated already. Hence, no violation of α-EJR can occur. □

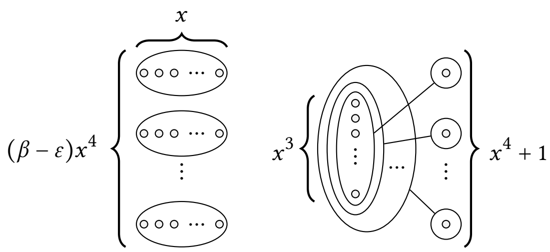

Figure 7: Illustration of the profile constructed in the proofs of Proposition 5.7 and Proposition 5.9. Block voters are on the left, central voters in the middle, and arm voters on the right. Block candidates are approved by x block voters each, whereas arm candidates are approved by all central voters and one arm voter each.

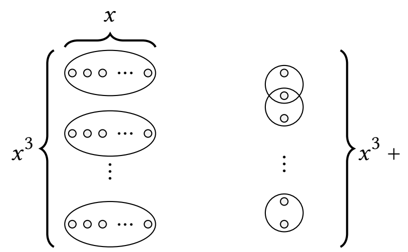

Figure 8: An illustration of the profile constructed in the proof of Proposition 5.8. Block voters are on the left, central voters on the right. Each central candidate only adds one pair, but choosing all central candidates yields a large connected component.

Proposition 5.7. *For every $\alpha, \beta \in [0, 1]$, if a voting rule satisfies α-Pairs and β-EJR, then $\alpha + \beta \leq 1$.*

Proof. Let f be a voting rule that satisfies β-EJR for some $\beta \in [0, 1]$. Let any $\varepsilon > 0$ be given with $\varepsilon < \beta$. Consider the profile from the proof in Proposition 5.4, but reduce the number of arms such that we only have $(\beta - \varepsilon)x^4$ instead of x^4 arm candidates and voters.

We claim that any committee satisfying β-EJR has to contain all block candidates. For this, note that there are $(\beta - \varepsilon)x^5$ block voters and $x^3 + x^4 + 1$ central and arm voters. Thus, $\frac{n}{k} = (\beta - \varepsilon)x + O(1)$, yielding $\frac{n}{\beta k} = \frac{(\beta-\varepsilon)}{\beta}x + O(1) < x$ for large enough x. Take any block candidate b. Since b is approved by x voters, at least one of the supporters obtains an approved candidate in the committee. The only such candidate is b itself, proving the claim.

Following from our claim, at most $(1 - \beta + \varepsilon)x^4$ candidates that are not block candidates can be contained in any committee satisfying β-EJR. With the same argument as for previous proofs, such a committee induces at most $(1 - \beta + \varepsilon)x^7 + O(x^6)$ pairs that share an approval, and $(1 - \beta + \varepsilon)^2 \frac{x^8}{2} + O(x^7)$ pairs that are connected in the hypergraph. The optimal number of pairs sharing an approval is $x^7 + O(x^6)$, and the optimal number of pairs that are connected is $\frac{x^8}{2} + O(x^7)$. Thus, any rule that satisfies β-EJR is at most a $1 - \beta + \varepsilon$ approximation of Pairs, and a $(1 - \beta + \varepsilon)^2$ approximation of Cons. Since ε can be chosen arbitrarily close to zero, letting ε tend to 0 concludes the proof. □

B.2 Proofs from Section 5.2

Proposition 5.8. *For every $\alpha, \beta \in [0, 1]$, if a voting rule satisfies α^2-Cons and β-AV, β-CC, or β-Pairs, then $\alpha + \beta \leq 1$. This already holds in the VI domain.*

Proof. For the proof of all three statements, consider the instance (V, A, x^3) depicted in Figure 8: There are x^3 blocks, each consisting of x block voters $(v^b_{i,j})_{i \in [x^3], j \in [x]}$. In addition, there are x^3 block candidates $(c_i)_{i \in [x^3]}$. All block voters $v^b_{i,j}$ from block $i \leq x^3$ approve block candidate c_i exclusively. Further, we have $x^3 + 1$ central voters $(v^q_i)_{i \in [x^3+1]}$, together with x^3 central candidates $(c^q_i)_{i \in [x^3]}$. Each central candidate c^q_i is exclusively approved by central voters v^q_i and v^q_{i+1}.

To show that this instance is in VI, we enumerate first the voters according to their blocks, and then the central candidates from first to last. By definition of the profile, each candidate is approved by an interval of voters.

Now, the proof consists of two parts. In part one, we show that to satisfy β-Pairs, CC, or AV, we require at least $\beta x^3 - \mathcal{O}(x^2)$ block candidates. In part two, we show that with the remaining candidates, we can obtain at most a $(1 - \beta)^2$ approximation of Cons.

Pairs: We start by considering how many block candidates are required to guarantee β-Pairs. Each block candidate creates $\frac{x^2}{2} + \mathcal{O}(x)$ direct pairs, each central candidate only one. Thus, clearly the optimal committee for Pairs consists of the x^3 block candidates with an objective value of $x^3(\frac{x^2}{2} - \frac{x}{2})$. Since we have to guarantee at least a β-fraction of this value, note that the chosen central candidates can in total yield at most x^3 pairs, negligible for the Pairs objective. Thus, if we choose y block candidates, then the total number of connected pairs is at most $x^3 + y(\frac{x^2}{2} - \frac{x}{2})$, hence our approximation guarantee implies $x^3 + y(\frac{x^2}{2} - \frac{x}{2}) \geq \beta x^3(\frac{x^2}{2} - \frac{x}{2})$ and thus

$$y \geq \beta x^3 - \frac{x^3}{(\frac{x^2}{2} - \frac{x}{2})} = \beta x^3 - \mathcal{O}(x^2).$$

Cons: We now show that with the remaining $(1 - \beta)x^3 + \mathcal{O}(x^2)$ candidates, we obtain at best a $(1 - \beta)^2$ approximation of Cons. For this objective, it is optimal to fill all remaining seats with central candidates. To see this, the block candidates have disjoint support sets and can together contribute at most $x^3\binom{x^2}{2}$ connections, negligible for Cons, but as soon as we choose at least x^2 central voters each further central voter contributes more than that each block voter to Cons. By adding the remaining $(1 - \beta)x^3 + \mathcal{O}(x^2)$ central candidates to the chosen $\beta x^3 - \mathcal{O}(x^2)$ block candidates, we hence obtain at most $\binom{(1-\beta)x^3 + \mathcal{O}(x^2)}{2} + \mathcal{O}(x^5) = (1 - \beta)^2 \frac{x^6}{2} + \mathcal{O}(x^5)$ connections. However, when choosing all x^3 central candidates, we obtain $\binom{x^3}{2} = \frac{x^6}{2} - \mathcal{O}(x^5)$ connections. For large x, hence the

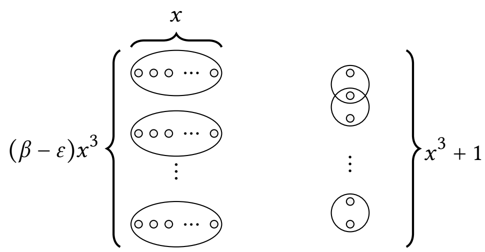

Figure 9: An illustration of the profile constructed in the proof of Proposition 5.9. By reducing the number of blocks in comparison to Figure 8, each block enforces via EJR that their block candidate is elected.

achieved fraction of the objective value converges to $(1-\beta)^2$. In other words, $\alpha \le (1-\beta)^2$. This yields $\sqrt{\alpha}+\beta \le 1$, concluding the proof for Pairs.

AV: Next, we consider committees satisfying β-AV. Note that each block candidate gives an AV-score of x, while each central candidate only contributes a score of 2. Thus, the optimal candidate consists of x^3 block candidates, with an AV-score of x^4. If we choose $y \le x^3$ block candidates, the total score is $yx + 2(x^3 - y)$. To satisfy β-AV, it must hold that $yx + 2(x^3 - y) \ge \beta x^4$, or equivalently $y \ge \frac{(\beta x^4 - 2x^3)}{(x-2)} \ge \frac{\beta x^4 - 2x^3}{x} = \beta x^3 - \mathcal{O}(x^2)$.

CC: If we choose $y \le x^3$ block candidates,the CC-score is $yx + (x^3 - y) + 1$. The optimal CC-score is x^4. Thus, all committees satisfying β-CC have to satisfy $yx + (x^3 - y) + 1 \ge \beta x^4$, which is equivalent to $y \ge \frac{\beta x^4 - x^3 - x^3 + 1}{x-1} \ge \beta x^3 - \mathcal{O}(x^2)$. We obtain once more that for x tending to ∞ we only obtain a $(1-\beta)^2$-approximation of Pairs, concluding the proof. ◻

Proposition 5.9. *For every $\alpha, \beta \in [0,1]$, if a voting rule satisfies α^2-Cons and β-EJR, then $\alpha + \beta \le 1$. This already holds in the VI domain.*

Proof. Fix some small rational number $\varepsilon > 0$ and and some, for now rational, $\beta \le 1$. Consider the instance (V, A, x^3) depicted in Figure 9: There are $(\beta - \varepsilon)x^3$ blocks (which is a natural number for infinitely many $x \in \mathbb{N}$) instead of x^3, the rest remains exactly the same as in Proposition 5.8.

Clearly, this instance remains in VI.

EJR: We start by showing that at least $(\beta-\varepsilon)x^3$ block candidates are required to guarantee β-EJR. Each block candidate has a support of x voters and the total number of voters is $(\beta - \varepsilon)x^4 + \mathcal{O}(x^3)$. Hence, for large enough x, $\frac{n}{k} = (\beta - \varepsilon)x + \mathcal{O}(1) < x$. Consequentially, EJR demands that in each block at least one voter approves a candidate of the winning committee. Since the voters in each block exclusively approve of the corresponding block candidate, all $(\beta - \varepsilon)x^3$ block candidates have to be selected by EJR.

Cons: It is clearly optimal for Cons to use the remaining $(1-\beta+\varepsilon)x^3$ seats of the committee to elect central candidates. This in total yields $\frac{(1-\beta+\varepsilon)^2 x^6}{2} + \mathcal{O}(x^5)$ connections for the elected committee. Since the optimal value still is $\frac{x^6}{2} - \mathcal{O}(x^5)$, obtained by choosing only central candidates, for $x \to \infty$ we obtain an upper bound on the Cons approximation of $(1-\beta+\varepsilon)^2$. Since we can choose ε arbitrarily close to 0, the upper bound becomes $(1-\beta)^2$, concluding the proof for rational β.

For irrational β, simply observe that if the rule satisfied β-EJR, then the rule also satisfies β_i-EJR for a sequence of rational $\beta_i < \beta$ that converge towards β. Since the inequality $\sqrt{\alpha} + \beta_i \le 1$ holds for all i, it also holds for β. ◻

Proposition 5.11. *For instances (V, A, k) in the VI domain with even k, there exists a committee satisfying $\frac{1}{2}$-AV, $\frac{1}{2}$-CC, $\frac{1}{2}$-EJR, or $\frac{1}{2}$-Pairs and $\frac{1}{4}$-Cons.*

Proof. Let (V, A, k) be an election instance in the VI domain as certified by the voter ordering $v_1, \dots, v_n$ and let k be even. It suffices to show that with $\frac{1}{2}k$ candidates, we can guarantee $\frac{1}{4}$-Cons. With the other $\frac{1}{2}k$ candidates, we can use the methods in Propositions 5.1 and 5.2 and Lemma 5.5 to obtain $\frac{1}{2}$-AV, $\frac{1}{2}$-CC, $\frac{1}{2}$-EJR, or $\frac{1}{2}$-Pairs.

The proof idea is to partition the candidates of the optimal solution for Cons into subsets that form intervals. Each interval formed by an even number of candidates is split into two halves and we choose the larger one. If some intervals are formed using an odd number of candidates, there must be an even number of such intervals. We show that for each pair of such intervals, we can always assign one candidate more than half to one interval and one candidate less than half to the other in a way that the $\frac{1}{4}$-approximation remains intact.

First, let $d_1, \dots, d_t \in C$ be candidates such that the voters approving at least one of them form an interval. Since for Cons it is never desirable to select candidates approved by the identical set of voters or dominated candidates, we can assume that in W there are no such candidates. Let $\ell(i)$ and $r(i)$ denote the index of the leftmost and rightmost voter approving d_i, respectively. By renaming candidates, we may assume without loss of generality that $\ell(1) \le \ell(2) \le \dots \le \ell(t)$. By the choice of our candidates, it follows that in fact $\ell(1) < \ell(2) < \dots < \ell(t)$, $r(1) < r(2) < \dots r(t)$ (no candidates approved by identical set of voters, no dominated candidates), and $r(i) \ge \ell(i+1)$ (the candidates form a connected subinterval). Without loss of generality, we assume that $\ell(1) = 1$ and $r(t) = x$ for some $x \in \mathbb{N}$. Hence, the connected subinterval connects x voters.

There are two cases. First, if $t = 2s$ is even, then we simply consider $r(s)$. If $r(s) > \frac{x}{2}$, then clearly we can choose $\{d_1, \dots, d_s\}$ which create an interval containing more than $\frac{x}{2}$ voters. Else, $\ell(s+1) \le r(s) \le \frac{x}{2}$. Hence, the candidates $\{d_{s+1}, \dots, d_t\}$ create an interval containing more than $\frac{x}{2}$ voters. Note that $\binom{y}{2} \ge \frac{1}{4}\binom{x}{2}$ for all $y \ge \frac{x}{2} + 1$, concluding the first part of the proof.

The remaining case is that $t = 2s+1$ is odd, where we say that candidate d_{s+1} is the *central candidate*. Since k is even, there must be a second interval consisting of y voters $\{w_1, \dots, w_y\}$, also formed by an odd number of candidates $e_1, \dots, e_{2s'+1}$. In both intervals we can only choose strictly less or strictly more than half of the candidates. E.g., in the first interval, we can choose s or $s+1$ candidates. Without loss of generality, we may assume that the left half of the interval including the central candidate contains strictly more than half of the voters. Hence, we can consider $h_x \in 0.5\mathbb{N}$ such that $\frac{x}{2} + h_x = r(s+1)$. Then, choosing $\{d_1, \dots, d_{s+1}\}$ yields an interval containing $\frac{x}{2} + h_x$ voters and consequently choosing $d_{s+2}, \dots, d_t$

yields an interval containing at least $x - \frac{x}{2} - h_x + 1 = \frac{x}{2} - h_x + 1$ voters, where the $+1$ comes from the fact that $\ell(s+2) \le r(s+1)$. Analogously, we define h_y such that half of the candidates e_i rounded up we connect a subinterval containing $\frac{y}{2} + h_y$ voters, and with half of the candidates rounded down we can still include $\frac{y}{2} - h_y + 1$ voters in our subinterval.

We now claim that if choosing $d_1, \dots d_s, d_{s+1}$ and $e_1, \dots e_{s'}$ does not yield a $\frac{1}{4}$-approximation of Cons, then we can choose the other half of candidates to achieve this. Clearly, the number of pairs that needs to be connected is

$$\frac{x^2}{8} - \frac{x}{8} + \frac{y^2}{8} - \frac{y}{8}.$$

First, the number of pairs induced by $d_1, \dots d_s, d_{s+1}$ is $\binom{\frac{x}{2}+h_x}{2}$, and the number of pairs induced by $e_1, \dots e_{s'}$, is $\binom{\frac{y}{2}-h_y+1}{2}$. Note that

$$\begin{aligned}\binom{\frac{x}{2}+h_x}{2} &= \frac{(\frac{x}{2}+h_x)^2 - (\frac{x}{2}+h_x)}{2} \\ &= \frac{\frac{x^2}{4} + xh_x + h_x^2 - \frac{x}{2} - h_x}{2} \\ &= \frac{x^2}{8} + \frac{(x+h_x-1)h_x}{2} - \frac{x}{4}.\end{aligned}$$

We want the sum of these to be at least as large as the number of pairs that needs to be connected, so we subtract the former from the latter obtaining

$$\binom{\frac{x}{2}+h_x}{2} - \left(\frac{x^2}{8} - \frac{x}{8}\right) = \frac{(x+h_x-1)h_x}{2} - \frac{x}{8}.$$

Doing the same for terms involving y, we obtain

$$\begin{aligned}\binom{\frac{y}{2}-h_y+1}{2} &= \frac{(\frac{y}{2}-h_y+1)^2 - (\frac{y}{2}-h_y+1)}{2} \\ &= \frac{\frac{y^2}{4} + h_y^2 + 1 - yh_y - 2h_y + y - \frac{y}{2} + h_y - 1}{2} \\ &= \frac{y^2}{8} - \frac{yh_y}{2} + \frac{y}{4} + \frac{h_y^2}{2} - \frac{h_y}{2}\end{aligned}$$

and hence

$$\binom{\frac{y}{2}-h_y+1}{2} - \left(\frac{y^2}{8} - \frac{y}{8}\right) = -\frac{yh_y}{2} + \frac{3}{8}y + \frac{h_y^2}{2} - \frac{h_y}{2}.$$

In total, by summing the two differences together we obtain

$$\binom{\frac{x}{2}+h_x}{2} + \binom{\frac{y}{2}-h_y+1}{2} - \left(\frac{x^2}{8} - \frac{x}{8} + \frac{y^2}{8} - \frac{y}{8}\right) = \frac{(x+h_x-1)h_x}{2} - \frac{x}{8} - \frac{yh_y}{2} + \frac{3}{8}y + \frac{h_y^2}{2} - \frac{h_y}{2}.$$

If this term is at least zero, then we can guarantee $\frac{1}{4}$ of the pairs that the two intervals connect by electing the subcommittee $\{d_1, \dots, d_s, d_{s+1}, e_1, \dots e_{s'}\}$, where $s + 1 + s'$ is precisely half of the $2s + 2s' + 2$ candidates required to connect the two intervals.

Else, the difference is strictly less than zero. By isolating yh_y in the inequality, we obtain

$$yh_y > (x + h_x - 1)h_x - \frac{x}{4} + \frac{3}{4}y + h_y^2 - h_y. \qquad (*)$$

We use this to claim that with the other half of the candidates, we can obtain the desired guarantee. Formally, consider the subcommittee $\{d_{s+2}, \dots, d_{2s+1}, e_{s'+1}, \dots, e_{2s'+1}\}$. Now, we only obtain $\frac{x}{2} - h_x + 1$ voters from the chosen candidates d_i, but in return obtain $\frac{y}{2} + h_y$ voters from the chosen e_i. The calculations are precisely symmetric to the ones we did before, just with x and y replaced by each other. Thus, if we denote the difference between actually connected pairs and the desired approximation by D, we obtain that

$$\begin{aligned}D &= \binom{\frac{x}{2}-h_x+1}{2} + \binom{\frac{y}{2}+h_y}{2} - \left(\frac{x^2}{8} - \frac{x}{8} + \frac{y^2}{8} - \frac{y}{8}\right) \\ &= \frac{(y+h_y-1)h_y}{2} - \frac{y}{8} - \frac{xh_x}{2} + \frac{3}{8}x + \frac{h_x^2}{2} - \frac{h_x}{2}.\end{aligned}$$

Then, we obtain

$$\begin{aligned}2D &= (y+h_y-1)h_y - \frac{y}{4} - xh_x + \frac{3}{4}x + h_x^2 - h_x \\ &= yh_y + h_y^2 - h_y - \frac{y}{4} - xh_x + \frac{3}{4}x + h_x^2 - h_x \\ &> (x+h_x-1)h_x - \frac{x}{4} + \frac{3}{4}y + h_y^2 - h_y \qquad \text{(by } (*)\text{)} \\ &\quad + h_y^2 - h_y - \frac{y}{4} - xh_x + \frac{3}{4}x + h_x^2 - h_x \\ &= 2h_x^2 - 2h_x + \frac{1}{2}x + \frac{1}{2}y + 2h_y^2 - 2h_y \\ &\ge 1 + 2h_x^2 - 2h_x + 2h_y^2 - 2h_y \qquad \text{(since } x, y \ge 1\text{)} \\ &\ge 0. \qquad \text{(as } z^2 - z \ge -\tfrac{1}{4} \text{ for all } z \in \mathbb{R}\text{)}\end{aligned}$$

Thus, we get $D > 0$, which concludes the proof. □