Technische Universität München

Department of Mathematics

Master's Thesis

Optimal Voting Rules for Few Candidates

Sophie Heilmaier

Supervisor: Prof. Dr. Felix Brandt

Advisor: Christian Saile

Submission Date: 01.07.2020

I hereby declare that this thesis is my own work and that no other sources have been used except those clearly indicated and referenced.

Garching,

Zusammenfassung

Die Auswahl einer Wahlregel, mit der die Gewinner einer Wahl bestimmt werden, ist eine wichtige Aufgabe, da die verwendete Regel großen Einfluss auf den Ausgang der Wahl haben kann. Daher ist es in jedermanns Interesse, eine möglichst attraktive Regel zu verwenden. In dieser Arbeit konzentrieren wir uns ausschließlich auf Wahlen mit drei und vier Kandidaten und wir bewerten die Qualität von gebräuchlichen Regeln unter mehreren Gesichtspunkten.

Wir untersuchen zum einen die Ähnlichkeit von Regeln und können so Regeln feststellen, die in ihrer Entscheidung übereinstimmen. Für Regeln, die nicht übereinstimmen, liefern wir minimale Präferenzprofile, die eine solche Abweichung der gewählten Kandidaten aufzeigen.

Jede Regel kann unerwünschte und paradoxe Situationen hervorrufen. Das Auftreten derartiger Paradoxe wird ebenfalls untersucht. Dabei betrachten wir nicht nur die theoretische Anfälligkeit für Paradoxe, sondern auch deren erwartete Häufigkeit. Wir nutzen dazu die Tatsache, dass das Auftreten eines Paradoxes häufig als lineares Ungleichungssystem beschrieben werden kann. So können wir Ehrhart-Theorie, eine Methode zum Zählen von ganzzahligen Punkten in Polyedern, auf die jeweiligen Ungleichungssysteme anwenden. Dieser Ansatz ermöglicht es uns aktuelle Resultate zu ergänzen und liefert ein gründliches Verständnis des Verhaltens von beliebten Regeln in Profilen mit drei und vier Kandidaten.

Abstract

Choosing a voting rule, with which the winner of an election is determined, is an important task because the used rule can have significant impact on the outcome of the election. Hence it is in everybody's interest to use an attractive rule. In this thesis we focus solely on elections that involve three and four candidates, and we evaluate the quality of established voting rules under several measures.

We investigate the similarity of rules, and determine rules that coincide in their decision. For rules that do not coincide we provide minimal preference profiles that showcase such a difference in elected winners.

Every voting rule can generate undesirable and paradoxical voting situations. We analyse the occurrence of such paradoxes as well. Not only the theoretical susceptibility to such paradoxes is taken into account, but also its expected frequency. We make use of the fact that occurrences of paradoxes can usually be described as linear inequalities. Therefore we can apply Ehrhart theory, a method for counting integer points inside polyhedra, to the respective linear inequality descriptions. This approach enables us to complement current results, and obtain a thorough understanding of the behaviour of common voting rules for three and four candidates.

Contents

1 Introduction

"Elections alone do not make true democracy.[1]"

The right to take part in political elections is a cornerstone of modern democracy. Although the history of political elections reaches back hundreds of years, some groups of society had to fight heavily in order to get the right to vote. In democratic states the right to take part in a political election is now a fundamental right for everyone. Apart from political elections, situations in which we have to elect one of many alternatives based on different preferences are part of our everyday life. Whether it is a rather small decision such as which film to watch in the cinema or which dessert to share in a restaurant, such situations are omnipresent. It seems to be clear that the preferences of the voters determine the winner of the election. An important aspect, that is often neglected in those discussions, is the question of how to aggregate the voters' preferences. Much to the surprise of most people the way the preferences are aggregated, can change the outcome of an election dramatically. Also situations can occur that are against human intuition, and seem rather undesirable. In this thesis we want to investigate how different voting rules behave in elections, how they interact and how prone they are to counter-intuitive outcomes.

As this is a wide field of study we restrict ourselves to a limited domain of elections. In this thesis we focus solely on the special case of voting procedures that only include three or four candidates. Such elections are not uncommon, and even in the political context there are often only few candidates that can be chosen upon. We get the motivation for this specific restriction by the even stricter case, where there are only two candidates to chose from. The question on how to aggregate preferences is quite straight-forward to answer in this case. It is known that the most natural decision rule - the majority rule - is also reasonable from a mathematical point of view as a lot of the commonly used voting rules coincide in this case, and have nice properties. Nevertheless, in profiles with three and more candidates the resulting preference relation can include majority cycles, and is therefore called intransitive. This does occur even if every voters' preference ranking is transitive. This circumstance is cause for most of the difficulties in voting theory. Hence it is unrealistic to expect such an obvious optimal voting rule as in the two candidate case, but there is still good cause for hope that the restriction of the candidates simplifies the search for a reasonable or even optimal voting rule.
We will investigate if a similar equivalence of rules can be shown for three and four candidates. To do so, we will analyse the interaction of several well-known voting rules, and check for equivalences and differences among them in chapter 3. We distinguish between three candidates in 3.2 and four candidates in 3.3.
Additional to that, we are interested in the susceptibility of voting rules to paradoxical situations. It has been shown through famous impossibility results that certain sets of desirable properties cannot occur simultaneously, and therefore basically every rule suffers some sort of paradox. We will give an overview of the literature in section 4.1. However, some of the impossibilities might not already materialise in the special case of a restricted

[1]Barack Obama in a speech at the University of Cairo in 2009

number of candidates. We will study which impossibility results can be softened by the restriction to a small number of candidates, and provide an overview in section 4.4.
Also, for paradoxical situations that can occur theoretically, the frequency of their occurrence certainly is a crucial argument. This is why we will also consider results and methods to calculate the probability of the occurrence of an undesirable situation. Methods of assessing the probability of paradoxes are introduced in 4.2. These frequencies can often be obtained with the use of Ehrhart theory, a method for counting integer points inside polyhedra, that has been developed in the sixties, and is now applied to such problems of voting theory. A short introduction and example for the use of Ehrhart theory can be found in section 4.3. New results of the application of Ehrhart theory are shown in sections 4.5 and 4.6.
By investigating both the behaviour of the rules themselves as well as their vulnerability to voting paradoxes we try to provide a good understanding as to which of the established rules are attractive or even optimal for the case of a small number of candidates. The conclusion in section 5. provides an overview of all the accomplishments of this thesis.
In order to set up the notation for all following investigations we will introduce all basic notions such as voting rules and voting paradoxes next.

2 Preliminaries

In this section the mathematical notation, that is used throughout the thesis, is introduced. Also the most important concepts are defined in this section. The definitions are mainly based on [Fel12] and [BCE+16].
Let $A = \{a, b, c, \dots\}$ be the set of candidates or alternatives of size m. In most cases m will be restricted to three or four candidates as this is the special case that this thesis focuses on. Let $N = \{1, 2, 3, \dots\}$ be the set of voters of size n. Every voter is endowed with a preference ranking $\succ_i$ over the candidates. This ranking is assumed to be a complete, asymmetric and transitive binary relation, $\succ_i \in A \times A$, which gives a strict preference order over all candidates. Hence $a \succ_i b$ means that voter i prefers candidate a strictly over candidate b.

Definition 2.1. *A preference profile $\succ$ specifies the preference relation of every individual voter $i \in N$, i.e. $\succ = (\succ_1, \dots, \succ_n)$. If a certain voter i is not submitting a preference ranking, we denote this by $\succ_{-i} = (\succ_1, \dots, \succ_{i-1}, \succ_{i+1}, \dots, \succ_n)$. Similarly if a set $I \subset N$ of voters is not submitting their rankings, we denote it as $\succ_{-I}$.*

Note that we assume that every voters' preference relation has to be transitive. Even though it can happen that the preference profile of the whole population is intransitive, and thus contains a majority cycle[2]. This is the root for most of the difficulties of voting theory as we will establish later.

Usually we are not interested in which voter is submitting which ranking. We say we treat preference profiles in an anonymous manner. An anonymous preference profile is unchanged under permutations of the voters, and all voters are therefore treated equally. Representing a preference profile in an anonymous fashion reduces to stating the numbers of voters with a certain preference ranking.

Example 2.1. As an example consider this anonymous preference profile on three candidates, and how it is usually depicted throughout the thesis.

1	1	1	4
a	a	b	c
b	c	a	a
c	b	c	b

Our example is a preference profile containing seven voters, one of them prefers candidate a before b and candidate b before c. Rankings with zero voters are omitted. Note that we do not distinguish between the voters, but only care about their preference ranking as we treat them anonymously.

Often we are not even interested in the preference profile but only care about the absolute numbers of voters that prefer one candidate over another. Hence we define the paired comparison margin and majority margin between two candidates in the following.

[2]This property is also known as the Condorcet paradox [Fel12].

Definition 2.2 (Paired comparison margin, Majority margin). *Let $\succ$ be a preference profile with a finite set of candidates A and a finite set of voters N. Let $a, b \in A$ and $i \in N$. Define*

$$n(a, b) := |\{i : a \succ_i b\}|$$

as the number of voters that prefer candidate a over candidate b. This margin will be called paired comparison margin.
The majority margin of candidate a and b is defined as

$$m(a, b) := n(a, b) - n(b, a).$$

A positive majority margin indicates that candidate a beats candidate b in a paired comparison, whereas a negative margin indicates a pairwise loss. Is the majority margin zero, then as many voters prefer candidate a over b as vice versa.

Example 2.1. In the above example it holds that $n(a, b) = 6$ as six voters prefer candidate a over candidate b. All paired comparison margins can be depicted in a matrix:

	a	b	c
a	$-$	6	3
b	1	$-$	2
c	4	5	$-$

As $n(b, a) = n - n(a, b) = 1$, the majority margin is $m(a, b) = 6 - 1 = 5$. From this positive majority margin we can see that candidate a wins over candidate b in a pairwise comparison. All majority margins can be depicted in a matrix as well:

	a	b	c
a	$-$	5	-1
b	-5	$-$	-3
c	1	3	$-$

One of the central concepts in social choice theory is the notion of a Condorcet winner. This notion is now defined:

Definition 2.3 (Condorcet Winner). *Let $\succ$ be a preference profile on a set of candidates A with $a, b \in A$. Candidate a is a Condorcet winner if and only if $m(a, b) > 0$ for every other candidate b, i.e. candidate a wins each pairwise comparison with every other candidate.*

Clearly, if there is a Condorcet winner in a preference profile, then it is unique. No two candidates can both win against all other candidates, as this would imply that one wins against the other, which contradicts the fact that both are Condorcet winners.
The definition of a Condorcet loser is now straight-forward.

Definition 2.4 (Condorcet Loser). *Let $\succ$ be a preference profile on a set of candidates A with $a, b \in A$. Candidate a is a Condorcet loser if and only if $m(b, a) > 0$ for every other candidate b, i.e. candidate a loses each pairwise comparison with every other candidate.*

Example 2.1. As one can read from either the paired comparison matrix or even more easily from the majority margin matrix, candidate c is the Condorcet winner in the above example as it wins the paired comparisons with both candidates a and b. On the other hand candidate b loses not only against c, but also against candidate a which makes it the Condorcet loser.

Note that neither Condorcet winner nor Condorcet loser have to exist in a preference profile. It is also possible that a Condorcet winner exists but no Condorcet loser and vice versa.

2.1 Voting Rules

Voting rules are functions that determine the outcome of an election. Therefore the choice of the voting rule can have a significant impact on the outcome of the election, and it should be chosen with great care and knowledge about the rules' properties. We will now introduce the general notion of a voting rule formally, and define some of the most commonly used rules.

Definition 2.5. *A voting rule f is a function that maps a preference profile $\succ$ to a nonempty set of candidates. The selected set of candidates $f(\succ)$ is called winning set or choice set. If this set is single-valued, i.e. $|f(\succ)| = 1$, then the selected element is called winner, and we sometimes directly write the selected element as outcome of the voting rule.*

We say that a voting rule f is *anonymous* if it is immune to permutations of the voters, i.e. $f(\succ_1) = f(\succ_2)$ if $\succ_1 = \pi(\succ_2)$ with $\pi : N \to N$ a permutation of the voters. Anonymity means that every voter is treated equal by the voting rule.
Hence for an anonymous voting rule it suffices to input an anonymous preference profile. All voting rules we will consider in this thesis are anonymous. Therefore from now on we can assume that we work with anonymous preference profiles only.

Another useful property in the context of voting rules is neutrality. We say that a voting rule is *neutral* if it is immune to permutations of the candidates, i.e. $\pi(f(\succ)) = f(\pi(\succ))$ holds with $\pi : A \to A$ a permutation of the candidates. Neutrality means that every candidate is treated equal. All voting rules that are defined in this thesis are also neutral.

An important property of voting rules is how they deal with preference profiles that have a Condorcet winner, and this is formalised in the following definition:

Definition 2.6 (Condorcet consistency)**.** *A voting rule f is called Condorcet consistent if and only if it elects the Condorcet winner whenever one exists. Condorcet consistent voting rules are also called Condorcet extensions.*

Now we will define voting rules that are examined throughout the thesis, and we already distinguish between Condorcet consistent and not Condorcet consistent rules. We will start with rules that are not Condorcet consistent.

Definition 2.7 (Plurality Rule). *The Plurality rule selects the candidate that is ranked first by the highest number of voters.*

$$f_{Plurality}(\succ) = arg\ max_{a \in A} |\{i : a \succ_i b, \forall b \in A \setminus \{a\}\}|$$

Example 2.2. As an example for the mechanisms of the voting rules consider the following anonymous preference profile with four candidates and seven voters:

3	1	1	2
a	b	d	b
b	c	c	c
c	a	a	d
d	d	b	a

The resulting paired comparison matrix then looks as follows:

	a	b	c	d
a	–	4	3	4
b	3	–	6	6
c	4	1	–	6
d	3	1	1	–

Candidates a and b are ranked first by three voters each, whereas candidate d is ranked first by only one voter, and candidate c is never ranked first. Hence the Plurality rule selects both candidates a and b as winners.

Definition 2.8 (Borda's Rule). *Borda's rule assigns scores from zero to $|A| - 1$ to the candidates according to their position in the voters' ranking, and elects the candidates with the highest Borda score sum as winners.*

$$f_{Borda}(\succ) = arg\ max_{a \in A} \sum_{b \in A \setminus \{a\}} n(a, b)$$

Example 2.2. In order to calculate the Borda scores from the paired comparison matrix one needs to add up the numbers of each row. Hence the Borda scores for the candidates a, b, c and d are 11, 15, 11 and 5 respectively, and hence candidate b is the unique Borda winner.

Definition 2.9 (Plurality with Runoff Rule). *The Plurality with Runoff rule is a runoff rule based on the plurality rule. If a candidate is ranked first by an absolute majority, then it is elected immediately. If there is no absolute majority winner, then the pairwise comparison between the two candidates that had the highest number of first ranks in the first round decides on the winner.*

Example 2.2. No candidate is ranked first by an absolute majority. Hence the two candidates that received the highest numbers of first ranks are compared. These candidates are a and b. Candidate a wins this pairwise comparison and is therefore the Plurality with Runoff winner.

Definition 2.10 (Instant Runoff)**.** *The Instant Runoff rule is again a rule that requires several steps. If there is a candidate that is ranked first by an absolute majority of the voters, then this candidate is elected immediately. If no such candidate exists, the candidate that is ranked first by the smallest number of voters is eliminated, and after the elimination it is again checked whether there is now a candidate ranked first by an absolute majority of the voters. This is continued until an absolute majority winner is found.*

Example 2.2. Candidate c is never ranked first, so it is discarded in the first step. This elimination did not produce an absolute majority winner. Hence candidate d has to be discarded in the second round as well, and leaves only candidates a with four first rankings and candidate b with three first rankings. This makes candidate a the Instant Runoff winner.

Definition 2.11 (Coombs' Rule)**.** *Coombs' rule is quite similar to the above defined Instant Runoff rule. Again, if there is an absolute majority winner in the first place, it is elected immediately. If this is not the case, then the candidate that is ranked last by the largest number of voters is eliminated (instead of the candidate that is ranked first by the smallest number of voters). This process is again continued until a candidate is found that is ranked first by an absolute majority of the voters.*

Example 2.2. The candidate that is ranked last by the highest number of voters is candidate d, thus it is discarded in the first step. Now both candidates a and b are ranked first by three voters each, and hence no absolute majority winner is found yet. It turns out that candidate a is now ranked last by the highest number of voters and is hence discarded next which makes candidate b the absolute majority winner and hence Coombs' winner.

Definition 2.12 (Bucklin's Rule)**.** *If a candidate exists, that is ranked first by an absolute majority, then it is elected by Bucklin's rule as well. If no such candidate exists, then the number of voters that rank the candidate second is added to the number of voters that rank the respective candidate first. We say these voters "support" the respective candidate. It is checked if now there is a candidate that is "supported" by an absolute majority. If this did not suffice, then also the number of voters that ranked the candidate third are added to the number of supporters. This is continued until a candidate is found that is supported by an absolute majority of the voters. If there are more candidates found to be supported by an absolute majority, the candidate with the highest number of supporters is elected.*

Example 2.2. As there is no absolute majority winner we need to add the voters that rank a certain candidate second to the group of supporting voters. Hence candidate a still has only three supporters, whereas b has now six, c has four, and d one supporter. Clearly candidate b is supported by an absolute majority and by the highest number of supporters, and it is therefore the Bucklin winner.
All following rules are Condorcet consistent, and hence they select the unique Condorcet winner whenever one exists.

Definition 2.13 (Maximin Rule)**.** *The Maximin rule elects the candidates as winners whose "worst loss is the least bad [3]". Hence it compares the smallest majority margins of each candidate, and elects the candidate with the highest among the minimal majority*

[3] [Fel12], page 28

margins. In case a Condorcet winner exists, this candidate will be elected as its smallest majority margin is positive contrary to the minimal margins of all other candidates.

$$f_{Maximin}(\succ) = arg\ max_{a \in A}\{min_{b \in A} m(a,b)\}$$

Example 2.2. Note that there is no Condorcet winner but a Condorcet loser in our toy example as there is a majority cycle $(a \succ b \succ c \succ a) \succ d$ where every candidate of the cycle beats candidate d. Note further that even though the Maximin winners are formally defined via the majority margins, also checking for the paired comparison margins suffices as those two notions are positive linearly related. From the paired comparison matrix one can read the row minimum of every candidate. These are 3 for candidates a and b and 1 for the other two candidates. As we are searching for the highest row minimum both candidates a and b are Maximin winners.

Definition 2.14 (Black's Rule)**.** *Blacks's rule is a so called hybrid rule as it elects the Condorcet winner whenever one exists, and in all other cases the winning set is determined using Borda's rule.*

Example 2.2. Because there is no Condorcet winner, the Borda winner has to be chosen. Hence Black's winner is candidate b.

Definition 2.15 (Kemeny's Rule)**.** *Kemeny's rule maximises the number of agreements between the voters' preference rankings and the elected ranking. The number of agreements is the number of pairs that are ranked in the same order in the two respective rankings. Hence the number of agreements between all theoretical possible rankings and the preference profile of every voter is counted, and the maximising ranking is elected. The winner is the top ranked alternative of the elected Kemeny ranking. Define the number of agreements between two rankings $\succ$ and $\succ^*$ as $agree(\succ, \succ^*)$.*

$$f_{Kemeny}(\succ) = arg\ max_{\succ^*} \sum_{i \in N} agree(\succ^*, \succ_i)$$

Example 2.2. In order to calculate the Kemeny winner one has to calculate the number of agreements between the voters' preferences and every possible ordering of candidates. Consider for example the ranking $a \succ b \succ c \succ d$. The first three voters agree with this ranking totally. The voter with preference $b \succ c \succ a \succ d$ disagrees with $a \succ b \succ c \succ d$ on the two pairs a, b and a, c. The next voter only agrees with the ranking of the pair a, b. The last three voters agree on three pairs. Hence in total the number of agreements for the ranking $a \succ b \succ c \succ d$ is 32. Now this procedure has to be done for all 24 possible rankings and it turns out that 32 is actually one of the maxima and is also reached by the ranking $b \succ c \succ a \succ d$. Hence we have two Kemeny winners and those are candidates a and b.

Definition 2.16 (Nanson's Rule)**.** *Nanson's rule is a runoff method related to Borda's rule. In the first round the Borda scores are calculated for every candidate. Then all candidates whose Borda scores are below or at the average Borda score are eliminated, and a revised Borda score is calculated for the remaining candidates. This is continued until a winner is found. In case the elimination should return the empty set, all candidates that had an average Borda score in the previous round are elected.*

Example 2.2. We know already that the Borda scores of the candidates are 11, 15, 11 and 5 for candidates a, b, c and d respectively, which makes an average Borda score of 10.5. Hence only candidate d has a score below the average and is discarded. After d's elimination the Borda scores are recalculated and now turn out to be 7, 9 and 5 for a, b and c. Hence the average Borda score in the second round is now 7, and candidates a and c are now eliminated which makes candidate b the Nanson winner.

Definition 2.17 (Young's Rule). *Young's rule selects the candidate that can be turned into a Condorcet winner by removing the smallest number of voters from the preference profile.*
Define the Young score of a candidate as the number of voters that have to be removed to make the respective candidate a Condorcet winner:

$$y(a) := min|\{i : i \in I, I \subset N, a \text{ is Condorcet winner in } \succ_{-I}\}|$$

Now the Young winner can be defined as the minimiser of the Young scores.

$$f_{Young}(\succ) = min_{a \in A} y(a)$$

Example 2.2. In order to make candidate a a Condorcet winner both of the voters that rank a last have to be removed. Thus $y(a) = 2$. In order for candidate b to become a Condorcet winner two voters have to be removed as well. But for candidates c and d it does not suffice to remove two voters only. The Young scores of those candidates are $y(c) = 4$ and $y(d) = 6$. Hence candidates a and b are the Young winners.

Definition 2.18 (Baldwin's Rule). *Baldwin's rule is closely related to Nanson's rule and is also a runoff procedure involving Borda scores. Just like in Nanson's rule again the Borda scores are calculated for every candidate. Now only the candidate with the lowest Borda score is eliminated, whereas for Nanson's rule all candidates with below average Borda scores are eliminated. After the deletion of the candidate from the ballots the Borda scores are recalculated and the process is continued until there is only one candidate left, or the Borda scores of all remaining candidates are the same. Either the only remaining candidate is elected as the winner, or the candidates with the same Borda scores are elected as winning set.*

Example 2.2. As before the Borda scores of the candidates are 11,15,11 and 5 respectively. The first candidate to be eliminated is d. The revised Borda scores are now 7,9 and 5 which makes us remove candidate c from the ballots. Now the paired comparison between a and b is checked, and as a wins this paired comparison it is elected to be the Baldwin winner.

Table 1 provides an overview of the results of all defined rules and our example preference profile. As one can clearly see the voting rules all support the statement that candidates a and b seem to be superior in the public opinion to the other two candidates. But the decision who of those candidates wins the election or if a tie is the best outcome, seems to be quite controversial. Note that in such a situation the choice of the voting rule used to determine the winner is highly critical. There are even examples where this situation is even worse and it can happen that the choice of the voting rule determines the winner

Voting Rules	**Outcome of election**
Plurality	a, b
Borda	b
Plurality with Runoff	a
Instant Runoff	a
Coombs	b
Bucklin	b
Maximin	a, b
Black	b
Kemeny	a, b
Nanson	b
Young	a, b
Baldwin	a

Table 1: Overview of Voting Results of Example 2.2

of an election[4].

In such situations it is important to have a profound understanding of how the voting rules interact. It seems to be a good indicator whether a lot of voting rules coincide in their choice or not. Therefore we will determine if a coincidence of voting rules happens in restricted domains as this would support the choice decision if a set of rules were to chose the same winners. Another interesting thing to check is if the rules are vulnerable to certain paradoxical situations. The investigated paradoxes are now defined in the next section.

2.2 Paradoxes

Paradoxes in Social Choice are outcomes produced by voting rules that seem counter-intuitive or undesirable. Being vulnerable to certain paradoxes is a strong argument against the respective voting rule. Nevertheless every voting rule suffers from some of those undesirable properties, thus it is a question of quantity, frequency and personal opinion, which paradoxes are to be considered bad or out of the question.
In this subsection the paradoxes are introduced that are examined in this thesis.
One of the most important paradoxes is the so called Condorcet winner paradox:

Definition 2.19 (Condorcet Winner Paradox)**.** *A voting rule f suffers the Condorcet winner paradox if there exists a Condorcet winner in a preference profile, but f fails to always elect it.*

[4]Such an example can be

7	7	6	1
a	b	c	a
c	c	b	b
b	a	a	c

where Plurality selects candidate a, Plurality with Runoff b and the Borda rule selects c. Hence depending on the choice of the voting rule every candidate could potentially win this election.

This paradox is one of the most prominent paradoxes. As we defined the notion of Condorcet consistency earlier, we already know that no Condorcet consistent rule suffers from the Condorcet winner paradox by definition.
The Condorcet loser paradox is now defined quite straight-forward.

Definition 2.20 (Condorcet Loser Paradox)**.** *A voting rule f suffers the Condorcet loser paradox if there exists a Condorcet loser in a preference profile, and it gets elected by f.*

The following paradoxes are not that straight-forward as they require a change in the underlying preference profile. Hence they showcase more a counter-intuitive consequence of a change in the underlying preference profile.

Definition 2.21 (Lack of Monotonicity Paradox)**.** *Let f be a voting rule and $\succ$ a preference profile. Let candidate x be the winner of the election under f. We say f suffers from Lack of Monotonicity if candidate x is no longer elected as winner after one or more voters increase their support of candidate x by moving it upwards in their preference ranking. More precisely this is known as the More-is-less-paradox* [5]*. Analogously the Less-is-more paradox describes the situation if the support for a candidate is decreased by some of the voters, but it is then elected, whereas before the change in support it was not. When mentioning the Lack of Monotonicity paradox without further specification we usually refer to the More-is-less-paradox.*

Another paradox that can occur when a change in voter behaviour happens is the following:

Definition 2.22 (No Show Paradox)**.** *A voting rule f suffers the No Show Paradox if a voter can benefit from abstaining the election, i.e. $f(\succ_{-i}) \succ_i f(\succ)$ for some $i \in N$. It is also possible that a group of voters abstains from the election, and every member of the group benefits from this decision.*

Lastly a paradox that can occur if there are elections in more than one district:

Definition 2.23 (Reinforcement Paradox)**.** *A voting rule f suffers the Reinforcement paradox if it elects the same candidate in two separate districts, but when the rankings of both districts are combined and treated as one joint election, a different candidate is elected.*

Note that this list of paradoxes is far from being complete. For a more extensive study of different paradoxes we refer to Felsenthal [Fel12]. For this thesis we decided to only examine these five introduced paradoxes since those are often considered severe or interesting, and have also been studied previously. These criteria are also met by other paradoxes, but in order to keep this thesis to a reasonable extent, we restricted ourselves to the mentioned five paradoxes.

[5] [LMS17]

2.3 Probabilistic Models for Preference Profiles

In order to get insights into the behaviour of voting rules we will sometimes assume randomness of the preference profiles in some way. Also when discussing experiments the generation of random preference profiles is important. We will now only briefly discuss different probabilistic assumptions that define the distribution in order to generate preference profiles. For a more thorough explanation we refer to Gehrlein and Lepelley [GL12] [GL11].

The Dual Culture Assumption The dual culture assumption is a special case of a multinomial probability model. The multinomial probability model assigns a probability p_i to the ith preference ranking of a randomly selected voter. For instance in a three candidate case the probability that a randomly selected voter ranks the candidates $a \succ b \succ c$ is p_1. Clearly the probabilities of all possible rankings must add up to 1, i.e. $\sum_{i=1}^{m!} p_i = 1$. We further assume that each voter's ranking can be obtained independently from the other voters' rankings. Obtaining a preference profile can now be seen as an urn experiment, where n many balls are drawn. Each ball corresponds to one voter's preference ranking. The proportion of balls of each colour in relation to the total number of balls corresponds to their probability p_i. The selection is done with replacement as the probabilities for observing a certain ranking are the same for each voter.
For the case of dual culture (DC) one now assumes that the probability of a ranking is identical to the probability of the inverted ranking. Hence it is equally likely that a randomly selected voter ranks $a \succ b \succ c$ or $c \succ b \succ a$, i.e. it now holds that $p_1 = p_6, p_2 = p_4$ and $p_3 = p_5$. Under DC there is no expected advantage for any candidate in the paired comparison with any other candidate. Hence there is expected balance when comparing two candidates.

The Impartial Culture Assumption Impartial Culture (IC) on the other hand is a special case of dual culture as it assumes that every ranking is equally likely for a randomly selected voter. Hence $p_i = 1/m!$ holds. Again the preferences of any voter are assumed to be independent of the preferences of all other voters' preferences. Under IC every candidate is equally likely to be the most preferred, last preferred or middle ranked candidate. As IC is a special case of DC, it still holds that there is also expected balance in the pairwise comparisons of any two candidates as well. Therefore it is often said that IC is the "purest" of all assumptions as no candidate has any expected advantage whatsoever.

Impartial Anonymous Culture In contrast to the previous models the Impartial Anonymous Culture (IAC) is not generated by assigning a probability to a randomly selected voter's preference ranking. Instead every preference profile with n voters is said to be equally likely. As shown by Gehrlein and Lepelley [GL11], IAC actually does not assume that the voters' preference rankings are independent of each other which is a noteworthy difference to the IC assumption. Gehrlein and Lepelley show that IAC can also be seen as an urn experiment. The setting is similar to the urn model described for IC, but the difference is that after each draw the ball is placed back in the urn together with one additional ball of the same colour. Repeating this draw and replacement action n times gives a preference profile with n voters under the IAC assumption. Therefore, if one colour is observed once, then the probability to observe this colour again is increased for the following draws. This indicates a voter interaction as there is some sort of dependency between the voters' preferences, and some rankings turn out to be more popular than others as voters influence each other. This model is also called Pólya-Eggenberger model with parameter $\alpha = 1$ whereas the IC assumption corresponds to a Pólya-Eggenberger model with $\alpha = 0$.

These cultural assumptions have been criticised for being unrealistic. In [RGMT06] it is shown for instance that the examined real world election data are not similar to preference profiles obtained by DC, IC or IAC, and because of that the threat of majority cycles is heavily overestimated when assuming the introduced probability models. Nevertheless, applying these cultural assumptions can still be useful because of several reasons. One is that large scale real world data is basically not available. Also DC,IC and IAC assumptions tend to exaggerate the probability of voting paradoxes, and hence can be seen as worst case analysis. If only a small frequency is observed under IC or IAC, then the probability of the paradox can be expected to be even smaller in real world elections. Also the relative difference between several voting rules and different paradoxes can be examined perfectly well. More reasons why using IC or IAC assumptions is useful can be found in [GL12]. Plassmann and Tideman [PT14] use a special distribution that is a spatial model which is fitted according to real world data. So it is especially designed to imitate real world voter behaviour. We will briefly refer to this model later.
Throughout this thesis we will only use the IAC assumption. This assumption can be used to obtain results with Ehrhart theory, a method for counting integer points in polyhedra. Ehrhart theory together with recently developed algorithms enables us to come up with new results. An introduction can be found in section 4.3. As this method requires the IAC assumption, it is reasonable to compare to other results that also make use of IAC. In the literature IAC is one of the most used cultural assumptions, and hence there are already some results that we can compare our results with. This is the main reason for choosing IAC as our preferred preference model. It comes in addition that IAC provides an amount of voter interaction which seems to be somewhat closer to reality than the IC assumption that requires independence of the voters' preferences.

3 Equivalences and Differences of Voting Rules

This thesis focuses on the case that there are only three of four candidates to chose from. Such a restriction can have effects on the voting rules themselves and also on their susceptibility to certain paradoxes. As a motivation one can consider the restriction to preference profiles that involve two candidates only. A famous result has been shown for this special case:

Theorem 3.1 (May's Theorem). *[May52] Let $|A| = 2$. The only voting rule that is anonymous, neutral, and does not suffer the Lack of Monotonicity paradox is the Plurality rule.*

This means also that every rule that is anonymous, neutral, and does not suffer the Lack of Monotonicity paradox in a domain with only two candidates reduces to the plurality rule. Hence a lot of voting rules coincide in such a scenario which indicates a good and natural fit of voting rules. Also those equivalent rules behave nicely in regards to vulnerability to voting paradoxes. This theorem provides good cause for choosing the Plurality rule in a profile with two candidates only.
We are now interested if a similar result can be obtained for the three or four candidate case as well, and if we are able to obtain equivalences of voting rules. If several rules coincide in their decision, this can be seen as a strong argument for the chosen candidate.

3.1 Finding minimal Preference Profiles using Linear Programming

We will now examine if certain voting rules turn out to be equivalent when restricting the domain to three or four candidates.
If rules do not coincide in their decision always, it is still interesting to find a preference profile in which such a discrepancy of the respective choice sets happens. We want to investigate if rules already differ for a very small number of voters. Therefore we search for preference profiles in which the respective pair of voting rules differs in their decision under the minimisation of the number of voters. As this is basically a minimisation problem, it is straight-forward to use linear programming. The standard form of a linear program is

$$\begin{aligned} \min\ & c^T x \\ s.t.\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

with $c \in \mathbb{R}^k, A \in \mathbb{R}^{j \times k}, b \in \mathbb{R}^j$ and the decision variables $x \in \mathbb{R}^k$.

In order to set up suitable linear programs we first have to define reasonable decision variables. We work solely with anonymous preference profiles. In order to specify such profiles it suffices to state the number of voters with a certain preference ranking. There are $m!$ many possible preference rankings of m many candidates, and we will order rankings lexicographically. Hence in the three candidate case every voter can have one of the following numbered rankings:

$$\begin{array}{ll} 1: a \succ b \succ c & 4: b \succ c \succ a \\ 2: a \succ c \succ b & 5: c \succ a \succ b \\ 3: b \succ a \succ c & 6: c \succ b \succ a \end{array}$$

We now define the decision variables n_i to be the number of voters with respective preference ranking i. These variables then specify an anonymous preference profile uniquely. Obviously the sum of all variables has to equal the total number of voters n, and clearly all variables have to be nonnegative and integer. Hence it always has to hold that

$$\sum_{i=1}^{m!} n_i = n$$
$$n_i \geq 0, \textit{integer}, \forall i \in \{1, \dots, m!\}.$$

We define the decision variables n_i to be integer from now on, unless indicated otherwise, and therefore we are faced with an integer linear program.
Note that the restriction of the variables to the natural numbers changes the hardness of the minimisation task quite dramatically as integer linear programs are known to be NP-complete, whereas linear programs can famously be solved in polynomial time[6]. This hardness result does not bother us too much as we have to deal with relatively small dimensions which can be quickly solved by an MILP solver. We decided to use the state-of-the-art solver FICO Xpress 8.8 [Xpr20].

For our purpose we need to find preference profiles in which the pair of voting rules differs in their respective choice sets, and the number of voters is minimised. The minimisation of the number of voters is stated in the objective function, and the constraints will ensure the discrepancy of elected winners.
As an example we show the integer linear program for a minimal preference profile with three candidates in which Plurality and Borda's rule select different winners. The associated integer linear program is:

$$\min \sum_{i=1}^{6} n_i \tag{1}$$

The objective function minimises the total number of voters.

$$-n_1 - n_2 + n_3 + n_4 < 0 \tag{2}$$
$$-n_1 - n_2 + n_5 + n_6 < 0 \tag{3}$$

Constraints (2) and (3) ensure that candidate a wins the plurality election as they demand that the number of voters that rank candidate a first is larger than the number of voters that rank candidates b and c first.

$$-n_1 - 2n_3 - 2n_4 - n_6 + 2n_1 + 2n_2 + n_3 + n_5 < 0 \tag{4}$$
$$-n_1 - 2n_3 - 2n_4 - n_6 + n_2 + n_4 + 2n_5 + 2n_6 < 0 \tag{5}$$
$$n_i \geq 0, i = 1, \dots, 6 \tag{6}$$

[6] [PS82]

The second set of inequalities (4) and (5) demands that candidate b has the highest Borda score. Hence the constraints ensure that candidate a is the Plurality winner and that candidate b is the Borda winner under minimisation of the total number of voters. Obviously we need to demand nonnegativity of the variables in (6).
The resulting preference profile can be seen in 3.2.

3.2 Rules for three Candidates

The goal of this chapter is to check for equivalence of voting rules. If a set of voting rules coincides in their election this usually indicates a good and natural fit of the chosen candidate. We also know from May's theorem 3.1 that in the restricted domain of only two candidates a lot of rules are equivalent to the Plurality rule. This provides good reasoning for the use of the Plurality rule in all two candidate elections.
Therefore we want to gain insights into whether some rules coincide for three candidates as well, and, if they do not, then we are interested in how many voters are needed to obtain an example that showcases the difference. We will start with Condorcet consistent rules so that they are certainly equivalent whenever a Condorcet winner exists. It turns out that actually several rules coincide in the three candidate case which makes this set of rules ever more attractive.

In this section it is shown that some commonly used Condorcet extensions coincide in the special case of only three candidates. Only two of the introduced Condorcet extensions differ from the others, namely Black's rule and Baldwin's rule. This section is heavily influenced by Courtin, Mbih and Moyouwou [CMM14] although the notation can differ, and some of the proofs are modified.
We start off stating the main result of this chapter and proving it.

Equivalent rules for three candidates

Theorem 3.2. *In a preference profile $\succ$ with three candidates the choice sets selected by Maximin, Kemeny's and Young's rule are equivalent.*

Proof. As all the rules are Condorcet consistent, they certainly select the same candidate whenever a Condorcet winner exists due to their Condorcet consistency. Therefore we can exclude this trivial case, and assume that there is no Condorcet winner in the preference profile.

The proof of the equivalence of the mentioned rules works as follows. Basically all rules will be shown to be equivalent to the Maximin rule which of course yields the equivalence of all rules. As mentioned beforehand we assume that there is no Condorcet winner. We will further distinguish two cases of preference profiles - one that have a majority cycle and those that have not. Note that if there is an odd number of voters, every profile that has no Condorcet winner has a majority cycle as no ties can occur.

Case 1: There is a majority cycle.
We now can assume a majority cycle, and due to the neutrality of all rules one can assume the cycle to be $a \succ c \succ b \succ a$ without loss of generality.

The Maximin winner is defined to be

$$f_{Maximin}(\succ) = \arg\max_{x \in A}\{\min_{y \in A\setminus\{x\}} n(x,y)\}$$

Define further the respective row minimum of each candidate as the so called Maximin score:

$$\text{Maximin score of candidate } x = \min_{y \in A\setminus\{x\}} n(x,y).$$

The winning alternative is then obtained by maximising the Maximin score.

As we have a majority cycle each candidate loses exactly one pairwise comparison against one other candidate. Hence it holds that

$$n(a,b) < {}^{n}\!/_{2}, \qquad n(c,a) < {}^{n}\!/_{2}, \qquad n(b,c) < {}^{n}\!/_{2}.$$

From this and the fact that $n(a,b) + n(b,a) = n$ it follows that

$$n(b,a) \geq {}^{n}\!/_{2}, \qquad n(a,c) \geq {}^{n}\!/_{2}, \qquad n(c,b) \geq {}^{n}\!/_{2}.$$

This yields the following Maximin scores:

$$\begin{aligned} &\text{Maximin score of candidate } a \\ = &\min_{x \in A\setminus\{a\}} n(a,x) \\ = &\min\{n(a,b), n(a,c)\} \\ = &n(a,b) \end{aligned}$$

Analogously

$$\begin{aligned} \text{Maximin score of candidate } b &= n(b,c) \\ \text{Maximin score of candidate } c &= n(c,a). \end{aligned}$$

As the Maximin winner is the maximiser of the Maximin scores, we have

$$f_{Maximin}(\succ) = \arg\max\{n(a,b), n(b,c), n(c,a)\}.$$

We will show equivalence of every voting rule to this expression.

Claim 1: Kemeny's Rule is equivalent to the Maximin Rule

Proof of Claim 1. The Kemeny rule maximises the number of agreements between the voters' preferences and the elected preference order. Every voter has one of the following possible rankings:

$\succ_1$	$a \succ b \succ c$
$\succ_2$	$a \succ c \succ b$
$\succ_3$	$b \succ a \succ c$
$\succ_4$	$b \succ c \succ a$
$\succ_5$	$c \succ a \succ b$
$\succ_6$	$c \succ b \succ a$

We again assign a score to each of these orders so that the winning preference ordering is then obtained by maximising the score. The score is chosen to be exactly the number of voters' agreements to fulfil the definition of Kemeny's rule:

$$\begin{aligned}s(\succ_1) &= n(a,b) + n(a,c) + n(b,c)\\ s(\succ_2) &= n(a,c) + n(a,b) + n(c,b)\\ s(\succ_3) &= n(b,a) + n(b,c) + n(a,c)\\ s(\succ_4) &= n(b,c) + n(b,a) + n(c,a)\\ s(\succ_5) &= n(c,a) + n(c,b) + n(a,b)\\ s(\succ_6) &= n(c,b) + n(c,a) + n(b,a)\end{aligned}$$

Using that our profile contains $a \succ c \succ b \succ a$, it holds that

$$n(b,a) > n(a,b), \qquad n(a,c) > n(c,a), \qquad n(c,b) > n(b,c).$$

This yields

$$s(\succ_1) < s(\succ_2), \qquad s(\succ_4) < s(\succ_3), \qquad s(\succ_5) < s(\succ_6).$$

This reduces the election process to determining the maximiser of $\{s(\succ_2), s(\succ_3), s(\succ_6)\}$. For example candidate a is therefore elected if $s(\succ_2) = \max\{s(\succ_2), s(\succ_3), s(\succ_6)\}$.
The scores can also be written in terms of the numbers of voters per preference profile. So for example $s(\succ_2) = 2n_1 + 3n_2 + 2n_5 + n_3 + n_6$ holds.

In order to show that Kemeny's rule now really coincides with the Maximin rule we consider the differences in scores. We now examine every pair of candidates and calculate their difference in scores. The alternative pair (a, b) is considered first, and thus we will calculate the Maximin score of candidate a and subtract the Maximin score of candidate b, and compare this to the difference in Kemeny scores of the two candidates.
The difference of Maximin score of candidate a to Maximin score of candidate b is

$$\begin{aligned}&n(a,b) - n(b,c)\\ =&n_1 + n_2 + n_5 - n_1 - n_3 - n_4\\ =&n_2 + n_5 - n_3 - n_4.\end{aligned}$$

Now the difference between Kemeny score of candidate a and Kemeny score of candidate b is

$$\begin{aligned}&s(\succ_2) - s(\succ_3)\\ =&n(a,c) + n(a,b) + n(c,b) - n(b,a) - n(b,c) - n(a,c)\\ =&n(a,b) + n(c,b) - n(b,a) - n(b,c)\\ =&n_1 + 2n_2 + 2n_5 + n_6 - (n_1 + 2n_3 + 2n_4 + n_6)\\ =&2(n_2 + n_5 - n_3 - n_4).\end{aligned}$$

Now the same calculation is carried out for candidates b and c. The difference in Maximin scores is

$$\begin{aligned}&n(b,c) - n(c,a)\\ =&n_1 + n_3 + n_4 - n_4 - n_5 - n_6\\ =&n_1 + n_3 - n_5 - n_6\end{aligned}$$

and in Kemeny scores

$$\begin{aligned}
&s(\succ_3) - s(\succ_6)\\
=&n(b,a) + n(b,c) + n(a,c) - n(c,b) - n(c,a) - n(b,a)\\
=&n(b,c) + n(a,c) - n(c,b) - n(c,a)\\
=&2n_1 + n_2 + 2n_3 + n_4 - (n_2 + n_4 + 2n_5 + 2n_6)\\
=&2(n_1 + n_3 - n_5 - n_6).
\end{aligned}$$

A similar observation can be made for candidates c and a, and therefore the differences in Maximin scores are precisely half of the differences in Kemeny scores. For deciding on a winning candidate we consider the maximiser of the respective score in both cases. This can be done for instance by checking the signs in the score differences, and electing the candidate that yields only positive differences. Note that the sign of all differences, whether in Maximin score or in Kemeny score, coincides. Therefore always the same set of candidates is elected, and the two rules are equivalent. □

Claim 2: Young's Rule is equivalent to the Maximin Rule

Proof of Claim 2. In Young's rule we check how many voters must be deleted from the ballots in order for a candidate to become a Condorcet winner. In case we have the majority cycle $a \succ c \succ b \succ a$ every alternative loses exactly one pairwise comparison. So we can use Proposition 3 from [CMM14], and get that we have to eliminate $n - 2n(a,b)$ many voters in order for candidate a to become the Condorcet winner. For b and c the numbers are $n - 2n(b,c)$ and $n - 2n(c,a)$ respectively. As the Young winner is the candidate that requires the smallest number of voters to be deleted from the ballots, it holds that

$$\begin{aligned}
&f_{Young}(\succ)\\
=&\arg\min\{n - 2n(a,b), n - 2n(b,c), n - 2n(c,a)\}\\
=&\arg\max\{n(a,b), n(b,c), n(c,a)\}\\
=&f_{Maximin}(\succ).
\end{aligned}$$

From this it is immediately clear that Young's rule and the Maximin rule are equivalent. □

Case 2: There is no majority cycle.

We now can exclude the cases that there is a majority cycle and the cases in which a Condorcet winner exists. Hence all the cases left include at least one tied paired comparison. Define a weak Condorcet winner as a candidate that does not lose any paired comparison. Note that in contrast to the Condorcet winner definition, a weak Condorcet winner can tie with other candidates. Every preference profile in which there is neither Condorcet winner nor a majority cycle, has at least one weak Condorcet winner. According to Fishburn [Fis77] Maximin and Young's rule satisfy the Strict Condorcet Principle, which means they select exactly the set of weak Condorcet winners if one or more weak Condorcet winners exist. Therefore in every preference profile that does not include a Condorcet winner or a majority cycle, they select the same set of weak Condorcet winners. Fishburn also showed that Kemeny's rule satisfies the Inclusive Condorcet Principle which means that every weak Condorcet winner is included in Kemeny's choice set. Nevertheless, it can additionally contain candidates that are not weak Condorcet winners in a general domain. In order to prove that Maximin, Young's and Kemeny's rule are equivalent for three candidates, it now suffices to show that Kemeny's rule excludes every candidate that is not a weak Condorcet winner in the special case of three candidates.

Let candidate a be the unique weak Condorcet winner. Therefore $b \succ a$ and $c \succ a$ cannot hold. Both candidates b and c have to lose at least one paired comparison. Without loss of generality say that c loses against b, and b must then lose against candidate a. Therefore it holds that

$$n(b,a) < {}^{n}\!/_{2}, \qquad n(c,b) < {}^{n}\!/_{2}, \qquad n(c,a) \leq {}^{n}\!/_{2}.$$

Using the Kemeny scores introduced earlier we get

$$\begin{aligned} s(\succ_3) &= n(b,a) + n(b,c) + n(a,c) < n(a,b) + n(a,c) + n(b,c) = s(\succ_1) \\ s(\succ_4) &= n(b,c) + n(b,a) + n(c,a) < n(a,b) + n(a,c) + n(b,c) = s(\succ_1) \end{aligned}$$

and hence candidate b cannot maximise the Kemeny score and is therefore not included in the choice set. The same holds for candidate c:

$$\begin{aligned} s(\succ_5) &= n(c,a) + n(c,b) + n(a,b) < n(a,b) + n(a,c) + n(b,c) = s(\succ_1) \\ s(\succ_6) &= n(c,b) + n(c,a) + n(b,a) < n(a,b) + n(a,c) + n(b,c) = s(\succ_1) \end{aligned}$$

Now consider the case that only candidate c is not a weak Condorcet winner. It has to hold that c cannot win any paired comparison, and it has to lose at least one. Without loss of generality assume that c loses against candidate a, therefore $n(c,a) < {}^{n}\!/_{2}$ and $n(c,b) \leq {}^{n}\!/_{2}$ holds. It follows that

$$\begin{aligned} s(\succ_5) &= n(c,a) + n(c,b) + n(a,b) < n(a,b) + n(a,c) + n(b,c) = s(\succ_1) \\ s(\succ_6) &= n(c,b) + n(c,a) + n(b,a) < n(b,a) + n(b,c) + n(a,c) = s(\succ_3) \end{aligned}$$

and therefore candidate c cannot be included in the Kemeny choice set. This shows that in presence of weak Condorcet winners no candidate that is not a weak Condorcet winner is elected using Kemeny's rule. As also every candidate that is a weak Condorcet winner is elected, according to Fishburn, it must hold that Kemeny's rule satisfies the Strict Condorcet Principle for three candidates just like Maximin and Young's rule.

Therefore in every profile in which there is a set of weak Condorcet winners Kemeny's, Maximin and Young's rule elect the same set of candidates. As the winning sets were shown to be equivalent for both the cases of a Condorcet winner presence and a majority cycle, this finishes the proof of equivalence of these rules for three candidates. □

In the following theorem another Condorcet extension is shown to be closely related to Maximin's rule and thus to Young's and Kemeny's rule as well.

Theorem 3.3. *In a preference profile with three candidates Nanson's choice set is always contained in the choice sets of the rules from Theorem 3.2.*

Proof. We have to show that Nanson's set is included in the choice set of the Maximin rule, and from this it follows that it is also included in Young's and Kemeny's set.
When determining the winning candidate using Nanson's rule one has to calculate the Borda scores of every alternative, and eliminate those candidates whose Borda scores are below or at average.
As a first observation note that the average Borda score always equals the number of voters n in the three alternative case. This is due to the fact that every voter induces three Borda points, two for the voter's most preferred candidate and one for the second highest. Thus the sum of all Borda scores equals $3n$. As we also have three candidates, the average Borda score for each candidate is $3n/3 = n$.

As before again consider the cases that there is a majority cycle and that there is not.
Case 1: There is a majority cycle.
Due to neutrality we can assume that we have the majority cycle $a \succ c \succ b \succ a$, and it holds that

$$n(b,a) > n/2, \qquad n(a,c) > n/2, \qquad n(c,b) > n/2.$$

Hence one can define positive $\epsilon_a > 0, \epsilon_b > 0$ and $\epsilon_c > 0$ such that the following holds:

$$\begin{aligned} n(b,a) &= n/2 + \epsilon_c \\ n(a,c) &= n/2 + \epsilon_b \\ n(c,b) &= n/2 + \epsilon_a \end{aligned}$$

The paired comparison matrix can be written like this:

	a	b	c
a	$-$	$n/2 - \epsilon_c$	$n/2 + \epsilon_b$
b	$n/2 + \epsilon_c$	$-$	$n/2 - \epsilon_a$
c	$n/2 - \epsilon_b$	$n/2 + \epsilon_a$	$-$

The Borda scores of the alternatives are the row sums in the above matrix and therefore $n - \epsilon_c + \epsilon_b$ for candidate a, $n - \epsilon_a + \epsilon_c$ for candidate b and $n - \epsilon_b + \epsilon_a$ for candidate c.

Now it is shown that a candidate that is selected by Nanson's rule also has to be in the choice set of the Maximin rule.
Assume without loss of generality that candidate a is a Nanson winner. Two subcases have to be considered. First assume that candidate a is a Nanson winner and candidate b is not. Then a has not been eliminated in the first step, and must have an above average Borda score, so

$$n - \epsilon_c + \epsilon_b > n$$

must hold. Also b must have been eliminated, and must hence have a below average Borda score:

$$n - \epsilon_a + \epsilon_c \leq n.$$

It follows that $\epsilon_c \leq \epsilon_a$ and $\epsilon_c < \epsilon_b$. This implies that

$$n(a,b) > n(c,a), \qquad n(a,b) \geq n(b,c)$$

and therefore $n(a,b)$ is a maximiser of the Maximin scores which ensures that candidate a is contained in the Maximin choice set.
Secondly assume that both candidates a and b are Nanson winners. It has to hold that also candidate c is a Nanson winner then. This is only possible if all candidates have the same average Borda score and hence

$$\epsilon_a = \epsilon_b = \epsilon_c.$$

This yields that every row minimum is equally large, and all candidates are also in the winning set of the Maximin rule. Thus the winning set of the Nanson rule is always a subset of the winning set of the Maximin rule.

For the reverse direction it is only possible to show that the winning sets coincide if a single candidate is chosen by the Maximin rule. Hence assume b is the unique Maximin winner. Then it has to hold that $n(b,c) > n(a,b)$ and $n(b,c) > n(c,a)$ or equivalently

$$n(c,b) < n(b,a), \qquad n(c,b) < n(a,c).$$

It follows that $\epsilon_a < \epsilon_b$ and $\epsilon_a < \epsilon_c$, and therefore candidate b's Borda score is above average and c's below. Hence c gets certainly eliminated in the first step. It is unknown if candidate a also gets deleted in the first step, but even if not, then candidate b wins the pairwise comparison against a in the second step, and is therefore also elected as Nanson winner.

Case 2: There is no majority cycle.
As discussed previously in a preference profile in which there is neither a Condorcet winner nor a majority cycle, there have to be one or more weak Condorcet winners. We proved before that exactly this set of weak Condorcet winners is selected by Maximin, Kemeny's and Young's rule. As we want to show that Nanson's choice set is included therein, we are left to show that Nanson's rule does not select a candidate that is not a weak Condorcet winner.
Let candidate a be the unique weak Condorcet winner. Both candidates b and c have to lose at least one paired comparison, say $b \succ c$ and $a \succ b$. Hence it holds that

$$\begin{aligned} n(a,b) &= {}^n\!/_2 + \epsilon_c \\ n(a,c) &= {}^n\!/_2 \\ n(b,c) &= {}^n\!/_2 + \epsilon_a \end{aligned}$$

with $\epsilon_c, \epsilon_a > 0$. It follows that the Borda scores of the candidates are $n + \epsilon_c$ for candidate a, which is above average, and $n - \epsilon_a$ for candidate c, which is below average. Candidate b's Borda score can be above or below average. In case it is below or at average, candidate b is eliminated immediately. In case it is above average, then a paired comparison with candidate a is carried out which is lost by b. So only the weak Condorcet winner is selected in this case.
Let now candidates a and b be both weak Condorcet winners. Hence candidate c has to lose at least one paired comparison and cannot win any. Without loss of generality

assume that a loss occurs in the comparison between a and c. Therefore $n(c,a) < n/2$ and $n(c,b) \leq n/2$. It can be written

$$n(c,a) = n/2 - \epsilon_b$$
$$n(c,b) = n/2 - \epsilon_a$$

with $\epsilon_b > 0$ and $\epsilon_a \geq 0$. Candidate c's Borda score is $n - \epsilon_b - \epsilon_a < n$ which is certainly below average, and hence candidate c is not already selected by Nanson's rule.
This shows that Nanson's rule satisfies the Exclusive Condorcet Principle for three candidates[7], and no additional candidate can be selected by Nanson that is not included in the choice set of Maximin, Kemeny and Young.
This finishes the proof of Theorem 3.3. □

Theorem 3.2 and Theorem 3.3 provide a nice set of Condorcet consistent rules that can always agree on a winning set. The fact that those rules agree in their decision is a strong argument for the selected candidates. Two other rules, that are not included in other investigations of this thesis, can also be shown to be equivalent to the Maximin rule. We will informally introduce them, and state the fact of their equivalence to the rules from Theorem 3.2.

Remark: Dodgson's and Schulze's rule Although they have not been introduced formally, it is worth mentioning that the choice sets selected by Dodgson's rule and Schulze's rule are equivalent to Maximin's, Kemeny's and Young's choice sets.
Dodgson's rule is related to the Young rule as it elects the Condorcet winner whenever it exists, and if no Condorcet winner exists, then Dodgson's rule selects the candidate that requires the smallest number of switches in the voters' rankings in order to become a Condorcet winner.

Schulze's rule is defined as follows: A *path* from one candidate x to another candidate y is a sequence of candidates $c(i), i \in \mathbb{N}$ so that the first is x, i.e. $c(1) = x$, and the last candidate after finitely many steps is y, i.e. $c(n) = y, 2 \leq n < \infty$. For every step on the path it has to hold that $c(i+1) \in A \setminus \{c(i)\}$. Every step on the path has the weight of the respective paired comparison margin $n(c(i), c(i+1))$.
The *strength* of a path is the smallest weight occurring on the path. Let $P(x,y)$ be the largest strength of all paths from candidate x to candidate y.
Define a binary relation O as follows: $xy \in O$ if and only if $P(x,y) > P(y,x)$. That means there exists a path from x to y that has a higher strength than the path with the highest strength from y to x. We say that candidate x dominates candidate y in this case.
The Schulze choice set S is the set of all candidates that are not dominated by another candidate, i.e. $S = \{x \in A \mid \forall y \in A \setminus \{x\} : yx \notin O\}$.

Corollary 3.1 (Dodgson's Rule)**.** *According to Courtin, Mbih and Moyouwou [CMM14] also Dodgson's rule is equivalent to Maximin, Young's and Kemeny's rule in a preference profile with three candidates.*

[7] Note that Fishburn [Fis77] defines the Nanson rule differently, and hence his results are not applicable here. His definition of Nanson's rule is equivalent to the usual definition of Baldwin's rule which can cause some confusion.

The proof that also Dodgson's rule is equivalent to the above mentioned rules is left out as this rule is not included in any further work of this thesis. The respective proof can be found in Propositions 1, 2 and 9 of [CMM14] and the rule should be mentioned for the sake of completeness.

Corollary 3.2 (Schulze's Rule)**.** *Schulze's rule is equivalent to Maximin, Young's and Kemeny's rule in a preference profile with three candidates.*

Proof. As we have already shown the equivalence of Young's, Kemeny's and Maximin rule, it suffices to show that Schulze's rule selects the same winning set as the Maximin rule as well. We again distinguish the two cases, one if there is a majority cycle and one if there is not.
Case 1: There is a majority cycle.
Assume without loss of generality that the majority cycle is $a \succ c \succ b \succ a$. Hence the following steps have weights bigger than $n/2$:

from	*to*	*weight*
a	c	$n(a,c)$
c	b	$n(c,b)$
b	a	$n(b,a)$

All other steps yield a weight smaller than $n/2$ and therefore paths including those steps yield a strength smaller than $n/2$. As these strengths are clearly smaller than paths that only use the above listed steps, these are left out in the following. Therefore it suffices to only consider the following paths with corresponding strengths:

from	*to a*	*to b*	*to c*
a	–	$\min\{n(a,c), n(c,b)\}$	$n(a,c)$
b	$n(b,a)$	–	$\min\{n(b,a), n(a,c)\}$
c	$\min\{n(c,b), n(b,a)\}$	$n(c,b)$	–

Assume that candidate a is a Schulze winner. Therefore ba and ca cannot be contained in O. In order for ba not to be contained in O, it has to hold that

$$n(b,a) \leq \min\{n(a,c), n(c,b)\}$$

and from this it follows that

$$n(b,a) \leq n(a,c)$$
$$n(b,a) \leq n(c,b).$$

These inequalities immediately give that candidate a is also a Maximin winner.
For the reverse direction assume that candidate a is a Maximin winner. Hence it has to hold that

$$n(a,b) \geq n(b,c)$$
$$n(a,b) \geq n(c,a)$$

or equivalently

$$n(b,a) \leq n(c,b)$$
$$n(b,a) \leq n(a,c).$$

From this it clearly follows that also

$$n(b,a) \leq \min\{n(c,b), n(a,c)\}$$

and hence ba cannot be in O. Also ca is not contained in O as

$$\min\{n(c,b), n(b,a)\} = n(b,a) \leq n(a,c).$$

Therefore candidate a is also selected by Schulze's rule. This shows the equivalence of Schulze's and Maximin rule in case of a majority cycle.

Case 2: There is no majority cycle.
As argued before if there is neither a Condorcet winner nor a majority cycle, there has to be a nonempty set of weak Condorcet winners. Schulze [Sch18b] himself has shown that every weak Condorcet winner is contained in the Schulze choice set. So it is left to show that for three candidates in the presence of a weak Condorcet winner no other candidate can be elected. Hence assume that candidate c looses at least one comparison, and say without loss of generality that $a \succ c$ holds. Therefore in the majority graph there is a directed path from a to c with weight $n(a,c) > n/2$. As we assumed that there is no majority cycle, $c \succ b$ and $b \succ a$ cannot both hold. Therefore a directed path from c to a can have strength at most $n/2$. But this means that the path from a to c has a strictly higher strength than from c to a which means that ac is contained in O. Therefore candidate c cannot be Schulze winner. Hence no candidate that is not a weak Condorcet winner is included in the Schulze set for three candidates. This shows that for three candidates, if there is no majority cycle, the Schulze rule selects the set of weak Condorcet winners as does the Maximin rule. □

Experimental results for Nanson's Rule and Maximin Rule We proved in Theorem 3.3 that Nanson's choice set is always included in the choice set of the Maximin rule. It is interesting to see how often Nanson's choice set is a real subset of the Maximin set, and how often they are actually equivalent. In order to get a rough understanding of this we decided to test several random preference profiles with a varying number of voters, and check for their winners. The profiles are generated with the help of PreflibTools [MW13a] under the IAC assumption. For every number of voters we tested 1000 profiles on their winners. The results of this experimental analysis are the following:

Number of voters	*Difference of Choice sets in %*	*95% - Confidence Interval in %*
100	1.1	$(0.5, 1.7)$
1000	0.1	$(-0.1, 0.3)$

We also tested profiles with 10,000 and 100,000 many voters, but there was no difference in the choice set of Maximin and Nanson's rule. This is due to the fact that ties are less common the higher the number of voters is. Therefore the higher the number of voters, the more likely it is that the Maximin rule selects a unique candidate, and Maximin's and Nanson's choice sets are equivalent then. This shows that for a large number of voters Nanson's rule is equivalent to Maximin, Kemeny's and Young's rule almost surely.

We will now consider two voting rules that are also Condorcet extensions, but can differ in the winners that they select.

Black's Rule It turns out that Black's rule is different to the above mentioned voting rules. To show that the Black rule does not always yield the same outcome as the above mentioned rules one needs to find a counterexample. It is interesting to check for a minimal such example. When following the approach described in chapter 3.1, this then yields the following minimal example where Black's and Maximin's winner differ in the three alternative case. The example involves eleven voters:

4	2	5
a	b	c
b	c	a
c	a	b

This profile yields the following pairwise majority matrix:

	a	b	c
a	–	9	4
b	2	–	6
c	7	5	–

This preference profile has no Condorcet winner as it contains the majority cycle $a \succ b \succ c \succ a$. This is why Black's rule says to choose the Borda winner here. The Borda scores for a, b and c are 13, 8 and 12 respectively. Therefore the Black winner is candidate a.
As one can read from the paired comparison matrix, the row minima for candidates a, b and c are 4, 2 and 5 respectively. Therefore candidate c is the unique Maximin winner. It follows from Theorem 3.2 and Theorem 3.3 that also Young's, Kemeny's and Nanson's rule uniquely select c, and hence those rules cannot coincide with Black's rule in the three candidate case.
Note that Nanson's rule is sometimes more decisive, and hence a minimal profile in which Nanson's and Black's rule differ, can already be obtained with only five involved voters[8].

Baldwin's Rule Despite its similarity to Nanson's rule Baldwin's rule does not coincide with the above mentioned rules for three candidates either. A minimal example for a preference profile with different Maximin and Baldwin winners involves eleven voters:

5	2	4
a	b	c
b	c	a
c	a	b

The following pairwise comparison matrix corresponds to the preference profile.

	a	b	c
a	–	9	5
b	2	–	7
c	6	4	–

[8] e.g. $n_1 = 2, n_4 = 1, n_5 = 2$ yields candidate a as Black winner and candidate c as Nanson winner

The Borda scores of the candidates a, b and c are 14, 9 and 10 respectively. In the first step of the Baldwin procedure only b is eliminated as it has the lowest score. In the second step one has to compare a and c pairwise, and it turns out that a loses this comparison, and hence c is the Baldwin winner. The row minima on the other hand are 5,2 and 4 for candidates a, b and c which makes candidate a the unique Maximin winner and hence also the unique Kemeny, Young and Nanson winner. This preference profile is also minimal when only considering Baldwin's and Nanson's rule.

Black's and Baldwin's Rule Black's and Baldwin's rule do not coincide either. A minimal example is obtained using an MILP solver, and involves five voters:

2	1	2
a	b	c
b	c	a
c	a	b

The preference profile in this example is cyclical. Hence Black's rule elects the Borda winner. The Borda scores of the candidates in this example are 6, 4 and 5 for a, b and c respectively, and a gets elected as the Black winner. As candidate b has the lowest Borda score, it is eliminated by Baldwin's rule in the first step, and then the pairwise comparison between the remaining alternatives is won by c.

We will now do similar investigations for rules that are not Condorcet consistent.

As we were able to obtain a set of four Condorcet consistent rules that always share a winner for the three candidate case, we now check the rules that are not Condorcet consistent. We first show that those rules are not even Condorcet consistent for three candidates by finding minimal profiles in which a Condorcet winner exists that is not elected. Hence no equivalence to a Condorcet extension can be obtained as they elect the Condorcet winner whenever one exists. The indicated profiles are taken from Schmidt [Sch18a] and the remaining profiles are obtained by following the exact same method as Schmidt.

Plurality Rule According to [Sch18a] a minimal example that shows that the Plurality rule is not Condorcet consistent for three alternatives is:

2	1	2	2
a	a	b	c
b	c	a	b
c	b	c	a

Clearly alternative a is the Plurality winner, but as four voters prefer b over a and also four voters prefer b over c, b is the Condorcet winner. This shows that the Plurality rule is not Condorcet consistent for three candidates which implies that it cannot coincide with a Condorcet extension even for only three candidates.

Borda's Rule Similar to the Plurality rule Borda's rule is not Condorcet consistent as well even for only few candidates. A minimal example according to [Sch18a] for three candidates is

2	3
a	b
c	a
b	c

Here b is the Condorcet winner, but candidate a is elected using the Borda count because it has a Borda score of 7, whereas b and c have a score of 6 and 2 respectively.

Plurality with Runoff Rule The Plurality with Runoff method is also not Condorcet consistent. With the same approach as in [Sch18a] a profile is determined that shows without ties that Plurality with Runoff is not a Condorcet extension, and it involves 5 voters:

1	2	2
a	b	c
b	a	a
c	c	b

Here a is the Condorcet winner as it wins both pairwise comparisons with b and c. Nevertheless a is eliminated in the first step of the runoff procedure as it is the candidate with the least top votes. In the second step b wins the pairwise comparison with c and is elected the winner here. This example also shows that Plurality with Runoff and Borda's rule do not coincide as a has the highest Borda score here.

Coombs' Rule It turns out that also Coombs' method is not Condorcet consistent even for three alternatives. A minimal example uses 13 voters:

3	3	1	3	1	2
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

Again alternative a is the Condorcet winner. But clearly a gets eliminated in the first step of Coombs' method as it is ranked last by a plurality of the voters. After this elimination b wins the pairwise comparison against c and is therefore elected.
This example also shows that the Plurality rule, Plurality with Runoff, Borda's rule and Coombs' rule do not coincide.

Bucklin's Rule In the same manner as above it can be established that the Bucklin method is not Condorcet consistent for three candidates. The minimal example involves seven voters:

3	1	2	1
a	b	b	c
b	a	c	a
c	c	a	b

As before candidate a is the Condorcet winner. No alternative is ranked first by a majority, therefore the voters who rank the alternative second are added to those voters who rank the alternative first. Then 6 voters rank b first or second, whereas only 5 and 3 voters rank a and c first or second. The candidate supported by the largest majority is elected, and in this case this is candidate b.
In this example also Plurality with Runoff and Coombs' method would have a different winner.

In total this subsection shows that none of the rules that are not Condorcet consistent can coincide with rules from the other subsection.

Also for the rules in this chapter it could happen that a subset of them coincide in the restricted domain of only three candidates. Unfortunately it turns out that this is not the case. For each pair of rules a counterexample can be found so that the winning alternatives are different. For all these examples we again minimised the number of voters involved.

Plurality and Borda's Rule A minimal example that shows that Plurality and Borda rule can elect different alternatives as winner involves four voters:

2	1	1
a	b	c
b	c	b
c	a	a

Clearly candidate a is the Plurality winner. The Borda scores of the candidates a, b and c are 4, 5 and 3 respectively, and therefore b is elected as winner.

Plurality and Plurality with Runoff Rule Plurality and Plurality with Runoff also do not yield the same winner in all preference profiles. A minimal counterexample involves nine voters:

4	3	2
a	b	c
b	a	b
c	c	a

Again candidate a is the Plurality winner as it gets the most top votes. But in the Plurality with Runoff procedure a second voting round is carried out as candidate a has no absolute majority. The two candidates with the highest number of top votes are a and b. So in the second voting round the pairwise comparison between those alternatives is made, and b is elected as winner.

Plurality and Coombs' Rule Similar to the above example also Plurality and Coombs' rule do not coincide for three candidates. A minimal example that shows their difference involves seven voters:

2	1	2	2
a	a	b	c
b	c	a	b
c	b	c	a

Again candidate a is the Plurality winner, but has no absolute majority. Therefore a is not immediately elected in Coombs' procedure. As a majority of voters has ranked c last in their preferences, candidate c is eliminated. In the pairwise comparison of the two remaining candidates a and b, a majority prefers b over a, and hence Coombs' winner is candidate b.

Plurality and Bucklin's Rule Also the comparison of Plurality rule and Bucklin's rule shows that they do not coincide with each other. A minimal example is obtained by solving the respective mixed integer linear program, and involves seven voters:

2	1	1	1	2
a	a	b	b	c
b	c	a	c	b
c	b	c	a	a

In this example candidate a is the Plurality winner, but it has no absolute majority of the top votes. Therefore for Bucklin's procedure the second and first votes of all ballots are added for each candidate, and then the majority winner is elected. It turns out that six voters ranked alternative b first or second, whereas a and c are ranked first or second by only four voters. Hence b is Bucklin's winner.

Borda's Rule and Plurality with Runoff/Coombs' Rule/Bucklin's Rule Borda's rule also differs from the mentioned rules. The following exemplary preference profile uses that the Borda rule does not have to elect an alternative that is ranked first by an absolute majority, whereas Plurality with Runoff, Coombs' and Bucklin's rule do. The minimal example involves five voters with only two different preference rankings:

3	2
a	b
b	c
c	a

The Borda counts of the candidates a, b and c are 6, 7 and 2 which is why candidate b is Borda winner. But clearly a is ranked first by a majority of voters. Hence candidate a is immediately elected from Plurality with Runoff, Coombs' and Bucklin's rule.

Plurality with Runoff and Coombs' Rule When comparing the Plurality with Runoff procedure and Coombs' rule, one can also find preference profiles with three candidates so that the respective selected winners from those two rules differ. The following is a minimal example with five voters:

2	2	1
a	b	c
c	c	a
b	a	b

The Plurality with Runoff procedure checks the pairwise comparison between candidates a and b as those are the alternatives that are ranked first by the most people, but have no absolute majority. This comparison is won by candidate a. Coombs' rule on the other hand compares alternatives a and c, because b is ranked last by the highest number of voters. In this comparison c is elected as winner.

Plurality with Runoff and Bucklin's Rule The following preference profile shows that Plurality with Runoff and Bucklin's rule do not always yield the same outcome in an election with three candidates:

2	2	1
a	b	c
b	c	a
c	a	b

In the Plurality with Runoff procedure candidate c is eliminated in the first step as a and b are ranked first by more voters. The following pairwise comparison is won by candidate a. But b is elected when Bucklin's rule is used, because b is supported by the largest number of voters when counting the first and second place rankings.

Coombs' Rule and Bucklin's Rule Coombs' Rule and Bucklin's Rule do not coincide in elections with only three alternatives. This can be seen in the following minimal example that involves seven voters:

3	1	2	1
a	b	b	c
b	a	c	a
c	c	a	b

Note that no alternative is ranked first by an absolute majority, so no candidate is elected immediately. As candidate c is ranked last by the most voters, it is eliminated when using Coombs' procedure. Then a wins as it is preferred over b by a majority. But candidate b is selected by Bucklin's rule, because it has the most first and second place rankings.

Plurality with Runoff and Instant Runoff It is immediately clear from the definitions that for three alternatives these rules are equivalent.

We will now move on to four candidates, and check whether any equivalences can be maintained in that case.

3.3 Rules for four Candidates

Firstly, note that if two rules do not coincide for three candidates, then they cannot coincide for four candidates or any bigger number of alternatives in general. That is due to the fact that one can model the three candidate case as a four candidate ranking with a candidate that is always ranked last, and therefore has no influence on the election. In that way all four alternative cases can be reduced to three alternatives cases, and it is shown that two rules that were different for three candidates are certainly different for four. Hence we are only left with the set of rules from Theorem 3.2 and Theorem 3.3.

3.3.1 Differences of Maximin, Kemeny's, Nanson's and Young's Rule

Maximin Rule and Nanson's Rule First note that according to Felsenthal [Fel12] the Maximin rule is vulnerable to the Condorcet Loser paradox, but Nanson's rule is not. So these rules can differ if there exists a Condorcet loser but no Condorcet winner. In an example where the Condorcet loser is uniquely selected by the Maximin rule 15 voters are needed:

3	1	5	1	5
a	a	c	d	d
b	d	a	a	b
c	b	b	b	c
d	c	d	c	a

In this example candidate d is the Condorcet loser. The pairwise comparison matrix looks as follows:

	a	b	c	d
a	–	10	5	9
b	5	–	10	8
c	10	5	–	8
d	6	7	7	–

Now the row minima are 5 for candidates a, b and c and 6 for candidate d which makes candidate d the unique Maximin winner, even though it is the Condorcet loser. The winner according to Nanson's rule is the set $\{a, b, c\}$.

Apart from the Maximin rule choosing a Condorcet loser the rules can also differ in other preference profiles that do not have a Condorcet winner. In such profiles there has to be a majority cycle involving all four alternatives. Therefore one can assume $a \succcurlyeq b \succcurlyeq c \succcurlyeq d \succcurlyeq a$ without loss of generality. One can assume, due to neutrality of the voting rules, that b is the Maximin winner. To ensure that b is eliminated by Nanson's rule there are several possibilities. Hence one has to distinguish for example between a preference profile where b is eliminated in the first step of the Nanson procedure and one where it is eliminated later. Calculating the minimal profiles for every possible elimination order yields the following minimal preference profile with seven voters:

2	1	2	2
b	b	c	d
a	c	d	a
c	d	a	b
d	a	b	c

The pairwise comparison matrix looks as follows:

	a	b	c	d
a	–	4	4	2
b	3	–	5	3
c	3	2	–	5
d	5	4	2	–

From this matrix it is clear that the row minima are 2 for candidates a, c and d and 3 for candidate b which makes b the unique Maximin winner. For Nanson's procedure the Borda scores have to be calculated. Candidates a and c have Borda score 10, whereas the other alternatives b and d have Borda score 11. Hence a and c are eliminated in the first round, and then b loses the pairwise comparison to d which makes candidate d the unique Nanson winner.

Young's Rule and Nanson's Rule According to [BGS] it is unclear how the definition of a Young winner can be translated into an inequality description. In order to carry out the calculation of a minimal profile, one has to increment the Young scores. By proceeding in such manner one can find a preference profile with different Nanson and Young winner involving seven voters:

2	1	2	2
a	a	c	d
b	c	d	a
c	d	a	b
d	b	b	c

The Borda scores of the candidates are 15, 6, 10 and 11 which makes up for an average Borda score of 10.5. Hence candidates b and c are discarded in the first round of Nanson voting. The pairwise comparison of the remaining alternatives a and d is lost by a which makes candidate d the Nanson winner. The Young winner on the other hand is a with a score of 2. Using the worst defeats of each candidate one can calculate lower bounds on the Young scores[9]. It turns out that all candidates have at least a Young score of 4 except candidate a who meets its lower bound 2 by deleting two voters with the preference ranking $c \succ d \succ a \succ b$.

Nanson's Rule and Kemeny's Rule Also Nanson's and Kemeny's rule differ when the number of candidates is increased to four. Again there cannot be a Condorcet winner, and therefore it was assumed that there is a majority cycle $a \succ b \succ c \succ d \succ a$. Candidate c was forced to be the Kemeny winner by some linear indicator constraints according to the explanation in [Sch18a]. We also forced Nanson's rule to eliminate the Kemeny winner c. To do so every possible elimination order of candidates including c was examined, and a minimal example could be obtained that involves seven voters with four different rankings only:

[9] [Sch18a]

2	2	1	2
a	b	c	c
b	c	a	d
c	d	b	a
d	a	d	b

The preference order with the lowest number of disagreements is $c \succ a \succ b \succ d$ and hence c really is the Kemeny winner. The Borda scores of the alternatives are 10, 11, 15 and 6 for a, b, c and d respectively. As the average Borda score is 10.5, candidates a and d are eliminated in the first step. This leaves b and c for a pairwise comparison which is won by b. Hence the Nanson winner is candidate b, whereas the Kemeny winner is c.

Maximin Rule and Kemeny's Rule The Maximin rule and Kemeny's rule do not coincide for more than three candidates. This is partly due to the fact that the Maximin rule can select a Condorcet loser which the Kemeny rule never does. The example that has been used in the section about Nanson's rule and Maximin rule can also be used to show the difference in winners between Maximin and Kemeny for the case that the Maximin rule elects a Condorcet loser. But apart from that case, there are also profiles where there is neither a Condorcet winner nor a Condorcet loser, and still the winning alternatives differ. For such an example consider the following profile which involves nine voters:

2	1	1	2	3
a	a	c	c	d
b	d	a	d	b
c	b	b	a	c
d	c	d	b	a

The selected order in this profile by Kemeny is $c \succ d \succ a \succ b$, and hence candidate c is the Kemeny winner again. The pairwise comparison matrix looks as follows:

	a	b	c	d
a	–	6	3	4
b	3	–	6	3
c	6	3	–	5
d	5	6	4	–

The row minima are clearly 3 for a, b, c and 4 for d. Hence candidate d is the Maximin winner, and therefore different from the Kemeny winner.

Young's Rule and Kemeny's Rule Young's rule and Kemeny's rule also differ for the four candidate case. As an inequality description of Young's rule is not known, one has to search for a Young winner by increasing the Young score one by one. It turns out that there is a profile in which Kemeny and Young winner are different, and the Young score is 2. For this example consider the following profile that involves nine voters:

3	1	3	2
a	a	c	d
b	d	d	b
c	b	a	c
d	c	b	a

The selected Kemeny ranking is $c \succ d \succ a \succ b$ which makes c the Kemeny winner. For determining the Young winner consider the pairwise comparison matrix:

	a	b	c	d
a	–	7	4	4
b	2	–	6	3
c	5	3	–	6
d	5	6	3	–

One can see from the pairwise comparison matrix that the lower bounds of the Young scores are 2, 6, 4 and 4 for candidates a, b, c and d respectively. In order to show that candidate a really is the Young winner, it suffices to show that a's Young score really is 2, as all other alternatives have certainly a score larger than 3. If the two voters with preference ranking $d \succ b \succ b \succ a$ are deleted from the ballots, then a is Condorcet winner. Hence candidate a's Young score really is equivalent to its lower bound 2. Therefore a is Young winner, whereas c is Kemeny winner.

3.3.2 Interaction of Young's Rule and Maximin Rule

Similar to the other rules also Young's rule and the Maximin rule are not equivalent for four candidates. Nevertheless we identified preference profiles in which Young's winner and Maximin winner always coincide. The condition for when the winners are equivalent is stated in the following theorem.

Theorem 3.4 (Sufficient condition for equivalence of Maximin and Young winners)**.** *Let $\succ$ be a preference profile with four candidates and an arbitrary number of voters. Let candidate a be the unique Maximin winner. We state the following properties of a preference profile $\succ$:*

(i) a loses against at least two candidates, i.e. $|x : x \succ a| \geq 2$ *, and*

(ii) the number of voters that rank a last is less or equal to its smallest margin of defeat, i.e. $n(a\ last) \leq \min_{x:x\succ a}(n(x,a) - n(a,x))$ *.*

If a preference profile $\succ$ does not satisfy the above condition (i) and (ii), then candidate a is also the unique Young winner and hence the winners coincide.

Note that the reverse does not hold. Hence if $\succ$ does satisfy the mentioned condition, the winners can still be equivalent.

Proof. The proof is a bit lengthy. First we prove an auxiliary condition that is stated in Lemma 3.1, and makes a statement about the Young scores of candidates in certain preference profiles. This can only be shown with a case distinction of the preference profiles. Having proved Lemma 3.1, it is then easy to conclude Theorem 3.4.

For the proof introduce the following notation: Let $\succ$ be the starting preference profile with an arbitrary number of voters. Denote $\succ^*$ as the preference profile in which a minimal number of voters is deleted in order to make candidate a the Condorcet winner. Hence one can assume that a is not a Condorcet winner in $\succ$ as this is a trivial case, but it is in $\succ^*$. Let $n(a, b)$ be the number of voters that prefer a over b. Assume due to neutrality that if candidate a loses against other candidates, then its worst loss happens against candidate d.

Lemma 3.1. *Assume that candidate d defeats candidate a with the highest number of voters. If $\succ$ is a preference profile with four candidates and $\succ$ does not satisfy (i) and (ii), then candidate a's Young score is* $n(d, a) - n(a, d) + 1$.

Proof of Lemma 3.1. In the following we show a way to delete specific voters in order to make candidate a the Condorcet winner. We will select $n(d, a) - n(a, d) + 1$ many voters, and thereby prove that an elimination of those will result in candidate a being the Condorcet winner. In order to keep notation short define $q = n(d, a) - n(a, d) + 1$ as the number of voters that we choose to delete, and let Q be the set of voters that we choose

to delete. Denote the profile in which the designated voters are removed with $\succ_{-Q}$.
As candidate a needs to win over candidate d in $\succ^*$, at least q voters need to be eliminated, and hence q yields a lower bound on the Young score of candidate a. It is now left to show that q is also an upper bound on candidate a's Young score. We do that by selecting q many voters to be removed, and claim that in the resulting profile $\succ_{-Q}$ candidate a wins every pairwise comparison.

One can distinguish two cases here.
Case 1: Let q be smaller or equal to the number of voters that rank a last; this will be denoted by $q \leq n(a\ last)$. Hence one can delete q voters that rank a last. Then it holds that

$$n(a, d, \succ_{-Q}) = n(a, d, \succ).$$

An elimination of voters that rank candidate a last has clearly no impact on the voters that rank candidate a before candidate d. On the other hand for voters that rank candidate d before candidate a, it holds that

$$n(d, a, \succ_{-Q}) = n(d, a, \succ) - q = n(d, a, \succ) - n(d, a, \succ) + n(a, d, \succ) - 1.$$

Using these two expressions one can now calculate the majority margin in the altered profile:

$$\begin{aligned}
&n(a, d, \succ_{-Q}) - n(d, a, \succ_{-Q}) \\
=&n(a, d, \succ) - n(a, d, \succ) + 1 \\
=&1
\end{aligned}$$

This indicates that candidate a defeats d with a margin of 1.
The majority margins for the other candidates can be calculated similarly. For candidate b it holds that

$$\begin{aligned}
&n(b, a, \succ_{-Q}) - n(a, b, \succ_{-Q}) \\
=&n(b, a, \succ) - q - n(a, b, \succ) \\
=&n(b, a, \succ) - n(d, a, \succ) + n(a, d, \succ) - 1 - n(a, b, \succ) \\
<&0.
\end{aligned}$$

Likewise for candidate c it holds that

$$\begin{aligned}
&n(c, a, \succ_{-Q}) - n(a, c, \succ_{-Q}) \\
=&n(c, a, \succ) - q - n(a, c, \succ) = \\
=&n(c, a, \succ) - n(d, a, \succ) + n(a, d, \succ) - 1 - n(a, c, \succ) \\
<&0
\end{aligned}$$

as $n(a, b) \geq n(a, d)$ and $n(d, a) \geq n(b, a)$, $n(a, c) \geq n(a, d)$ and $n(d, a) \geq n(c, a)$, and all deleted voters preferred both b and c over a. Hence in this case the elimination of q voters suffices to make candidate a win every paired comparison, and hence a's Young score is upper bounded by q. Therefore we already know that for preference profiles in which

$q \leq n(a\ last)$ the Young score of candidate a is exactly q.

<u>Case 2:</u> The second case is that there are less voters that rank candidate a last than q. Then q can be set to $q = n(a\ last) + r$, for some $r \in \mathbb{N}, r > 0$. Hence we are aiming for a preference profile where we delete all voters that rank candidate a last and some more voters - namely r many - so that after their elimination candidate a is the Condorcet winner. The voters that have to be deleted additionally have to rank d before a, but they do not rank a last as by assumption these are already deleted and are less than q. So they have to have one of the following preferences:

r_1		r_2		r_3	
d	b	d	c	d	d
b	d	c	d	a	a
a	a	a	a	b	c
c	c	b	b	c	b

In these tables one can also already see that we can refine r even further, and hence we define r_1 as voters with the above stated preferences and similarly for r_2 and r_3. Note that now one can write $r = r_1 + r_2 + r_3$. Note further that if r such voters are deleted together with all voters that ranked candidate a last, then a wins over d with margin 1. Now one can compare the majority margins of candidates b and c with candidate a before and after the elimination.

$$
\begin{aligned}
&n(a, c, \succ_{-Q}) \\
=&n(a, c, \succ) - r_1 - r_3 \\
=&n(a, c, \succ) + n(a\ last) - q + r_2 \\
=&n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8 + n_{11} + n_{19} + n_{20} + n_{21} \\
&+ n_{10} + n_{12} + n_{16} + n_{18} + n_{22} + _{24} \\
&- n(d, a, \succ) + n(a, d, \succ) - 1 + r_2 \\
=&2(n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8) - n_{17} - n_{23} + n_9 + n_{13} + n_{14} + n_{15} - 1 + r_2
\end{aligned}
$$

$$
\begin{aligned}
&n(c, a, \succ_{-Q}) \\
=&n(c, a, \succ) - n(a\ last) - r_2 \\
=&n_9 + n_{10} + n_{12} + n_{13} + n_{14} + n_{15} + n_{16} + n_{17} + n_{18} + n_{22} + n_{23} + n_{24} \\
&- n_{10} - n_{12} - n_{16} - n_{18} - n_{22} - n_{24} - r_2 \\
=&n_9 + n_{13} + n_{14} + n_{15} + n_{17} + n_{23} - r_2
\end{aligned}
$$

In order to calculate the majority margin of candidates a and c one has to subtract the above expressions from one another:

$$
\begin{aligned}
&n(a, c, \succ_{-Q}) - n(c, a, \succ_{-Q}) \\
=&2(n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8) - 2n_{17} - 2n_{23} + 2r_2 - 1
\end{aligned}
$$

Similar calculations can be done for the paired comparison between candidates b and a, and this yields:

$$\begin{aligned}&n(a,b,\succ_{-Q})\\=&2(n_1+n_2+n_3+n_4+n_5+n_6+n_{13}+n_{14})-n_{11}-n_{21}+n_7+n_8+n_9+n_{15}-1+r_1\end{aligned}$$

$$n(b,a,\succ_{-Q})=n_7+n_8+n_9+n_{11}+n_{15}+n_{21}-r_1$$

$$\begin{aligned}&n(a,b,\succ_{-Q})-n(b,a,\succ_{-Q})\\=&2(n_1+n_2+n_3+n_4+n_5+n_6+n_{13}+n_{14})-2n_{11}-2n_{21}+2r_1-1\end{aligned}$$

As candidate a needs to win over c in the altered preference profile, the majority margin that has been calculated above has to be positive. Therefore we can choose r_2 in such a way that a beats c by a margin of 1 when r_2 voters are removed whenever this is possible. Hence choose

$$\begin{aligned}r_2&=\max\{0,-(n_1+n_2+n_3+n_4+n_5+n_6+n_7+n_8)+n_{17}+n_{23}+1\}\\r_3&=0.\end{aligned}$$

Then for the paired comparison between candidates a and b and the case that $r_2>0$ was previously chosen it follows:

$$\begin{aligned}&n(a,b,\succ_{-Q})-n(b,a,\succ_{-Q})\\=&2(n_1+n_2+n_3+n_4+n_5+n_6+n_{13}+n_{14}-n_{11}-n_{21}+q-n(a\ last)-r_2)\\=&2(n_1+n_2+n_3+n_4+n_5+n_6+n_{13}+n_{14}-n_{11}-n_{21}\\&-n_{10}-n_{12}-n_{16}-n_{18}-n_{22}-n_{24}\\&+n_{10}+n_{11}+n_{12}+n_{16}+n_{17}+n_{18}+n_{19}+n_{20}+n_{21}+n_{22}+n_{23}+n_{24}\\&-n_1+n_2-n_3-n_4-n_5-n_6-n_7-n_8-n_9-n_{13}-n_{14}-n_{15}\\&+n_1+n_2+n_3+n_4+n_5+n_6+n_7+n_8-n_{17}-n_{23})+1-1\\=&2(n_1+n_2+n_3+n_4+n_5+n_6+n_{19}+n_{20}-n_9-n_{15})\end{aligned}$$

For the choice $r_2=0=r_3$ it holds that

$$\begin{aligned}&n(a,b,\succ_{-Q})-n(b,a,\succ_{-Q})\\=&2(n_1+n_2+n_3+n_4+n_5+n_6+n_{13}+n_{14}-n_{11}-n_{21}+q-n(a\ last))\\=&n_{17}+n_{19}+n_{20}+n_{23}-n_7-n_8-n_9-n_{15}+1.\end{aligned}$$

From here on in the help of an LP-solver is needed. We need to show that our choice was suited to make candidate a win every paired comparison provided that we have a preference profile that does not satisfy (i) and (ii). We do this by contradiction, so we assume that our selection of voters that ought to be eliminated was not sufficient to make candidate a a Condorcet winner, and hence that a still loses one paired comparison. As we defined r in such a way that candidate a surely wins the paired comparisons with candidates d and c after the elimination of the designated voters, we can only assume for contradiction that a loses against candidate b. We derived an expression of the majority margin between candidates a and b in $\succ_{-Q}$ dependent on the choice of r_2 above. Hence we consider the following sets of linear inequalities to provoke a contradiction:

$$-(n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8) + n_{17} + n_{23} + 1 > 0 \quad (7)$$
$$2(n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_{19} + n_{20} - n_9 - n_{15}) \leq 0 \quad (8)$$

Inequality (7) describes the case that r_2 can be chosen to be a positive number as one can see from the definition above. The resulting majority margin was also derived above, and in (8) it is stated that precisely this margin is less or equal to zero which means that candidate a does not win over b. If r_2 does not turn out to be positive, then (9) has to hold:

$$(n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8) + n_{17} + n_{23} + 1 \leq 0 \quad (9)$$
$$n_{17} + n_{19} + n_{20} + n_{23} - n_7 - n_8 - n_9 - n_{15} + 1 \leq 0 \quad (10)$$

Similar to above we also add the inequality (10) that states that the respective majority margin of candidate a to candidate b is less or equal to zero in order to provoke the contradiction. Of course we always have to consider both sets of inequalities to cover both cases of r_2.
Additional to those two sets of linear inequalities we now need inequalities that describe the starting preference profiles so that they violate either (i) or (ii). In such preference profiles candidate a either loses against only one other candidate, or it loses against more than one other candidate, but the smallest margin of defeat is smaller than the number of voters that rank candidate a last. Hence these are the cases that we need to distinguish in the following. Together with either (7) and (8) or (9) and (10) the inequalities that describe the preference profile form a linear program. As we hope for a contradiction these linear programs should be infeasible as then (8) and (10) turn out to be untrue, and this results in candidate a being indeed the Condorcet winner. To obtain the infeasibility results an MILP-solver is used.
Claim 1: If candidate a loses against one candidate only, the describing inequalities together with each (7) and (8) and with (9) and (10) form an infeasible linear program.

Proof of Claim 1. We can assume without loss of generality that the candidate that a loses against is candidate d. The starting profile can then be described as follows:

$$n(a,b) > {}^{n}\!/_{2} \quad (11)$$
$$n(b,c) \geq {}^{n}\!/_{2} \quad (12)$$
$$n(c,d) \geq {}^{n}\!/_{2} \quad (13)$$
$$n(d,a) > {}^{n}\!/_{2} \quad (14)$$
$$n(a,c) > {}^{n}\!/_{2} \quad (15)$$

The first four inequalities describe the majority cycle $a \succ b \succ c \succ d \succ a$, and the last inequality states that candidate a also wins against candidate c. Now the two sets of inequalities are added independently of each other. In both linear programs an infeasibility is obtained which means that our selection of voters that are removed in the altered profile $\succ_{-Q}$ was successful in order to make candidate a become a Condorcet winner. □

Claim 2: If candidate a loses against both candidates d and c and $n(a\ last) \geq n(c, a) - n(a, c)$ holds, then the describing inequalities together with each (7) and (8) and with (9) and (10) form an infeasible linear program.

Proof of Claim 2. We proceed as above, and represent the preference profile as a linear program. We have to assume that candidate a's highest defeater is candidate d.

$$n(a, b) > {}^{n}\!/_{2} \tag{16}$$
$$n(b, c) \geq {}^{n}\!/_{2} \tag{17}$$
$$n(c, d) \geq {}^{n}\!/_{2} \tag{18}$$
$$n(d, a) > {}^{n}\!/_{2} \tag{19}$$
$$n(a, c) < {}^{n}\!/_{2} \tag{20}$$
$$n(a, d) \leq n(a, c) \tag{21}$$
$$n(a\ last) \geq n(c, a) - n(a, c) \tag{22}$$

Inequalities (16) - (21) describe the preference profile. The last inequality describes the assumption that there are more or equally many voters that rank candidate a last than the smallest margin of a's defeats. Again both sets of inequalities are added to the linear program, and with the help of the MILP-solver infeasible programs are detected. Therefore here it is possible as well to turn candidate a into a Condorcet winner whilst eliminating only q many voters. □

Claim 3: If candidate a loses against all other alternatives and $n(a\ last) \geq n(b, a) - n(a, b)$ holds, then the describing inequalities together with each (7) and (8) and with (9) and (10) form an infeasible linear program.

Proof of Claim 3. Again we describe the preference profile as a linear program, and assume that the candidate that beats candidate a with the smallest margin of defeat is candidate b.

$$n(a, b) < {}^{n}\!/_{2} \tag{23}$$
$$n(a, c) \leq n(a, b) \tag{24}$$
$$n(a, d) \leq n(a, c) \tag{25}$$
$$n(b, c) \geq {}^{n}\!/_{2} \tag{26}$$
$$n(c, d) \geq {}^{n}\!/_{2} \tag{27}$$
$$n(d, b) \geq {}^{n}\!/_{2} \tag{28}$$
$$n(a\ last) \geq n(b, a) - n(a, b) \tag{29}$$

The first six inequalities (23) - (28) describe the starting profile. The last inequality (29) is our assumption that the number of voters that rank candidate a last is sufficiently large. As above both pairs of inequalities are again added, and in both cases an infeasible linear program is the result which contradicts the starting assumption that a is not the Condorcet winner in the altered profile. □

Together the three claims show that the choice of voters ought to be eliminated in order to turn candidate a into a Condorcet winner was suited. Hence we managed to make candidate a win every paired comparison whilst only removing $q = n(d, a) - n(a, d) + 1$ many voters. This shows that q is also an upper bound on the Young score of candidate a, and hence the Young score of a is precisely $n(d, a) - n(a, d) + 1$ which concludes the proof of Lemma 3.1. □

It is left to show that now Young winner and Maximin winner are equivalent if the preference profile does not satisfy conditions (i) and (ii).

Remark: Note that in the argumentation so far we have not used that a is the Maximin winner. Hence it holds for every candidate x that does not satisfy conditions (i) and (ii), that its Young score is $n(y, x) - n(x, y) + 1$ where y is the candidate that defeats x by the highest number of voters.

Let canidate a be the unique Maximin winner in the preference profile $\succ$, and assume $\succ$ does not satisfy conditions (i) and (ii). Then, as established in Lemma 3.1, candidate a's Young score is $n(d, a) - n(a, d) + 1$ if alternative d is again the candidate that defeats a by the highest number of voters. As a is the Maximin winner, it holds that $n(a, d) > n(b, x)$ where x is the alternative that defeats b with the highest number of voters and is hence b's row minimum. It follows that

$$\begin{aligned}
&y(a)\\
=&n(d, a) - n(a, d) + 1\\
=&n - n(a, d) - n(a, d) + 1\\
=&-2n(a, d) + n + 1\\
<&-2n(b, x) + n + 1\\
=&n(x, b) - n(b, x) + 1\\
\leq&y(b)
\end{aligned}$$

and similar for the other alternatives. Therefore candidate a is also the unique Young winner in this scenario. For the reverse direction assume a is not the Maximin winner. Then there is another unique Maximin winner that is by the above argument also the unique Young winner and hence a cannot be Young winner as well.
This finishes the proof of Theorem 3.4. □

Hence we have established that for a broad class of preference profiles Young winner and Maximin winner do coincide. Nevertheless, there are profiles in which the winners differ, and two such cases are now shown as an example:
As an example for a preference profile in which the Maximin winner loses against two other candidates consider the following:

1	1	4	5	1	3
a	a	b	c	d	d
b	d	c	d	a	b
c	b	a	a	b	a
d	c	d	b	c	c

The majority matrix looks as follows:

	a	b	c	d
a	–	8	6	6
b	7	–	10	5
c	9	5	–	10
d	9	10	5	–

The row minima are 6 for candidate a and 5 for all other candidates, which makes a the unique Maximin winner. But in order to make candidate a become a Condorcet winner one has to eliminate seven voters, whereas for alternative b for example an elimination of six voters suffices. Hence a is certainly not the Young winner.
The second example is an example in which the Maximin winner is also the Condorcet loser. Hence this in an instance in which the Maximin rule suffers the Condorcet loser paradox, but the Young rule does not.

1	1	2	3	2
a	b	b	c	d
d	a	c	d	b
b	c	a	a	a
c	d	d	b	c

The majority matrix looks as follows:

	a	b	c	d
a	–	4	4	4
b	5	–	6	3
c	5	3	–	6
d	5	6	3	–

Clearly a is the Maximin winner as its row minimum is the highest among all candidates. But in order to make candidate a a Condorcet winner one has to eliminate more than four voters, whereas an elimination of four rankings suffices to make for example candidate c become a Condorcet winner.

In this context it is worth mentioning that the previous proof settles a question that came up in [Str15] where it remained open, whether the Condorcet loser paradox occurs for the Maximin rule whenever it occurs for the Young rule at least for the four candidate case.

Corollary 3.3. *Whenever the Young rule suffers the Condorcet Loser paradox in a preference profile with four candidates, then so does the Maximin rule but not vice versa.*

Proof. This can easily be seen using Lemma 3.1. Assume that candidate a is the Condorcet loser, candidate d is its highest defeater, and candidates b, c and d form a majority cycle. Assume further that candidate a is chosen by the Young rule. Hence it holds that

$$y(a) \geq n(d, a) - n(a, d) + 1.$$

Assume for contradiction that the Maximin rule does not simultaneously suffer the Condorcet loser paradox, and picks candidate b instead. Note that b loses against one alternative only, say c, and therefore its Young score is by the above argument $y(b) = n(c,b) - n(b,c) + 1$. Note further that $n(b,c)$ is also b's row minimum as c is the only candidate that it loses to. As candidate a is the Young winner the following holds:

$$\begin{aligned} y(a) &< y(b) \\ n(d,a) - n(a,d) + 1 &< n(c,b) - n(b,c) + 1 \\ 2n(d,a) - n &< 2n(c,b) - n \\ n(d,a) &< n(c,b) \\ n(a,d) &> n(b,c) \end{aligned}$$

and $n(a,d)$ is a's row minimum, and $n(b,c)$ is b's row minimum. This now contradicts the fact that candidate b is Maximin winner, and hence whenever the Young rule chooses the Condorcet loser then so does Maximin. The reverse is obviously not true as the above example shows. Hence the frequency of the Condorcet loser paradox when using the Maximin rule is an upper bound for the frequency of the Condorcet Loser paradox when using the Young rule in the four candidate case. □

3.3.3 Experimental results about Nanson's Rule and Maximin Rule

We have obtained in Theorem 3.2 and Theorem 3.3 that Maximin, Young's, Kemeny's and Nanson's rule always share at least one winning candidate. This does not hold for the four candidate case. As one would expect that the rules are still similar, we decided to test their similarity experimentally. In order to gain more insights into the interaction of Nanson's rule and the Maximin rule we decided to compare those two rules for four candidates.
Using PreflibTools [MW13a] it is easy to generate random preference profiles according to the IAC assumption. For the explanation of randomness and definition of cultural assumptions we refer to section 2.3.
For every generated profile we calculated both Nanson's and Maximin choice sets, and compared them. We distinguish between different choice sets, which means that they are not equivalent, and disjoint choice sets, which indicates an empty intersection of the respective sets. For every number of voters we did 1000 runs, and obtained the following results.

Number of voters	*Difference of Choice sets in %*	*95% - Confidence interval in %*
101	5.6	$(4.2, 7.0)$
1001	3.8	$(2.6, 5.0)$
10001	3.6	$(2.4, 4.8)$
100001	4.1	$(2.9, 5.3)$

For the disjointness of the choice sets we obtained the following percentages:

Number of Voters	*Disjointness of Choice sets in %*	*95% - Confidence interval in %*
101	3.3	$(2.2, 4.4)$
1001	3.5	$(2.4, 4.6)$
10001	3.6	$(2.4, 4.8)$
100001	4.1	$(2.9, 5.3)$

These results are rather unsurprising. Overall it seems to be the case that the rules are still quite similar, as one would suspect. Naturally ties occur more often the smaller the number of voters involved is, and as the Nanson rule seems to be more decisive this yields a higher percentage in the differing choice sets. If the number of voter is increased this effect fades as the occurrence of ties is less common.
Also when looking at the choice sets that are disjoint, the percentages that indicate that Nanson's and Maximin rule cannot agree on a winning candidate are still rather small. They seem to rise a bit the more voters are involved.
Hence one can expect the outcomes of Nanson's and Maximin rule to still be similar in the four candidate case.

In this section we have shown that no pair of voting rules coincides for the four candidate case, whereas we found a set of rules that are equivalent in the three candidate case. Unfortunately this did not carry over to the four candidate restriction. In the next chapter we will now take voting paradoxes as defined in 2.2 into account in order to obtain an understanding of how vulnerable voting rules are to certain paradoxical situations.

4 Paradoxical Behaviour of Voting Rules

For the definition of paradoxes we refer to 2.2. At first we will consider general results about voting paradoxes such as important impossibility results, that indicate that some sets of properties are incompatible. We will briefly introduce general methods that can be used to gain knowledge about the frequency of paradoxical voting situations. One such method, that is used later on, is Ehrhart theory, that will be introduced in 4.3. We will also give an overview of recent results of research in this area, and complement these results with our own work.

4.1 Results about Paradoxes

Several results have been shown that deal with the interaction of voting paradoxes and voting rules, and with the properties of voting rules in general. As we want to understand the vulnerability of certain voting rules to paradoxes better, these general results are an important starting point. A lot of impossibility results have already been proven. Hence it should not come as a surprise that there simply is no flawless voting rule. All impossibilities are commonly shown to hold on a general domain. This raises the question whether a restriction of the preference profiles results in a softening of the impossibility results. As we deal with domains that have a restricted number of candidates such behaviour could come in handy for us.
We want to briefly mention one of the most famous results: the impossibility theorem of Gibbard [Gib73] and Satterthwaite [Sat75]. The theorem introduces the notion of strategyproof voting rules which means that no voter can improve the outcome of the election by lying about his true preferences. The theorem says that certain common properties cannot be combined with strategyproofness without turning the rule into a dictatorial rule. Dictatorial rules always elect the most preferred candidate of one respective voter. It has been shown by Gibbard and Satterthwaite that every single valued, non-imposing voting rule is strategyproof if and only if it is dictatorial. Single valued means that only one candidate is selected which can be achieved by a tie-breaking mechanism for example, and non-imposing means that every candidate can be elected under some sort of preference profile. These two properties are quite common and reasonable.
Hence the Gibbard-Satterthwaite-Theorem is quite strong because in a society with democratic principles a dictatorial voting rule is usually not desired. All rules that we introduced are not dictatorial and non-imposing. We can turn all voting rules into single valued rules by linking them to a tie breaking mechanism. This means that all rules we consider throughout this thesis are not strategyproof, and voters can benefit from deviating from their genuine preference ranking and reporting a dishonest ranking. The theorem of Gibbard and Satterthwaite shows that this flaw is quite common and very hard to avoid. Therefore we will not consider strategyproofness in the forthcoming.

Another important statement that deals with the relationship between Condorcet consistency and the susceptibility to the No Show paradox is the following:

Theorem 4.1 (Moulin's Theorem)**.** *[Mou88]*

1. *For $|A| \geq 4$ and $|N| \geq 25$, every Condorcet consistent voting rule is susceptible to the No Show paradox.*

2. *If* $|A| \leq 3$*, there exist Condorcet consistent voting rules that do not violate the No Show paradox.*

Brandt, Geist and Peters [BGP17] tightened the bound, and have shown that starting from twelve voters already no Condorcet extension exists that also does not violate the No Show paradox in the case of four or more candidates. This theorem shows that the restriction of the domain is an opportunity to soften certain impossibility results. Whereas in general the combination of Condorcet consistency and immunity to the No Show paradox is impossible, it can be reached in a three candidate domain. Hence when restricting ourselves to three candidates, we can hope for a rather attractive voting rule that is Condorcet consistent and does not violate the No Show paradox, whereas this will not happen for the four candidate case.

The following theorem considers the interaction between Condorcet consistency and the Reinforcement paradox.

Theorem 4.2 (Young's Theorem)**.** *[You75] [BCE+16] Let* f *be a Condorcet consistent voting rule and* $|A| \geq 3$*. Then* f *violates the Reinforcement paradox.*

From this theorem we immediately know that we cannot find a Condorcet consistent voting rule that does not violate Reinforcement in preference profiles with more than 2 voters.

These theorems give an overview on what we can hope for when searching for attractive rules. Which paradoxes are more severe than others is controversially discussed.
Additional to the vulnerability to paradoxical situations other arguments have to be taken into account as well. For instance in most elections it is important that the rule is rather simple so that it can be quickly understood by the voters and also the communication is easy. If this is not the case and the voter does not understand how the winner is selected, this might discourage him or her from taking part in the election.
Another aspect is the difficulty of the rule itself as there are some rules that are computationally quite demanding and others where the calculations are more straight-forward and easy.
Nevertheless these arguments are hard to quantify which is why we will mainly concentrate on the equivalences of rules and their vulnerability to the mentioned paradoxes. Apart from the number of paradoxes that a rule is susceptible to, there seems to be a consensus that the severity to a certain paradox is influenced by the frequency with which the paradox occurs. Therefore we are interested in the frequencies of paradoxes in randomly selected preference profiles. Obtaining the probability of a voting paradox can be done by following one of three approaches, that are introduced in the next section.

4.2 Methods of Assessing the Probability of Voting Paradoxes

We want to shortly introduce the most common three approaches on how to assess the probability of a certain voting paradox under a voting rule. Each of those three approaches obviously has advantages and drawbacks, that we also want to discuss briefly.

The Analytical Approach When determining probabilities with the analytical approach we use mathematical calculations to describe the probability of the voting paradox exactly. In order to do this, we need to specify the cultural assumptions as discussed in 2.3, and under those assumptions calculations can then be carried out to precisely determine the wanted frequency. Clearly the assumptions determine the usefulness of the results, and how close they are to real world examples. As stated before, using IAC is a fair assumption for the cause of this thesis, but will most likely not represent frequencies of paradoxes in reality. Nevertheless, it is especially useful to compare different results that also assume IAC. There are already quite a lot of such results as IAC is one of the most prominent cultural assumptions.
The major advantage of the analytical approach is that it creates reproducible results. On the other hand in a lot of cases the analytical approach seems to be infeasible. Most of the results obtained analytically are limited to three candidates. Only recent improvements of algorithms have made it possible to determine results for the four candidate case. Results in this situation are still quite rare, and also some of the problems seem to remain infeasible despite the positive development of computational power. To the best of our knowledge analytical results for five of more candidates have not been obtained yet. As this thesis focuses on preference profiles with a small amount of candidates, analytical results are still very interesting for us.

The Experimental Approach In the experimental approach preference profiles are created randomly. For these random profiles it can then be checked if a certain paradox occurs under a specified voting rule, and thus the frequency of the voting paradox is determined. As stressed before the random creation of preference profiles also requires a cultural assumption that heavily influences the outcome. Which of the probability models is most suited is controversially discussed. Even though it sounds temptingly easy, checking for voting paradoxes can be computationally very demanding as well, and even determining winning sets under certain rules is demanding in itself as there are some prominent voting rules that are in NP[10]. Also the number of runs has to be carefully selected as it determines the statistical significance of the obtained result[11].
Nevertheless this approach is much more versatile than the analytical approach, and results are not restricted to a small number of candidates, but can be obtained for various combinations of numbers of candidates and numbers of voters. Due to this flexibility of the experimental approach there is a variety of papers published on frequencies of voting paradoxes. Strobl [Str15] provides a summary of these results and the respective sources.

The Empirical Approach When aiming for results that are likely to be observed in reality, the empirical approach is ideal without a doubt. When using this approach, one only examines real world data. There is no need for choosing a probabilistic model that restricts the general applicability of the result. The biggest flaw of this approach is obviously that real world data is quite sparse. Also some data sets are either incomplete or inaccurate. This is due to several difficulties that arise when acquiring data. Often the collection and preprocessing of the data is the main difficulty as for instance medical data

[10] [BCE+16]
[11] [Str15]

needs to be carefully anonymised, and must be available in larger scales to ensure the personal rights and privacy of the patients, and to obey the current laws for data processing. Also some researchers are afraid to share their data sets in order to maintain exclusivity. Therefore the success of this approach bases solely on the quality and availability of the data sets.
Although in general it is quite hard to acquire suitable data, the library PreflibTools [MW13b] provides access to a collection of different data sets. These data sets are provided by a community of researchers that believe in sharing their findings. As this thesis focuses on a small set of candidates, only data sets that fulfil this requirement come into question, which again restricts the availability of suitable data.

We will now introduce a method following the analytical approach. Using this method we are able to obtain new results about frequencies of some paradoxes and voting rules.

4.3 Introduction to Ehrhart Theory

Ehrhart theory is a method firstly developed by the French mathematician Eugène Ehrhart [Ehr62] for counting integer points in polyhedra. Luckily we are able to make use of these exact analytical tools for our purposes under the assumption of IAC. Assuming IAC means that we assume that every anonymous voting situation is equally likely which enables us to create Laplace experiments. So if we are interested in the probability of a voting paradox, we can obtain it by counting the number of paradoxical profiles and dividing it by the number of all possible profiles.
As mentioned before voting situations can typically be described as sets of linear inequalities, and hence the problem corresponds to computing the number of integer solutions to these inequality descriptions. It has been shown by Ehrhart in the sixties that the number of integer solutions to a linear inequality system can be described as a quasi-polynomial in n with periodic coefficients. Only recently one realised that this result comes in handy for problems from voting theory such as the frequency of voting paradoxes. The following is a brief introduction to Ehrhart theory based on Lepelley, Louichi and Smaoui [LLS08].

$\mathbb{R}^d$ is the Euclidean d-space of all d-dimensional vectors with real entries, i.e. $(x_1, ..., x_d) \in \mathbb{R}^d$ with $x_i \in \mathbb{R}$. Then $\mathbb{Z}^d$ is the integer lattice and as such a subset of $\mathbb{R}^d$. All d-dimensional vectors with integer coordinates are contained in $\mathbb{Z}^d$.

Definition 4.1 (Rational polyhedron, rational polytope)**.** *A rational polyhedron P of dimension d is a subset of the Euclidean d-space that is defined as the solution of a system of linear inequalities.*

$$P = \{x \in \mathbb{R}^d : Ax \leq b\} \subset \mathbb{R}^d$$

with $A \in \mathbb{Z}^{m \times d}, b \in \mathbb{Z}^m$ and m the number of linear inequalities.
If the polyhedron is bounded, it is called a polytope.

Hence the problem of counting integer solutions to a finite set of linear inequalities is equivalent to counting integer points in the corresponding rational polyhedron.

Definition 4.2 (Dilated polytope). *Let $n \geq 1$ be an integer parameter and $P \subset \mathbb{R}^d$ a d-dimensional polytope. The polytope P_n defined by*

$$P_n := \{nx : x \in P\}$$

is called the dilation of P. Then $|P_n \cap \mathbb{Z}^d|$ is the number of integer points that lie in the dilation of P.

Definition 4.3 (Quasi-polynomial). *A function $f : \mathbb{Z} \to \mathbb{Q}$ is a quasi-polynomial if there are q many polynomials g_i such that $f(n) = g_i(n)$ if $n \equiv i \mod q$.*

Theorem 4.3 (Ehrhart's Theorem [LLS08]). *Let P be a rational polytope in dimension d. The number of integer points that lie in the dilation of P is given by a quasipolynomial of degree d. The coefficient of the leading term is independent of n and equivalent to the Euclidean volume of P.*

Every anonymous preference profile is uniquely specified by an integer vector of dimension $m!$ when m is the number of candidates. The paradox, that is ought to be examined, needs to be described as set of inequalities. Then counting the integer solutions to the induced linear program is equivalent to the number of voting situations in which the paradox occurs. As we want to obtain a frequency, we have to divide this number by the number of all possible voting situations. Because we assume IAC, every voting situation is equally likely, and thus this approach fulfils its purpose.
The number of all possible voting situations that can occur is known to be $\binom{m!+n-1}{m!-1}$.
It follows that the probability of occurrence of a voting paradox with n many voters described as polytope P is given as

$$\mathbb{P}(n) = \frac{|P_n \cap \mathbb{Z}^d|}{\binom{m!+n-1}{m!-1}}.$$

Ehrhart's theorem states that the numerator is given as a quasipolynomial of degree d. Finding these quasipolynomials can only be done due to recent improvements in algorithms, and luckily there are now computer programs such as NORMALIZ [BIR$^+$]. Computing the quasipolynomial is computationally very demanding which limits this analytical tool to a rather small number of candidates. This is due to the fact that the number of rankings and hence the dimension of the polytope grows exponentially in the number of candidates. As far as we are aware, not more than four candidates can be tackled by now, and even cases with four candidates are not always feasible. As NORMALIZ has previously been used in problems where four candidate cases were successfully examined, we decided to use this program as well.
One is often especially interested in the behaviour of a voting rule if a large number of voters is involved. Fortunately restricting computations to finding the leading coefficient of the quasipolynomials suffices in order to obtain the limit probability for a fixed number of candidates and $n \to \infty$, which in itself is a nice and helpful result. In the following we present an example that shows an application of Ehrhart theory in the voting theory context.

Example 4.1 (Probability that no Condorcet winner exists). As the notion of a Condorcet winner is essential in social choice we are interested in the probability that no

Condorcet winner exists in a preference profile with a large number of voters and three candidates. As we want to follow the above described approach, and determine the limit probability under IAC with Ehrhart theory, we have to describe the voting situation as system of linear inequalities. As it appears to be easier, we will determine the complementary probability, and therefore describe the situation in which candidate a is the Condorcet winner:

$$n_1 + n_2 + n_5 - n_3 - n_4 - n_6 > 0 \tag{30}$$
$$n_1 + n_2 + n_3 - n_4 - n_5 - n_6 > 0 \tag{31}$$
$$n_i \geq 0, i = 1, \ldots, 6 \tag{32}$$

The first inequality describes that candidate a wins pairwise against candidate b, and the second inequality states that candidate a also wins over candidate c. Hence in the described voting situation candidate a has to be the Condorcet winner. We implicitly assume that the total number of voters is n.
Entering this system of linear inequalities into NORMALIZ [BIR$^+$] immediately gives the desired limit probability:

$$\mathbb{P}(\textit{candidate a is Condorcet winner } | m = 3, n \to \infty) = \frac{5}{16}$$

Note that the voting situations that have candidates b or c as Condorcet winner are symmetric to the described situation which allows us to multiply the result by 3 to account for the three possibilities in choosing a Condorcet winner. We then obtain that the limit probability of having a Condorcet winner in a random preference profile is $^{15}/_{16}$. Therefore the complementary event of having no Condorcet winner is determined to be $^{1}/_{16} = 6.25\%$. This limit probability was already derived in the seventies by Fishburn and Gehrlein [FG76] [Geh82]. We are even able to describe the probability that no Condorcet winner exists as a quasipolynomial in the number of voters. The quasipolynomial has period 2 and degree 5 and looks as follows:

$$g_0(n) = 1 - 3(\frac{1}{384}n^5 + \frac{12}{384}n^4 + \frac{52}{384}n^3 + \frac{96}{384}n^2 + \frac{64}{384}n)$$
$$g_1(n) = 1 - 3(\frac{1}{384}n^5 + \frac{15}{384}n^4 + \frac{86}{384}n^3 + \frac{234}{384}n^2 + \frac{297}{384}n + \frac{135}{384})$$

Hence the probability that a preference profile with three candidates under IAC has no Condorcet winner is

$$\frac{g_0(n)}{\binom{n+5}{5}} \textit{ for even } n \qquad \frac{g_1(n)}{\binom{n+5}{5}} \textit{ for odd } n.$$

In Figure 1 one can see the probabilities that there is no Condorcet winner in dependency of the number of voters in a preference profile with three candidates assuming IAC. It is obvious from the figure and the quasipolynomial that in profiles with an even number of voters the probability that there is no Condorcet winner is higher than in profiles with an odd number of voters as also the possibility for ties is higher in profiles with an even number of voters.

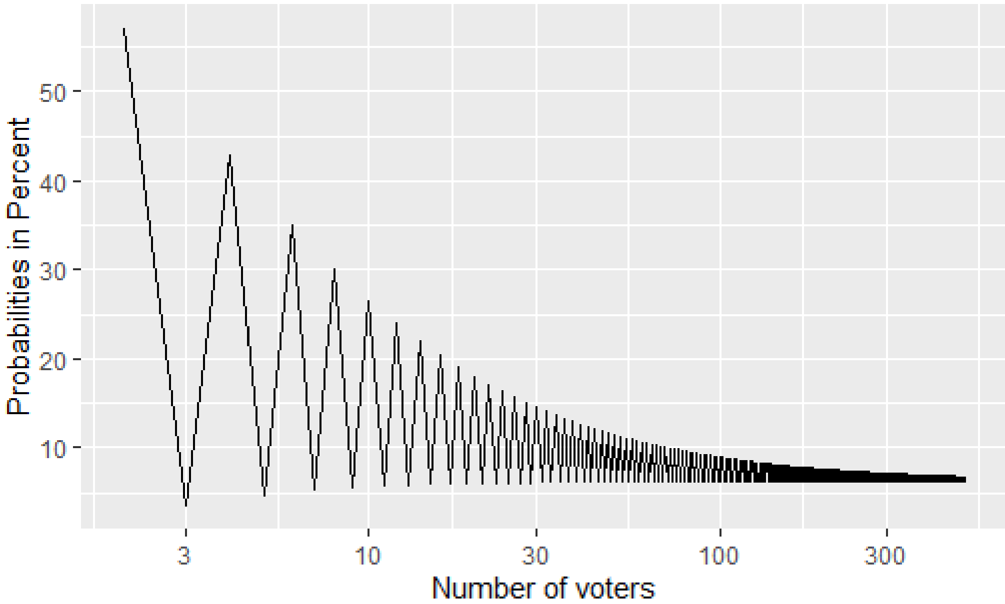

Figure 1: Probability that there is no Condorcet winner

4.4 Axiomatic Overview

For the following axiomatic consideration some rules introduced earlier are left out. We chose to concentrate on the simplest, most popular and most attractive rules. Because of their simplicity and popularity we decided to examine Plurality, Plurality with Runoff and Borda's rule further. Due to Theorem 3.2 we have good cause to suspect that Maximin, Kemeny's, Nanson's and Young's rule are an attractive choice for a small set of candidates, and therefore we are interested to see how they behave concerning paradoxical situations. In contrast we decided to also examine Black's rule further as it is a Condorcet extension but not equivalent to the previous mentioned set of rules.

Additional to before we will now also take paradoxes into consideration. Paradoxes are unwanted outcomes that can result from elections that use a certain voting rule. These outcomes are counter-intuitive and undesirable. Several different paradoxes were introduced and defined in 2.2. It is quite commonly known that there is no voting rule that is immune to all paradoxes, so every voting rule violates some axioms. The discussion about which paradoxes are more "severe" than others, or which paradoxes should be avoided at all cost is ongoing with a lot of different opinions. Some people consider voting rules that suffer the Condorcet winner paradox as inappropriate.
We will again distinguish between the three and four candidate case. Not all rules that are in general vulnerable for a certain paradox already exhibit this vulnerability in profiles with a small number of voters only. Minimal examples - in the number of candidates and voters - for pairs of voting rules and paradoxes are calculated by Schmidt [Sch18a].
A wide consensus is reached that even if the voting rule suffers a certain paradox, the probability of such an undesirable outcome should be assessed in order to gain knowledge about the frequency with which one can expect a paradox to occur. It is clear that if the frequency that a certain paradox occurs is quite small, then also the severity of this

paradox is not as big. To gain insights into these probabilities three main methods can be used. One is gaining theoretical results using Ehrhart theory, the second method is to gain experimental frequencies using computer simulations and randomly generated profiles, and the third method requires real world election data that can then be examined for paradoxes. All three methods have been introduced in 4.2.

Table 2 provides an overview of the results that have been gained in the last years for three candidates. In the table different pairs of voting rules and paradoxes that are vulnerable to the respective paradox are listed. The column "minimal" provides the smallest number of voters for which a profile with three candidates exhibits the paradox without the use of a tie-breaking mechanism. Most of the results are provided in [Sch18a]. If not calculated there, we followed the identical approach to find the minimal examples ourselves. The other columns deal with the results that have been found on the frequencies of paradoxes. The column "n" states the number of voters that was used in the calculation of the probabilities, whereby "∞ " indicates that a limit probability for an infinitely large amount of voters has been found. In the column "distribution" it is stated which kind of probabilistic model was assumed. Usually we focused on the IAC distribution as this is a widely used assumption, and makes the use of Ehrhart theory possible. Nevertheless, we found interesting results by Plassmann and Tideman [PT14] who used a spatial distribution, that is more realistic than the IAC assumption, and thus we also included these results in the table. The method that was used is also stated, whereby "Sim." indicates computer simulations, and of course the results are stated as well. The last column states the source of the frequency calculations.
It turns out that even for a relatively small number of voters most paradoxes can already occur. Especially the Condorcet winner paradox that is known to be quite severe, can occur for preference profiles with only five voters. Also the limit probability with which the Condorcet winner paradox occurs is not that low. Here it is worth mentioning that the Plurality with Runoff rules exhibits this paradox significantly less often than the Plurality rule. No results using Ehrhart theory have been found for the Reinforcement paradox, but the computer simulations suggest that the frequencies are rather low for the Reinforcement paradox to occur. This supports our assessment from before that the set of related rules from Theorem 3.2, namely Maximin, Kemeny's, Nanson's and Young's rule, are a quite attractive choice for an election with three candidates. We will still take a closer look to the occurrence of the Reinforcement paradox in 4.6.
In summary for the not Condorcet consistent rules Borda's rule seems to be a good choice as it is only vulnerable to the Condorcet Winner paradox, whereas Plurality suffers two severe paradoxes - the Condorcet Winner and the Condorcet Loser paradox. We will have a closer look at Plurality with Runoff in 4.5 as it is quite popular in political elections. Concerning the Condorcet extensions the set of equivalent rules remains attractive. Contrary to those rules, Black's rule suffers more often from the Reinforcement paradox, and is additionally also vulnerable to the No Show paradox.

For four candidates far less results have been obtained until now due to the immense computational power that is needed to tackle the exponential growth in dimension between three candidates and four candidates. The overview can be seen in table 3. Borda's rule is still only vulnerable to the Condorcet Winner paradox, and hence remains a good choice for a not Condorcet consistent rule. But for the Condorcet consistent rules it is much harder to decide on an attractive rule. All Condorcet consistent rules are inevitable vulnerable to both the No Show and the Reinforcement paradox due to Theorem 4.1 and Theorem 4.2. The only rule that is susceptible to those two paradoxes only is Kemeny's. Nanson's rule is additionally also vulnerable to the Lack of Monotonicity paradox. Both Young's and Maximin rule are vulnerable to the No Show paradox, the Reinforcement paradox and the Condorcet Loser paradox. As they are shown to be quite similar in a four candidate election also the frequencies can be expected to be similar. Hence between those two rules Maximin should be preferred as it is computationally easier. Apart from that the suitability of the rules has to be individually discussed and also arguments like computational hardness and simplicity as mentioned in 4.1 have to be taken into account.

Paradox	Voting Rule	minimal[12]	n	distribution	method	probability	source
Monotonicity	Pl. with Runoff	17	∞	IAC	Ehrhart	5%	[LMS17]
No Show	Pl. with Runoff	11	∞	IAC	Sim.	4.1%	[LM01]
No Show	Black	9	∞, 1 M	IAC, spatial	Ehrhart, Sim.	0.1%, 0%	[BHS19], [PT14]
Condorcet Winner	Pl. with Runoff	5	∞	IAC	Ehrhart	4% [13]	[OLS19]
Condorcet Winner	Borda	5	∞, 1 M	IAC, spatial	Ehrhart, Sim.	9% [13], 2.6%	[OLS19], [PT14]
Condorcet Winner	Plurality	5	∞, 1 M	IAC, spatial	Ehrhart, Sim.	12% [13], 4.5%	[OLS19], [PT14]
Reinforcement	Pl. with Runoff	26					
Reinforcement	Maximin	15	100, 1 M	IAC, spatial	Sim., Sim.	0.2%, 0%	[CMM14], [PT14]
Reinforcement	Kemeny	15	100, 1 M	IAC, spatial	Sim., Sim.	0.2%, 0%	[CMM14], [PT14]
Reinforcement	Nanson	11	100, 1 M	IAC, spatial	Sim., Sim.	0.2%, 0%	[CMM14], [PT14]
Reinforcement	Young	15	100, 1 M	IAC, spatial	Sim., Sim.	0.2%, 0% [14]	[CMM14], [PT14]
Reinforcement	Black	9	100, 1 M	IAC, spatial	Sim., Sim.	1.3%, 0%	[CMM14], [PT14]
Condorcet Loser	Plurality	7	∞, 1 M	IAC, spatial	Ehrhart, Sim.	3%[15], 0.3%	[OLS19], [PT14]

Table 2: Overview of the axiomatic properties for three candidates

[12] most examples from [Sch18a]

[13] provided a Condorcet winner exists

[14] It should not come as a surprise that the probabilities of a paradox are quite similar for Maximin, Kemeny, Nanson and Young rule as we proved in 3.2 that these rules are quite similar in their decision for three candidates. The decisiveness of Nanson's is also the reason why the minimal examples are not equivalent.

[15] provided a Condorcet loser exists

Paradox	Voting Rule	minimal[12]	n	distribution	method	probability	source
Monotonicity	Pl. with Runoff	13					
Monotonicity	Nanson	7					
No Show	Pl. with Runoff	11					
No Show	Nanson	13	∞	IAC	Sim.	3% (max[14])	[BHS19]
No Show	Maximin	9	∞	IAC	Ehrhart	0.55% (max[14])	[Str15], [BHS19]
No Show	Kemeny	tie breaking					[BGP17]
No Show	Black	6	∞	IAC	Sim.	4% (max[14])	[BHS19]
No Show	Young	9					
Condorcet Winner	Pl. with Runoff	5	∞	IAC	Ehrhart	9%[13]	[BIS19]
Condorcet Winner	Borda	5	∞	IAC	Ehrhart	13%[13]	[OLS19]
Condorcet Winner	Plurality	5	∞	IAC	Ehrhart	25%[13]	[BIS19]
Reinforcement	Pl. with Runoff	26					
Reinforcement	Maximin	11					
Reinforcement	Kemeny	9					
Reinforcement	Nanson	9					
Reinforcement	Young	11					
Reinforcement	Black	5					
Condorcet Loser	Maximin	9	∞	IAC	Ehrhart	0.06%	[Str15]
Condorcet Loser	Young	15	∞	IAC	Sim.	0.06% (max[15])	[Str15]
Condorcet Loser	Plurality	5	∞	IAC	Ehrhart	2.3%[15]	[OLS19]

Table 3: Overview of the axiomatic properties for four candidates

4.5 Analysis of the Plurality with Runoff Rule

As this particular voting rule is considered to be attractive for a small number of candidates, we want to take a closer look at the axiomatic properties of this rule. A lot of political elections use rules that are quite similar to the Plurality with Runoff procedure. So for instance during the municipal elections in Bavaria mayors and district administrators are elected with a similar rule to the Plurality with Runoff rule[18]. According to [FN19] the President of 40 countries is elected in a similar manner so for instance the President of state in France[19] or Austria[20]. This shows that the Plurality with Runoff rule is quite commonly used in essential political elections despite its vulnerability to four of our five selected voting paradoxes. This is good cause for a more thorough analysis of the properties and behaviour of the Plurality with Runoff rule in paradoxical voting situations, and whether they can be avoided.

4.5.1 Domain Restrictions

We will consider two different domain restrictions. We have seen that the Plurality with Runoff rule is in general susceptible for all of the selected voting paradoxes but the Condorcet loser paradox even for three candidates already. This might not be the case if we restrict the domain of preference profiles further. As Plurality with Runoff is often used in a political context, it can be a fair assumption to assume that voter preferences can be somehow ordered along a line from left-wing to right-wing. This property is known as single peakedness and is formalised in Definition 4.4. Felsenthal and Nurmi [FN19] make a different assumption as they consider a quite stable starting preference profile in which a Condorcet winner exists, and is also selected by the respective voting rule. As Plurality with Runoff is no Condorcet extension, it might not always select the Condorcet winner.

First we consider the restricted domain of voters' preference rankings for when they are single peaked:

Definition 4.4. *A domain is single peaked if for every voter i and all pairs of candidates $a, b \in A$ for which $a \succ_i b$, there exists an ordering of the alternatives such that either candidate a is the most preferred candidate, or a and b are on opposite sides of the most preferred candidate, or a and b are on the same side of the most preferred candidate and a is closer to it than b.*

In figure 2 one can see two different domains. The left domain has single peaked preferences, whereas the other has not. The right domain is not single peaked as the voter i indicated by the green line has the preference $c \succ a \succ b$. Consider now the pair a, b. It holds that $a \succ_i b$, but candidate a is not the most preferred candidate, a and b are on the left to the most preferred candidate and hence not on opposite sides, and candidate

[14]Limit probability tends to zero and is always below the mentioned percentage.

[15]As show in Corollary 3.3 the frequency of the Condorcet Loser paradox for the Young rule is upper bounded by the frequency of the Condorcet Loser paradox for the Maximin rule.

[18][Sta20]

[19][Pre20b]

[20][Pre20a]

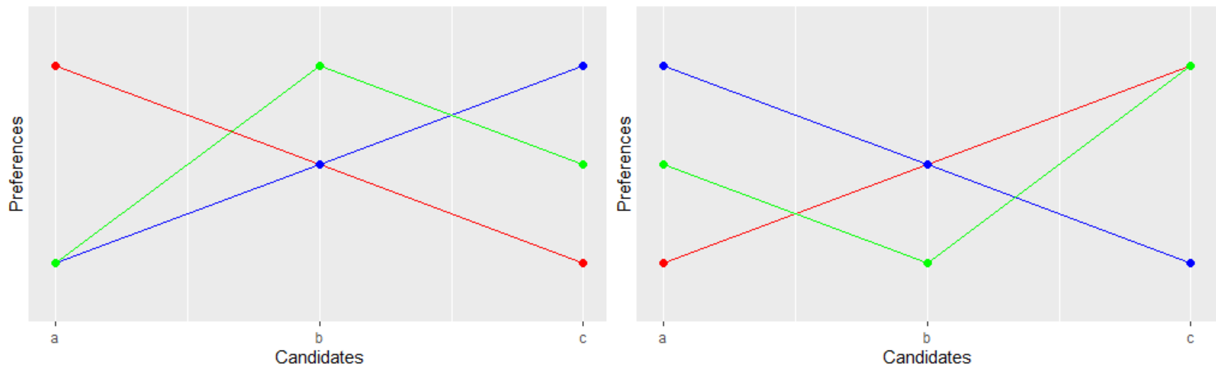

Figure 2: Single peaked and not single peaked domain

b is ordered closer to the most preferred candidate than a. Hence this ordering of the candidates does not fulfil the needed properties. Note that one has to check every possible ordering of the candidates, and it turns out that every ordering violates the defined properties. Hence the domain is not single peaked.
Unfortunately it turns out that even when restricting ourselves to three candidates and single peaked domains only, Plurality with Runoff is still vulnerable to most of the paradoxes.

Example 4.2. *We consider an example preference profile with three candidates. The preference rankings of the voters are either $a \succ b \succ c, b \succ a \succ c$ or $c \succ a \succ b$. Then this domain is single peaked. In this domain the No Show and the Reinforcement paradox can be observed.*

3	4	4
a	b	c
b	a	a
c	c	b

In this preference profile candidate b is the Plurality with Runoff winner. If now two voters with preference $c \succ a \succ b$ abstain from the election, then candidate a is elected which is preferred from the abstainers.
The following is an example of the Reinforcement paradox within the single peaked domain. The first two profiles describe the separate districts, and the third is the joint district that comes up when the two districts are merged into one:

4	6	3
a	b	c
b	a	a
c	c	b

4	3	6
a	b	c
b	a	a
c	c	b

8	9	9
a	b	c
b	a	a
c	c	b

In the two separate districts candidate a wins the elections, whereas when combining the districts candidate b wins.
In a quite similar domain where the preference rankings $a \succ b \succ c, b \succ a \succ c$ and $c \succ b \succ a$ are used the Condorcet loser and the Lack of Monotonicity paradox can be observed. This domain is also single peaked.

2	1	2
a	b	c
b	a	b
c	c	a

This preference profile is an example of the Condorcet winner paradox in a single peaked domain. Candidate a wins the election using the Plurality with Runoff rule, but candidate b is the Condorcet winner.

6	5	6
a	b	c
b	a	b
c	c	a

6	5	4	2
a	b	c	a
b	a	b	c
c	c	a	b

In this example two voters with preference ranking $c \succ b \succ a$ manipulated by moving candidate a up in the ranking. Before their manipulation candidate a won, but including the new preference ranking now candidate b wins. Note that the manipulated domain is no longer single peaked. But as the manipulation of the individual preference ranking can be seen as a dishonest report of the genuine individual preference, one can argue that also the Monotonicity paradox can happen in a single peaked domain.

This shows that even such a strong restriction does not suffice to make the Plurality with Runoff rule less vulnerable to voting paradoxes. Note that this observation holds for four candidates as well, as we can easily turn the above examples to a four candidate election by ranking the fourth candidate last for every voter. This does not change the assumption of single peakedness. Hence also for four voters the paradoxes, that the Plurality with Runoff rule is vulnerable to, can happen in a single peaked domain.
However, in [FN19] several other domain restrictions are considered, whereas the number of candidates is unrestricted. By doing so Felsenthal and Nurmi can show that Plurality with Runoff is not longer vulnerable to the No Show paradox if there is a Condorcet winner in the preference profile that is also elected by the Plurality with Runoff procedure. Also the Monotonicity paradox cannot happen in a domain where there exists a Condorcet winner that is simultaneously the Plurality with Runoff winner. On the other hand the Reinforcement paradox cannot be tackled by this approach. So even if in both districts there is a Condorcet winner that is elected by the Plurality with Runoff procedure, it can still happen that another candidate is elected by Plurality with Runoff in the joint district as the following example from [FN19] shows.

Example 4.3. *Consider the following two districts:*

5	4	4	3
a	b	c	c
b	a	a	b
c	c	b	a

4	3	3	3
a	b	b	c
c	a	c	a
b	c	a	b

In both districts candidate a is the Condorcet winner. Also candidate a is elected as the unique Plurality with Runoff winner. This seems like a rather stable position which is why it is even more astonishing what happens if the two districts are merged:

5	4	7	3	7	3
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

Clearly candidate a is still the Condorcet winner. But now candidate a is eliminated in the first round of the Plurality with Runoff procedure already as it is ranked first by the smallest number of voters, and hence we witnessed a rather surprising instance of the Reinforcement paradox.

We see that even this strict restriction that demands the existence of a seemingly stable profile does not suffice to make the Plurality with Runoff procedure immune to all paradoxes. One also has to add that Plurality with Runoff might even not elect the Condorcet winner if one exists.

In the following we will now focus on a unrestricted domain with three or four candidates again but we now assume a large amount of voters. As we have already seen that paradoxes can happen for the Plurality with Runoff procedure already for a rather small number of voters, we are now interested in seeing how this develops when the amount of voters is rather large. Hence we are interested in the limit probability of certain paradoxes when the number of candidates is fixed, and the number of voters tends to infinity. As listed above if the number of candidates is fixed to be three, some research is already done and some theoretical results are determined. Hence the limit probability for an instance of the Monotonicity paradox is 5%[21] and for the Condorcet Winner paradox it only happens in 4% of the preference profiles in which Condorcet winners exist[22], which is a rather small probability. Especially in comparison to the also very popular and simple Plurality rule this small probability for the Condorcet Winner paradox is a valid argument for the Plurality with Runoff rule. The Plurality rule selects a candidate different to the Condorcet winner in 12% of all profiles in which a Condorcet winner exists, which is three times the frequency of the Plurality with Runoff rule. As many consider the Condorcet Winner paradox as quite severe, and an instance of this paradox is often quite obvious for the voters to see, this could be the cause why the Plurality with Runoff rule is chosen over the Plurality rule despite its susceptibility to many other paradoxes. As we did not find such results, we decided to examine the frequency of the Lack of Monotonicity paradox and the frequency of the Reinforcement paradox ourselves following the introduced approach with Ehrhart theory.

[21] [LMS17]
[22] [OLS19]

4.5.2 Analytical Results for the No Show paradox with Ehrhart Theory

In order to make use of Ehrhart theory and the NORMALIZ [BIR+] software one has to describe the preference profile in which the No Show paradox appears for the Plurality with Runoff rule as system of inequalities. This inequality description has already been determined by Lepelley and Merlin [LM01]. Due to the neutrality of the voting rule we can assume for now that candidate c wins the election, and candidate a is eliminated in the first step. Note that it is not possible that no candidate is eliminated in the first step as this would imply that candidate c is ranked first by an absolute majority of the voters. Such an absolute majority win can only be changed if some of the voters that rank c first do not show up, but as they are content with the outcome of the election they have no incentive to abstain the election. Therefore only elections where one of the candidates is eliminated in the first step come into question.

$$n_5 + n_6 \leq 1/2 \sum_{i=1}^{6} n_i \tag{33}$$

$$n_3 + n_4 \leq 1/2 \sum_{i=1}^{6} n_i \tag{34}$$

Due to the above mentioned reasons neither candidate b nor candidate c can be ranked first by an absolute majority as they would then be immediately elected. This is ensured by the above two inequalities.

$$n_5 + n_6 > n_1 + n_2 \tag{35}$$

$$n_3 + n_4 > n_1 + n_2 \tag{36}$$

$$n_2 + n_5 + n_6 > n_1 + n_3 + n_4 \tag{37}$$

Inequalities (35) and (36) state that candidate a gets eliminated in the first step as it has the smallest number of first places. After a's removal the pairwise comparison between candidates b and c is carried out which returns candidate c as the Plurality with Runoff winner due to (37).

$$n_1 + n_2 > n_4 \tag{38}$$

In this situation only voters with preference ranking $b \succ a \succ c$ can benefit from abstaining the election as they can change the elimination process in their favour. The number of voters of this type is encoded as n_3. We need to make sure that there are enough voters of this type present at the election. Hence when removing all voters with ranking $b \succ a \succ c$, which corresponds to setting $n_3 = 0$, candidate b has to be eliminated instead of candidate a. This is ensured by (38).

It is now left to demand that deleting $k, k \in \mathbb{N}, 0 < k \leq n_3$ many voters succeeds to make candidate a the new Plurality with Runoff winner. Hence the following two inequalities have to hold:

$$n_1 + n_2 > n_3 - k + n_4 \tag{39}$$

$$n_1 + n_2 + n_3 - k > n_4 + n_5 + n_6 \tag{40}$$

Inequality (39) states that after the removal of k many voters with ranking $b \succ c \succ a$, candidate b gets now eliminated instead of candidate a as it lost first ranks through the abstention of voters. Now candidate a wins the paired comparison against candidate c as ensured by (40), and is therefore the new Plurality with Runoff winner. Hence the abstention of k many voters led to their benefit which characterises the No Show paradox. Fortunately adding those two inequalities (39) and (40) results in

$$2n_1 + 2n_2 - 2n_4 - n_5 - n_6 > 0 \tag{41}$$

which is independent of k. And therefore adding (41) to (33) - (38) describes a general instance of the No Show paradox.
A proof that this inequality description is suited for its purpose can be found in [LM01].

Note that all three candidates can be chosen to be the winner in the original preference profile, and then still two candidates are available to be eliminated in the first step. Hence in total there are six possibilities in which the No Show Paradox happens, but only one possibility is described as a set of inequalities. Due to symmetry of the cases it suffices to multiply the result by 6 to account for the six possible cases.
With the help of NORMALIZ [BIR$^+$] we get:

Theorem 4.4. *The limit probability that an instance of the No Show paradox happens in a preference profile with three candidates under the assumption of IAC and under the use of the Plurality with Runoff rule is*

$$\mathbb{P}(\textit{No Show paradox for Plurality with Runoff}\,|m = 3, n \to \infty) = 6 \cdot \frac{47}{6912} \approx 4.0799\%.$$

This result is in perfect accordance to the result of Lepelley and Merlin [LM01] who used integration in order to obtain the polyhedral volume and thus the limit probability.
Additionally we can even obtain the probability for the No Show paradox in dependency of the number of voters.

Theorem 4.5. *The probability of the No Show paradox combined with the Plurality with Runoff voting rule in a preference profile with three candidates under the assumption of IAC can be described by a quasipolynomial with period 24 and degree 5.*

In figure 3 one can see the probability in dependency of the voters.

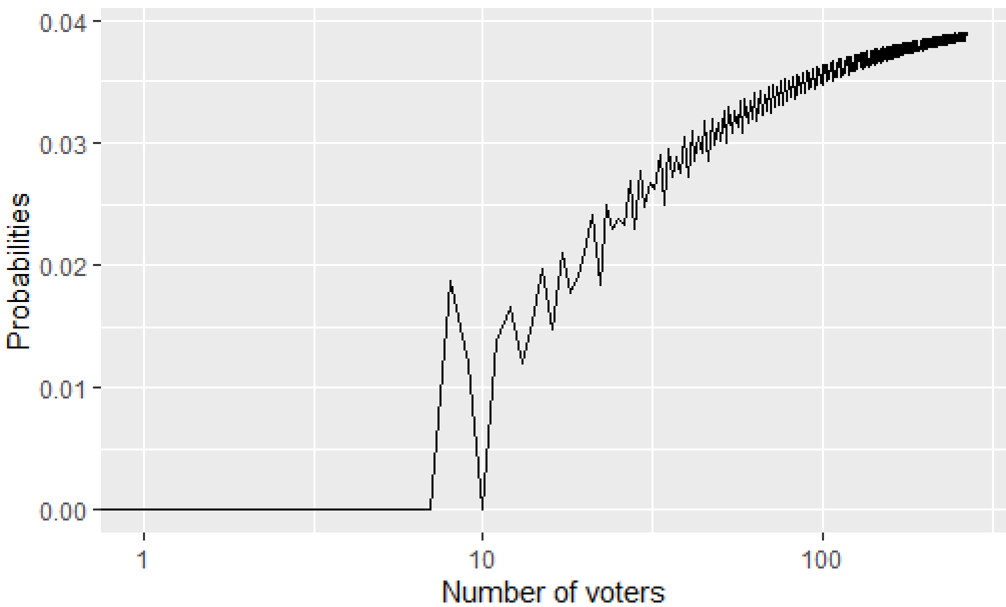

Figure 3: Probability of the No Show paradox under the Plurality with Runoff rule

4.5.3 Analytical Results for the Reinforcement paradox with Ehrhart Theory

The Plurality with Runoff rule is vulnerable to the Reinforcement paradox for three candidates as well. Hence we decided to also use Ehrhart theory to obtain the probability with which one can expect the Reinforcement paradox to happen when the Plurality with Runoff rule is used on randomly selected profiles with three candidates. In order to do so we need to find an inequality description again. As the Plurality with Runoff rule can select a winner in either of its two steps we have to make a case distinction. We will again assume, due to the neutrality of the rule, that candidate a is selected as winner. Note that it is not possible that candidate a has an absolute majority in both districts as then it certainly also has an absolute majority in the combined district. It can also not happen that in both districts the same elimination order happens as this will only produce the identical elimination order in the joint district, and therefore no different winner can occur there. So the cases left to consider are, if in one of the districts candidate a has an absolute majority and it has not in the other district, and if there are different elimination orders in the districts.
Consider first the case that in one of the districts a is absolute majority winner. Let $x_i \in \mathbb{N}$ correspond to the number of voters with ranking i in the first district and $y_i \in \mathbb{N}$ the number of voters with ranking i in the second district. The rankings are lexicographically ordered.

$$x_1 + x_2 > x_5 + x_6 \tag{42}$$
$$x_3 + x_4 > x_5 + x_6 \tag{43}$$
$$x_1 + x_2 + x_5 > x_3 + x_4 + x_6 \tag{44}$$

Inequalities (42) and (43) ensure that candidate c is eliminated in the first step as it has the smallest number of first ranks. The constraint (44) ensures that candidate a then wins over b in a paired comparison in the first district.

$$y_1 + y_2 > {}^1\!/_2 \sum_{i=1}^{6} y_i \tag{45}$$

This inequality makes sure that in the second district candidate a is the absolute majority winner.

$$x_1 + x_2 + y_1 + y_2 > x_3 + x_4 + y_3 + y_4 \tag{46}$$
$$x_5 + x_6 + y_5 + y_6 > x_3 + x_4 + y_3 + y_4 \tag{47}$$
$$x_4 + x_5 + x_6 + y_4 + y_5 + y_6 > x_1 + x_2 + x_3 + y_1 + y_2 + y_3 \tag{48}$$

Now for the joint district a different elimination order has to happen in order to obtain a different winner. Hence now candidate b has to be eliminated in the first step. This is ensured by inequalities (46) and (47). Therefore now candidates a and c reach the second step, and c has to win the paired comparison as demanded in (48).
Combining all inequalities then yields a linear program that describes an instance of the

Reinforcement paradox. Note that we have to account for the symmetric possibilities of obtaining a Reinforcement paradox. There are three possibilities as to which candidate is the winner in the separate districts and two possibilities which candidate is eliminated in the first district. This then specifies who can be the winner in the joint district. Also it can happen that the first district has the absolute majority winner and the second has not, which is symmetric, and therefore has equal probability.
Similar to this inequality description one can proceed for the case that in none of the districts there is an absolute majority winner, and instead in the first district candidate c is eliminated in the first step, and in the second district candidate b is eliminated. Using both these linear programs and NORMALIZ [BIR+] one can obtain the limit probability for the Reinforcement paradox under the Plurality with Runoff rule. Again the symmetric cases have to be considered. There are three possibilities as to which candidate wins both separate districts. The ordering of the two separate districts is symmetric and the elimination order also induces two possibilities. Hence multiplying the sum of the probabilities obtained through both of the linear programs by 12 accounts for symmetries.

Theorem 4.6. *The limit probability that an instance of the Reinforcement paradox happens in a profile with three candidates under the assumption of IAC and under the use of the Plurality with Runoff rule is*

$$\mathbb{P}(\textit{Reinforcement paradox for Plurality with Runoff}\ |m = 3, n \to \infty) \approx 0.59\%.$$

As mentioned in the introduction of Ehrhart Theory in chapter 4.3 we divide the number of all paradoxical profiles by the number of all possible profiles. In this case one can argue that, when considering the Reinforcement paradox, dividing by the number of all possible profiles is not very reasonable. This includes dividing by the number of all profiles in which the two separate districts do not even coincide in their winning candidates, although in such profiles an occurrence of the Reinforcement paradox is simply impossible. This of course decreases the probability of the Reinforcement paradox in an unnatural manner. So by this reasoning dividing by the number of all profiles in which the winner of the two separate districts coincides is more natural.

We are now left with determining the probability that the two winners of the separate districts coincide. Assuming IAC gives no candidate an expected advantage over one another, and hence the probability that candidate a wins in a randomly selected profile is $1/3$. As the districts are assumed to be independent of another, the event that candidate a wins in the second district simultaneously is $1/9$. As there are three possible candidates the overall probability that there is the same winner in both districts is $1/3$. Therefore we can now determine the probability of the Reinforcement paradox if both districts elect the same candidate.

Corollary 4.1. *The limit probability that an instance of the Reinforcement paradox happens in a profile with three candidates under the assumption of IAC and under the use of the Plurality with Runoff rule, if the winners of the separate districts coincide, is*

$$\mathbb{P}(\textit{Reinforcement paradox}\ |m = 3, n \to \infty, \textit{same winner in both districts}) \approx 1.77\%.$$

We have studied the Plurality with Runoff rule quite thoroughly. We have seen that even in single-peaked domains Plurality with Runoff can suffer from paradoxical situations.

For all considered paradoxes, that can happen in the three candidate case, we obtained frequencies of their occurrences. We can conclude that Plurality with Runoff suffers significantly less often from the Condorcet winner paradox than other rules that are prone to this paradox. For the Lack of Monotonicity paradox the expected frequency is 5% and for the No Show paradox it is 4%. These frequencies are not dramatically huge, but are also not that small that they can be ignored. The Reinforcement paradox happens not very often, and the frequency will later be compared to the frequencies of other rules. Overall it is important to stress that Plurality with Runoff does have more problems with the occurrence of paradoxical situations than other rules that are considered here.

4.6 Analysis of the Reinforcement Paradox with Ehrhart Theory

Because Ehrhart theory has, at least to our knowledge, not been applied to the Reinforcement paradox yet, we decided to investigate this paradox with the use of Ehrhart theory. Due to Young's Theorem 4.2 we know that the Reinforcement paradox cannot be avoided for Condorcet consistent rule, and such rules play an important part in voting theory which makes it valuable to learn about the frequency of this paradox. Given the frequency is reasonably low the susceptibility to the Reinforcement paradox is possibly not a big threat to the attractiveness of Condorcet extensions.

4.6.1 Black's Rule

Black's rule is a Condorcet consistent rule that differs from the set of rules from Theorem 3.2. Due to Young's Theorem it is also susceptible to the Reinforcement paradox, and apart from this vulnerability the No Show paradox can occur as well. As shown from Brandt, Hofbauer and Strobl [BHS19] with the help of Ehrhart theory the No Show paradox is not very likely to happen. Therefore one can argue that the theoretical possibility of a No Show paradox occurrence is so low that this is not a strong argument against Black's rule in comparison to Maximin, Nanson's, Kemeny's and Young's rule which are not vulnerable to the No Show paradox at all. It is left to compare the frequencies of the Reinforcement paradox of Black's rule to the aforementioned rules. As we want to apply Ehrhart theory, we have to formulate a paradoxical preference profile under the use of Black's rule as a linear program.
As Black's rule elects the Condorcet winner whenever it exists and the Borda winner in all other cases, we have to distinguish these two cases. Due to neutrality one can assume that candidate a is the winner in the separate districts, and candidate b is the winner of the joint district. Note that if in both districts the same winner is elected, and it is a Condorcet winner, then the same candidate will be a Condorcet winner in the joint district, and therefore no Reinforcement paradox can occur in this case. Also if in both districts there is the same majority cycle, this carries over to the joint district, and hence the winner cannot differ. So we are left with two cases. The first is that in one of the districts candidate a is elected as it is Condorcet winner, and in the other district there is no Condorcet winner, and candidate a is elected as Borda winner. The second case is that there are no Condorcet winners in both districts, and different majority cycles happen. We have to make a case distinction for the joint district as well, and again distinguish between a district in which a Condorcet winner exists and one in which it does not. As an example consider the case that there is a Condorcet winner in one of the districts, and in the other there is not, but in the joint district again a Condorcet winner is elected. Due to the symmetry of the districts one can assume that the Condorcet winner occurs in the first district, and the majority cycle in the second. The following linear program assumes further that the majority cycle is $a \succ c \succ b \succ a$ in the second district. Let the variables that correspond to the number of voters of ranking i in the first district be denoted as $x_i \in \mathbb{N}$ and $y_i \in \mathbb{N}$ as the respective variables in the second district.

$$x_1 + x_2 + x_5 > x_3 + x_4 + x_6 \tag{49}$$
$$x_1 + x_2 + x_3 > x_4 + x_5 + x_6 \tag{50}$$

These two inequalities state that candidate a is the Condorcet winner in the first district. For the second district it has to hold:

$$y_1 + y_2 + y_3 > y_4 + y_5 + y_5 \tag{51}$$
$$y_3 + y_4 + y_6 > y_1 + y_2 + y_5 \tag{52}$$
$$y_2 + y_5 + y_6 > y_1 + y_3 + y_4 \tag{53}$$
$$2y_1 + 2y_2 + y_3 + y_5 > 2y_3 + 2y_4 + y_1 + y_6 \tag{54}$$
$$2y_1 + 2y_2 + y_3 + y_5 > 2y_5 + 2y_6 + y_2 + y_4 \tag{55}$$

Inequalities (51) - (53) describe the majority cycle. Therefore Black's rule has to elect the Borda winner which is also candidate a as ensured by (54) and (55).
The following inequalities describe the joint district.

$$x_3 + y_3 + x_4 + y_4 + x_6 + y_6 > x_1 + y_1 + x_2 + y_2 + x_5 + y_5 \tag{56}$$
$$x_3 + y_3 + x_4 + y_4 + x_1 + y_1 > x_2 + y_2 + x_5 + y_5 + x_6 + y_6 \tag{57}$$

These two constraints demand that candidate b is Condorcet winner in the joint district which corresponds to an occurrence of the Reinforcement paradox.

In the following combinations of districts we obtained a positive probability for the case that candidate a is elected in both the separate districts and candidate b in the joint.

District 1	**District 2**	**Joint District**	**Probability for Reinforcement**
Condorcet winner	$a \succ c \succ b \succ a$	*Condorcet winner*	0.01756%
Condorcet winner	$a \succ b \succ c \succ a$	$a \succ b \succ c \succ a$	0.00096%
Condorcet winner	$a \succ c \succ b \succ a$	$a \succ c \succ b \succ a$	0.00003%
$a \succ b \succ c \succ a$	$a \succ c \succ b \succ c$	*Condorcet winner*	0.00025%

Note that in order to obtain the total probability of the Reinforcement paradox one has to account for the symmetric possibilities. There are three possible winners of the elections in the separate districts, two possible winners of the joint district and two possibilities of the ordering of the above noted districts. Hence one has to multiply the sum of the aforementioned probabilities by 12 to obtain the total probability.

Theorem 4.7. *The limit probability that an instance of the Reinforcement paradox happens in a preference profile with three candidates, under the assumption of IAC and under the use of Black's rule is*

$$\mathbb{P}(\textit{Reinforcement paradox for Black's rule } | m = 3, n \to \infty) = 12 \cdot 0.0188 \approx 0.23\%$$

Corollary 4.2. *The limit probability that an instance of the Reinforcement paradox happens in a preference profile with three candidates, under the assumption of IAC and under the use of Black's rule, if the winner in the separate districts coincide, is*

$$\mathbb{P}(\textit{Reinforcement paradox } | m = 3, n \to \infty, \textit{same winner in both districts }) \approx 0.69\%.$$

4.6.2 Maximin Rule

As stated in Theorem 3.2 the choice sets of Maximin, Kemeny's, Young's and Nanson's rule are intersecting in the three candidate case which makes this set of voting rules quite attractive. Therefore we are especially interested in the frequency of the Reinforcement paradox when using these voting rules. Hence we decided to apply the analytical approach using Ehrhart theory to this problem setting. We will use the Maximin rule for our calculations as it is computationally not that demanding in contrast to Young's and Kemeny's rule. Also it does not require a runoff procedure and revised calculations in contrast to Nanson's rule which makes it the most suitable rule in order to be described as system of linear inequalities.

We will assume due to neutrality that candidate a is elected in both separate districts, and candidate b wins in the joint district.
In order to describe the Reinforcement paradox under the Maximin rule, define m_i as the paired comparisons in the first district and g_i as the paired comparisons of the second district. Then obviously the comparisons in the joint district are the sums of the respective m_i and g_i.
In order to describe the Maximin winners with an inequality description, it is crucial which pairwise comparison constitutes the row minimum for each candidate. As for every of the three candidates two pairwise comparisons are listed, there are eight possibilities on how the row minima are distributed within the paired comparison matrix. These possibilities are listed in the following, whereby the row minima are denoted as boxed entries of the matrices:

1.

	a	b	c
a	–	$\boxed{m_1}$	m_2
b	$\boxed{m_3}$	–	m_4
c	$\boxed{m_5}$	m_6	–

2.

	a	b	c
a	–	$\boxed{m_1}$	m_2
b	$\boxed{m_3}$	–	m_4
c	m_5	$\boxed{m_6}$	–

3.

	a	b	c
a	–	$\boxed{m_1}$	m_2
b	m_3	–	$\boxed{m_4}$
c	$\boxed{m_5}$	m_6	–

4.

	a	b	c
a	–	$\boxed{m_1}$	m_2
b	m_3	–	$\boxed{m_4}$
c	m_5	$\boxed{m_6}$	–

5.

	a	b	c
a	–	m_1	$\boxed{m_2}$
b	$\boxed{m_3}$	–	m_4
c	$\boxed{m_5}$	m_6	–

6.

	a	b	c
a	–	m_1	$\boxed{m_2}$
b	$\boxed{m_3}$	–	m_4
c	m_5	$\boxed{m_6}$	–

7.

	a	b	c
a	–	m_1	$\boxed{m_2}$
b	m_3	–	$\boxed{m_4}$
c	$\boxed{m_5}$	m_6	–

8.

	a	b	c
a	–	m_1	$\boxed{m_2}$
b	m_3	–	$\boxed{m_4}$
c	m_5	$\boxed{m_6}$	–

As mentioned above, we will assume that in both districts candidate a is the Maximin winner. If the districts have either a majority margin matrix with pattern 4. or 8., candidate a cannot be the Maximin winner, and such districts can therefore be excluded. Similarly if the joint district turns out to have a majority margin with pattern 5. or 7., then candidate b cannot be the Maximin winner in the joint district.
Note also that if the row minima happen at the same matrix entry, so for example both m_1 and g_1 are the row minima of candidate a in the two separate districts, then $m_1 + g_1$

will be the row minimum of candidate a in the joint majority matrix. Therefore we can exclude all joint districts that do not have $m_1 + g_1$ as the row minimum of candidate a in the joint district if m_1 is row minimum in the first district and g_1 in the second.
This is also the reason why the Reinforcement paradox cannot happen if both districts have the same pattern of row minima because the row minima and their relations are transferred to the joint majority matrix by adding the margins. Hence we can exclude the case that both districts have the same row minimum pattern.

The following linear program is an example for the combination of two separate districts in which the Reinforcement paradox takes place. We assume, as stated above, that candidate a wins in both of the separate districts, but candidate b wins in the joint district. The row minima of the first district are distributed as in pattern 1.:

$$m_1 - m_2 \leq 0 \tag{58}$$
$$m_3 - m_4 \leq 0 \tag{59}$$
$$m_5 - m_6 \leq 0 \tag{60}$$

These inequalities state the entries of the paired comparison matrix that constitute the row minimum for each candidate in the first district. For example candidate a's row minimum is admitted at m_1 due to (58).

$$m_3 - m_1 < 0 \tag{61}$$
$$m_5 - m_1 < 0 \tag{62}$$

Through these inequalities it is ensured that candidate a is the Maximin winner in the first district as its row minimum is the largest of all candidates.

$$g_1 - g_2 \leq 0 \tag{63}$$
$$g_4 - g_3 \leq 0 \tag{64}$$
$$g_5 - g_6 \leq 0 \tag{65}$$
$$g_4 - g_1 < 0 \tag{66}$$
$$g_5 - g_1 < 0 \tag{67}$$

Similarly to the first district now the second district is described through inequalities (63) - (67). As explained above the second district cannot also have a row minimum pattern 1. as the first district. We decided to demand a row minimum pattern as in matrix 3. Hence candidate b's row minimum in the second district is set to m_4, whereas it is m_3 in the first district. The row minima of the candidates a and c coincide in their matrix entries to those from the first district, and therefore they must be transferred to the paired comparison matrix of the joint district. Thus the row minimum of candidate a in the joint district must occur at m_1+g_1 as it occurred at m_1 in the first district and at g_1 in the second. Similarly also candidate c's row minimum in the joint district is determined by the separate districts. Only the row minimum of candidate b is not pre-determined already. The only two patterns that can occur in the paired comparison matrix of the joint district are 1. and 3. We decided to state the case that pattern 1. occurs in the joint district as ensured by the following inequalities:

$$m_1 + g_1 - m_2 + g_2 \leq 0 \quad (68)$$
$$m_3 + g_3 - m_4 - g_4 \leq 0 \quad (69)$$
$$m_5 + g_5 - m_6 - g_6 \leq 0 \quad (70)$$

In this last set of inequalities we state that candidate b is the Maximin winner in the joint district:

$$m_1 + g_1 - m_3 - g_3 < 0 \quad (71)$$
$$m_5 + g_5 - m_3 + g_3 < 0 \quad (72)$$

Note that every paired comparison margin can be translated in terms of number of voters with a certain preference ranking, so for example $m_1 = x_1 + x_2 + x_5$ as m_1 is the number of voters that prefer candidate a over candidate b, and the preference rankings 1,2 and 5 rank a before b. As we already did before we will again use the number of voters of each possible ranking as our decision variables, and for the implementation we translated every of the above inequalities in terms of those variables. This formulation has the big advantage that it uniquely specifies the anonymous preference profile that causes the instance of the Reinforcement paradox. Therefore we have six variables in each district, and it follows that the dimension of the above - with inequalities - described polyhedron is 12.

Apart from the described restrictions one has to take into account all possible combinations of two separate districts with majority margins of pattern 1, 2, 3, 5, 6 or 7 and a resulting joint district. Depending on the entries of the row minima the inequality description has to be altered. So for every of the above paired comparison matrices a different inequality description is used in order to describe the Maximin winner. Hence for every combination of districts and resulting joint districts a special inequality description has to be implemented. With the use of NORMALIZ [BIR+] all those cases were examined, and eight cases were identified in which the Reinforcement paradox can happen with a positive probability. These cases and the respective probabilities are the following. The number of the districts refer to the pattern of their paired comparison matrix.

District 1	**District 2**	**Joint District**	**Probability for Reinforcement**
1	3	1	0.00244%
1	3	3	0.00163%
2	3	1	0.00382%
2	3	2	0.00081%
2	3	3	0.00015%
3	6	1	0.00054%
3	6	2	0.00027%
3	6	3	0.00054%

As the incidents of the Reinforcement paradox happen symmetrically, the probability of Reinforcement is also 0.00244% if the first district has pattern 3., and both the second district and the joint district have pattern 1. Hence the total probability of the Reinforcement paradox is double the amount of the above listed probabilities and hence 0.0204%. We have assumed that candidate a wins in the separate districts and candidate b in the joint district. Hence we have to multiply the above number by 6 to take into account that there are three possible winners for the separate districts and two for the joint district then. Hence the following turns out:

Theorem 4.8. *The limit probability that an instance of the Reinforcement paradox happens in a preference profile with three candidates, under the assumption of IAC and under the use of the Maximin rule is*

$$\mathbb{P}(\textit{Reinforcement for the Maximin Rule } | m = 3, n \to \infty) = 6 \cdot 0.0204 = 0.1224\%.$$

In [CMM14] Monte Carlo simulations are done in order to obtain probabilities for the Reinforcement paradox up to 100 voters. Due to this moderate number of voters the results are not perfectly comparable, but it is still nice to see that the order of magnitude seems to be similar. In [CMM14] even the maximal probabilities, that seem to be more likely for smaller electorates, are well below 1% and their result for 100 voters is even below 0.2%. Hence it is fair to say that our theoretical limit probability is in accordance to their work.

Corollary 4.3. *The limit probability that an instance of the Reinforcement paradox happens in a preference profile with three candidates, under the assumption of IAC and under the use of Maximin rule, if the winner of the separate districts coincide, is*

$$\mathbb{P}(\textit{Reinforcement paradox } | m = 3, n \to \infty, \textit{same winner in both districts }) \approx 0.37\%.$$

4.6.3 Comparison of the Probabilities for the Reinforcement paradox

We applied Ehrhart theory to three different rules in order to find out about the frequency of the Reinforcement paradox. As the Maximin rule is only vulnerable to the Reinforcement paradox, and turns out to be equivalent to some other voting rules for three candidates, we were especially interested in this rule. In order to have reasonable trademarks, we decided to compare to the Plurality with Runoff rule and Black's rule. Plurality with Runoff has previously been investigated because of its popularity. As we also wanted to compare to a Condorcet consistent rule, we chose to take Black's rule into account. We obtained the following results for the limit probabilities:

Voting Rule	Frequency of Reinforcement given both districts elect equivalently
Pl. with Runoff	1.77%
Black's rule	0.69%
Maximin Rule	0.37%

Overall the frequencies are quite low, so the Reinforcement paradox should not be a big problem in elections with a large number of voters. Nevertheless, Plurality with Runoff suffers the Reinforcement paradox more often than the other rules do. As we have seen before, even domain restrictions do not prevent the Reinforcement paradox. Black's rule and Maximin rule both exhibit the Reinforcement paradox very rarely, whereby the

Maximin rule suffers even less often than Black's rule. This presents another argument why the Maximin rule and its related and equivalent rules are a good choice in the three candidate case.

4.7 Further Ideas and Limitations

Not only paradoxical behaviour can be investigated with Ehrhart theory. Also an analysis of the similarity of voting rules can be done with this approach. It is an interesting question with what frequency a pair of rules selects different or disjoint winning sets similar to what we did previously for the Maximin rule and the Nanson rule following an experimental approach in 3.2 and 3.3. This task can also be approached with Ehrhart theory as we can often describe the situation that one rule selects one candidate, and the other rule does not select the same candidate as linear program. Obviously we could apply this approach for almost every pair of here considered rules, but this would surely go beyond the scope of this thesis.

Unfortunately this task showcases the limitations of the approach with Ehrhart theory. It has already been established by other researchers that investigating frequencies of paradoxical behaviour with Ehrhart theory for four candidates can lead to infeasible problems. This is due to the fact that the dimension of variables grows exponential in the number of candidates, and when investigating problems with four candidates the number of variables is already 24, whereas it was 6 for the three candidate case.

Also the problem of comparing voting rules is likely to be quite demanding. If you consider the Plurality and the Plurality with Runoff rule as an example, these two rather simple rules already show how dramatically the running times change when moving from three to four candidates. Say for instance we are interested in the frequency of how often Plurality and Plurality with Runoff select different winners. For the three candidate case the inequality description looks as follows:

$$n_1 + n_2 > n_3 + n_4 \tag{73}$$
$$n_3 + n_4 > n_5 + n_6 \tag{74}$$
$$n_1 + n_2 + n_5 < n_3 + n_4 + n_6 \tag{75}$$

We assumed that candidate a is the Plurality winner, whereas candidate b gets elected by Plurality with Runoff. Using this inequality description in NORMALIZ [BIR$^+$] we get that the limit frequency of this specific situation is 2.05% and hence 12.3% when taking all symmetric possibilities into account. The calculation of the describing quasipolynomial terminated in under a second. Also Lepelley, Louichi and Smaoui [LLS08] were able to calculate this within a few seconds. In the case of four candidates they concluded that it was computationally too demanding and hence infeasible. This does show that the running time can increase dramatically when we step up to four candidates.

Still, we can make use of the fact that we are often only interested in the limit probability. This has been used by De Loera et al.[LDK$^+$13], and hence they were able to come up with the limit probability that Plurality and Plurality with Runoff select different winners in an election with four candidates. Also in our implementation of the four candidate case we are able to obtain the limit probability within seconds. We can conclude that Plurality and Plurality with Runoff choose different winners in 24.5% of large elections with four

candidates. Nevertheless, the calculation of the whole describing quasipolynomial remains infeasible.
Yet another improvement has been found by Schürmann [Sch13]. He shows that one can exploit appearing symmetries, and further decrease the dimension which also reduces the computation time. He claims that this approach is hundred times faster than if the full dimensional inequality description is used.
As another example we also considered Plurality and Borda's rule. In contrast to runoff procedures there is no case distinction needed here, and the inequality description consists of only six inequalities. We were able to obtain the limit probability that Plurality and Borda's rule do not coincide in their decision for a four candidate election, which turns out to be 27.5%. Nevertheless even restricting ourselves to the limit probability resulted in a running time of above two hours. In this case an exploitation of symmetries is not as easily possible as before due to the specific inequalities, and hence it is hard to improve the running time in this case.

We noted that often the running times are higher when using strict inequalities in the inequality description. Usually strict inequalities are needed in order to describe the desired event, because inequalities that are not strict, can result in ties that should not be included. But when restricting ourselves to the limit probability, we can make use of the fact that the probability for ties vanishes as the number of voters grows higher. This knowledge allows us to use not strict inequalities as they only include events with probability zero to the limit calculations. We believe that this idea should be further investigated, and that it could improve calculations with Ehrhart theory.

Note that in contrast to Maximin and Nanson's rule the above mentioned rules are rather easy to describe as linear inequalities as they do not induce a case distinction such as Nanson's rule. The Maximin rule needs several cases as explained in section 4.6.2 as well. Also exploiting symmetries of the inequalities might not always be possible depending on the specific rule, not to mentioned that some of the rules simply do not allow for an inequality description. Hence the chance of obtaining feasible results for example of the similarity of Maximin's and Nanson's rule in the four candidate case must be expected to be rather low. This of course is unfortunate as these rules have drawn our attention due to their similarity and their nice behaviour concerning paradoxes.
Nevertheless, using Ehrhart theory to obtain results on the similarity of rules and how often they coincide in their decision is an interesting task that can surely be applied for the three candidate cases, and is very interesting to research further for four candidates.

5 Conclusion and Outlook

The aim of this master thesis is to get insights into which rules are attractive in three and four candidate elections. By working towards this goal we obtained four major accomplishments:

- **A detailed investigation of equivalences and differences of voting rules for three and four candidates**
  A set of voting rules was identified to be equivalent in the three candidate case. We were able to prove the equivalence of Maximin, Kemeny's and Young's rule. Although not further investigated also Dodgson's and Schulze's rule have been proven to be equivalent in that case. Furthermore it has been shown that the choice set of the Nanson rule is always contained inside the choice set of the aforementioned rules. Hence four Condorcet consistent rules provide a nonempty intersection of their choice sets in the three candidate case. For the four candidate case it turned out that no such set of rules exists. We were only able to identify properties that lead to an equivalence of the Maximin and Young's rule.
  For all other pairs of rules - whether in the three candidate or in the four candidate case - we provided preference profiles in which the respective rules have choice sets with empty intersection, and thus do not agree on a winning candidate. In order to gain insights into the behaviour of voting rules for a small number of voters we minimised the number of voters while searching for such profiles. It can be seen that almost all these examples occur for under ten voters already. Note that the number of voters for which the choice sets differ, but may have a nonempty intersection can only be smaller, and therefore differences in rules can occur for an even smaller number of voters already.

- **An overview of the axiomatic properties for selected voting rules and paradoxes for three and four candidates**
  Not only the voting rules themselves but also their behaviour in paradoxical voting situations play an important role when investigating the quality of rules. Hence we provided an overview on existing results in this area. It is quite well known which rules are vulnerable to which paradoxes in general. For every rule that proves to be vulnerable to a certain paradox for three or four candidates we distinguished between a small and a large number of voters. Hence we stated which is the smallest number of voters for which the respective paradox can occur. When considering a large number of voters, we were especially interested in statements about the limit case where the number of voters tends to infinity. Therefore we listed results for this case that have been obtained with the use of Ehrhart theory. As applying Ehrhart theory is sometimes infeasible, we also considered experimental results that use a rather high number of voters.

- **A thorough analysis of the axiomatic properties of the Plurality with Runoff rule**
  Due to its simplicity and enormous popularity in political elections we decided to take a closer look at the Plurality with Runoff rule. As Plurality with Runoff is vulnerable to the No Show paradox, the Reinforcement paradox, the Lack of Monotonicity paradox and the Condorcet Winner paradox for three candidates already,

several domain restrictions were considered. Results about the frequency of the Condorcet Winner paradox and the Lack of Monotonicity paradox for three candidates have previously been obtained with the use of Ehrhart theory. In order to complement these results we managed to apply Ehrhart theory to the No Show paradox and the Reinforcement paradox, and have therefore obtained a thorough understanding of the behaviour of the Plurality with Runoff procedure in the three candidate case. Our results suggest that there are several other rules that perform better than Plurality with Runoff considering the introduced quality measures and that are equally simple.

- **An analytic study of the Reinforcement paradox for three candidates** Due to the previously mentioned result that Maximin, Kemeny's and Young's rule are equivalent for the three candidate case we were especially interested in their axiomatic behaviour. The only paradox out of the five paradoxes under consideration, that can happen for those rules in a three candidate election, is the Reinforcement paradox. We decided to study this paradox thoroughly. We applied Ehrhart theory to the Maximin rule, Black's rule and Plurality with Runoff rule to get insights into the frequency of the Reinforcement paradox for different rules. To the best of our knowledge an application of Ehrhart theory to the Reinforcement paradox has not been made before, and we were able to obtain new results here. These results support our assessment of the Maximin rule and the related rules as very attractive because the frequency of the Reinforcement paradox is pleasantly small.

The goal of this master thesis was to identify attractive rules for three and four candidate elections. In order to measure the attractiveness of a voting rule we considered the following criteria. Firstly, studying the similarity and equivalence of rules provides information if certain rules agree in their decision, which indicates a reasonable choice. Secondly, a rule can be considered more attractive if it is hardly susceptible to paradoxical situations. Other criteria, that should be kept in mind, are simplicity of the rules, computational hardness and decisiveness.
In the three candidate case we succeeded in finding a set of rules that seem to be attractive in many ways. The fact that Maximin, Kemeny's, Young's and Nanson's sets are never disjoint, already indicates a good fit. Furthermore these rules satisfy Moulin's Theorem as they are Condorcet consistent and do not suffer the No Show paradox for three candidates which is impossible for four or more candidates. Therefore out of the five considered paradoxes they are only vulnerable to the Reinforcement paradox. The smallest instances for the occurrence of the Reinforcement paradox involve at least eleven voters. For a very small number of voters, the rules are indeed immune to every paradox that has been considered here. Through the application of Ehrhart theory results for a large, in fact infinitely large, number of voters have been obtained. These results show that the frequency with which an instance of the Reinforcement paradox can be expected is rather low. Also in comparison with other rules that suffer the Reinforcement paradox for three candidates - we have used a Condorcet consistent and a not Condorcet consistent rule as trademark - the frequency is desirably small. Additionally, Nanson's and the Maximin rule are easy to compute and reasonably simple so that they can be quickly understood by the voters. Nanson's rule is not always equivalent to Maximin, Kemeny's and Young's rule, but it differs in rare occasions. Nevertheless its choice set

is still included in the choice sets of the other equivalent rules. Hence Nanson's rule is the most decisive rule among our set of four rules but still provides all good qualities that have been stated above. As decisiveness is often desired and due to the other listed reasons the recommendation for the three candidate case is Nanson's rule.

The picture is not as clear for the four candidate case. The first drawback in this case was the fact that no pair of rules turned out to be equivalent. Also not nearly as much results about axiomatic properties have been obtained with the use of Ehrhart theory. This is due to the increased dimension of the problems that causes infeasibility as it demands exorbitant computational power. So unfortunately we, as many others, were not able to contribute to analytical results in this area. Nevertheless the results that have been obtained in the past provide hints as to which rules are attractive in this case. Again Maximin and Nanson's rule seem to be a good choice due to their simplicity. We also showed experimentally that these rules are still quite similar in the four candidate case, and a lot of their decisions coincide. But differences appear in their axiomatic behaviour. Additionally to the Reinforcement paradox the Maximin rule can also suffer from the No Show paradox and the Condorcet Loser paradox for four candidates although the frequencies of the latter two have been proven to be quite small. Nanson's rule on the other hand can suffer the Lack of Monotonicity, the No Show and the Reinforcement paradox. The frequency for the No Show paradox seems to be higher than for the Maximin rule. Still, for some a vulnerability to the Condorcet Loser paradox can understandably be an exclusion criterion and they might thus prefer Nanson's rule over the Maximin rule. A voting rule that avoids this conflict of axioms is Kemeny's rule as it is only vulnerable to the Reinforcement and the No Show paradox. Obviously the computation is harder for this rule, and it is also not quite as straight-forward to understand. We would confidently say that it comes down to these three rules, and among them it is a question of personal taste and opinion.

There are certainly attractive rules and important voting paradoxes that have not been considered in this thesis. Therefore all the investigations we made can be adapted for new voting rules and different paradoxes. Especially Schulze's rule is certainly worth further investigations as it has been shown that it is equivalent to attractive rules in the three candidate case, and satisfies many desirable properties.
Note that a lot of the rules we considered are runoff rules of some sort. Due to several rounds of eliminations that are often needed when using those rules, stepping up to four candidates increases the number of possible elimination orders. This comes often hand in hand with a high number of cases that have to be distinguished. Due to these case distinctions and the computational issues we did not apply Ehrhart theory to any four candidate problem. A way to decrease the dimension of the problems is to make use of possibly occurring symmetries as described by Schürmann [Sch13]. If computation was still too demanding, then also experimental results would be quite helpful to quantify the vulnerability to paradoxical situations.
Also analysis of the similarity of rules as mentioned in 4.7 is an interesting application of Ehrhart theory, and can provide new insights into which rules are related to one another. As often criticised for being unrealistic, the IAC assumption can be changed for a more realistic probability model. One must not forget that assuming IAC enabled us to make

use of Ehrhart theory, and come up with reproducible analytical results. So ditching this assumption means losing a nice analytical tool. Obviously studying real election data is a nice way to obtain empirical results. Note that the PrefLib library [MW13b] is a brilliant source of real world data, that would surely be worth investigating. With that comes the major problem that in most elections not strict preference rankings are submitted in contrast to our assumption throughout this thesis. Hence one would be required to pre-process the data reasonably or to adapt the computations to also allow for incomplete or partial rankings and ties within the rankings.

A Quasipolynomial of the No Show paradox with the Plurality with Runoff rule

The following are the coefficients of the quasipolynomial that describes the No Show paradox in a preference profile with three candidates under the assumption of IAC and under the use of the Plurality with Runoff rule. It has period 24 and degree 5. The common denominator is 829,440.

	x^0	x^1	x^2	x^3	x^4	x^5
0	0	4608	10080	700	−40	47
1	2085	11583	−10730	−3150	165	47
2	15520	1568	−6280	700	−40	47
3	−22275	7983	−1170	−590	165	47
4	−99840	−5312	12640	−1860	−40	47
5	−5915	−31617	−490	−590	165	47
6	272160	−8352	−19080	700	−40	47
7	180285	56943	−13970	−3150	165	47
8	−204800	40448	22880	700	−40	47
9	−174555	−11457	2070	−590	165	47
10	−86880	−70112	−16520	−1860	−40	47
11	−35075	−38097	−3730	−590	165	47
12	207360	30528	10080	700	−40	47
13	183525	37503	−10730	−3150	165	47
14	67360	27488	−6280	700	−40	47
15	3645	33903	−1170	−590	165	47
16	−307200	−31232	12640	−1860	−40	47
17	−187355	−57537	−490	−590	165	47
18	220320	−34272	−19080	700	−40	47
19	154365	31023	−13970	−3150	165	47
20	2560	66368	22880	700	−40	47
21	6885	14463	2070	−590	165	47
22	−35040	−44192	−16520	−1860	−40	47
23	−9155	−12177	−3730	−590	165	47

References

[BCE+16] Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D Procaccia. *Handbook of Computational Social Choice.* Cambridge University Press, 2016.

[BGP17] Felix Brandt, Christian Geist, and Dominik Peters. Optimal bounds on the no-show paradox via sat solving. *Mathematical Social Sciences 90, 18-27*, 2017.

[BGS] Felix Brandt, Christian Geist, and Martin Strobel. Analyzing the practical relevance of voting paradoxed via ehrhart theory, computer simulation and empirical data.

[BHS19] Felix Brandt, Johannes Hofbauer, and Martin Strobl. Exploring the no-show paradox for condorcet extensions using ehrhart theory and computer simulations. *Proc. of the 18th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2019), Montreal, Canada, May 13–17*, 2019.

[BIR+] Winfired Bruns, Bogdan Ichim, Tim Römer, Richard Sieg, and Christof Söger. Normaliz. algorithms for rational cones and affine monoids. Available at `https://www.normaliz.uni-osnabrueck.de`.

[BIS19] Winfried Bruns, Bogdan Ichim, and Christof Söger. Computations of volumes and ehrhart series in four candidates elections. *Annals of Operations Research*, 2019.

[CMM14] Sébastien Courtin, Boniface Mbih, and Issofa Moyouwou. Are condorcet procedures so bad according to the reinforcement axiom? *Social Choice and Welfare, Springer Verlag*, 2014.

[Ehr62] Eugène Ehrhart. Sur les polyèdres rationnels homothétiques à n dimensions. *Compte-rendus de l'académie des sciences, Paris 254, 616-618,*, 1962.

[Fel12] Dan S. Felsenthal. Review of paradoxes afflicting procedures for electing a single candidate. *Electoral Systems*, 2012.

[FG76] P.C. Fishburn and William V. Gehrlein. Condorcet's paradox and anonymous preference profiles. *Public Choice 26, 1–18 (1976)*, 1976.

[Fis77] Peter C. Fishburn. Condorcet social choice functions. *SIAM Journal on Applied Mathematics, Vol. 33, No. 3 (Nov., 1977), pp. 469-489*, 1977.

[FN19] Dan S. Felsenthal and Hannu Nurmi. *Voting Procedures Under a Restricted Domain: An Examination of the (In)Vulnerability of 20 Voting Procedures to Five Main Paradoxes.* SpringerBriefs in Economic, 2019.

[Geh82] William Gehrlein. The frequency of condorcet's paradox in large groups. April 1982.

[Gib73] Allen Gibbard. Manipulation of voting schemes: A general result. *Econometrica, Vol. 41, No. 4 (Jul., 1973), pp. 587-601*, 1973.

[GL11] William V. Gehrlein and Dominique Lepelley. *Voting Paradoxes and Group Coherence : The Condorcet Efficiency of Voting Rules*. Springer Verlag Berlin Heidelberg, 2011.

[GL12] William V. Gehrlein and Dominique Lepelley. *Electoral Systems: Paradoxes, Assumptions, and Procedures*, chapter The Value of Research Based on Simple Assumptions about Voters' Preferences, pages 173–200. 2012.

[LDK+13] J.A. De Loera, B. Dutra, M. Köppe, S. Moreinis, G. Pinto, and J. Wu. Software for exact integration of polynomials over polyhedra. *Computational Geometry: Theory and Applications, 46(3)*, 2013.

[LLS08] Dominique Lepelley, Ahmed Louichi, and Hatem Smaoui. On ehrhart polynomials and probability calculations in voting theory. *Social Choice and Welfare (2008) 30:363–383*, 2008.

[LM01] Dominique Lepelley and Vincent Merlin. Scoring run-off paradoxes for variable electorates. *Economic Theory 17, 53–80 (2001)*, 2001.

[LMS17] Dominique Lepelley, Issofa Moyouwou, and Hatem Smaoui. Monotonicity paradoxes in three-candidate elections using scoring elimination rules. *Social Chioce and Welfare*, 2017.

[May52] Kenneth O. May. A set of independent necessary and sufficient conditions for simple majority decision. *Econometrica, Vol. 20, No. 4 (Oct., 1952), pp. 680-684*, 1952.

[Mou88] Hervé Moulin. Condorcet's principle implies the no show paradox. *Journal of Economic Theory 45, 53-64*, 1988.

[MW13a] Nicholas Mattei and Toby Walsh. Preflib: A library of preference data `http://preflib.org`. In *Proceedings of the 3rd International Conference on Algorithmic Decision Theory (ADT 2013)*, Lecture Notes in Artificial Intelligence. Springer, 2013.

[MW13b] Nicholas Mattei and Toby Walsh. Preflib: A library of preference data `http://preflib.org`. In *Proceedings of the 3rd International Conference on Algorithmic Decision Theory (ADT 2013)*, Lecture Notes in Artificial Intelligence. Springer, 2013.

[OLS19] Abdelhalim El Ouafdi, Dominique Lepelley, and Hatem Smaoui. Probabilities of electoral outcomes: from three-candidate to four-candidate elections. *Theory and Decision*, 2019.

[Pre20a] Bundespräsidentenwahlgesetz 1971. `https://de.wikipedia.org/wiki/Bundespr%C3%A4sidentenwahlgesetz_1971#Wahlrecht`, April 2020. Accessed on 01.04.2020.

[Pre20b] Presidential elections in france. `https://en.wikipedia.org/wiki/Presidential_elections_in_France`, March 2020. Accessed on 01.04.2020.

[PS82] Christos Papadimitriou and Kenneth Steiglitz. *Combinatorial Optimization: Algorithms and Complexity.* 1982.

[PT14] Florenz Plassmann and T.Nicolaus Tideman. How frequently do different voting rules encounter voting paradoxes in three-candidate elections? *Social Choice and Welfare*, 2014.

[RGMT06] Michel Regenwetter, Bernard N. Grofman, A.A.J. Marley, and Ilia Tsetlin. *Behavioral Social Choice.* Cambridge University Press, 2006.

[Sat75] Mark Allen Satterthwaite. Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory 10, 187-217*, 1975.

[Sch13] Achill Schürmann. Exploiting polyhedral symmetries in social choice. *Social Choice and Welfare (2013) 40:1097–1110*, 2013.

[Sch18a] Marie Schmidt. Using optimization software to find minimal voting paradoxes. Master's thesis, Technische Universität München, 2018.

[Sch18b] Markus Schulze. The schulze method of voting. *arXiv e-prints*, 2018.

[Sta20] Bayrische Staatskanzlei. Bayglkrwg, art.46. https://www.gesetze-bayern.de/Content/Document/BayGLKrWG-46, 2020. Accessed on 01.04.2020.

[Str15] Martin Rüdiger Strobl. A quantitative study of the axiomatic properties of selected voting rules. Master's thesis, Technische Universität München, 2015.

[Xpr20] Fico xpress optimization. Available at `http://www.fico.com/en/products/fico-xpress-optimization-suite/`, 2020.

[You75] H.P. Young. Social choice scoring functions. *SIAM Journal of Applied Mathematics, Vol. 28, No. 4:824 - 838*, 1975.