Network Creation with Homophilic Agents

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Abstract

Network Creation Games are an important framework for understanding the formation of real-world networks. These games usually assume a set of indistinguishable agents strategically buying edges at a uniform price, which leads to the formation of a network among them. However, in real life, agents are heterogeneous and their relationships often display a bias towards similar agents, say of the same ethnic group. This homophilic behavior on the agent level can then lead to the emergent global phenomenon of social segregation. We study Network Creation Games with multiple types of homophilic agents and non-uniform edge cost, introducing two models focusing on the perception of same-type and different-type neighbors, respectively. Despite their different initial conditions, both our theoretical and experimental analysis show that both the composition and segregation strength of the resulting stable networks are almost identical, indicating a robust structure of social networks under homophily.

1 Introduction

Networks play an eminent role in today's world. They are crucial for our energy supply (power grid networks), our information exchange (the Internet and the World Wide Web), and our social relationships (friendship networks, email exchange, or co-author networks). There exists an abundance of approaches to provide formal frameworks for modeling networks, see, for example, the books by Jackson (2010) and Newman (2018). In many of these models, the nodes of the network correspond to agents that strategically create connections, which is particularly suitable for our main focus of modeling social networks. One such stream of research considers variants of the *Network Creation Game (NCG)* as proposed by Fabrikant *et al.* (2003). There, selfish agents create edges to form a network among themselves. Forming edges is costly and hence agents try to create only the most useful edges. On the other hand, the force that causes agents to form edges at all is well-connectivity within the network, captured by a desire to occupy a central position.

The NCG is a stylized model of social interaction, providing valuable insight to agents' decision processes when interacting with each other. However, it is important to refine the basic model to spotlight specific details of this decision making. In this sense, we study network creation under the additional assumption that agents are separated into various *types* that model ethnic groups or affiliations.

Our goal is to contribute a new perspective on the simple causes that lead to the segregation of a society, similar to Schelling's checkerboard model for residential segregation (Schelling, 1969, 1971). Therefore, our agents' cost functions have a bias towards the creation of relationships with agents of the same type. Specifically, we study two models based on two seemingly orthogonal treatments of other agents. In the first model, agents incur a fixed cost for every created edge and a variable cost that only depends on the number of edges towards same-type agents. In the second model, edges towards different-type agents are initially more expensive but their cost drops with an inverse linear decay. Both models give a different point-of-view on the same underlying principle, namely homophily of agents, i.e., the tendency to form connections with like-minded people. This is often summarized with the proverb "birds of a feather flock together", a dominant intrinsic force repeatedly observed in the creation of social networks, see McPherson *et al.* (2001) for a survey on the extensive sociological research on homophily in social networks. While our first model expresses homophily explicitly by an increasing comfort among friends, the second model incorporates homophily indirectly by accounting for a decreasing effort of integration once first contact is established. The latter paradigm is closely related to the well-known effect in social sciences called the "contact hypothesis" which states that stereotypes and prejudices between ethnic groups can be weakened by intensified contact (Allport *et al.*, 1954; Amir, 1969; Dovidio *et al.*, 2003).

We measure the desirability of networks by means of stability. Since we consider social networks, we assume a bilateral model where two agents have to cooperate to connect. Consequently, we use pairwise stability (Jackson and Wolinsky, 1996) as solution concept, rather than, for instance, Nash stability which is more appropriate for unilateral models.

Interestingly, we find an almost identical structure of stable networks for both models. This hints at a robust structure of networks created under homophily incentives. Naturally, a very small edge cost causes extremely high connectivity. For moderately small edge cost, we provide characterizations of stable networks which are all highly segregated. We interpret this as identifying a sweet spot of high sensitivity towards agent types. For larger edge cost, stability causes a large spectrum of networks to form with respect to segregation strength. We accompany this theoretical limitation with an average-case analysis by detailed simulations of a simple distributed dynamics, where agents perform improvements towards stable networks. It would be plausible if a generally high edge cost causes less distinction of agent types. While this is sometimes confirmed, we also identify contrasting tendencies towards extreme segregation. An important driver for the different behavior is the initial segregation level. In fact, segregation can be avoided by a high initial effort without permanent further interaction.

2 Related Work

In the original NCG the cost of every edge is α, where α is a parameter of the game that allows adjusting the tradeoff between the agents' cost for creating edges and their cost for the centrality in the network, e.g., the sum of distances to all other nodes. Stable networks always exist, in particular, for $\alpha < 1$, only cliques are stable, whereas for $1 \leq \alpha < n$ stars, other trees and also non-tree networks can be stable (Mamageishvili *et al.*, 2015). For $\alpha \geq n$ it is conjectured that all stable networks are trees and a recent line of works has proven this for $\alpha > 3n - 3$ (Àlvarez and Messegué, 2017; Bilò and Lenzner, 2020; Dippel and Vetta, 2021). Bilateral NCGs with uniform edge price have been introduced by Corbo and Parkes (2005) and recently this framework was extended by Friedrich *et al.* (2023). Also variants of the NCG with non-uniform edge cost have been studied: a version where edges of differing quality can be bought (Cord-Landwehr *et al.*, 2014), and NCGs where the edge cost depends on the node degrees (Chauhan *et al.*, 2017), on the length of the edges in a geometric setting (Bilò *et al.*, 2019), or on the hop-distance of the endpoints (Bilò *et al.*, 2021). The latter is motivated by social networks, and bilateral edge formation with pairwise stability as a solution concept is considered. The NCG variant by Meirom *et al.* (2014) features different types of agents and different but fixed edge costs for each agent type.

Closest to our work is the model proposed by Martí and Zenou (2017) that is a variant of the connections model (Jackson and Wolinsky, 1996) with different types of agents. Similar to our model, the cost for maintaining an other-type connection depends on the homogeneity of the neighborhoods of the involved agents. In contrast to us, the cost for same-type edges is fixed and the distance cost is defined differently. The authors study the existence and structure of equilibria but do not investigate segregation quantitatively. The latter has been done by Henry *et al.* (2011) using a stochastic process that starts with a randomly drawn network with nodes of different types. Then edges are randomly rewired with a built-in bias toward favoring same-type edges. As the main result, the authors show that the network strongly segregates over time, even if the built-in bias is very low.

Residential segregation has recently received a lot of attention by a stream of research developing a game-theoretic framework based on Schelling's checkerboard model (Chauhan *et al.*, 2018; Agarwal *et al.*, 2021; Echzell *et al.*, 2019; Bilò *et al.*, 2020; Kanellopoulos *et al.*, 2021; Bullinger *et al.*, 2021). There, agents of several types strategically select positions on a given *fixed* network and they individually aim for having at least a τ-fraction of same-type neighbors, for some $0 < \tau \leq 1$.

Also, certain classes of coalition formation games have a similar flavor. In hedonic diversity games (Bredereck *et al.*, 2019; Boehmer and Elkind, 2020; Darmann, 2021; Brandt *et al.*, 2023), there are two types of agents and the utility of an agent within some coalition depends on the type distribution of her coalition. Moreover, there exist classes of hedonic games, where the preferences depend on distinguishing friends and enemies (see, e.g., Dimitrov *et al.*, 2006; Kerkmann *et al.*, 2020).

3 Preliminaries and Model

We consider a set $V = \{1, \ldots, n\}$ of n agents partitioned into $k \geq 2$ disjoint *types*. The set of types is denoted by $\mathcal{T}$, and for every type $T \in \mathcal{T}$, let V_T be the set of agents of type T, i.e., $V = \bigcup_{T \in \mathcal{T}} V_T$ and $V_T \cap V_{T'} = \emptyset$ for $T, T' \in \mathcal{T}$, with $T \neq T'$. For an agent $u \in V$, we denote by $\mathcal{T}(u)$ her type, i.e., we have that $u \in V_{\mathcal{T}(u)}$. Given a type $T \in \mathcal{T}$, let $n_T = |V_T|$ denote the number of agents of type T. We identify types with colors and assume that there are specific types B and R of *blue* and *red* agents, respectively, which are associated with an agent type having the smallest and largest number of agents, respectively. Thus, for every type $T \in \mathcal{T}$, we have $n_B \leq n_T \leq n_R$. In particular, with exactly two agent types we have precisely a blue minority and a red majority type.

In a network creation game agents will buy edges to eventually form a network, which is an undirected graph $G = (V, E)$. Therefore, it is useful to introduce some common concepts and notation from graph theory. Consider an undirected graph $G = (V, E)$ together with vertices $u, v \in V$. We denote the (potential) edge between u and v by uv (whether it is present or not). For two agents $u, v \in V$, the edge uv is called *monochromatic* if u and v are of the same type, and *bichromatic*, otherwise. If $uv \in E$, we use the notation $G - uv := (V, E \setminus \{uv\})$, otherwise we use $G + uv := (V, E \cup \{uv\})$. Further, let $N_G(u) := \{v \in V : uv \in E\}$ denote the *neighborhood* of u in G, let $\deg_G(u) := |N_G(u)|$ be the *degree* of u in G, i.e., the size of its neighborhood, and let $\mathrm{d}_G(u, v)$ be the *distance* from u to v in G, i.e., the length of a shortest path from u to v in G. The *diameter* of G is defined as $\mathrm{diam}(G) := \max_{u,v \in V} \mathrm{d}_G(u, v)$, i.e., the maximum length of any shortest path in G. Finally, a useful measure for the centrality of a vertex in a network is its distance to a set of vertices. Given a subset $V' \subseteq V$ of vertices, let $\mathrm{d}_G(u, V') := \sum_{v \in V'} \mathrm{d}_G(u, v)$ denote the sum of distances from u to all vertices in V'. Also, given a subset of agents $C \subseteq V$, we denote by $G[C]$ the subgraph of G *induced by* C, i.e., $G[C] := (C, F)$, where $F = \{uv \in E : u, v, \in C\}$.

Before formally defining our network creation model, we introduce some relevant special types of graphs. The graph $\mathbf{K}_n = (V, E)$ is called *complete* if $E = \{uv : u, v, \in V\}$, i.e., all possible edges are present. Further, $\mathbf{S}_n = (V, E)$ is called *star* if there exists $u \in V$ such that $E = \{uv : v \in V \setminus \{u\}\}$. We also define networks for the special case of two types. Given two agents $u \in V_B$ and $v \in V_R$, the network $\mathbf{DS}_n = (V, E)$ is called *double star* if $E = uv \cup \{uw : w \in V_B\} \cup \{vw : w \in V_R\}$ and $\mathbf{DSX}_n = (V, E)$ is called *double star with switched centers* if $E = uv \cup \{uw : w \in V_R\} \cup \{vw : w \in V_B\}$. An undirected graph G is called *complete*, *star*, *double star*, or *double star with exchanged centers* if it is isomorphic to $\mathbf{K}_n$, $\mathbf{S}_n$, $\mathbf{DS}_n$, or $\mathbf{DSX}_n$, respectively (where isomorphisms have to preserve agent types).

Network Creation Games with Homophilic Agents We study network creation within a cost-oriented bilateral model à la Corbo and Parkes (2005), where the agent cost is separated into a neighborhood cost encompassing the cost of sponsoring edges and a distance cost encompassing the cost of the agents' centrality. In both of our models, a created network G has a *distance cost* for agent u of $\mathrm{d}_G(u) := \mathrm{d}_G(u, V)$, i.e., the sum of agent u's distances to all other agents. The neighborhood cost is different in our two models and will be specified in the definition of our network creation games.

To model the cost dependency on the types of neighbors, we define the set of same-type agents in the neighborhood of agent u as $\mathrm{F}_G(u) := V_T \cap N_G(u)$, if $u \in V_T$. We will sometimes call the set of same-type neighbors of an agents as her *friends*. The set of other-type neighbors is defined as $\mathrm{E}_G(u) := N_G(u) \setminus \mathrm{F}_G(u)$. We denote the cardinalities of these sets by $\mathrm{f}_G(u) := |\mathrm{F}_G(u)|$ and $\mathrm{e}_G(u) := |\mathrm{E}_G(u)|$, respectively.

Now we define our network creation games. A *network creation game with increasing comfort among friends* (ICF-NCG) with cost parameter $\alpha > 0$ is a network creation game where the neighborhood cost is given by

$$a_G^{ICF}(u) = \deg_G(u) \cdot \alpha \left(1 + \frac{1}{\mathrm{f}_G(u) + 1}\right),$$

i.e., there is a fixed cost of α for every edge and an additional cost that decreases with an increasing number of same-type neighbors.

A *network creation game with decreasing effort of integration* (DEI-NCG) with cost parameter $\alpha > 0$ is a network creation game where the neighborhood cost ist given by

$$a_G^{DEI}(u) = \alpha \left(\deg_G(u) + \sum_{k=1}^{\mathrm{e}_G(u)} \frac{1}{k}\right).$$

Hence, there is a fixed edge cost of α for every edge to an agent in the neighborhood together with a harmonically decreasing additional cost for edges towards other-type agents. Note that the sum is empty for $\mathrm{e}_G(u) = 0$, and therefore, the game is identical to the single-type bilateral network creation game by Corbo and Parkes (2005) if $k = 1$.

For the neighborhood cost, we omit the superscript indicating the type of network creation game, whenever this is clear from the context. Also, for both of our models, we define the *cost of an agent u in a network G* as $c_G(u) := a_G(u) + \mathrm{d}_G(u)$.

The cost functions mimic the two effects that we want to model, namely a general homophilic behavior via the ICF-NCG and diminishing prejudices with intensified contact via the DEI-NCG. In both models, edge costs have a similar decay structure and identical range of $[\alpha, 2\alpha]$. In the ICF-NCG, the cost of edges is 2α for each edge if an agent has no friends, and the edge cost is approaching α when the number of neighboring same-type agents is growing. In the DEI-NCG, the cost of edges to friends is always α and the variable cost only affects other-type agents, where we approach α with a harmonic decay starting at a cost of 2α for the first other-type agent.

Measures for Desirable Networks We analyze networks by the incentives of agents to maintain the network in terms of stability and by the diversity of their neighborhood with respect to other agent types. Following Jackson and Wolinsky (1996), a network $G = (V, E)$ is called *pairwise stable* if the following two properties hold:

(i) for all agents $u \in V$ and neighbors $v \in N_G(u)$, it holds that $c_G(u) \le c_{G-uv}(u)$, i.e., no agent can benefit from unilaterally severing an edge, and

(ii) for all agents $u \in V$ and non-neighbors $v \notin N_G(u)$, it holds that $c_G(u) \leq c_{G+uv}(u)$ or $c_G(v) \leq c_{G+uv}(v)$, i.e., no pair of agents can bilaterally create an edge such that the individual cost for both agents decreases.

Connectivity is an important aspect in network analysis. With multiple agent types, the internal connectivity per type deserves special consideration. Formally, a network $G = (V, E)$ is called *fully intra-connected* if, for every pair $u, v \in V$ of same-type agents, it holds that $uv \in E$. Further, G is *fully connected* if G is complete.

For the evaluation of diversity, we consider two segregation measures. Given a network $G = (V, E)$, its *local segregation*, denoted by $LS(G)$, is defined as the average fraction of agents of the same type, i.e.,

$$LS(G) = \frac{1}{|V|} \sum_{u \in V} \frac{\mathrm{f}_G(u)}{\deg_G(u)}.$$

The *global segregation*, called $GS(G)$, is the proportion of monochromatic edges, i.e.,

$$GS(G) = \frac{\sum_{u \in V} \mathrm{f}_G(u)}{2|E|}.$$

Note that $\frac{1}{2} \sum_{u \in V} \mathrm{f}_G(u)$ is the number of monochromatic edges, i.e., in the numerator of $GS(G)$, we count each such edge twice. LS and GS are (related to) standard measures in social sciences to capture the agents' *exposure* (Massey and Denton, 1988). LS is used by Paolillo and Lorenz (2018) and GS is used in the simulation framework Netlogo (Wilensky, 1997) and by Zhang (2011).

Finally, the minimum willingness to integrate of an agent can be evaluated by checking if she entertains any bichromatic edge. Therefore, we call an agent *curious* if she is part of a bichromatic edge. Similarly, a type of agents is called *curious* if it solely consists of curious agents. Note that this concept is related to the degree of integration, which is identical to the number of curious agents and has been studied in game-theoretic models for residential segregation (Agarwal *et al.*, 2021).

4 Increasing Comfort among Friends

In this section we perform our theoretical analysis of the ICF-NCG. Unless explicitly stated otherwise, all statements hold for an arbitrary number of types. All missing proofs here and in the subsequent sections can be found in the appendix.

We start by gathering some statements concerning structural properties and simple pairwise stable networks. Their proof follows by a careful analysis of the cost difference after the creation and deletion of edges.

Proposition 4.1. *For the ICF-NCG the following hold:*

1. *If $\alpha < \frac{6}{7}$, then every pairwise stable network is fully intra-connected.*
2. *If $\alpha < \frac{4}{3}$, then $\mathrm{diam}(G) \leq 2$ for every pairwise stable network G. In particular, G contains a curious type.*
3. *Let $\alpha < 1$, G a pairwise stable network, and $C \subseteq V$ such that every agent in C is curious and $C \subseteq V_T$ for some type $T \in \mathcal{T}$. Then, $G[C]$ is a clique. In particular, every curious type of agents is fully intra-connected.*
4. *If $\alpha \leq \frac{n_B}{n_B+1}$, then the complete network $\mathbf{K}_n$ is pairwise stable. Moreover for $\alpha < \min\{\frac{6}{7}, \frac{n_B}{n_B+1}\}$, $\mathbf{K}_n$ is the unique pairwise stable network.*
5. *If $\alpha \geq 1$, then the star $\mathbf{S}_n$ is pairwise stable.*

The uniqueness in Proposition 4.1(4) excludes the parameter range $\frac{6}{7} \leq \alpha \leq \frac{n_B}{n_B+1}$, which can only happen for sufficiently many blue agents. In fact, there the uniqueness ceases to hold, as we show in the next example.

Example 4.2. Consider an ICF-NCG with two agent types. Let $n_B \geq 6$ and $\frac{6}{7} \leq \alpha \leq \frac{n_R}{n_R+1}$. We fix a specific red agent $r^* \in V_R$ and consider the network $G = (V, E)$ with $E = \{vw : v, w \in V_R\} \cup \{vr^* : v \in V \setminus \{r^*\}\}$, i.e., the red type is fully intra-connected and there is a special agent r^* to which all agents are connected. The structure of this network is depicted in Figure 1. If

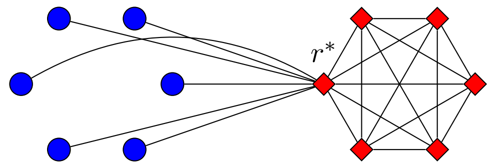

Figure 1: Pairwise stable network for $\frac{6}{7} \leq \alpha \leq \frac{n_R}{n_R+1}$ with $n_B = 6$ and $n_R = 6$ blue and red agents, respectively.

$\frac{6}{7} \leq \alpha \leq \frac{n_B}{n_B+1}$, it is even possible to interchange the roles of the two agent types. Pairwise stability of this network follows by straightforward considerations. $\lhd$

For the existence of stable networks, we still have to consider the intermediate parameter range $\frac{n_B}{n_B+1} < \alpha < 1$. We fill this gap by identifying two similar types of stable networks for this range. We illustrate the construction for two agent types. The general case is covered in the appendix.

Proposition 4.3. *In the ICF-NCG, there exists a pairwise stable network for every* $\frac{n_B}{n_B+1} \leq \alpha < 1$.

Proof. Consider an instance of the ICF-NCG and let $\frac{n_B}{n_B+1} \leq \alpha < 1$. We will define a stable network for α dependent on the threshold $\tau = \frac{n_B(n_B+1)}{n_B(n_B+1)+1}$. Note that $\frac{n_B}{n_B+1} < \tau < 1$, as $n_B(n_B+1) > n_B$.

We assume $V_B = \{b_1, \ldots, b_{n_B}\}$ and $V_R = \{r_1, \ldots, r_{n_R}\}$ and define the edge set of the graph $G = (V, E)$ as follows:

- $\{x_i, x_j\} \in E$, for $x \in \{b, r\}, i, j \in \{1, \ldots, n_B\}, i \neq j$,
- $\{r_i, b_i\} \in E$, for $i \in \{1, \ldots, n_B\}$,
- $\{r_i, r_j\} \in E$, for $i \in \{1, \ldots, n_B\}$ and $j \in \{n_B + 1, \ldots, n_R\}$,
- if $\alpha < \tau$, then $\{r_i, r_j\} \in E$, for $i, j \in \{n_B + 1, \ldots, n_R\}$, $i \neq j$, and no further edges are in E,
- otherwise, no further edges are in E.

The network G is illustrated in Figure 2. We claim that G is pairwise stable.

First, we show that no agent can sever an edge. Let $i \in \{1, \ldots, n_B\}$ and $j, k \in \{n_B + 1, \ldots, n_R\}$.

If agent b_i severs an edge to an agent of her type, the distance cost is increased by 1 while the neighborhood cost is decreased by $\alpha < 1$ (which can be computed using Lemma A.1). If a connection to a red agent is severed, then the distance to this neighbor increases by 2 while the

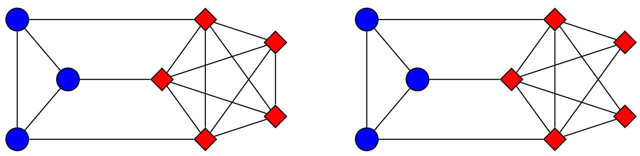

Figure 2: Pairwise stable networks for $\frac{n_B}{n_B+1} \leq \alpha < \tau$ (left) and $\tau \leq \alpha < 1$ (right).

neighborhood cost is decreased by $\alpha\left(1+\frac{1}{n_B}\right) < 2\alpha < 2$. The same considerations show that agents r_i cannot sever edges to red and blue agents, respectively.

Next, the red agent r_j cannot sever the edge towards agent r_i, because this improves the neighborhood cost by less than 2 while it increases the distance to both r_i and b_i by 1 each.

Finally, consider the case that $\alpha < \tau$. Then, r_j cannot sever $r_j r_k$ for $k \neq j$. Indeed, this would increase the distance cost by 1 while saving a neighborhood cost of

$$\alpha\left(1+\frac{1}{n_R(n_R-1)}\right) \leq \alpha\left(1+\frac{1}{(n_B+1)n_B}\right).$$

Here, we use that such an edge can only exist if $n_R \geq n_B + 1$. Hence, the total increase in cost is at least

$$1-\alpha\left(1+\frac{1}{(n_B+1)n_B}\right) = 1-\alpha\frac{n_B(n_B+1)+1}{(n_B+1)n_B} > 1-\tau\frac{n_B(n_B+1)+1}{(n_B+1)n_B} = 0.$$

Next, we show that it is also not possible to add edges. Let $i \in \{1,\dots,n_B\}$ and $j \in \{1,\dots,n_R\}$ with $i \neq j$. Then, agent b_i does not benefit from creating the edge $b_i r_j$. Indeed, this decreases her distance cost by exactly 1 while it increases her neighborhood cost by $\alpha\frac{n_B+1}{n_B} \geq 1$, using the lower bound on α.

It remains the case of missing edges between red agents for large edge cost. Assume therefore $\alpha \geq \tau$ and let $i, j \in \{n_B+1,\dots,n_R\}$, $i \neq j$. Adding the edge $r_i r_j$ decreases the distance cost for r_i by 1 while increasing her neighborhood cost by

$$\alpha\left(1+\frac{1}{n_B(n_B+1)}\right) \geq \tau\frac{n_B(n_B+1)+1}{(n_B+1)n_B} = 1.$$

Hence, creating this edge is not beneficial for r_i.

Together, we have established stable networks for $\frac{n_B}{n_B+1} \leq \alpha < 1$. $\square$

Interestingly, the stable networks constructed in the previous proof give an almost full characterization of stable networks for the considered range of edge costs when $k = 2$.

Theorem 4.4. *Consider the ICF-NCG with parameter α and $k = 2$ agent types. Let $\frac{n_R}{n_R+1} < \alpha < 1$ and assume that G is pairwise stable. Then, the blue agents are fully intra-connected, the bichromatic edges form a matching of size n_B, and curious red agents are connected to all other red agents.*

Proof. Let $\frac{n_R}{n_R+1} < \alpha < 1$ and assume that G is pairwise stable network in the ICF-NCG with cost parameter α. By Proposition 4.1(2), the diameter of G is bounded by 2 and there exists a curious type of agents. By Proposition 4.1(3), the curious type of agents forms a clique C and the curious agents of the other type form a clique as well.

Assume towards a contradiction that the bichromatic edges form no matching. Assume that there is an agent $x \in C$ that maintains bichromatic edges with two different agents y and z. We will show that agent y has an incentive to sever the edge xy. Consider therefore the network $G' = G - xy$. First, the distance cost of y decreases by at most 1. Indeed, since all agents of the type of x are still curious in G' and since y forms edges to all curious agents of her type, the distance to all these agents is 2 in G' and 1 to agents other than x to which a bichromatic edge exists in G. Also, since y is connected to all curious agents of her type, the shortest paths to agents of her own type in G cannot use x and still exist after severing the edge xy. Now, the neighborhood cost decreases by

$$\alpha\left(1+\frac{1}{\mathrm{f}_G(y)+1}\right) \geq \alpha\left(1+\frac{1}{n_R}\right) > 1.$$

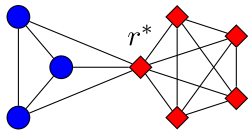

Figure 3: Pairwise stable network for $\frac{n_B}{n_B+1} \leq \alpha \leq \frac{n_R}{n_R+1}$.

Hence, no agent in C maintains more than one bichromatic edge.

Next, assume that two agents $w, x \in C$ maintain a bichromatic edge to the same agent y. It is quickly checked that severing xy increases the distance cost by 1 for y and her neighborhood cost decreases by more than 1, as above.

Together, the bichromatic edges form a matching. Hence, only a minority type can be a curious type and we can conclude that the blue agents are fully intra-connected and that the matching of bichromatic edges is of size n_B. It remains to show that all curious red agents maintain edges with non-curious red agents. Assume that y is a curious red agent forming a bichromatic edge to the blue agent x and that there is no edge to a non-curious red agent z, i.e., yz is not present in G. But then, $\mathrm{d}_G(x,z) \geq 3$, contradicting Proposition 4.1(2). □

Example 4.5. The characterization encountered in Theorem 4.4 does not cover the whole range of Proposition 4.3. In fact, it does not hold for $\frac{n_B}{n_B+1} \leq \alpha \leq \frac{n_R}{n_R+1}$, and further pairwise stable networks exist. Assume that $n_R \geq 2$ and let $r^* \in V_R$. Consider the network $G = (V, E)$, where

$$E = \{\{v,w\} \colon v,w \in V_R\} \cup \{\{v,w\} \colon v,w \in V_B\} \cup \{\{v,r^*\} \colon v \in V_B\},$$

i.e., the network is fully intra-connected and there is a special agent r^* to which all blue agents are connected. The structure of this network is depicted in Figure 3. It is straightforward to check that the network is pairwise stable.

Moreover, recall that Proposition 4.1(1) implies full intra-connectivity for $\alpha < \frac{6}{7}$. If this is not the case, i.e., $\frac{6}{7} \leq \alpha \leq \frac{n_R}{n_R+1}$ (which implies $n_R \geq 6$, i.e., a sufficiently large number of agents), then there exist even pairwise stable networks where most agents of one type have exactly one neighbor (recall Example 4.2). However, it is necessarily the case that the agents of the other type are fully intra-connected. ◁

Until now, we set our focus on the existence of pairwise stable networks. In the remainder of the section, we want to consider the segregation of pairwise stable networks. First, Theorem 4.4 yields very high segregation for $\frac{n_R}{n_R+1} < \alpha < 1$. The corollary follows from a direct computation based on the characterization of Theorem 4.4.

Corollary 4.6. *Consider the ICF-NCG with parameter α and $k = 2$ agent types. Let $\frac{n_R}{n_R+1} < \alpha < 1$ and assume that G is pairwise stable. Then, $GS(G) \geq 1 - \frac{1}{n_R}$ and $LS(G) \geq 1 - \frac{2}{n}$.*

Proof. Let $\frac{n_R}{n_R+1} < \alpha < 1$ and assume that $G = (V, E)$ is a pairwise stable network for an ICF-NCG with cost parameter α.

We start with computing the global segregation. By Theorem 4.4, there are n_B bichromatic edges. Additionally,

$$|E| \geq n_B + 2\binom{n_B}{2} + n_B(n_R - n_B) = n_B n_R.$$

Hence,

$$GS = \frac{|E| - n_B}{|E|} \geq 1 - \frac{1}{n_R}.$$

For the local segregation, we need to compute the quantity $\frac{\mathrm{f}_G(u)}{\deg_G(u)}$ for every agent u. We can apply the characterization of Theorem 4.4 again to find

$$\frac{\mathrm{f}_G(u)}{\deg_G(u)} = \begin{cases} \frac{n_B-1}{n_B} & \text{if } u \text{ blue,} \\ \frac{n_R-1}{n_R} & \text{if } u \text{ red and curious,} \\ 1 & \text{otherwise.} \end{cases}$$

Consequently,

$$\begin{aligned} LS(G) &= \frac{1}{n}\left(n_B\frac{n_B-1}{n_B} + n_B\frac{n_R-1}{n_R} + (n_R - n_B)\right) \\ &= \frac{1}{n}\left(n - 1 - \frac{n_B}{n_R}\right) \geq 1 - \frac{2}{n}\ . \end{aligned}$$

□

We know that segregation is low for sufficiently low parameter α, where cliques are (uniquely) pairwise stable. Then, there is a transition at $\alpha = \frac{n_R}{n_R+1}$, where segregation is provably high regardless of further parameters like the distribution of agents into types. Once, the cost parameter increases to $\alpha \geq 1$, the picture becomes less clear. Stars yield very high or very low segregation.

Proposition 4.7. *Consider the ICF-NCG with parameter $\alpha \geq 1$. Then, for every $n \geq 2$, there exist pairwise stable networks G and G' on n nodes such that $GS(G) = LS(G) = 1$ and $GS(G') = LS(G') = \frac{1}{n-1}$.*

Proof. Note that in the considered parameter range, the star $\mathbf{S}_n$ is pairwise stable according to Proposition 4.1(5). If there are only agents of one type, then $G = \mathbf{S}_n$ fulfills $GS(G), LS(G) = 1$. On the other hand, if there are 2 blue agents and $n-2$ red agents, consider $G' = \mathbf{S}_n$ where the center agent is blue. Then $GS(G'), LS(G') = \frac{1}{n-1}$. □

The networks in the previous proposition have the drawback that we need to fix the exact numbers of agents of each type to obtain the desired segregation. By contrast, for $\alpha \geq \frac{4}{3}$, the double star is always highly segregated.

Proposition 4.8. *Consider the ICF-NCG with $\alpha \geq \frac{4}{3}$. Then, the double star $\mathbf{DS}_n$ is a pairwise stable network with $GS(\mathbf{DS}_n) = 1 - \frac{1}{n-1}$ and $LS(\mathbf{DS}_n) \geq 1 - \frac{2}{n}$.*

Proof. Consider the double star $\mathbf{DS}_n$ and let c_B and c_R be the blue and red star center, respectively.

Note that no agent can sever an edge, because this would disconnect the network. Also, no edge between a star center and a leaf node can be created, because it is not profitable for the center node. Indeed, consider a pair of nodes $v \in V_R$ and the central node c_B. Adding the edge c_Bv improves the distance to only one node for the agent c_B, while the neighborhood cost increases by

$$\begin{aligned} &a_{\mathbf{DS}_n + c_Bv}(c_B) - a_{\mathbf{DS}_n}(c_B) \\ &= \alpha\left((\deg_{\mathbf{DS}_n}(c_B) + 1)\left(1 + \frac{1}{n_B}\right) - \deg_{\mathbf{DS}_n}(c_B) \cdot \left(1 + \frac{1}{n_B}\right)\right) = \alpha\left(\frac{n_B+1}{n_B}\right) \geq 1. \end{aligned}$$

Hence, the edge c_Bv will be rejected by the agent c_B. Analogously, a new edge between the center node c_R and a node $v \in V_B$ is not profitable for the center node c_R, because it increases the neighborhood cost by $\alpha\left(1 + \frac{1}{n_R}\right) \geq 1$ and decreases the distance cost by 1.

Next, consider the case of creating a bichromatic edge between two leave nodes. Then, the distance cost is decreased by 2, while the neighborhood cost is increased by $\frac{3}{2}\alpha \geq 2$.

Finally, consider the creation of an edge between two nodes u, v of the same type, say type R. The new edge improves the distance cost by 1 for both agents but increases the neighborhood cost by $\alpha\left(2\cdot\left(1+\frac{1}{2+1}\right)-1-\frac{1}{2}\right)=\frac{7\alpha}{6}\geq 1$. Hence, $\mathbf{DS}_n$ is pairwise stable for any $\alpha\geq\frac{4}{3}$.

It remains to compute the segregation measures for the double star.

First,

$$GS(\mathbf{DS}_n)=\frac{n_B-1+n_R-1}{n-1}=1-\frac{1}{n-1}.$$

Second,

$$LS(\mathbf{DS}_n)=\frac{1}{n}\left(n_B-1+n_R-1+\frac{n_B-1}{n_B}+\frac{n_R-1}{n_R}\right)=1-\frac{1}{n}\left(\frac{1}{n_B}+\frac{1}{n_R}\right)\geq 1-\frac{2}{n}.$$

□

5 Decreasing Effort of Integration

We consider the DEI-NCG. We start by collecting some results determining simple stable networks for sufficiently small and large values of α, respectively. Recall that we implicitly assume the restriction to two agent types when considering the networks $\mathbf{DS}_n$ and $\mathbf{DSX}_n$. All other statements hold for an arbitrary number of agent types.

Proposition 5.1. *For the DEI-NCG the following holds:*

1. *If $\alpha<\frac{1}{2}$, then $\mathbf{K}_n$ is the unique pairwise stable network.*
2. *If $\alpha<1$, then every pairwise stable network is fully intra-connected.*
3. *If $\alpha<1$, then every pairwise stable network G satisfies* $\operatorname{diam}(G)\leq 2$.
4. *The network $\mathbf{K}_n$ is pairwise stable if $\alpha\leq\frac{n-n_R}{n-n_R+1}$.*
5. *If $\alpha\geq 1$, then $\mathbf{S}_n$ and $\mathbf{DS}_n$ are pairwise stable networks.*
6. *If $\alpha\geq\frac{4}{3}$, then $\mathbf{DSX}_n$ is a pairwise stable network.*

Proposition 5.1(2) and Proposition 5.1(3) imply that, for $\alpha<1$, every pairwise stable network consists of two monochromatic cliques and one type of agents is curious. Still, there are highly segregated pairwise stable networks. Also, the upper bound in Proposition 5.1(4) is equal to $\frac{n_B}{n_B+1}$ in the case of two agent types. Moreover, the highly integrated clique investigated in this statement is not the unique stable network for $\alpha\geq\frac{1}{2}$, as the next example shows for the case $k=2$. Similar examples exist for more than 2 types.

Example 5.2. Assume $k=2$ and $\frac{1}{2}\leq\alpha\leq\frac{n_R}{n_R+1}$. Recall that n_R is the size of the majority type of agents. In particular, this covers the case $\alpha\leq\frac{n_B}{n_B+1}=\frac{n-n_R}{n-n_R+1}$. Assume that $n_B\geq 2$ and let b^* be some fixed blue agent, i.e., an agent from the minority type. Consider the network $G=(V,E)$ with

$$E=\{vw\colon v,w\in R\}\cup\{vw\colon v,w\in B\}\cup\{vb^*\colon v\in R\},$$

i.e., the network is fully intra-connected and there is a special blue agent b^* to which all red agents are connected. There are no further bichromatic edges. For an illustration of the network, see Figure 4.

We prove pairwise stability of the network. First, no agent can sever a monochromatic edge. Red agents cannot sever the bichromatic edge, because this decreases the distance to every blue agent by 1. The blue agent b^* cannot sever a bichromatic edge, because this increases her cost by $1-\alpha\frac{n_R+1}{n_R}\geq 0$. Also, further bichromatic edges cannot be added since their cost is more than 1 for a blue agent while decreasing the distance cost only by 1. ◁

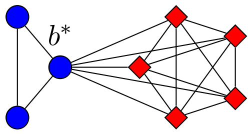

Figure 4: Pairwise stable network for $\frac{1}{2} \leq \alpha \leq \frac{n_R}{n_R+1}$.

In the previous example, it was still possible to simultaneously have full intra-connectivity while there are agents entertaining several bichromatic edges. This is not possible anymore if we further increase α.

Lemma 5.3. *Let $k = 2$ in the DEI-NCG. Consider a fully intra-connected and pairwise stable network G.*

1. *If $\alpha > \frac{n_B}{n_B+1}$, then every red agent in G entertains at most one bichromatic edge.*
2. *If $\alpha > \frac{n_R}{n_R+1}$, then every agent in G entertains at most one bichromatic edge.*

As a consequence, we can even characterize all pairwise stable networks for $\frac{n_R}{n_R+1} < \alpha < 1$ and $k = 2$.

Theorem 5.4. *Let $k = 2$ in the DEI-NCG. Assume that $\frac{n_R}{n_R+1} < \alpha < 1$ and consider a network G. Then, G is pairwise stable if and only if it is fully intra-connected and its bichromatic edges form a matching covering V_B.*

Proof. Clearly, if $k = 2$ and $n_R = 1$, then the unique stable network consists of a neighboring blue and red agent. Hence, the assertion is true. Thus, we may assume that $n_R \geq 2$.

Let $\frac{n_R}{n_R+1} < \alpha < 1$ and assume first that G is a pairwise stable network. By Proposition 5.1(2), the network is fully intra-connected. By Lemma 5.3, the bichromatic edges form a matching. Finally, by Proposition 5.1(3), one type of the agents must be curious, and therefore the matching covers the minority type of agents.

Conversely, assume that G is a fully intra-connected network such that the bichromatic edges form a matching covering one type of agents. Then, no edge can be severed because monochromatic edges only decrease the neighborhood cost by $\alpha < 1$ while increasing the distance cost by 1. Also, bichromatic edges decrease the neighborhood cost by $2\alpha < 2$ while increasing the distance cost by 2. Finally, it is impossible to create another bichromatic edge. This edge would be the second bichromatic edge incident to its endpoint from the minority type of agents. This agent would only decrease her distance cost by 1 while increasing her neighborhood cost by $\frac{3}{2}\alpha \geq \frac{3}{2}\frac{n_R}{n_R+1} \geq 1$, where we use $n_R \geq 2$ in the last step. $\square$

The second part of the above proof shows that the networks characterized in the theorem are even stable for $\frac{2}{3} \leq \alpha < 1$. Putting together Proposition 5.1, Example 5.2, and Theorem 5.4, we have proved the existence of pairwise stable networks for almost every DEI-NCG if $k = 2$ (except a limit case when $n_B = 1$). By generalizing the encountered networks, we can show the existence of stable networks for an arbitrary number of types in the next theorem. The generalization of the network in Example 5.2 is straightforward, maintaining the property that there exists one specific agent entertaining all bichromatic edges. However, the generalization of the network in Theorem 5.4 is a bit disguised. We define the network by providing an efficient algorithm. This algorithm initially considers a fully intra-connected network and adds edges by having agents create bichromatic edges via specific better responses. In the special case of $k = 2$, this results precisely in the matchings encountered in Theorem 5.4.

Theorem 5.5. *In the DEI-NCG pairwise stable networks always exist.*

Proof. Suppose that $\mathcal{T} = \{T_1, \dots, T_k\}$ with $n_{T_1} \leq \dots \leq n_{T_k}$ and, for each $1 \leq j \leq k$, $V_{T_j} = \{t_j^1, \dots, t_j^{n_{T_j}}\}$. By Proposition 5.1, it suffices to consider the parameter range $\frac{n-n_R}{n-n_R+1} < \alpha < 1$. We will construct pairwise stable networks for this parameter range.

First, we will generalize the network of Example 5.2 to an arbitrary number of agent types. Let $j^* = \min(\{1 \leq j \leq k \colon n_{T_j} \geq 2\} \cup \{k\})$, i.e., the index of the smallest type of size at least 2 or the index of the last type if there exists exactly one agent per type. Consider the network $G = (V, E)$ with edge set defined by

- $\{t_j^i, t_j^l\} \in E$ for $1 \leq j \leq k$, $1 \leq i < l \leq n_{T_j}$,
- $\{t_{j^*}^1, t_j^i\} \in E$ for $1 \leq j \leq k$, $j \neq j^*$, $1 \leq i \leq n_{T_j}$, and
- no further edges are in E.

We provide now conditions, under which the network G is pairwise stable.

Lemma 5.6. *The network G is pairwise stable if*

(i) $j^* = k$ *and* $\frac{2}{3} \leq \alpha \leq 1$,

(ii) $k = 2$, $j^* = k$, $n_{T_k} \geq 2$ *and* $\frac{1}{2} \leq \alpha \leq 1$, *or*

(iii) $\frac{2}{3} \leq \alpha \leq \frac{n - n_{T_{j^*}}}{n - n_{T_{j^*}} + 1}$.

Proof. *(i)* Assume that $j^* = k$ and $\frac{2}{3} \leq \alpha \leq 1$. Then, no monochromatic edge can be severed because of $\alpha \leq 1$. Since $j^* = k$, bichromatic edges cannot be severed due to connectivity. Also, and creating an edge costs $\frac{3}{2}\alpha \geq 1$ for an agent of type different to k while it decreases her distance cost by exactly 1.

(ii) Next, consider the case that $k = 2$, $j^* = k$, $n_{T_k} \geq 2$ and $\frac{1}{2} \leq \alpha \leq 1$. Then, again, no monochromatic edge can be severed because of $\alpha \leq 1$. The unique bichromatic edge cannot be severed as this would disconnect the network. Also, adding another bichromatic edge must include a non-curious red agent. This agent would increase her neighborhood cost by $2\alpha \geq 1$ while only decreasing her distance cost by 1.

(iii) Now, assume that $\frac{2}{3} \leq \alpha \leq \frac{n - n_{T_{j^*}}}{n - n_{T_{j^*}} + 1}$. Again, monochromatic edges cannot be severed as $\alpha < 1$. Further, bichromatic edges incident to an agent t_j^1 for $1 \leq j \leq j^* - 1$ cannot be severed as this would disconnect the network. Next, agent $t_{j^*}^1$ cannot sever another bichromatic edge, because this would increase her cost by $1 - \alpha \frac{n - n_{T_{j^*}} + 1}{n - n_{T_{j^*}}} \geq 0$. Also, for $j^* < j \leq k$ and $1 \leq i \leq n_{T_j}$, agent t_j^i cannot sever $\{t_{j^*}^1, t_j^i\}$, because this increases the distance to at least $n_{T_{j^*}} \geq 2$ agents (in T_{j^*}) by 1 while decreasing the neighborhood cost by 2.

It remains to consider the creation of edges. Every agent in $V \setminus V_{T_{j^*}}$ entertains exactly one bichromatic edge. Creating a second bichromatic edge costs $\frac{3}{2}\alpha \geq 1$ while it decreases the distance cost by exactly 1. Together, the network is pairwise stable.

□

Second, we generalize the network from Theorem 5.4. To this end, we design an algorithm that constructs pairwise stable networks. In the special case of 2 agent types, it yields the networks encountered in Theorem 5.4. Note that this must specifically hold for the parameter range where the uniqueness of the theorem applies.

Therefore, consider the network $G' = (V, E')$ where the edge set E' is computed according to Algorithm 1.

The algorithm starts with the fully intra-connected network without any bichromatic edges. Then, bichromatic edges are added whenever the distance between two agents is too large. Clearly, this algorithm has to terminate by returning E' after at most $\binom{n}{2}$ executions of the while loop.

Algorithm 1: Determination of Edge Set for Network G'

Input: Set of agents V.
Output: Edge set E'.

$E' \leftarrow \{\{t_j^i, t_j^l\}: 1 \leq j \leq k, 1 \leq i < l \leq n_{T_j}\}$;
while *there exist* $u, v \in V$ *with* $\mathrm{d}_{(V,E')}(u,v) \geq 3$ **do**
  $E' \leftarrow E' \cup \{uv\}$;
return E'

Lemma 5.7. *The following properties are valid.*

- *The diameter of G' satisfies* $\mathrm{diam}(G') \leq 2$.
- *Every triangle*[1] *in G' consists of monochromatic edges only.*
- *Every agent is incident to at most $k-1$ bichromatic edges in G'.*

Proof. The first property is immediate from the definition of the while loop. We prove the second property by contradiction. Assume that G' contains a triangle containing agents u, v, and w of at least two different types. Assume that uv is the last edge that was added by the algorithm. At this point uw and vw were already present, so $\mathrm{d}_{(V,E')}(u,v) \leq 2$, which is a contradiction to adding uv.

For the third property, we observe that every agent can add at most one bichromatic edge to an agent of each fixed type. Once this edge is added, the distance to all agents of this type is at most 2 due to the intra-connectivity of the network. As there are at most $k-1$ other types, the assertion follows. □

It is easy to deduce the pairwise stability of G'.

Lemma 5.8. *The network G' is pairwise stable for* $\frac{k}{k+1} \leq \alpha \leq 1$.

Proof. As in previous networks, monochromatic edges cannot be severed because of $\alpha \leq 1$. Now, consider a bichromatic edge uv. Then, $\mathrm{d}_{G-uv}(u,v) \geq 3$. Indeed, if $\mathrm{d}_{G-uv}(u,v) = 2$, then uv is part of a triangle, contradicting the second statement in Lemma 5.7. Hence, severing uv increases the distance cost for uv by at least 2 while saving a neighborhood cost of at most 2.

It remains to consider the creation of edges. As the network is fully intra-connected, only bichromatic edges can be created. Hence, consider the creation of a bichromatic edge uv. Its creation decreases the distance cost for u by exactly 1. Indeed, as $\mathrm{diam}(G') \leq 2$, the distance to v is decreased by exactly 1, and the distance to other agents is no shorter. On the other hand, as u is incident to at most $k-1$ bichromatic edges, the creation of uv costs at least $\alpha\left(1+\frac{1}{k}\right) \geq 1$. Hence, the total cost for u cannot have decreased. □

To conclude the proof, we want to argue that we can cover the whole parameter range of α. First, we cover the range until $\alpha = \frac{2}{3}$. According to Proposition 5.1(4), this is covered by $\mathbf{K}_n$ if $n - n_{T_k} \geq 2$. In particular, this is the case if $k \geq 3$ or $n_{T_1} \geq 2$. If $k = 2$ and $n_{T_1} = 1$, we can apply case *(ii)* of Lemma 5.6 if $n_{T_k} \geq 2$. If $k = 2$ and $n_{T_k} = 1$, then the network consisting of two agents of different types, connected by an edge, is pairwise stable.

Finally, consider the parameter range $\frac{2}{3} \leq \alpha \leq 1$. If $j^* = k$, then case *(i)* of Lemma 5.6 applies. Otherwise, $j^* < k$, and therefore $n - n_{T_{j^*}} \geq k$. This implies that $\frac{n-n_{T_{j^*}}}{n-n_{T_{j^*}}+1} \geq \frac{k}{k+1}$, and the parameter range is covered by case *(iii)* of Lemma 5.6 and Lemma 5.8. □

[1]A triangle is defined as a complete subnetwork induced by three vertices.

Finally, we want to consider the segregation of pairwise stable networks in the DEI-NCG. Clearly, segregation only depends on the networks, not on the type of NCG. Hence, based on our investigation of ICF-NCGs, we already know that that cliques provide low segregation for small α and stars provide high or low segregation for higher α, but require a specific distribution of agents into types. Independently of this distribution, double stars provide high segregation and it is clear that $GS(\mathbf{DSX}_n) = LS(\mathbf{DSX}_n) = 0$. Finally, for an intermediate range of α, high segregation is inevitable.

Corollary 5.9. *Let $k = 2$ and $\frac{n_R}{n_R+1} < \alpha < 1$. Then, every pairwise stable network G in the DEI-NCG with parameter α satisfies $GS(G) \geq 1 - \frac{2}{n}$ and $LS(G) \geq 1 - \frac{2}{n}$.*

Proof. Consider a network $G = (V, E)$ satisfying the assumptions of the corollary. We start with the global segregation measure. According to Theorem 5.4, there are n_B bichromatic edges and a total of

$$n_B + n_B(n_B - 1)/2 + n_R(n_R - 1)/2 \geq n_B + n_B(n_B - 1)/2 + n_B(n_R - 1)/2 = n_B n/2$$

edges. Hence,

$$GS(G) = \frac{|E| - n_B}{|E|} = 1 - \frac{n_B}{|E|} \geq 1 - \frac{n_B}{n_B n/2} = 1 - \frac{2}{n}.$$

Using the characterization in Theorem 5.4 once again, the computation of the local segregation measure is identical as in the proof of Corollary 4.6. □

6 Experimental Analysis

While our theoretical results indicate a clear structure of stable networks for $\alpha \leq 1$, there is a broad range of possibilities for larger α. Therefore, we support our theoretical findings for $\alpha > 1$ by a detailed experimental analysis. To this end, we simulate a simple dynamic process based on distributed and strategic edge creation and deletion over time, incentivized by agents optimizing their individual cost functions in our two models.

The dynamics start with sparse initial networks (spanning trees or grids) and distribute agents of two equally-sized types such that the segregation of the initial network is either very low or very high. In each step, a single agent is activated uniformly at random and can either create or delete an edge, performing a best response with respect to the cost function under consideration. In particular, we also study an *add-only* variant of the model, where agents can only create edges. This dynamics is particularly natural when modeling social networks, as confirmed by the observation that many real-world social networks get denser over time (Leskovec *et al.*, 2005). In both variants, if no improvement is possible, then the active agent's strategy remains unchanged and we call the agent *content*. This process is iterated until eventually all agents are content and, hence, a pairwise stable network is found. Finally, we measure the segregation strength in the obtained stable networks.

See the appendix for a detailed discussion of our experimental setup and further results. An exemplary consideration of the dynamics based on the cost function of the DEI-NCG can be found in Figure 5 and Figure 6 for the general version and for the add-only version, respectively.[2]

Interestingly, as shown in the appendix, the results for the ICF-NCG are qualitatively the same and this even holds if the segregation strength is measured with the global segregation measure instead. The experiments indicate that the segregation strength is proportional to α, with low segregation for low α, despite the theoretical necessity of high segregation for α close

[2] As discussed in the appendix, for computational efficiency, we consider convergence to 1.01-approximate pairwise stable states if we simulate the variant where edge removals are allowed. Such states are qualitatively similar to pairwise stable states.

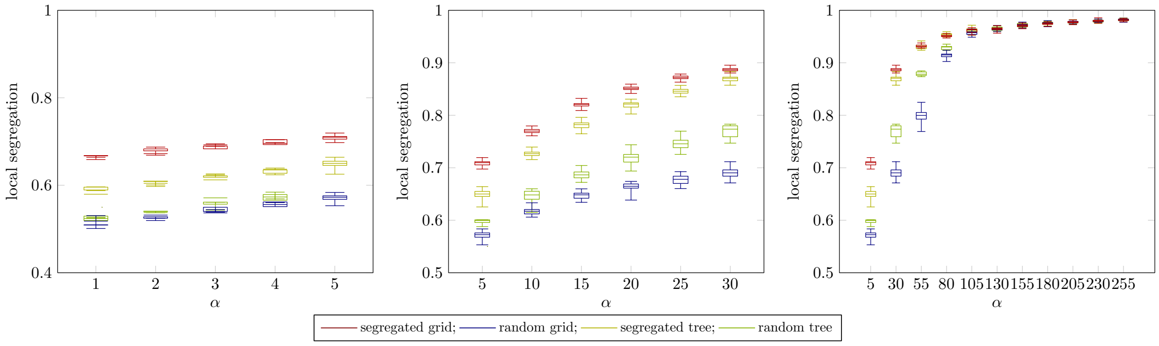

Figure 5: Local segregation of 1.01-approximate stable networks in the DEI-NCG obtained by sequential iterative best response moves for $n = 1000$ over 50 runs starting on a random spanning tree or a grid as initial graph and having a uniformly random or already strongly segregated initial distribution of the agent types. Note that a uniformly random initial type distribution yields very low segregation. E.g., "segregated tree" is the case where the initial graph is a random spanning tree and the initial type distribution of the agents is strongly segregated.

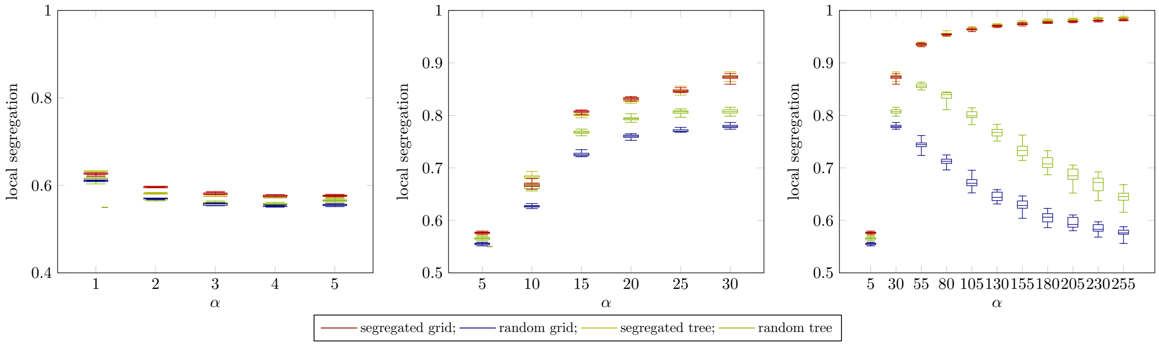

Figure 6: Local segregation of pairwise stable networks in the DEI-NCG obtained by iterative best add-only moves for $n = 1000$ over 50 runs starting on a random spanning tree or a grid as initial graph and having a uniformly random or already strongly segregated initial distribution of the agent types..

to 1.[3] Moreover, except for high α, the initial agent distribution significantly influences the segregation strength, with higher observed segregation strength when starting on already segregated initial states. The structure of the initial network seems less important for the qualitative behavior. Interestingly, the add-only version displays a similar behavior for low α, but the behavior changes drastically for moderately high α. Instead of high segregation, we find that initially integrated networks converge to only moderately segregated states, whereas this is not true for initially segregated networks, suggesting an escape route from segregation.

7 Conclusion

We have investigated two network creation games that consider heterogeneous edge creation of agents acting according to homophily. Our main goal was to analyze segregation within reasonable networks measured by pairwise stability. Our theoretical results are summarized in Figure 7. Even though our two game models feature two seemingly orthogonal perspectives based on a direct and an indirect consideration of homophily, their qualitative behavior is surprisingly similar.

Clearly, stable networks are highly integrated for a very small edge cost, when agents can

[3]The provably high segregation for $\alpha < 1$ close to 1 is not contradicting the experimental results. Just before we reach a cost parameter of α, we hit the sweet spot where buying monochromatic edges is desirable while buying bichromatic edges is not.

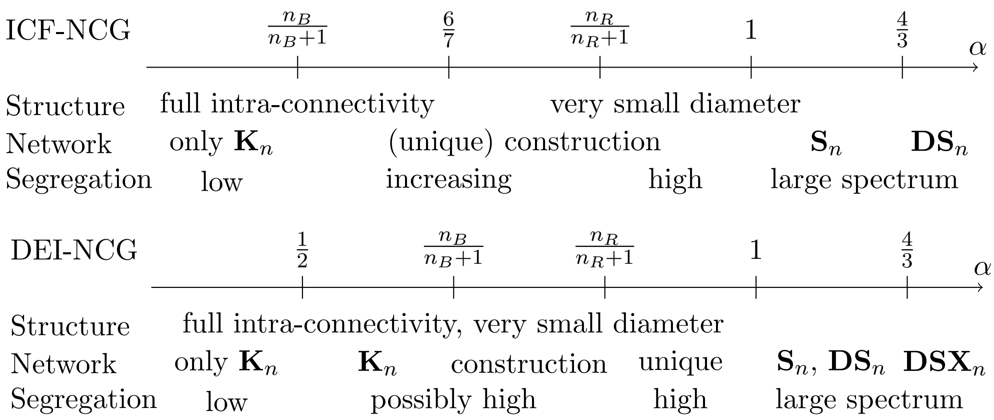

Figure 7: Overview of our theoretical results. We display structural properties of pairwise stable networks, explicit pairwise stable networks and findings about the segregation of pairwise stable networks. The two models behave surprisingly similar.

afford to buy all available edges. Once our cost parameter reaches the sweet spot where agents need to balance neighborhood and distance cost, there is provably high segregation, following from characterizations of stable networks. For slightly higher edge cost, our theoretical results cannot give a clear tendency of the segregation strength. In principle, both low and high segregation can be achieved by stable networks. Therefore, we performed an average-case analysis by running extensive simulation experiments. These experiments provide general tendencies about segregation contrasting the large theoretical spectrum for $\alpha \geq 1$. Most importantly, except for high edge price α, we consistently observe lower obtained segregation under integrated initial conditions. While this difference seems to vanish for high α when edges can also be deleted, in the add-only setting we even see a drastically increasing difference in the obtained segregation strength for high edge price α. This yields a possible escape route from segregation: by a high initial investment in integrated initial states and by incentivizing agents to keep their established connections, permanent integration might be reached.

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Appendix

A Missing Proofs

In this appendix, we provide missing proofs.

A.1 Increasing Comfort among Friends

For the analysis of pairwise stability, we frequently have to compute an agent's cost change after creating or severing one edge. To clarify the calculations, we gather the respective formulae in a technical lemma.

Lemma A.1. *Consider a network $G = (V, E)$ and an agent $u \in V$ in the ICF-NCG. Consider an agent $v \in V_{\mathcal{T}(u)}$ of the same type and an agent $w \in V \setminus V_{\mathcal{T}(u)}$ of a different type. Then, the following statements hold:*

1. $a_{G+uv}(u) - a_G(u) = \alpha\left(1 + \frac{\mathrm{f}_G(u) - \deg_G(u)+1}{(\mathrm{f}_G(u)+1)(\mathrm{f}_G(u)+2)}\right)$ *if $uv \notin E$ (creation of a monochromatic edge),*
2. $a_{G-uv}(u) - a_G(u) = -\alpha\left(1 + \frac{\mathrm{f}_G(u) - \deg_G(u)+1}{(\mathrm{f}_G(u)+1)\mathrm{f}_G(u)}\right)$ *if $uv \in E$ (deletion of a monochromatic edge),*
3. $a_{G+uw}(u) - a_G(u) = \alpha\left(1 + \frac{1}{\mathrm{f}_G(u)+1}\right)$ *if $uw \notin E$ (creation of a bichromatic edge), and*
4. $a_{G-uw}(u) - a_G(u) = -\alpha\left(1 + \frac{1}{\mathrm{f}_G(u)+1}\right)$ *if $uw \in E$ (deletion of a bichromatic edge).*

Proof. We perform the calculations for each case accordingly. Let G' be the network after the respective edge creation or deletion.

1. Creation of a monochromatic edge: $a_{G'}(u) - a_G(u) = (\deg_G(u)+1)\cdot\alpha\left(1+\frac{1}{\mathrm{f}_G(u)+2}\right) - \deg_G(u)\cdot\alpha\left(1+\frac{1}{\mathrm{f}_G(u)+1}\right) = \alpha\left(1+\frac{\mathrm{f}_G(u)-\deg_G(u)+1}{(\mathrm{f}_G(u)+1)(\mathrm{f}_G(u)+2)}\right)$.
2. Deletion of a monochromatic edge: $a_{G'}(u) - a_G(u) = (\deg_G(u)-1)\cdot\alpha\left(1+\frac{1}{\mathrm{f}_G(u)}\right) - \deg_G(u)\cdot\alpha\left(1+\frac{1}{\mathrm{f}_G(u)+1}\right) = -\alpha\left(1+\frac{\mathrm{f}_G(u)-\deg_G(u)+1}{(\mathrm{f}_G(u)+1)\mathrm{f}_G(u)}\right)$.
3. Creation of a bichromatic edge: $a_{G'}(u) - a_G(u) = (\deg_G(u)+1)\cdot\alpha\left(1+\frac{1}{\mathrm{f}_G(u)+1}\right) - \deg_G(u)\cdot\alpha\left(1+\frac{1}{\mathrm{f}_G(u)+1}\right) = \alpha\left(1+\frac{1}{\mathrm{f}_G(u)+1}\right)$.
4. Deletion of a bichromatic edge: $a_{G'}(u) - a_G(u) = (\deg_G(u)-1)\cdot\alpha\left(1+\frac{1}{\mathrm{f}_G(u)+1}\right) - \deg_G(u)\cdot\alpha\left(1+\frac{1}{\mathrm{f}_G(u)+1}\right) = -\alpha\left(1+\frac{1}{\mathrm{f}_G(u)+1}\right)$.

□

Next, we provide proofs for the collected statements about ICF-NCGs concerning structural properties of pairwise stable networks and simple pairwise stable networks.

Proposition 4.1. *For the ICF-NCG the following hold:*

1. *If $\alpha < \frac{6}{7}$, then every pairwise stable network is fully intra-connected.*
2. *If $\alpha < \frac{4}{3}$, then $\operatorname{diam}(G) \leq 2$ for every pairwise stable network G. In particular, G contains a curious type.*

3. *Let $\alpha < 1$, G a pairwise stable network, and $C \subseteq V$ such that every agent in C is curious and $C \subseteq V_T$ for some type $T \in \mathcal{T}$. Then, $G[C]$ is a clique. In particular, every curious type of agents is fully intra-connected.*
4. *If $\alpha \leq \frac{n_B}{n_B+1}$, then the complete network $\mathbf{K}_n$ is pairwise stable. Moreover for $\alpha < \min\{\frac{6}{7}, \frac{n_B}{n_B+1}\}$, $\mathbf{K}_n$ is the unique pairwise stable network.*
5. *If $\alpha \geq 1$, then the star $\mathbf{S}_n$ is pairwise stable.*

Proof. We prove the statements one after another.

1. Let $\alpha < \frac{6}{7}$. Assume that a network $G = (V, E)$ is given that is not fully intra-connected. Let $u, v \in V$ be agents of the same type with $uv \notin E$. Define $G' = G + uv$. We will show that $c_{G'}(u) - c_G(u) < 0$ (the computation for v is identical). We can assume that $\deg_G(u) \geq 1$, because otherwise agent u's cost would be infinite and adding uv would be beneficial. We compute the difference in the neighborhood cost, using Lemma A.1 in the first equality.

$$\begin{aligned} a_{G'}(u) - a_G(u) &= \alpha\left(1 + \frac{\mathrm{f}_G(u) - \deg_G(u) + 1}{(\mathrm{f}_G(u)+1)(\mathrm{f}_G(u)+2)}\right) \\ &= \alpha\left(\frac{\mathrm{f}_G(u)+3}{\mathrm{f}_G(u)+2} - \deg_G(u)\frac{1}{(\mathrm{f}_G(u)+2)(\mathrm{f}_G(u)+1)}\right) \\ &\leq \alpha\left(\frac{\mathrm{f}_G(u)+3}{\mathrm{f}_G(u)+2} - \frac{1}{(\mathrm{f}_G(u)+2)(\mathrm{f}_G(u)+1)}\right). \end{aligned}$$

Now, consider the function $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$, $f(x) = \frac{x+3}{x+2} - \frac{1}{(x+2)(x+1)}$. This function attains its maximum for $x = \sqrt{2}$ and is monotonically increasing for $0 \leq x \leq \sqrt{2}$ and monotonically decreasing for $x \geq \sqrt{2}$. Moreover, $f(1) = f(2) = \frac{7}{6}$. Hence, the maximum attained by integer values is $\frac{7}{6}$. We conclude that $a_{G'}(u) - a_G(u) \leq \frac{7}{6}\alpha < 1$. Since $\mathrm{d}_{G'}(u) - \mathrm{d}_G(u) \leq -1$, we obtain $c_{G'}(u) - c_G(u) < 0$. Hence, creation of the edge uv is beneficial for u.

2. Let $\alpha < \frac{4}{3}$ and consider a pairwise stable network G. In particular, G is connected. Assume that there are agents v and w of distance at least 3. We will show that $G' = G + vw$ is better for both of these agents, contradicting the pairwise stability of G.

   The same computations as in the proof of the first property show that the neighborhood cost increases by at most $\frac{7}{6}\alpha$ if vw is monochromatic. On the other hand, if vw is bichromatic, then the neighborhood cost increases by at most $\frac{3}{2}\alpha$. Since the distance cost decreases by at least 2, we conclude that $c_{G'}(x) - c_G(x) < 0$ for $\alpha < \frac{4}{3}$ and $x \in \{v, w\}$.

   The curiosity of one agent type follows from the fact that two agents from different types, which are both not curious, must have distance at least 3.

3. Let $\alpha < 1$ and assume that v is a curious agent of a network $G = (V, E)$. Consider an agent w of the same type such that $vw \notin E$. Then,

$$\begin{aligned} &a_{G'}(u) - a_G(u) \\ &\quad= \alpha\left(\frac{\mathrm{f}_G(u)+3}{\mathrm{f}_G(u)+2} - \frac{\deg_G(u)}{(\mathrm{f}_G(u)+2)(\mathrm{f}_G(u)+1)}\right) \\ &\quad\leq \alpha\left(\frac{\mathrm{f}_G(u)+3}{\mathrm{f}_G(u)+2} - \frac{\mathrm{f}_G(u)+1}{(\mathrm{f}_G(u)+2)(\mathrm{f}_G(u)+1)}\right) = \alpha < 1. \end{aligned}$$

   The first equality is derived by the same computations as in the proof of the first property. Consequently, $c_{G'}(u) - c_G(u) < 0$. Hence, if v and w are both curious agents of the same type, then the edge vw must be present in any pairwise stable network.

4. We start to show that $\mathbf{K}_n$ is pairwise stable for $\alpha \leq \frac{n_B}{n_B+1}$.

   To this end, we show that no edge can be deleted by one of its endpoints. Consider a pair of agents $u, v \in V$. If they are of the same type, then severing the edge uv by u decreases her cost by

$$\begin{aligned}
c_{G-uv}(u) - c_G(u) &= -\alpha\left(1 + \frac{\mathrm{f}_G(u) - \deg_G(u) + 1}{(\mathrm{f}_G(u)+1)\mathrm{f}_G(u)}\right) \\
&= -\alpha\left(1 + \frac{\mathrm{f}_G(u) + 2 - n}{\mathrm{f}_G(u)(\mathrm{f}_G(u)+1)}\right) + 1 \\
&\geq -\frac{n_B}{n_B+1} \cdot \frac{n^2 - n + 1}{(n-1)n} + 1 \\
&\geq -\frac{n}{n+1} \cdot \frac{n^2 - n + 1}{(n-1)n} + 1 \geq 0.
\end{aligned}$$

   Hence, no agent can improve her strategy by severing an edge to an agent of the same color.

   If u and v have different colors, the cost decrease is

$$\begin{aligned}
c_{G-uv}(u) - c_G(u) &= -\alpha\left(1 + \frac{1}{\mathrm{f}_G(u)+1}\right) + 1 \\
&\geq -\alpha\left(1 + \frac{1}{n_B}\right) + 1 \geq -\frac{n_B}{n_B+1} \cdot \frac{n_B+1}{n_B} + 1 = 0.
\end{aligned}$$

   Therefore, there is no improving move for any agent in the network, which implies that $\mathbf{K}_n$ is pairwise stable.

   For the uniqueness, consider any pairwise stable network $G = (V, E)$ and assume that $\alpha < \min\{\frac{6}{7}, \frac{n_B}{n_B+1}\}$. Note that G is fully intra-connected according to Proposition 4.1(1). Assume for contradiction that there are two agents $u, v \in V$ with $uv \notin E$ which have a different type.

   Then, creating the edge uv increases the neighborhood cost for each involved agent by at most

$$\alpha\left(1 + \frac{1}{\mathrm{f}_G(u)+1}\right) \leq \alpha\left(1 + \frac{1}{n_B}\right) < 1,$$

   while it decreases the distance to at least one node, a contradiction. Hence, $uv \in E$, which implies that G is a clique.

5. Consider a star graph $\mathbf{S}_n$ with central node c. To show that $\mathbf{S}_n$ is pairwise stable, we need to prove that no two leaves can jointly create an edge. Consider two leafs u and v. There can be a few possible situations. The first two cases cover the case that c and one of u and v are of the same color, say $u \in V_{\mathcal{T}(c)}$. If $v \in V_{\mathcal{T}(c)}$, then creating uv causes an increase in neighborhood cost of $a_{\mathbf{S}_n+uv}(u) - a_{\mathbf{S}_n}(u) = \alpha\left(1 + \frac{1}{6}\right) = \frac{7}{6}\alpha$, while the distance cost is only decreased by 1. Hence, for $\alpha \geq 1$, creating the edge uv is not beneficial for u. If v has a different color, then $a_{\mathbf{S}_n+uv}(u) - a_{\mathbf{S}_n}(u) = \frac{3}{2}\alpha$, and u would again prevent the creation of uv.

   It remains that u and v both have a different color from c. If $v \in V_{\mathcal{T}(u)}$, then creating the edge uv increases the neighborhood cost by α and decreases the distance cost by 1 for both u and v. Thus, since $\alpha \geq 1$, this is not beneficial.

   If all three nodes u, v, and c have different colors, then the creation of the edge uv increases the neighborhood cost of u by $2\alpha \geq 2$ and decreases her distance cost by only 1.

   Therefore, no pair of nodes can create an edge to improve their cost. Clearly, also no edge can be unilaterally deleted. The assertion follows.

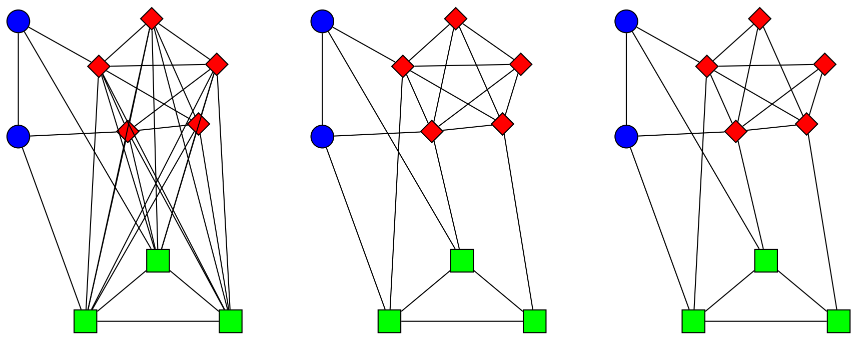

Figure 8: Illustration of the proof of Proposition 4.3. We consider an ICF-NCG with 3 types containing 3, 4, and 6 agents, respectively. Hence, we consider the parameter range $\frac{n_B}{n_B+1} = \frac{2}{3} \leq \alpha < 1$. The pairwise stable networks are dependent on the thresholds $\tau_2 = \frac{3}{4}$ and $\tau = \frac{12}{13}$. We then find the pairwise stable networks for $\frac{n_B}{n_B+1} \leq \alpha < \tau_2$ (left), $\tau_2 \leq \alpha < \tau$ (middle), and $\tau \leq \alpha < 1$ (right).

□

The proof of existence of pairwise stable networks for multiple types and an intermediate range of α has a similar structure as the special case of two types. In particular, the structure of the subnetwork induced by the agents in $V_B \cup V_T$ for any type $T \in \mathcal{T}$ with $T \neq B$ is essentially the same. However, dependent on α, agents from larger communities might have an incentive to maintain further bichromatic edges.

Proposition 4.3. *In the ICF-NCG, there exists a pairwise stable network for every $\frac{n_B}{n_B+1} \leq \alpha < 1$.*

Proof. Consider an instance of ICF-NCG and let $\frac{n_B}{n_B+1} \leq \alpha < 1$. Assume that we have ordered the types in increasing size, i.e., $\mathcal{T} = \{T_1, \dots, T_k\}$ where $T_1 = B$, $T_k = R$ and $n_{T_1} \leq \cdots \leq n_{T_k}$. Suppose that $V_{T_j} = \{t_j^1, \dots, t_j^{n_{T_j}}\}$. We will define a stable network for α dependent on several thresholds for α. In particular, there is a threshold $\tau = \frac{n_{T_{k-1}}(n_{T_{k-1}}+1)}{n_{T_{k-1}}(n_{T_{k-1}}+1)+1}$, which plays a similar role as in the case of 2 types. However, we have to consider further threshold values. Let therefore $2 \leq j \leq k-1$, and define $\tau_j = \frac{n_{T_j}}{n_{T_j}+1}$. Note that $\frac{n_B}{n_B+1} \leq \tau_2 \leq \tau_3 \leq \cdots \leq \tau_{k-1} < \tau < 1$ as $n_{T_{k-1}}(n_{T_{k-1}}+1) > n_{T_{k-1}}$.

We define the network $G = (V, E)$ with edges given as follows:

- $\{t_j^i, t_j^l\} \in E$ for $1 \leq j \leq k$, $1 \leq i < l \leq \min\{n_{T_j}, n_{T_{k-1}}\}$,
- $\{t_j^i, t_l^i\} \in E$ for $1 \leq j < l \leq k$, $1 \leq i \leq n_{T_j}$,
- $\{t_k^i, t_k^l\} \in E$ for $1 \leq i \leq n_{T_{k-1}}$ and $n_{T_{k-1}}+1 \leq l \leq n_{T_k}$,
- for each $2 \leq j \leq k-1$, if $\alpha < \tau_j$, then $\{t_j^i, t_l^m\} \in E$ for $j < l \leq k$, $1 \leq i \leq n_{T_j}$, and $1 \leq m \leq n_{T_l}$,
- if $\alpha < \tau$, then $\{t_k^i, t_k^l\} \in E$ for $n_{T_{k-1}}+1 \leq i < l \leq n_{T_k}$, and
- no further edges are in E.

The two cases for the network G are illustrated in Figure 8.

We claim that G is pairwise stable. First, we show that no agent can sever an edge. Let $1 \leq j \leq k$, $1 \leq i \leq n_{T_{k-1}}$, and $n_{T_{k-1}}+1 \leq l, m \leq n_{T_k}$.

If agent t_j^i severs an edge to an agent of her type, the distance cost is increased by 1 while the neighborhood cost is decreased by $\alpha\left(1 + \frac{\mathrm{f}_G(u) - \deg_G(u)+1}{(\mathrm{f}_G(u)+1)\mathrm{f}_G(u)}\right) \leq \alpha < 1$ (which can be computed with the aid of Lemma A.1).

In the next two paragraphs, we show that no agent can sever a bichromatic edge between an agent in V_{T_j} an agent of type T_p for $j+1 \leq p \leq k$. First, t_1^i cannot sever a bichromatic edge, because then the distance to the adjacent neighbor increases by 2 while the neighborhood cost is decreased by $\alpha\left(1+\frac{1}{n_{T_1}}\right) < 2\alpha < 2$. For the same reason, the unique neighbor of t_1^i in V_{T_p} for $2 \leq p \leq k$ cannot sever the edge to t_1^i.

Next consider the case that $2 \leq j \leq k-1$. If $\alpha < \tau_j$, then severing an edge to a neighbor in V_{T_p} for $2 \leq p \leq k$, because this increases the distance cost by 1 while saving only a neighborhood cost of $\alpha\left(1+\frac{1}{n_{T_j}}\right) < \tau_j\left(1+\frac{1}{n_{T_j}}\right) = 1$. The neighbors in V_{T_p} have (weakly) more friends and would save even less neighborhood cost. In the case $\alpha \geq \tau_j$, there is again a unique neighbor of type T_p and the case is analogous to the case for agents of type T_1. Thus, we have considered all bichromatic edges.

The red agent t_k^l cannot sever the edge towards agent t_k^i, because this improves the neighborhood cost by less than 2 while it increases the distance to both t_k^i and t_1^i by 1 each.

Finally, consider the case that $\alpha < \tau$. Then, t_k^l cannot sever $t_k^l t_k^m$ for $l \neq m$. Indeed, this would increase the distance cost by 1 while saving a neighborhood cost of $\alpha\left(1+\frac{1}{n_{T_k}(n_{T_k}-1)}\right) \leq \alpha\left(1+\frac{1}{(n_{T_{k-1}}+1)n_{T_{k-1}}}\right)$. Here, we use that such an edge can only exist if $n_{T_k} \geq n_{T_{k-1}}+1$. Hence, the total increase in cost is at least $1-\alpha\left(1+\frac{1}{(n_{T_{k-1}}+1)n_{T_{k-1}}}\right) = 1-\alpha\frac{n_{T_{k-1}}(n_{T_{k-1}}+1)+1}{(n_{T_{k-1}}+1)n_{T_{k-1}}} > 1-\tau\frac{n_{T_{k-1}}(n_{T_{k-1}}+1)+1}{(n_{T_{k-1}}+1)n_{T_{k-1}}} = 0$.

Next, we show that it is also not possible to create edges.

As a first step, we show that agents cannot create bichromatic edges. Let therefore $1 \leq j < p \leq k$ and let $1 \leq i \leq n_{T_j}$ and $1 \leq l \leq n_{T_p}$ with $i \neq j$. Note that $t_j^i t_p^l$ is present if $\alpha < \tau_j$ and $j \geq 2$. Hence, we assume that $\alpha \geq \tau_j$ if $j \geq 2$. Then, t_j^i does not benefit from creating the edge $t_j^i t_p^l$. Indeed, this decreases her distance cost by exactly 1 while it increases her neighborhood cost by $\alpha\frac{n_{T_j}+1}{n_{T_j}} \geq 1$. There, we use that $\alpha \geq \frac{n_B}{n_B+1}$ if $j=1$ and $\alpha \geq \tau_j$ if $j \geq 2$.

It remains the case of missing edges between red agents for large edge cost. Assume therefore $\alpha \geq \tau$ and let $n_{T_{k-1}}+1 \leq i, l \in n_{T_k}$. Adding the edge $t_k^i t_k^l$ decreases the distance cost for t_k^i by 1 while increasing her neighborhood cost by $\alpha\left(1+\frac{1}{n_{T_{k-1}}(n_{T_{k-1}}+1)}\right) \geq \tau\frac{n_{T_{k-1}}(n_{T_{k-1}}+1)+1}{(n_{T_{k-1}}+1)n_{T_{k-1}}} = 1$. Hence, creating this edge is not beneficial for t_k^i.

Together, we have found stable networks for α in the desired range. □

A.2 Decreasing Effort of Integration

We start with the proofs of the statements collected in Proposition 5.1.

Proposition 5.1. *For the DEI-NCG the following holds:*

1. *If $\alpha < \frac{1}{2}$, then $\mathbf{K}_n$ is the unique pairwise stable network.*
2. *If $\alpha < 1$, then every pairwise stable network is fully intra-connected.*
3. *If $\alpha < 1$, then every pairwise stable network G satisfies* $\operatorname{diam}(G) \leq 2$.
4. *The network $\mathbf{K}_n$ is pairwise stable if $\alpha \leq \frac{n-n_R}{n-n_R+1}$.*
5. *If $\alpha \geq 1$, then $\mathbf{S}_n$ and $\mathbf{DS}_n$ are pairwise stable networks.*
6. *If $\alpha \geq \frac{4}{3}$, then $\mathbf{DSX}_n$ is a pairwise stable network.*

Proof. We prove the statements one by one.

1. If some edge is not present, it has cost at most $2\alpha < 1$ and creating it decreases the distance cost by at least 1.

2. Creating a monochromatic edge has cost $\alpha < 1$ and decreases the distance cost by at least 1.

3. Let $\alpha < 1$. Assume that there are agents $u, v \in V$ with $\mathrm{d}_G(u, v) \geq 3$. Then, creating uv increases the neighborhood cost by at most $2\alpha < 2$, while decreasing the distance cost by at least 2 for each of its endpoints. Hence, G is not pairwise stable.

4. Clearly, no monochromatic edge can be severed. Now, consider a bichromatic edge uv. Then, severing uv increases the total cost for v by $1 - \alpha\left(1 + \frac{1}{n - n_{\mathcal{T}(v)}}\right) \geq 1 - \alpha\left(1 + \frac{1}{n - n_R}\right) \geq 1 - \frac{n - n_R}{n - n_R + 1}\left(1 + \frac{1}{n - n_R}\right) = 0$. Hence, also bichromatic edges cannot be severed.

5. No edge can be severed, because these networks are trees. Due to the sufficiently large distance cost, no agent favors to create an edge if this only improves the distance cost by 1. Hence, $\mathbf{S}_n$ is stable, the two centers of $\mathbf{DS}_n$ will not agree to build further edges, and leaves of $\mathbf{DS}_n$ will not agree to create further monochromatic edges. Finally, the cost for creating an edge between two leaves of different types is $2\alpha \geq 2$ which does not make up for a distance improvement of 2.

6. As for $\mathbf{DS}_n$, no edges can be severed, and the centers will not benefit from creating further edges. Also, leaves have no incentive to create monochromatic edges. Finally, the cost for a bichromatic edge between leaves of different types is $\frac{3}{2}\alpha \geq 2$, but creating such an edge yields only a distance improvement of 2.

□

Lemma 5.3. *Let $k = 2$ in the DEI-NCG. Consider a fully intra-connected and pairwise stable network G.*
1. If $\alpha > \frac{n_B}{n_B+1}$, then every red agent in G entertains at most one bichromatic edge.
2. If $\alpha > \frac{n_R}{n_R+1}$, then every agent in G entertains at most one bichromatic edge.

Proof. The proof of both statements follows from a unified approach. Let $G = (V, E)$ be a fully intra-connected and pairwise stable network. Let $u \in V$. By full intra-connectivity, severing one of several bichromatic edges incident to u, increases the distance cost of u by exactly 1 while decreasing the neighborhood cost by $\Delta = \alpha\frac{\mathrm{e}_G(u)+1}{\mathrm{e}_G(u)}$. If $\alpha > \frac{n_R}{n_R+1}$, then $\Delta > 1$ and severing a bichromatic edge is beneficial for u. This proves the second statement. If even $\alpha > \frac{n_B}{n_B+1}$ and u is an agent of the majority type, then $\mathrm{e}_G(u) \leq n_B$, and $\Delta \geq \alpha\frac{n_B+1}{n_B} > 1$. □

B Detailed Experimental Analysis

In this section we provide a more detailed experimental analysis complementing Section 6.

B.1 Details about the Experimental Setup

For our simulation experiments we first generated an initial network and an intitial agent-type distribution. Then agents are activated and compute a best possible edge addition or edge deletion. This sequential activation process is then run until no agent has an improving move and a pairwise stable network is found. We now discuss the details of this setup.

General Setup Our experiments considered 1000 agents partitioned into two types with 500 agents each. For each run we chose

- a random spanning tree or a grid as initial network,
- an integrated or perfectly segregated inital agent distribution,
- if best response moves or if best add-only moves are performed,

- if the segregation strength is measured via the local segregation measure LS or via the global segregation measure GS, and
- the value of α in 19 steps between 1 and 255.

In total this yielded $2^4 * 19 = 304$ different configurations and for every configuration we simulated 50 runs, yielding a total number of 15200 considered networks.

Generating the Initial Networks We considered random spanning trees and grids as initial networks. We used grids of size 20×50. Moreover, we sampled the random spanning trees by the following scheme: starting from a single node, we add nodes one-by-one, and each new arriving node attaches to one of the existing nodes chosen uniformly at random.

Generating the Initial Agent Distribution We focus on two cases: perfectly segregated and integrated initial states, respectively. An integrated initial state is sampled by a uniformly random type assignment to each node. To generate a perfectly segregated spanning tree, we generate two one-type spanning trees of 500 nodes and join them by connecting the initial nodes of each tree. A perfectly segregated grid is sampled by assigning one type to all 500 nodes in the first ten rows and another type to the rest.

Random Activation of the Agents We start with marking all nodes as willing to improve. In each step of the algorithm, one agent is chosen from the set of the marked nodes uniformly at random. This active agent is searching for a best allowed move. If no move is possible, the agent is unmarked. If the agent has an improving move, the new strategy is applied to the network, and all agents move back to the set of the marked nodes to be ready to become activated again. The algorithm stops when the last agent is unmarked.

Convergence Criteria Figure 9 shows a representative timeline of the local segregation of the obtained networks in each step of the best move dynamics starting from a random tree with a random color distribution. We observe that the segregation value quickly reaches a high value and remains in the interval $[0.8, 1]$ until the end of the execution of the dynamics. It illustrates the need for relaxation of the solution concept to avoid long calculations. Therefore, our experimental study of the best move dynamics uses 1.01-approximate pairwise stable states as solution concept. We say that a network is a 1.01-*approximate pairwise stable* if no agent can improve her cost by more than a factor of 1.01. The approximation factor is chosen empirically to minimize the convergence time and the approximation gap.

Note that for the add-only move dynamics, the process naturally stops at the latest when a complete network is reached. Hence, the computation time is rather low compared to the best move dynamics and we could consider exact pairwise stable networks.

Visualization of Our Results In the next section we show box-and-whiskers plots of the local and global segregation for the networks obtained by the best move dynamics for $n = 1000$ over 50 runs. Lower and upper whiskers are the minimal and maximal local segregation values over 50 runs of the algorithm. The middle lines are the median values, while the bottom and top of the boxes represent the first and the third quartiles.

B.2 Additional Experiments Regarding the Local Segregation Measure

This section provides additional simulation results for the local segregation measure for the ICF-NCG and the DEI-NCG.

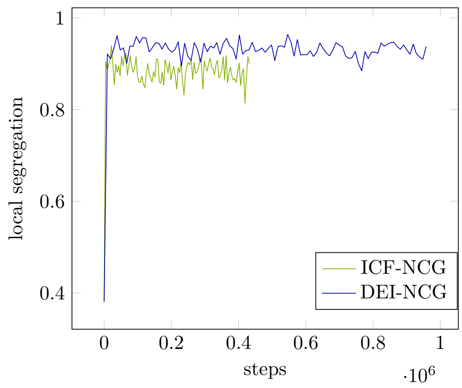

Figure 9: Trajectory of the local segregation of a network obtained by the best response dynamic for $n = 50$, $\alpha = 15$ starting from a tree with random color distribution in the ICF-NCG and DEI-NCG. The x-axis displays the number of steps taken in the best response dynamics.

B.2.1 Results for the ICF-NCG

The following figures are box-and-whiskers plots showing the obtained local segregation in our experiments for the sequential process of the ICF-NCG. The plots in Figure 10 and Figure 11 show that high segregation of stable networks can be avoided by a lower cost of the connections ($\alpha < 30$) and if started from an initially integrated state. Moreover, as shown in Figure 11, this even holds for high connection cost if the add-only process starts with an initially integrated network.

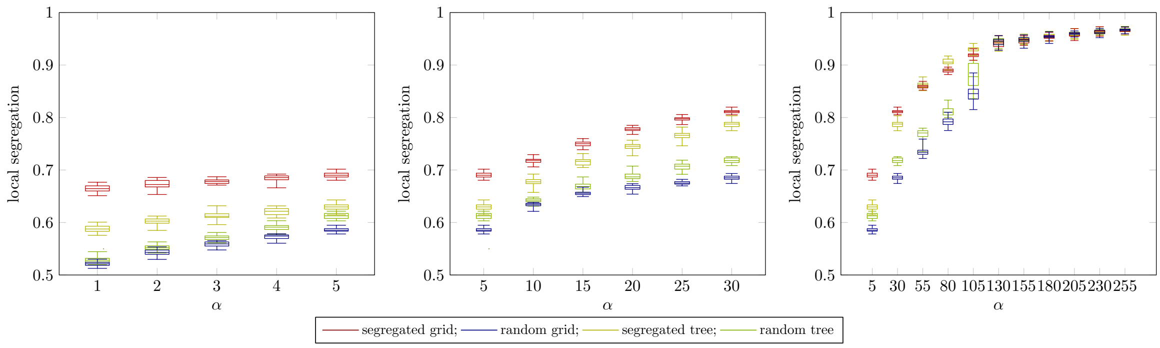

Figure 10: Local segregation of 1.01-approximate pairwise stable networks in the ICF-NCG obtained by the best move dynamic for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids.

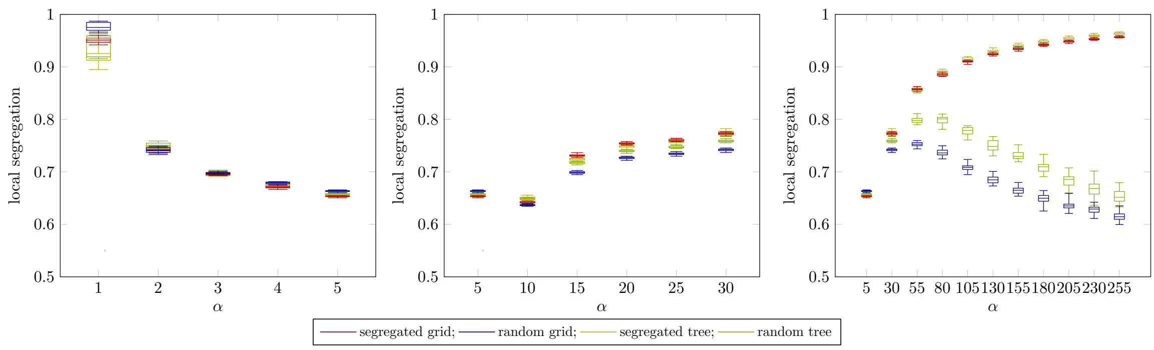

Figure 11: Local segregation of pairwise stable networks in the add-only ICF-NCG obtained by the best move dynamic for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids.

Results for the DEI-NCG

For sake of comparison, we include the results for the local segregation measure for the DEI-NCG again. The following two plots are identical to the respective plots in the main body of the paper. Compared to the results in Figure 10 and Figure 11, this clearly shows the similarities of the respective results. In particular, the tendency of decreasing segregation in case of the add-only version of the dynamics with integrated initial networks is observed for both games.

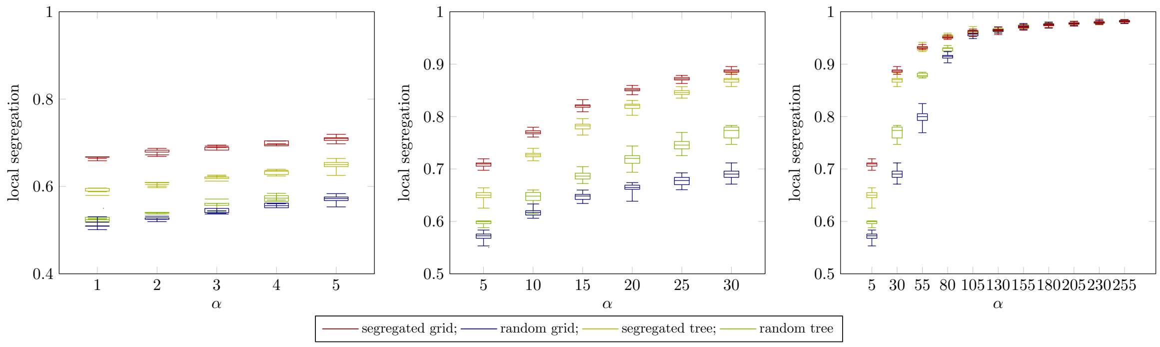

Figure 12: Local segregation of 1.01-approximate pairwise stable networks in the DEI-NCG obtained by the best move dynamic for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids.

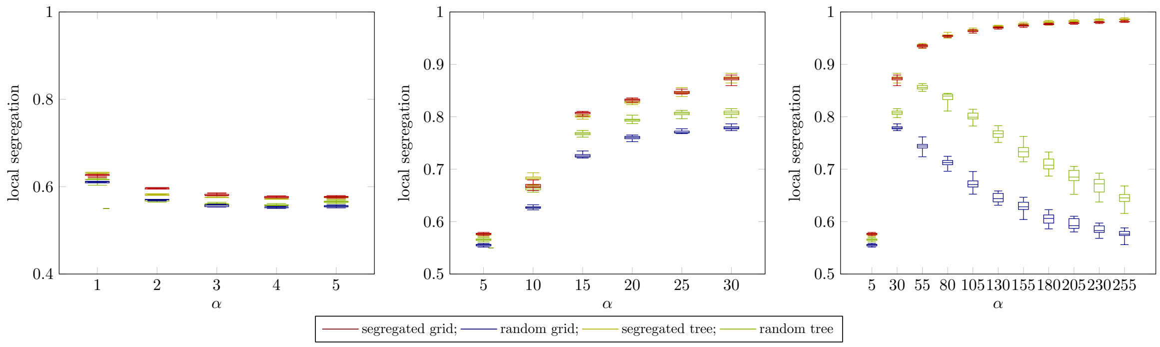

Figure 13: Local segregation of pairwise stable networks in the add-only DEI-NCG obtained by the best move dynamic for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids.

B.3 Experiments Regarding the Global Segregation Measure

This section illustrates the dependence of the global segregation measure on the parameter α and the initial state in the DEI-NCG and ICF-NCG. The observations are similar as for the local segregation measure, highlighting the robustness of our results.

B.3.1 Results for the ICF-NCG

The results for the global segregation measure for 1.01-approximate networks in the ICF-NCG and pairwise stable networks in the add-only ICF-NCG are presented in Figure 14 and Figure 15.

B.3.2 Results for the DEI-NCG

The results for the global segregation measure for 1.01-approximate pairwise stable networks in the DEI-NCG and pairwise stable networks in the add-only DEI-NCG are presented in

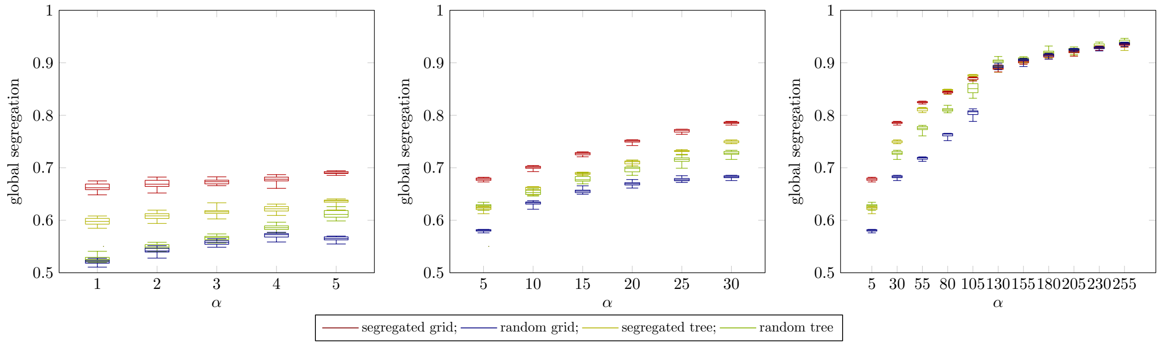

Figure 14: Global segregation of 1.01-approximate pairwise stable networks in the ICF-NCG obtained by the best move dynamic for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids.

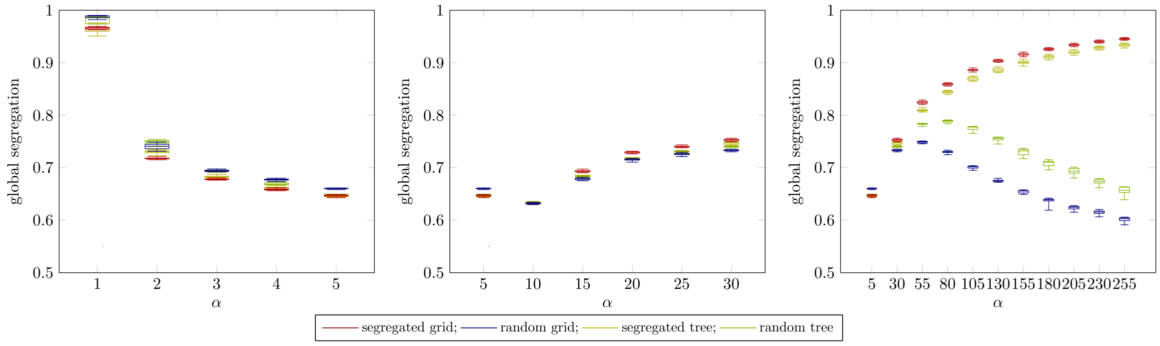

Figure 15: Global segregation of pairwise stable networks in the add-only ICF-NCG obtained by the best move dynamic for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids.

Figure 16 and Figure 17. Also these results are in-line with the corresponding results for the local segregation measure.

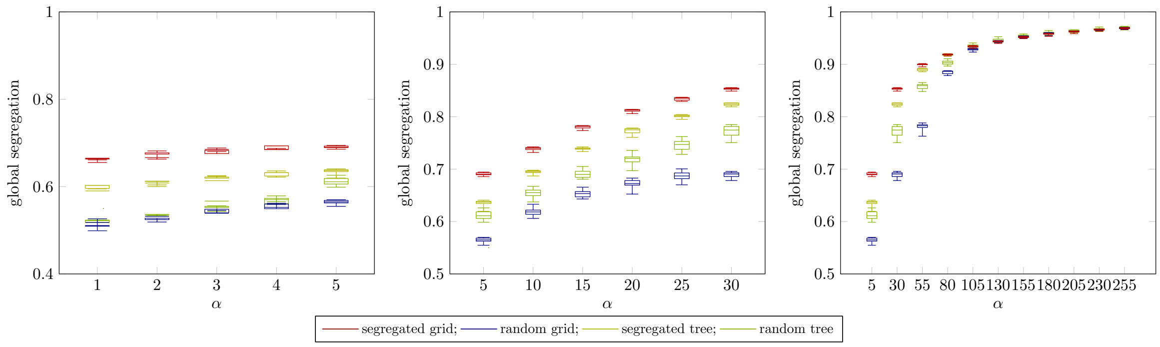

Figure 16: Global segregation of 1.01-approximate pairwise stable networks in the DEI-NCG obtained by the best move dynamic for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids.

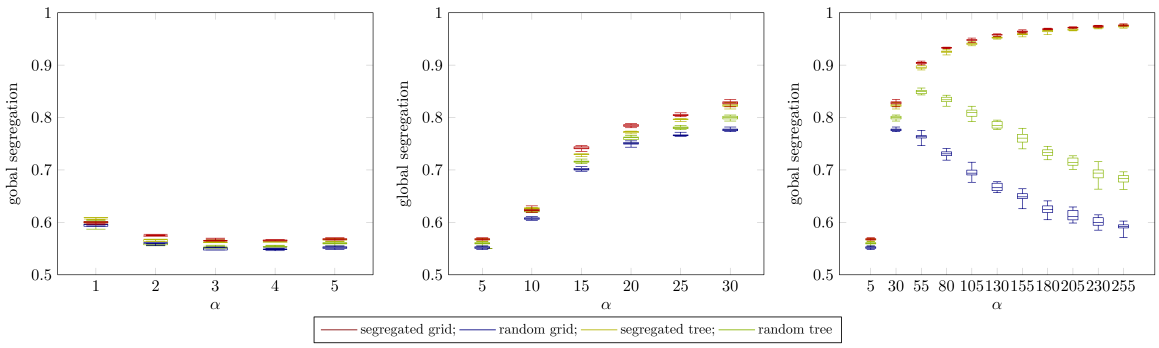

Figure 17: Global segregation of pairwise stable networks in the add-only DEI-NCG obtained by the best move dynamic for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids.