Local Rationalizability and Choice Consistency

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We introduce a new notion of rationalizability, where the rationalizing relation may depend on the set of feasible alternatives. More precisely, we say that a choice function is locally rationalizable if it is rationalized by a *family* of rationalizing relations such that a strict preference between two alternatives in some feasible set is preserved when removing other alternatives. We then show that a choice function is locally rationalizable if and only if it satisfies Sen's γ and give similar characterizations for local rationalizability via transitive, PIP-transitive, and quasi-transitive relations. Local rationalizability is realized via families of local revealed preference relations that are sandwiched in between the base relation and the revealed preference relation of a choice function. Our results permit a unified perspective on social choice functions that satisfy γ, including classic ones such as the top cycle and the uncovered set as well as new ones such as two-stage majoritarian choice and split cycle. We also provide simple axiomatic characterizations of the former two using local rationalizability.

1 Introduction

Rational choice theory studies functions that associate each feasible set of alternatives with a nonempty subset of chosen alternatives. Typically, the choices of rational agents are supposed to be rationalizable in the sense that there exists an underlying binary preference relation on the set of all alternatives such that the choice from any feasible subset consists precisely of the elements that are maximal according to this relation. When the number of feasible alternatives is finite, a relation can rationalize a choice function if and only if its strict part is acyclic. Stronger notions of rationalizability can be obtained by furthermore requiring the rationalizing relation to be quasi-transitive or transitive.

As Arrow (1951) and others have famously demonstrated, these notions of rationalizable choice can hardly be sustained in the context of *social* choice. There are several impossibility results showing that even the weakest form of rationalizability is incompatible with mild conditions that are considered indispensable for collective decision-making (for an overview of the extensive literature, see Kelly, 1978; Sen, 1977, 1986; Schwartz, 1986; Campbell and Kelly, 2002). Sen (1971) has characterized rationalizable choice functions using two consistency conditions that relate choices from varying feasible sets to each other, namely conditions α and γ (aka contraction and expansion consistency). α demands that every chosen alternative remains chosen in every feasible subset in which it is contained, and γ requires that every alternative that is chosen from two feasible sets will also be chosen from the union of both sets. While α has been identified as the main culprit of the impossibility results in social choice (Sen, 1977, 1986), γ and even strengthenings of γ such as β and β^+ appear to be much less harmful. As a matter of fact, two majoritarian social choice functions—the top cycle and the uncovered set—are known to satisfy β^+ and γ, respectively.

The goal of this paper is to improve our understanding of choice functions that satisfy γ and its stronger siblings. For one, this opens an avenue for escaping from the notorious Arrovian impossibilities. Secondly, γ and related consistency conditions are natural and appealing in their own right. Only few social choice functions were known to satisfy γ: variants of the top cycle and the uncovered set (see, e.g., Bordes, 1983), and two recently proposed functions called split cycle (Holliday and Pacuit, 2020) and two-stage majoritarian choice (Horan and Sprumont, 2022). Our results illuminate the similarities of these functions and enable the definition of new functions that satisfy γ.

The key to our characterizations of choice functions that satisfy γ is a weakened notion of rationalizability, where the rationalizing relation may vary depending on the feasible set. Clearly, without further restrictions any choice function C would be rationalizable in this relaxed setting by letting the rationalizing relation R^A for feasible set A be the relation in which each element of $C(A)$ is strictly preferred to each element of $A \setminus C(A)$. We therefore impose the following natural restriction on families of rationalizing relations: for two feasible sets A, B with $B \subseteq A$ and two alternatives $x, y \in B$ with $x\ P^A\ y$, we demand that $x\ P^B\ y$. In other words, a strict preference between two alternatives is preserved when reducing the feasible set. Since rationalizing relations are complete, this is equivalent to demanding that $x\ R^B\ y$ implies $x\ R^A\ y$. We then say that a choice function C is locally rationalizable if there is a family of acyclic and complete relations $(R^A)_A$ that rationalize C in this sense.

Our main results are as follows.

- A choice function satisfies γ if and only if it is locally rationalizable (Theorem 1).
- A choice function satisfies γ and ε^+ if and only if it is locally rationalized by a family of quasi-transitive relations (Theorem 3).
- A choice function satisfies β^+ if and only if it is locally rationalized by a family of transitive relations (Theorem 4).

As corollaries of these results, we obtain classic characterizations of rationalizability, quasi-transitive rationalizability, and transitive rationalizability by Sen (1971), Schwartz (1976), and Arrow (1959). We also give a characterization of choice functions that are locally rationalizable via PIP-transitive relations, an intermediate transitivity notion proposed by Schwartz (1976) that lies in between quasi-transitivity and full transitivity. Such functions are characterized by γ^+, a new natural strengthening of γ, which requires that for all feasible sets A, B, $C(A) \subseteq C(A \cup B)$ or $C(B) \subseteq C(A \cup B)$.

All of the above results make reference to families of local revealed preference relations that are sandwiched in between the base relation and the revealed preference relation of the choice function at hand. We introduce the γ-*hull* of a given choice function C as the finest coarsening of C that satisfies γ. In Theorem 2, we prove that the γ-hull of C is locally rationalized by the family of local revealed preference relations of C.

Local rationalizability allows us to give a simple characterization of the top cycle based on an observation by Bordes (1976): the top cycle is the finest choice function satisfying transitive local rationalizability. We also give a new characterization of Gillies' uncovered set as the finest choice function satisfying quasi-transitive local rationalizability and a condition we call weak idempotency ($C(C(A)) = C(A)$ whenever $|C(A)| = 2$). By introducing further technical axioms, we obtain characterizations of other variants of the uncovered set due to Bordes, McKelvey, and Duggan.

The notion of local rationalizability enables a unified perspective on social choice functions that satisfy γ. The idea is to view weak majority rule as a directed graph on the set of alternatives, break majority cycles in the feasible set by removing edges of this graph, and then return the maximal elements of the feasible set according to the resulting acyclic graph. Different rules specifying which strict edges ought to be removed lead to different social choice functions.

- *Top cycle*: Remove all edges that lie on cycles.
- *Uncovered set*: Remove edges that lie on three-cycles.[1]
- *Split cycle*: Remove all edges with minimal majority margins from each cycle.

Moreover, for any given choice function C, we define a new choice function, the *γ-core of C*, which satisfies γ. Here, in each cycle, all ingoing edges of alternatives that are selected by C are removed from the majority graph.

Note that the rule specifying which edges should be deleted must only depend on the alternatives contained in the cycle. Otherwise, the resulting social choice function is not guaranteed to satisfy γ. This is, for example, the case for Kemeny's rule, ranked pairs, and Schulze's rule, which share the same idea of taking maximal elements after breaking majority cycles (see, e.g., Fischer et al., 2016, for definitions). However, the cycles are not broken locally, which results in functions that violate γ.

The remainder of the paper is structured as follows. The choice theory model as well as standard notions of rationalizability and consistency are introduced in Section 2. We introduce local rationalizability and families of local revealed preference relations in Section 3. Finally, Section 4 shows that each of the four notions of local rationalizability is equivalent to an expansion consistency condition and that the γ-hull of a choice function is locally rationalized by its family of local revealed preference relations. We illustrate our main findings with various examples from social choice theory and give axiomatic characterizations of the top cycle and the uncovered set.

[1] Which edges are to be deleted from each three-cycle depends on which variant of the uncovered set is considered. For example, for Duggan's deep uncovered set, all strict edges are deleted whereas for McKelvey's uncovered set, only strict edges that lie on three-cycles in which at least two edges are strict are deleted. In the absence of majority ties, all variants coincide.

2 Preliminaries

Let U be a nonempty universe of alternatives. In this paper, every nonempty and finite subset of U is a *feasible set*. The set of all feasible sets is denoted by $\mathcal{U}$. A *choice function* C maps each feasible set $A \in \mathcal{U}$ to a nonempty subset of A. The set of all choice functions is denoted by $\mathcal{C}$. For two choice functions $C, \widehat{C}$, C is a *refinement* of $\widehat{C}$ if $C(A) \subseteq \widehat{C}(A)$ for all $A \in \mathcal{U}$. Analogously, $\widehat{C}$ is a *coarsening* of C. We say that $C \in \widehat{\mathcal{C}} \subseteq \mathcal{C}$ is the *finest* choice function in $\widehat{\mathcal{C}}$ if C is a refinement of every $\widehat{C} \in \widehat{\mathcal{C}}$.

If R is a relation, we denote its strict part by P and its symmetric part by I. R is *transitive* (*quasi-transitive*, *acyclic*, resp.) if for all $x, y, z, x_1, \dots, x_k \in U$,

$$x \; R \; y \text{ and } y \; R \; z \text{ implies } x \; R \; z, \tag{transitivity}$$

$$x \; P \; y \text{ and } y \; P \; z \text{ implies } x \; P \; z, \tag{quasi-transitivity}$$

$$x_1 \; P \; x_2, \dots, x_{k-1} \; P \; x_k \text{ implies } x_1 \; R \; x_k. \tag{acyclicity}$$

Transitivity implies quasi-transitivity, which implies acyclicity.

Let $R \subseteq U \times U$ be a relation on U and A a feasible set. The set of *maximal elements* in A with respect to R is defined by

$$\max_R A = \{x \in A \colon y \; P \; x \text{ for no } y \in A\}.$$

Note that $\max_R A$ is nonempty for all $A \in \mathcal{U}$ if and only if R is acyclic.

A choice function C is *rationalizable* (*quasi-transitively rationalizable*, *transitively rationalizable*, resp.) if there is an acyclic (quasi-transitive, transitive, resp.) and complete relation R on U such that for all feasible sets A,

$$C(A) = \max_R A. \tag{rationalizability}$$

In this case we say that C is rationalized by R. Two natural candidates for the rationalizing relation are the *base relation* $\overline{R}_C$ (Herzberger, 1973) and the *revealed preference relation* R_C (Samuelson, 1938; Houthakker, 1950), which, for all alternatives x and y, are given by

$$x \; \overline{R}_C \; y \quad \text{iff} \quad x \in C(\{x, y\}), \tag{base relation}$$

$$x \; R_C \; y \quad \text{iff} \quad x \in C(A) \text{ for some } A \in \mathcal{U} \text{ with } y \in A. \tag{revealed preference relation}$$

The revealed preference relation relates x to y if x is chosen in the presence of y and possibly other alternatives, whereas the base relation only relates x to y if x is chosen in the exclusive presence of y. Both the base relation and the revealed preference relation are complete by definition. The revealed preference relation is furthermore guaranteed to be acyclic. Whenever C is rationalizable, the base relation and the revealed preference relation coincide and rationalize C.

We now define four choice consistency conditions. For all feasible sets A and B,

$$
\begin{array}{rll}
(\alpha): & \text{if } B \subseteq A\text{, then } C(A) \cap B \subseteq C(B) & \text{(Chernoff, 1954)} \\
(\gamma): & C(A) \cap C(B) \subseteq C(A \cup B) & \text{(Sen, 1971)} \\
(\varepsilon^+): & \text{if } C(A) \subseteq B \subseteq A\text{, then } C(B) \subseteq C(A) & \text{(Bordes, 1983)} \\
(\beta^+): & \text{if } B \subseteq A \text{ and } C(A) \cap B \neq \emptyset\text{, then } C(B) \subseteq C(A) & \text{(Bordes, 1976)}
\end{array}
$$

Contraction consistency conditions specify under which circumstances an alternative chosen from some feasible set is still chosen from a feasible *subset*. Similarly, *expansion consistency* conditions specify under which circumstances a chosen alternative is chosen from a feasible *superset*. While α is a contraction consistency condition, γ, ε^+ and β^+ are expansion consistency conditions.[2] β^+ is typically seen as the strongest (non-trivial) expansion consistency condition. It implies both γ and ε^+, which are logically independent from each other. Choice theory has identified a number of fundamental relationships between consistency and rationalizability for any choice function C.

$$
\begin{array}{rcll}
C \text{ satisfies } \alpha \text{ and } \gamma & \text{iff} & C \text{ is rationalizable.} & \text{(Sen, 1971)} \\
C \text{ satisfies } \alpha, \gamma\text{, and } \varepsilon^+ & \text{iff} & C \text{ is quasi-transitively rationalizable.} & \text{(Schwartz, 1976)} \\
C \text{ satisfies } \alpha \text{ and } \beta^+ & \text{iff} & C \text{ is transitively rationalizable.} & \text{(Bordes, 1976)}
\end{array}
$$

[2] β^+ was first considered by Bordes (1976). It is stronger than Sen's β, but equivalent to β in the presence of α. Bordes (1983) introduced ε^+ as a strengthening of ε by Blair et al. (1976). It is also known as Aïzerman (Moulin, 1986), the weak superset property (Brandt, 2011), and $\hat{\alpha}_{\subseteq}$ (Brandt et al., 2018). ε^+ can be derived as the "expansion part" of Postulate 5* introduced by Chernoff (1954).

3 Local Rationalizability and Local Revealed Preference

In this section, we introduce a weakening of rationalizability, which allows the rationalizing relation to vary depending on the feasible set. Formally, there is a family of rationalizing relations $(R^A)_{A\in\mathcal{U}}$, one for each feasible set, such that $R^B \subseteq R^A$ whenever $B \subseteq A$. We also define, for every choice function C, a corresponding family of revealed preference relations $(R^A_C)_{A\in\mathcal{U}}$ by restricting the witness for a preference in R^A_C to subsets of A.

Definition 1 (Local rationalizability)**.** A choice function C is *locally rationalizable* if there is a family of relations $(R^A)_{A\in\mathcal{U}}$ such that for all feasible sets A,

(i) $R^A \subseteq A \times A$ is acyclic and complete,

(ii) $C(A) = \max_{R^A} A$, and

(iii) $R^B \subseteq R^A$ for all feasible sets $B \subseteq A$.

In this case, we say that C is *locally rationalized by* $(R^A)_A$.

Conditions *(i)* and *(ii)* are analogous to the case of standard rationalizability. Condition *(iii)* implies that a strict preference between two alternatives in some feasible set is preserved when removing other alternatives. Since all relations R^A are complete and hence $x\ R^A\ y$ if and only if not $y\ P^A\ x$, Condition *(iii)* can be reformulated as

(iii') $P^A \cap (B \times B) \subseteq P^B$ for all feasible sets $B \subseteq A$.

The crucial difference to classic rationalizability is that a strict preference between two alternatives can be revoked by introducing new alternatives. We believe that this assumption is quite natural, and it is satisfied by prominent (social) choice functions such as common variants of the top cycle and the uncovered set. Without Condition *(iii)*, all choice functions would satisfy local rationalizability. Standard rationalizability via a global relation R implies local rationalizability by letting $R^A = R|_A$ for each feasible set A.[3]

[3] Remarkably, if we replace the subset in Condition *(iii)* with a superset, then we obtain a dual definition that characterizes α for countable U. However, the results for our definition of local rationalizability can be structured much more elegantly than the corresponding theory for α. For example, there is not always a unique inclusion minimal family of rationalizing relations, and quasi-transitivity does not add anything to acyclicity in this context.

At first glance, it seems difficult to verify whether a choice function is locally rationalizable because there is a very large number of families of potentially rationalizing relations. It turns out that a rather natural candidate for such a family is obtained by extending the concept of revealed preference to families of relations that depend on the feasible set.

Definition 2 (Local Revealed Preference Relations)**.** Let C be a choice function and A a feasible set with $x, y \in A$. We write $x\ R_C^A\ y$ if and only if there is some feasible set $B \subseteq A$ with $x \in C(B)$ and $y \in B$. We call R_C^A the *local revealed preference relation* on A and $(R_C^A)_{A \in \mathcal{U}}$ the *family of local revealed preference relations.*

The difference to classic revealed preference is that we now locally restrict our witness B to be a subset of A, while the classic notion allows for arbitrary witnesses.

It follows from Definition 2 that a *strict* local revealed preference between alternatives x and y in A holds if and only if y is not chosen in any subset that contains x, i.e.,

$$x\ P_C^A\ y \quad \text{iff} \quad y \notin C(B) \text{ for all } B \subseteq A \text{ with } x, y \in B. \tag{1}$$

It is easily seen that the base relation of C is equivalent to the local revealed preference relations of two-element sets, that is, for all $x, y \in U$, $x\ \overline{R}_C\ y$ if and only if $x\ R_C^{\{x,y\}}\ y$. Moreover, all local revealed preference relations are sandwiched in between the base relation and the revealed preference relation, i.e., for all $A \in \mathcal{U}$,

$$\overline{R}_C \cap (A \times A) \quad \subseteq \quad R_C^A \quad \subseteq \quad R_C. \tag{2}$$

The first inclusion holds because every two-element feasible set can serve as a witness, and the second follows from $R_C^U = R_C$.

The following lemma shows that the family of local revealed preference relations already satisfies all but one of the conditions required for local rationalizability.

Lemma 1. *Let C be a choice function. Then, $(R_C^A)_A$ satisfies Condition (i) and (iii) of Definition 1.*[4]

Proof. Completeness of R_C^A follows directly from Equation (2) and the completeness of the base relation.

[4]On top of that, the inclusion from left to right in Condition *(ii)* ($C(A) \subseteq \max_{R^A} A$) is satisfied.

To show acyclicity of R_C^A, let A be a feasible set and $x_1, \dots, x_k \in A$ such that $x_i \ P_C^A \ x_{i+1}$ for all $0 < i < k$. We know that $B \coloneqq \{x_i \colon 1 \leq i \leq k\} \subseteq A$. For each $y \neq x_1$, there is some element $x \in B$ with $x \ P_C^A \ y$. Applying Equation (1) to $B \subseteq A$, it follows that $y \notin C(B)$. By nonemptiness we conclude $\{x_1\} = C(B)$. Hence, B is a witness for $x_1 \ R_C^A \ x_k$.

To show Condition *(iii)* of Definition 1, let A, B be feasible sets such that $B \subseteq A$ and $x, y \in B$ such that $x \ R_C^B \ y$. Now let $D \subseteq B$ be the witness containing y with $x \in C(D)$. Then of course $y \in D \subseteq A$ and still $x \in C(D)$. By definition, this implies $x \ R_C^A \ y$. Hence, $R_C^B \subseteq R_C^A$. □

Example. Let $U = \{x, y, z\}$ and C be defined on all non-singleton subsets as follows.

A	$C(A)$
$\{x, y, z\}$	$\{x, y\}$
$\{x, y\}$	$\{x\}$
$\{y, z\}$	$\{y\}$
$\{x, z\}$	$\{x\}$

For $B \coloneqq \{x, y\}$, we have $x \ P_C^B \ y$. On the other hand, we have $y \ R_C \ x$, since y is chosen from $A \coloneqq \{x, y, z\}$. Hence, we have $y \in \max_{R_C} B$, but $y \notin \max_{R_C^B} B$. Despite $x \ P_C^B \ y$, we still have $y \ I_C^A \ x$ and thus $y \in \max_{R_C^A} A$. In summary, C is not rationalizable but does satisfy local rationalizability.

It is easily seen that when assuming α, all revealed preference relations are restrictions of the same relation to the given feasible set, and our generalized notion of revealed preference coincides with the classic one.

Lemma 2. *Let C be a choice function that satisfies α. Then, $R_C^A = R_C \cap (A \times A)$ for all feasible sets A.*

Proof. By definition we have $R_C^A \subseteq R_C \cap (A \times A)$. Now let $x \ R_C \ y$ with $x, y \in A$. Then there is some witness $D \in \mathcal{U}$ with $x \in C(D)$, $y \in D$. By α, we have $x \in C(\{x, y\})$. Since $\{x, y\} \subseteq A$, this yields $x \ R_C^A \ y$. □

transitive local rationalizability	iff	β^+
PIP-transitive local rationalizability	iff	γ^+
quasi-transitive local rationalizability	iff	γ and ε^+
(acyclic) local rationalizability	iff	γ

Figure 1: Equivalences between local rationalizability and expansion consistency.

4 Results

We are now ready to state our results. The unifying theme of these results is that expansion consistency is deeply intertwined with local rationalizability, even without imposing contraction consistency. An overview of these results is given in Figure 1. As corollaries, we obtain three classic characterizations of rationalizability.

4.1 Local rationalizability

We start with the most central result, showing the equivalence between local rationalizability and γ. This equivalence can be leveraged to construct (social) choice functions that satisfy γ.

Theorem 1. *A choice function satisfies γ if and only if it is locally rationalizable. Moreover, any such function is rationalized by its family of local revealed preference relations.*

Proof. Let C be a choice function. For the direction from right to left, assume that $(R^A)_A$ locally rationalizes C and let $x \in C(A) \cap C(B)$. Now consider an arbitrary $y \in A \cup B$. If $y \in A$, then $x \in C(A)$ implies $x \; R^A \; y$. Otherwise, $y \in B$, and $x \; R^B \; y$ because $x \in C(B)$. In both cases, we can apply condition *(iii)* of Definition 1 to obtain $x \; R^{A\cup B} \; y$. Since y was arbitrary, we have $x \; R^{A\cup B} \; y$ for all $y \in A \cup B$. Hence, $x \in \max_{R^{A\cup B}} A \cup B = C(A \cup B)$.

For the direction from left to right, we will leverage the family of local revealed preference relations $(R^A_C)_A$. Assume that C satisfies γ and let A be a feasible set. By Lemma 1 we only need to show that $C(A) = \max_{R^A_C} A$. For the inclusion from left to right, let $x \in C(A)$. We then have by definition that $x \; R^A_C \; y$ for all $y \in A$

and consequently that $x \in \max_{R_C^A} A$. For the inclusion from right to left, let x be maximal in A. Now, let $y \in A$ be given. By maximality of x and completeness of R_C^A, we know that $x \; R_C^A \; y$. Hence, there is some $B_y \subseteq A$ such that $x \in C(B)$ and $y \in B_y$. Since A is finite and y was arbitrary, we can repeatedly apply γ to obtain $x \in C(\cup_{y \in A} B_y) = C(A)$. □

The family of local revealed preference relations is not the only family of relations locally rationalizing a choice function. This is hardly surprising because locally rationalizing relations for larger feasible sets may contain additional indifferences that are irrelevant for the set of maximal elements. Note that these indifferences are not in conflict with Condition *(iii)* of Definition 1.

It can be shown that the family of local revealed preference relations is the *finest* family of locally rationalizing relations in the following sense: we say that $(R^A)_A$ *is finer than* $(\tilde{R}^A)_A$, if $R^A \subseteq \tilde{R}^A$ for all $A \in \mathcal{U}$. Thus, when restricting attention to locally rationalizing relations that are minimal in this sense, uniqueness of local revealed preference is retained.

We can use the characterization of γ to obtain a classic result of Sen as a corollary. Since the latter involves α, we need the additional observation that rationalizability implies α. To see this, let R be a relation that rationalizes choice function C and $x \in C(A) \cap B$. Then, for all $y \in B$, $x \; R \; y$. Hence, $x \in C(B)$.

Lemma 3. *Every rationalizable choice function satisfies α.*

We now obtain Sen's characterization of rationalizable choice functions as follows. For any rationalizable choice function C, Theorem 1 and Lemma 3 imply that C satisfies α and γ. Conversely, if C satisfies α and γ, Theorem 1 and Lemma 2 imply that R_C rationalizes C.

Corollary 1 (Sen, 1971)**.** *A choice function satisfies α and γ if and only if it is rationalizable. Moreover, any such function is rationalized by its revealed preference relation.*

Whenever a choice function does not satisfy γ, we can 'repair' inconsistencies by adding elements to the choice sets. It turns out that there is a unique minimal way of turning any choice function into one that satisfies γ. To this end, first let $\mathcal{G}$ be nonempty set of choice functions such that $C^*(A) := \bigcap_{C \in G} C(A) \neq \emptyset$ for all $A \in \mathcal{U}$. Then, C^* is a well-defined choice function that satisfies γ if all choice functions contained in $\mathcal{G}$ do. To see this, let $x \in C^*(A) \cap C^*(B)$ for some $A, B \in \mathcal{U}$. Now, consider an arbitrary $C \in \mathcal{G}$. By definition, $x \in C(A) \cap C(B)$. Since C satisfies γ, we have $x \in C(A \cup B)$, and since C was arbitrary, we conclude $x \in C^*(A \cup B)$. Note that C^* is a refinement of all choice functions in $\mathcal{G}$. We are now ready to define the γ-hull of a choice function C, i.e., the unique finest coarsening of C that satisfies γ.

Definition 3 (γ-hull)**.** Let C be a choice function and $\mathcal{G} := \{\widehat{C} \in \mathcal{C} \colon \widehat{C}$ satisfies γ and $C \subseteq \widehat{C}\}$. Then, the γ-*hull* C^γ is a choice function given by

$$C^\gamma(A) := \bigcap_{\widehat{C} \in \mathcal{G}} \widehat{C}(A) \text{ for all } A \in \mathcal{U}.$$

While the existence and uniqueness of such a coarsening are already noteworthy, there is a very simple characterization of the γ-hull: for any choice function C, the γ-hull of C is locally rationalized by the family of local revealed preference relations $(R_C^A)_A$.

Theorem 2. *For any choice function C and feasible set A, $C^\gamma(A) = \max_{R_C^A} A$.*

Proof. For the inclusion from left to right, Lemma 1 and Theorem 1 imply that $\max_{R_C^A}$ is locally rationalized by $(R_C^A)_A$ and hence satisfies γ. Since $C \subseteq C^\gamma$, it suffices to show that $\max_{R_C^A} \subseteq C$. To this end, let $x \in C(A)$ for some feasible set A. Then, $x \; R_C^A \; y$ for all $y \in A$. In other words, $x \in \max_{R_C^A}(A)$.

For the inclusion from right to left, let $x \in \max_{R_C^A}(A)$. Then, x is maximal with respect to R_C^A and for all $y \in A$, there is a feasible set B_y such that $y \in B_y \subseteq A$ and $x \in C(B_y)$. It follows that $x \in C(B_y) \subseteq C^\gamma(B_y)$ since $C \subseteq C^\gamma$. Moreover, since C^γ satisfies γ, $x \in C^\gamma(\bigcup_{y \in A} B_y) = C^\gamma(A)$. □

Applications to Social Choice

In social choice theory, which studies choice functions that are based on the preference relations of multiple agents, choices between *pairs* of alternatives—or, equivalently, the base relation—are typically given by majority rule. The problem is how these choices should be extended to arbitrary feasible sets in some reasonable and consistent way (see, e.g., Laslier, 1997; Brandt et al., 2016). Few social choice functions are known to satisfy γ. The notion of local rationalizability sheds more light on why this is the case. Any such function is induced by a family of locally rationalizing relations satisfying the conditions given in Definition 1. Formally, let $\overline{R}$ be a complete relation on $\mathcal{U}$ and C be a choice function. We say that C is *based* on $\overline{R}$ if $\overline{R}_C = \overline{R}$.

Two-stage majoritarian choice. Let $\overline{R}$ be majority rule for a given preference profile and $\geq \subseteq U \times U$ a fixed strict order over U. Now define the local rationalizing relation for feasible set A by letting

$$x \; R^A \; y \quad \text{iff} \quad x \; \overline{R} \; y \text{ or } (x > y \text{ and there is } z \in A \text{ s.t. } z \; \overline{P} \; y \text{ and } z > y).$$

Clearly, for two feasible sets A and B with $B \subseteq A$, $x \; R^B \; y$ implies $x \; R^A \; y$. For the strict part of the local preference relation, we have $x \; P^A \; y$ if and only if $x \; \overline{P} \; y$ and ($x > y$ or $x \in \max_{\overline{P} \cap >}(A)$). Now assume for contradiction that R^A is not acyclic for some A. Then we have $x_1 \; P^A \; \cdots \; P^A \; x_k \; P^A \; x_1 = x_{k+1}$. For $B \coloneqq \{x_1, \ldots, x_k\}$, we still have $x_1 \; P^B \; \cdots \; P^B \; x_k \; P^B \; x_1$ by Condition *(iii')*. Now consider two cases. First, assume there is some i with $x_i > x_{i+1}$. By renaming the alternatives we can assume without loss of generality that $i = 1$. Then, $x_2 \notin \max_{\overline{P} \cap >}(B)$ and hence $x_2 > x_3$. It follows by induction that $x_2 > x_3 > \cdots > x_k > x_1$, which contradicts the acyclicity of $\geq$. Otherwise, $x_i \in \max_{\overline{P} \cap >}(B)$ for all i with $1 \leq i \leq k$. Strictness of $\geq$ then implies that $x_1 < x_2 < \cdots < x_k < x_1$, which again contradicts the acyclicity of $\geq$. The choice function locally rationalized by $(R^A)_A$ is based on $\overline{R}$. It was recently introduced by Horan and Sprumont (2022) as *two-stage majoritarian choice* and satisfies γ by Theorem 1. When $\overline{R}$ is anti-symmetric, the choice function is single-valued for all feasible sets.

Split cycle. For a fixed profile of voters' preferences, let $\overline{R}$ be majority rule and $\geq \subseteq (U \times U)^2$ the natural ordering of pairs of alternatives according to their majority margin, i.e., the number of agents who prefer the former to the latter minus the number of agents who prefer the latter to the former. Then we can define a locally rationalizing relation for feasible set A by letting

$$\begin{aligned} x \; R^A \; y \quad \text{iff} \quad & x \; \overline{R} \; y \text{ or there are } x_1, \ldots, x_k \in A \text{ s.t. } x = x_1 \; \overline{R} \cdots \overline{R} \; x_k = y \text{ and} \\ & (x_i, x_{i+1}) \geq (y, x) \text{ for all } i. \end{aligned}$$

By contraposition, this means $x \; P^A \; y$ if and only if $x \; \overline{P} \; y$ and for all weak base relation cycles $(x_1, \ldots, x_k)$ in A with $x_1 = y$, $x_k = x$, there is some i with $(x, y) > (x_i, x_{i+1})$.

Clearly, if $x\ P^A\ y$, then $x\ P^B\ y$ for all $B \subseteq A$ and $x, y \in A$. Further, P^A is acyclic. To see this, let $x_1\ P^A \cdots P^A\ x_k$. Assume for contradiction $x_k\ P^A\ x_1$. Then we have a base relation cycle $(x_1, \dots x_k)$ in A. Setting $x_{k+1} = x_1$, one of the pairs (x_i, x_{i+1}) must be minimal with respect to $\geq$, a contradiction to $x_i\ {P^A}_{i+1}$. The choice function locally rationalized by $(R^A)_A$, based on $\overline{R}$, is called *split cycle* and was recently proposed by Holliday and Pacuit (2020). Split cycle is similar to Tideman's ranked pairs (Tideman, 1987) and Schulze's rule (Schulze, 2011), but satisfies some additional desirable properties, most notably γ. Theorem 1 provides an intuitive explanation of this behavior.

Gamma hull. The γ-hull, as defined in Definition 3 and characterized in Theorem 2, allows us to turn any social choice function that violates γ into its finest coarsening that satisfies γ. Perhaps surprisingly, the γ-hulls of many common social choice functions are different from their known coarsenings that satisfy γ and thus yield new social choice functions that have not been studied yet. This is, for example, the case for the γ-hulls of Borda's rule, Copeland's rule, and the essential set, all of which are different from each other and any rule known to us (see, e.g., Brandt et al., 2016; Fischer et al., 2016, for definitions). It follows from the well-known inclusion of Copeland's rule in the top cycle that the γ-hull of Copeland's rule is contained in the top cycle (see Section 4.3). Furthermore, it can be shown that the γ-hull of the essential set is sandwiched in between the Gillies uncovered set and the McKelvey uncovered set (see Section 4.2). On the other hand, the γ-hull of the omninomination rule (which returns all top-ranked alternatives) is the Pareto rule. This example illustrates two observations. First, every social choice function that satisfies α has a rationalizable γ-hull. Secondly, if a social choice function satisfies ε^+, then so does its γ-hull.

Gamma core. There are two natural ways to define refinements of the γ-hull of a choice function C that still satisfy γ and comply with $\overline{R}_C$, but are not coarsenings of C itself. For the first, define a locally rationalizing relation for feasible set A by letting

$$x\ R^A\ y \quad \text{iff} \quad x\ \overline{R}_C\ y \text{ or there are } x_1, \dots, x_k \in A \text{ s.t. } x = x_1\ \overline{R}_C \cdots \overline{R}_C\ x_k = y \text{ and } x \in C(\{x_1, \dots, x_k\}).$$

$(R^A)_A$ satisfies the conditions given in Definition 1 and thus locally rationalizes the choice function $\dot{C}(A) \coloneqq \max_{R^A} A$, which satisfies γ and is based on $\overline{R}_C$. We refer to $\dot{C}$ as the *weak γ-core* of C. The idea is to break every base relation cycle by removing the ingoing strict edges of the alternatives that C selects from the cycle. $\dot{C}$ is a refinement of both C^γ and the top cycle with respect to $\overline{R}_C$ (see Section 4.3).

It is possible to define R^A to be even more selective while still satisfying Condition *(iii)* of Definition 1.

$$x \; R^A \; y \quad \text{iff} \quad x \; \overline{R}_C \; y \text{ or there are } x_1, \dots, x_k \in A \text{ s.t. } x = x_1 \; \overline{P}_C \cdots \overline{P}_C \; x_k = y \text{ and } x \in C(\{x_1, \dots, x_k\}).$$

The resulting choice function $\ddot{C}(A) \coloneqq \max_{R^A} A$ is called the *strict γ-core* of C. We then have $\ddot{C} \subseteq \dot{C} \subseteq C^\gamma$ and three different kinds of C-induced choice functions that satisfy γ. Note that in general $C \not\subseteq \dot{C}, \ddot{C}$.

4.2 Quasi-Transitive Local Rationalizability

As we show in this section, requiring that all locally rationalizing relations are quasi-transitive leads to choice functions that not only satisfy γ but also ε^+.

Theorem 3. *A choice function satisfies γ and ε^+ if and only if it is locally rationalized by a family of quasi-transitive relations.*

Proof. For the direction from left to right, let C be a choice function satisfying γ and ε^+. By Theorem 1, we already know that the local revealed preference relations locally rationalize C. In addition, we now show that all these relations are quasi-transitive. Let A be a feasible set and $x, y, z \in A$ such that $x \; P_C^A \; y$ and $y \; P_C^A \; z$. It needs to be shown that $x \; P_C^A \; z$ or, in other words, that z is not chosen from any subset of A that contains x. For this, consider an arbitrary $B \subseteq A$ with $x, z \in B$. Note that if $y \in B$, we directly obtain $z \notin C(B)$ from Definition 2. Otherwise, set $B_y \coloneqq B \cup \{y\}$. By $x, y \in B_y$, we have $y \notin C(B_y)$, which implies $C(B_y) \subseteq B \subseteq B_y$. It follows from ε^+ that $C(B) \subseteq C(B_y)$. Since $B_y \subseteq A$ and $y, z \in B_y$, we have $z \notin C(B_y)$. Hence we can conclude $z \notin C(B)$. Since B was arbitrary, we obtain $x \; P_C^A \; z$.

For the other direction, let $(R^A)_A$ be a family of quasi-transitive preference relations which locally rationalizes choice function C. We already know that C satisfies γ by Theorem 1. For ε^+, let A, B be feasible sets such that $C(A) \subseteq B \subseteq A$. It needs to be shown that $C(B) \subseteq C(A)$. In other words, it suffices to show that for any $z \in B \setminus C(A)$, it holds that $z \notin C(B)$. Since $z \notin C(A)$, there must be some $x_1 \in A$, such that $x_1 \; P^A \; z$. If $x_1 \in B$, then by local rationalizability $x_1 \; P^B \; z$ and hence $z \notin C(B)$. Otherwise, by $C(A) \subseteq B$, it must be that $x_1 \notin C(A)$. Hence there must be $x_2 \in A$ with $x_2 \; P^A \; x_1$. It follows from the quasi-transitivity of R^A that $x_2 \; P^A \; z$. Using induction, quasi-transitivity of R^A, and finiteness of A, there must eventually be some $x_\ell \in B$ with $x_\ell \; P^A \; z$. Hence, $x_\ell \; P^B \; z$ and thus $z \notin C(B)$. □

It follows from Theorem 1 that every quasi-transitively locally rationalizable choice function is rationalized by its family of local revealed preference relations. Aditionally, this constitutes the finest family of quasi-transitive locally rationalizing relations. Note that other families of rationalizing relations may not be quasi-transitive.

Again, we obtain a classic characterization as a corollary. Let C be a quasi-transitively rationalizable choice function. By Theorem 3 and Lemma 3, C satisfies α, γ and ε^+. For the converse direction, let C satisfy α, γ and ε^+. Then by Theorem 3 and Lemma 2 we have that R_C is quasi-transitive and rationalizes C.

Corollary 2 (Schwartz, 1976)**.** *A choice function satisfies α, γ and ε^+ if and only if it is quasi-transitively rationalizable.*

Applications to Social Choice

As in the previous section, consider the special case where the base relation $\overline{R}$ is given (say, by majority rule). We now define a family of rationalizing relations that induces a choice function based on $\overline{R}$.

$$x \; R^A \; y \quad \text{iff} \quad x \; \overline{R} \; y \text{ or there is some } z \in A \text{ with } x \; \overline{R} \; z \; \overline{P} \; y. \qquad \text{(Covering relation)}$$

This relation is known as the Gillies covering relation (Gillies, 1959; Duggan, 2013) and locally rationalizes a choice function known as the *(Gillies) uncovered set*. For each feasible set, the uncovered set contains the maximal elements with respect to the corresponding covering relation and is thus based on $\overline{R}$. Moreover, we see that for some given x, y, we have $x \; P^A \; y$ ("x *covers* y in A"), if and only if $x \; \overline{P} \; y$ and for all $z \in A$ with $z \; \overline{P} \; x$ we have that $z \; \overline{P} \; y$. Thus, if some x covers some y in some feasible set, then the same holds true for all feasible subsets containing x and y. By definition, all covering relations are quasi-transitive.

We now give a characterization of the uncovered set using quasi-transitive local rationalizability and one additional axiom.[5] We say that a choice function C is *weakly idempotent* if

$$C(C(A)) = C(A) \text{ for all } A \in \mathcal{U} \text{ with } |C(A)| = 2. \qquad \text{(Weak idempotency)}$$

The stronger—unrestricted— version of idempotency ($C(C(A)) = C(A)$ for all $A \in \mathcal{U}$) is not satisfied by the uncovered set (it is, for example, satisfied by the much cruder top cycle; see Section 4.3)

Proposition 1. *Let $\overline{R}$ be a complete relation on $\mathcal{U}$. Among all choice functions based on $\overline{R}$, the uncovered set is the finest one satisfying quasi-transitive local rationalizability and weak idempotency.*

Proof. We first show that the uncovered set satisfies quasi-transitive local rationalizability and weak idempotency. By definition, the family of covering relations $(R^A)_A$ locally rationalizes the uncovered set. To show that R^A is quasi-transitive, let A be a feasible set and $x, y, z \in A$ with $x \; P^A \; y$ and $y \; P^A \; z$. It then needs to be shown that $x \; P^A \; z$. Since y covers z and $x \; \overline{P} \; y$, we have that $x \; \overline{P} \; z$. Further, let w be given with $w \; \overline{P} \; x$. Since x covers y, we also have that $w \; \overline{P} \; y$. Moreover, since y covers z, $w \; \overline{P} \; z$. Hence, $x \; P^A \; z$. Now, assume for contradiction that the uncovered set violates weak idempotency. This implies that $\{x, y\}$ is the uncovered set in some feasible set A, even though $x \; \overline{P} \; y$. Since x cannot cover y in A, there has to be some $z \in A$ such that $y \; \overline{R} \; z \; \overline{P} \; x$. Moreover, since z is not in the uncovered set, it has to be covered by some alternative in A. Quasi-transitivity of the covering relation implies that $x \; P^A \; z$ or $y \; P^A \; z$. However, both cases are at variance with $z \; \overline{P} \; x \; \overline{P} \; y$.

In order to show that the uncovered set is the *finest* choice function satisfying the desired axioms, let C be a choice function based on $\overline{R}$ that satisfies quasi-transitive local rationalizability and weak idempotency. The family of quasi-transitive locally rationalizing relations is given by $(R^A)_A$. It then needs to be shown that the uncovered set is a refinement of C. To this end, let x be in the uncovered set of A. We will show that $x \in C(A)$ by repeatedly applying γ on a collection of 2-element and 3-element sets in which x is selected and whose union is A. First note that for all $y \in A$ with

[5] Other characterizations of the uncovered set using inclusion-minimality, either for strict base relations or McKelvey's variant of the uncovered set, were given by Moulin (1986), Dutta and Laslier (1999), and Peris and Subiza (1999).

$x \,\overline{R}\, y$, $x \in C(\{x,y\})$. The more difficult case is that of $y \in A$ with $y \,\overline{P}\, x$. It follows from the assumption that x is uncovered that there is $z \in A$ such that $z \,\overline{P}\, y$ and $x \,\overline{R}\, z$. Now, let $B \coloneqq \{x,y,z\}$. If $C(B) = \{y\}$, then transitivity of P^B implies $y \, P^B \, z$, which contradicts $z \,\overline{P}\, y$. Similarly, $C(B) = \{z\}$ entails $z \, P^B \, x$, which contradicts $x \,\overline{R}\, z$. Moreover, it follows from weak idempotency that $C(B) \neq \{y,z\}$. As a consequence, $x \in C(B)$. We have thus found a feasible set B_y for every alternative $y \in A$ such that $x, y \in B_y$ and $x \in C(B_y)$. Repeated application of γ then shows that $x \in C(A)$. $\square$

Further variants of the uncovered set. Several variants of covering relations and uncovered sets have been proposed in the literature (see, e.g., Bordes, 1983; Brandt and Fischer, 2008; Duggan, 2013). We follow Duggan's terminology here. The *Bordes uncovered set* is defined by replacing the second condition in the definition of the covering relation with '$x \,\overline{P}\, z \,\overline{R}\, y$'. Similar to the proof of Proposition 1, it can be shown that the Bordes uncovered set is the finest choice function satisfying quasi-transitive local rationalizability and a technical condition requiring that for all $A \in \mathcal{U}$ with $|A| = 3$,

$$\text{if } y \in C(A) \text{ and } x \,\overline{P}_C\, y, \quad \text{then } x \in C(A). \tag{$*$}$$

The *McKelvey uncovered set* is defined by replacing the second condition in the definition of the covering relation with '$x \,\overline{R}\, z \,\overline{P}\, y$ or $x \,\overline{P}\, z \,\overline{R}\, y$'. It follows from a characterization by Peris and Subiza (1999) that the McKelvey uncovered set is the finest choice function satisfying quasi-transitive local rationalizability, weak idempotency, and $(*)$. The McKelvey uncovered set contains both the Gillies and the Bordes uncovered set. Duggan (2013) introduced a coarsening of the McKelvey uncovered set that he calls the *deep uncovered set*. Here, the second condition in the definition of the covering relation is replaced with '$x \,\overline{R}\, z \,\overline{R}\, y$'. Proposition 1 can be adapted to characterize the deep uncovered set: it is the finest choice function satisfying quasi-transitive local rationalizability and a strengthening of $(*)$ which requires that for all $A \in \mathcal{U}$ with $|A| = 3$, $y \in C(A)$ and $x \,\overline{R}_C\, y$ implies $x \in C(A)$.

It turns out that the weak γ-core of the uncovered set (and each of its variants) is the uncovered set itself. Since for any choice function, the weak γ-core is always a subset of the γ-hull, the weak γ-core of the uncovered set is a subset of the uncovered set. Further, for every x that is uncovered in some A, by definition for all y with

$y\ \overline{P}\ x$, there must be a three-cycle on A in which y is present and x is chosen, which proves the claim.

4.3 Transitive Local Rationalizability

It turns out that choice functions that are locally rationalized via transitive relations are characterized by the strongest expansion consistency condition β^+.

Theorem 4. *A choice function satisfies β^+ if and only if it is locally rationalized by a family of transitive relations.*

Proof. For the direction from left to right, let C be a choice function satisfying β^+. Since β^+ implies γ, Theorem 1 applies and C must be locally rationalized by its family of local revealed preference relations. Now, let A be a feasible set. We show that R^A_C is transitive. Let x, y, z be given such that there are feasible sets $B_1, B_2 \subseteq A$ with $x \in C(B_1)$, $y \in C(B_2)$, $y \in B_1$, $z \in B_2$. It needs to be shown that there is some feasible $D \subseteq A$ such that D contains z and x is chosen from D. Let $D \coloneqq B_1 \cup B_2$. Observe that by β^+ and $B_1 \subseteq D$, if $C(D) \cap B_1 \neq \emptyset$, then $x \in C(D)$. If $C(D) \cap B_2 = \emptyset$, this follows from nonemptiness of choice sets. Otherwise, β^+ applied on $B_2 \subseteq D$ and $y \in C(B_2)$ implies $y \in C(D)$. Since $y \in B_1$, this concludes the first half of the proof.

For the direction from right to left, let $(R^A)_A$ locally rationalize C such that all relations are transitive. Now, let A, B be feasible sets such that $B \subseteq A$ and fix $y \in C(A) \cap B \neq \emptyset$. Moreover, let $x \in C(B)$. Then by completeness of R^B and maximality of x in B we obtain $x\ R^B\ y$. By local rationalizability, we also have $x\ R^A\ y$. Maximality of y in A yields $y\ R^A\ z$ for all $z \in A$. Transitivity now implies $x\ R^A\ z$ for all $z \in A$. Hence x is maximal in A and it follows that $x \in C(A)$. $\square$

It follows from Theorem 1 that every transitively locally rationalizable choice function is rationalized by its family of local revealed preference relations. Again, this constitutes the finest family of transitive locally rationalizing relations. Note that other families of rationalizing relations may fail to be transitive. Interestingly, Bordes (1976) already observed that β^+ implies that the revealed preference relation is transitive.

Again, we obtain a classic result as a corollary. Let C be a transitively rationalizable choice function. By Theorem 4 and Lemma 3, C satisfies α and β^+. Conversely, let C satisfy α and β^+. Then by Theorem 4 and Lemma 2 we have that R_C is transitive and C is rationalized by R_C.

Corollary 3 (Arrow, 1959; Bordes, 1976). *A choice function satisfies α and β^+ if and only if it is transitively rationalizable.*

Applications to Social Choice

In order to circumvent Arrow's impossibility, Bordes (1976) has studied a setting where pairwise choices are given and the expansion part of transitive rationalizability (β^+) extends these choices to feasible sets of cardinality greater than two. Bordes postulates that "the essence of the rationality concept is that choices on large sets must depend 'in a positive way' on choices on two-element sets." Eventually, Bordes arrives at a characterization of a social choice function known as the top cycle. The top cycle has been reinvented several times and is known under various names such as Good set (Good, 1971), Smith set (Smith, 1973), weak closure maximality (Sen, 1977), and GETCHA (Schwartz, 1986). We simplify and rewrite Bordes' characterization using transitive local rationalizability.

Let the base relation $\overline{R}$ be given and define the local rationalizing relation for a feasible set A as follows.

$$x \; R^A \; y \quad \text{iff} \quad \text{there are } x_1, \ldots, x_k \in A \text{ s.t. } x = x_1 \; \overline{R} \cdots \overline{R} \; x_k = y. \tag{Transitive closure}$$

This family of relations locally rationalizes the *top cycle*. Since R^A is defined as the transitive closure of $\overline{R}$, the top cycle naturally satisfies β^+ and is based on $\overline{R}$. Moreover, it is the finest choice function satisfying β^+.

Proposition 2. *Let $\overline{R}$ be a complete relation on $\mathcal{U}$. Among all choice functions based on $\overline{R}$, the top cycle is the finest choice function satisfying transitive local rationalizability.*

Proof. Let R^A be the transitive closure of $\overline{R}$ in A as defined above. By definition, each of these relations is transitive. Now let $(\widehat{R}^A)_A$ transitively locally rationalize some choice function C based on $\overline{R}$. Let $x, y \in A$ such that $x \; R^A \; y$. Then, there is a path $x = x_1 \; \overline{R} \cdots \overline{R} \; x_k = y$ in A. Since $\overline{R} \cap (A \times A) \subseteq \widehat{R}^A$, this directly implies $x = x_1 \; \widehat{R}^A \; x_2 \; \widehat{R}^A \ldots \widehat{R}^A \; x_k = y$. By transitivity of $\widehat{R}^A$, we thus have $x \; \widehat{R}^A \; y$. This shows that $R^A \subseteq \widehat{R}^A$ and in particular implies that the top cycle is a refinement of C. $\square$

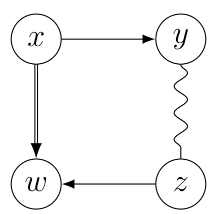
4.4 PIP-Transitive Local Rationalizability

Schwartz (1976) has proposed a transitivity notion called PIP-transitivity that lies in between quasi-transitivity and full transitivity. He argues that, in some cases, this notion represents human behavior more accurately than standard transitivity. Schwartz (1986) states that while transitivity is equivalent to representation by a utility function u, PIP-transitivity is equivalent to representation by a utility function u and a non-negative discriminatory function δ. The idea is that some a is only perceived to be strictly better than some b, if the increase in utility is noticeable, which is modeled by $u(a) > u(b) + \delta(b)$.

Let R be a relation on U. We say that R is *PIP-transitive* if for all (not necessarily distinct) $x, y, z, w \in U$,

$$x \; P \; y, \; y \; I \; z, \text{ and } z \; P \; w \text{ implies } x \; P \; w. \qquad \text{(PIP-transitivity)}$$

Graphically, this condition can be represented as below. The wiggly line represents indifference while the arrows stand for strict preference. The double edge from x to w denotes the consequence of the implication.



We now propose a new expansion consistency condition called γ^+ and show that it is equivalent to local rationalizability by families of PIP-transitive relations.

Definition 4. A choice function C satisfies γ^+ if for all feasible sets A, B,

$$C(A) \subseteq C(A \cup B) \quad \text{or} \quad C(B) \subseteq C(A \cup B). \qquad (\gamma^+)$$

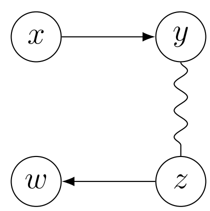
Clearly, γ^+ implies γ since $C(A) \cap C(B)$ is a subset of both $C(A)$ and $C(B)$. Moreover, γ^+ implies ε^+ and is stronger than the conjunction of γ and $W4$, a condition that was introduced by Schwartz (1976) for characterizing PIP-transitivity.[6]

[6] $W4$ is equivalent to a weakening of γ^+, which has the additional restriction that only disjoint A, B are considered. Schwartz (1976) uses α, γ, and $W4$ to characterize PIP-transitive rationalizability. γ^+ is strictly stronger than both γ and $W4$, but equivalent to their conjunction when assuming α.

Local rationalizability via PIP-transitive relations is precisely characterized by γ^+.

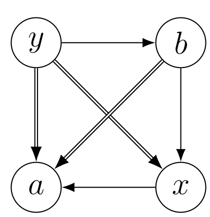
Theorem 5. *A choice function satisfies γ^+ if and only if it is locally rationalized by a family of PIP-transitive relations.*

Proof. Let C be a choice function. For the direction from left to right, let C satisfy γ^+. Since γ^+ implies γ, C is locally rationalized by $(R^A_C)_A$. Let A be a feasible set. We now show that the local revealed preference relation R^A_C is PIP-transitive. Let $x\ P^A_C\ y$, $y\ I^A_C\ z$ and $z\ P^A_C\ w$.

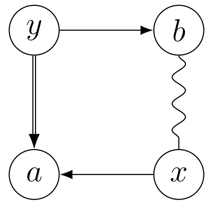


It follows from Equation (1) that w cannot be chosen from subsets of A when z is present, and the same holds for y when x is present. We now need to show that $x\ P^A_C\ w$, i.e., w cannot be chosen from subsets of A in the presence of x. To this end, let $\widehat{A} \subseteq A$ with $x, w \in \widehat{A}$. Further, let $\widehat{B} \subseteq A$ be a witness for $y\ I^A_C\ z$, i.e., $y \in C(\widehat{B})$ and $z \in \widehat{B}$. Now, apply γ^+ on $\widehat{A}$ and $\widehat{B}$. Since $y \in C(\widehat{B})$ and x is present in $\widehat{A}$, it cannot be that $C(\widehat{B}) \subseteq C(\widehat{A} \cup \widehat{B})$. Therefore, it has to be that $C(\widehat{A}) \subseteq C(\widehat{A} \cup \widehat{B})$. Since $w \notin C(\widehat{A} \cup \widehat{B})$ due to the presence of z, we conclude that $w \notin C(\widehat{A})$.

For the other direction, let $(R^A)_A$ locally rationalize C and let all relations be PIP-transitive. Let A, B be feasible sets and assume for contradiction that neither $C(A)$ nor $C(B)$ is a subset of $C(A \cup B)$. Then, there are $a \in C(A) \setminus C(A \cup B)$ and $b \in C(B) \setminus C(A \cup B)$. Hence there is some $x \in A \cup B$ with $x\ P^{A\cup B}\ a$ and some $y \in A \cup B$ with $y\ P^{A\cup B}\ b$. Since $a \in C(A)$, it must be that $x \in B$, and since $b \in C(B)$, we have that $b\ R^B\ x$. By local rationalizability, we even have $b\ R^{A\cup B}\ x$. There are now two possibilities. First, it could be that $b\ P^{A\cup B}\ x$. By applying quasi-transitivity, we then obtain the following relationships for $R^{A\cup B}$.



Otherwise, we have $b\ I^{A\cup B}\ x$, which entails the following relationships for $R^{A\cup B}$ since PIP-transitivity can be applied.



Since $b \in C(B)$, it must be that $y \in A$. Moreover, it follows from $a \in C(A)$ that $a\ R^A\ y$. By local rationalizability, we thus get $a\ R^{A\cup B}\ y$, the desired contradiction to $y\ P^{A\cup B}\ a$ in both cases. $\square$

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