# Funding Public Goods via Best-Response Dynamics

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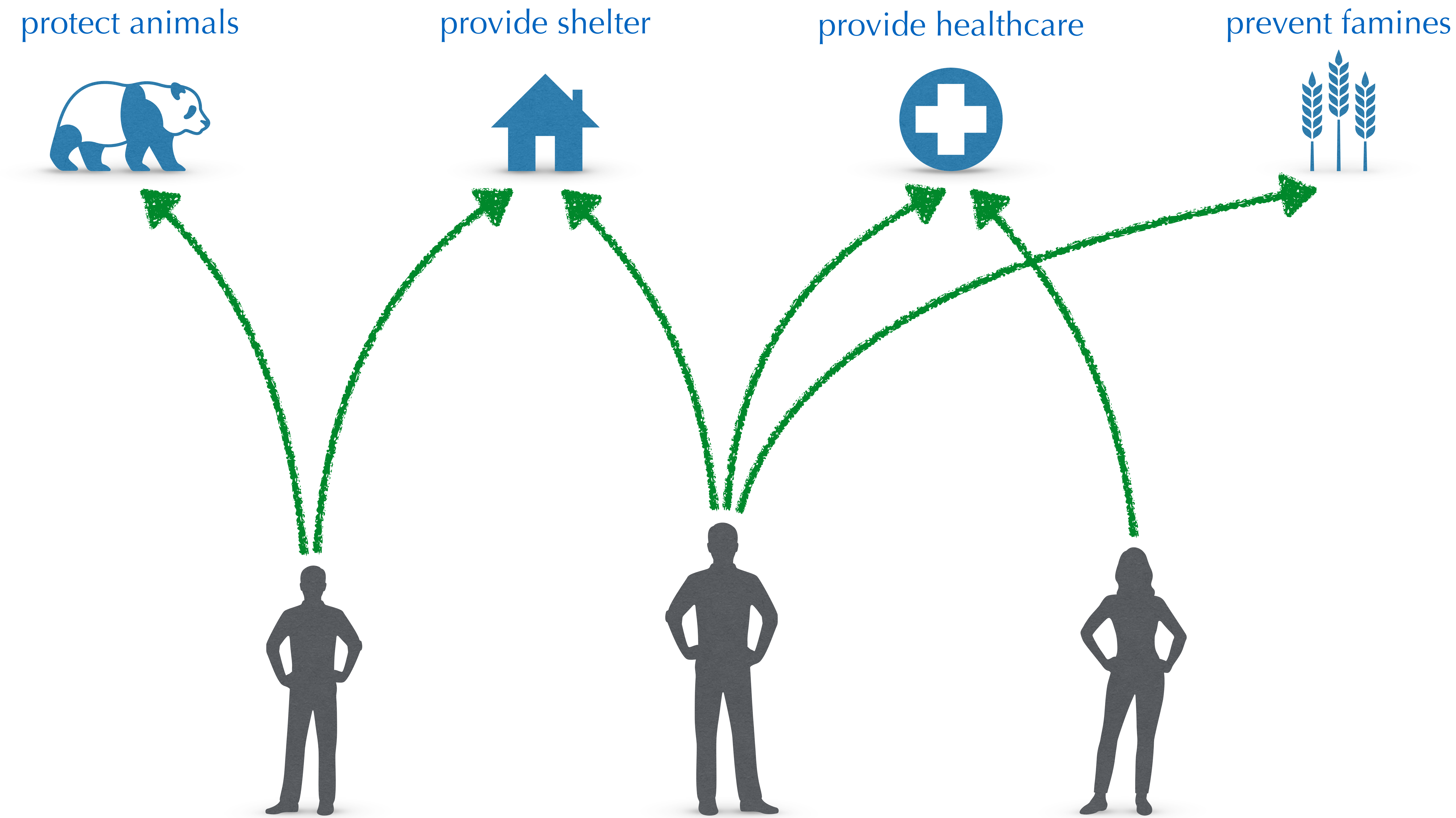
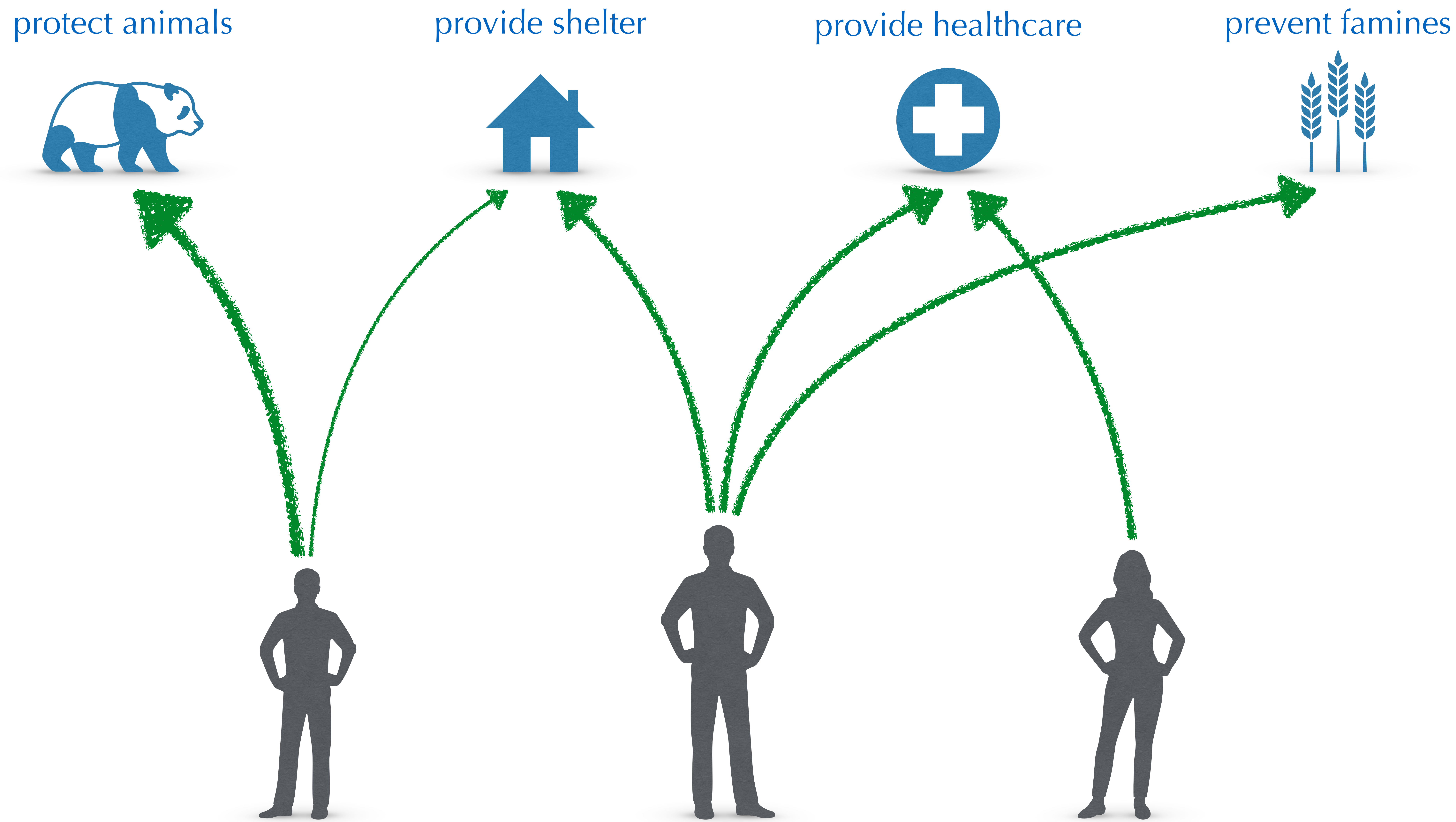
DSS
Decision Sciences & Systems


protect animals
provide shelter
provide healthcare
prevent famines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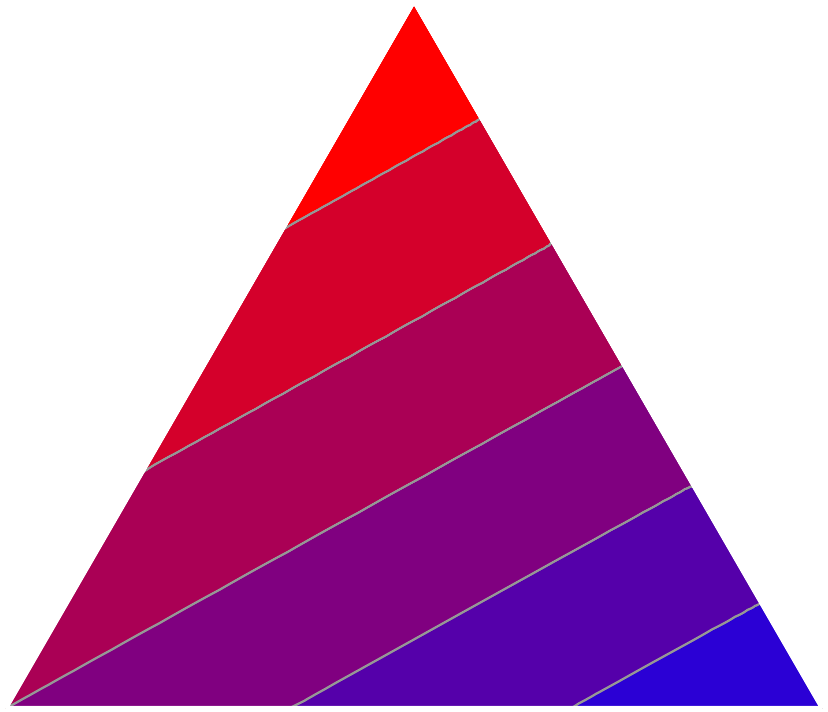
# The Model

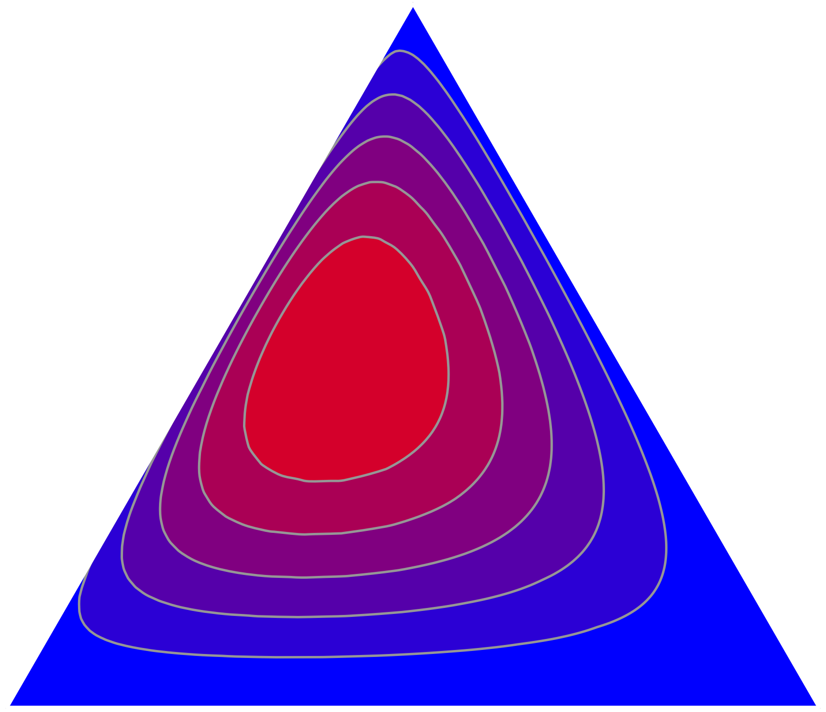
- Each agent $i \in N$ distributes amount $C_i > 0$ of a divisible and homogeneous resource (e.g., money) among a set $A$ of public goods (e.g., charities).
- A distribution $\delta_i \in [0,C_i]^A$ is a function with $\sum_{x\in A} \delta_i(x) = C_i$.
  - The set of all distributions of $C_i$ is denoted by $\Delta(C_i)$.
- $C = \sum_{i\in N} C_i$ is called the endowment.
- $\delta = \sum_{i\in N} \delta_i \in \Delta(C)$ is the collective distribution of the endowment $C$.
- Agent $i$ receives utility $u_i(\delta) \in \mathbb{R}$ from collective distribution $\delta$.

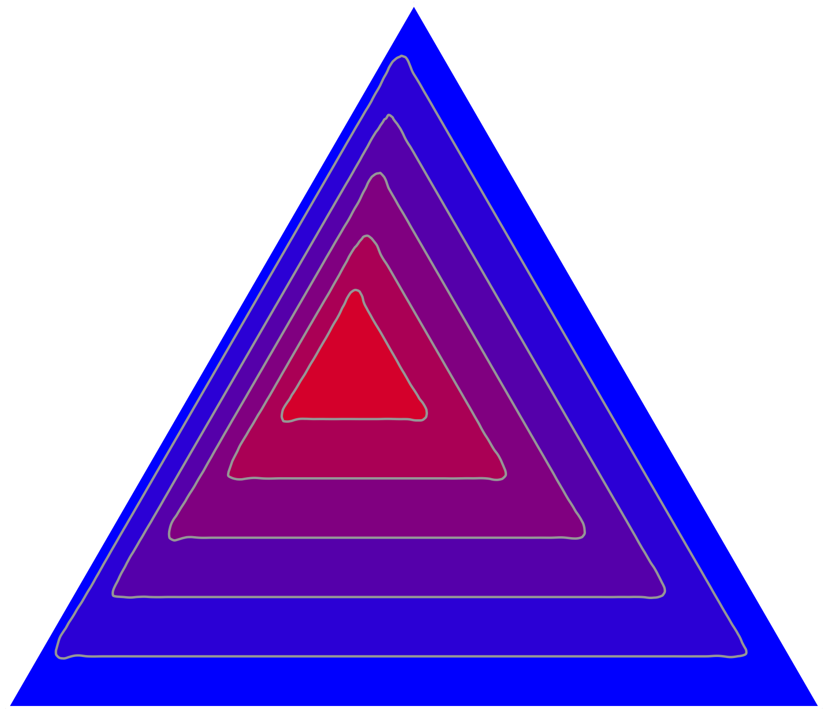
# Related Models

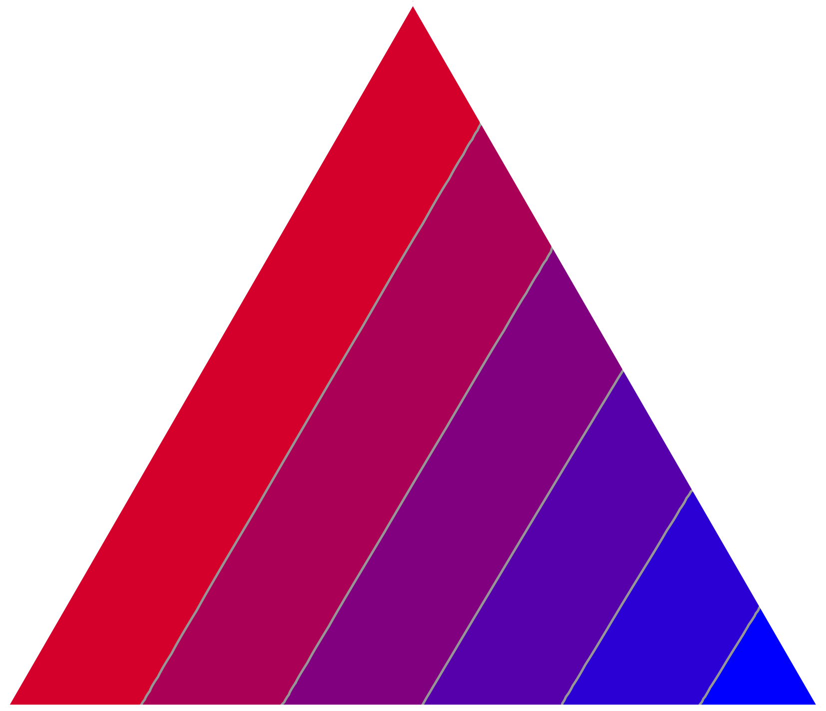
- Private provision of public goods (e.g., Bergstrom, Blume, and Varian, 1986)
  - agents distribute their wealth between a private and a public good
  - no preferences over different public goods
- Probabilistic social choice/ fair mixing (e.g., Gibbard, 1977; Bogomolnaia et al., 2005)
  - ordinal, linear, or dichotomous preferences
  - exogenous fixed "endowment" of probability mass 1
- Participatory budgeting (e.g., Cabannes, 2004)
  - typically fixed costs for projects, which are either fully funded or not at all
  - exogenous endowment
- Budget aggregation (e.g., Freeman et al., 2021)
  - norm-based preferences ($\ell_1$)
  - exogenous endowment

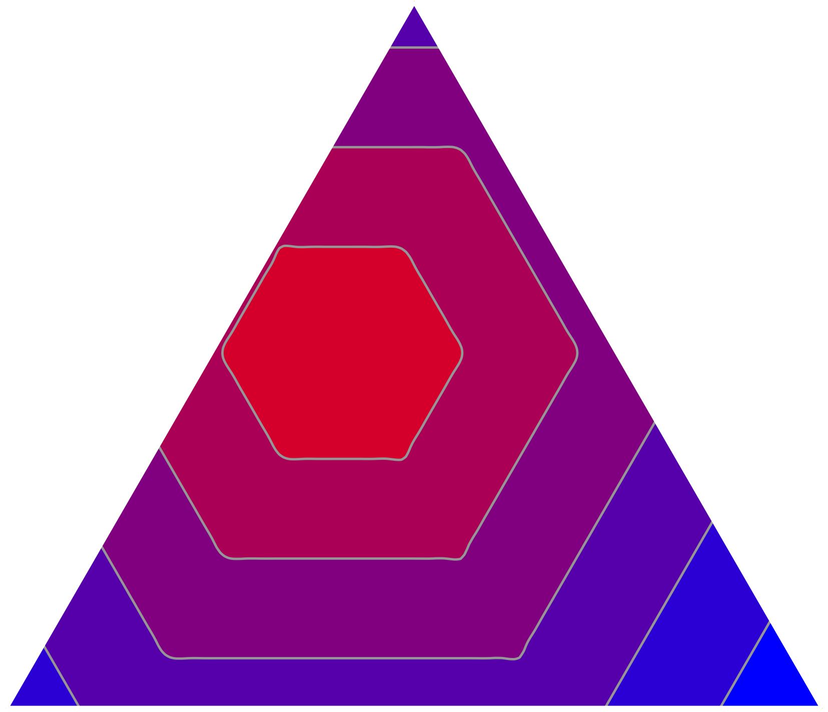
# Potential Utility Functions

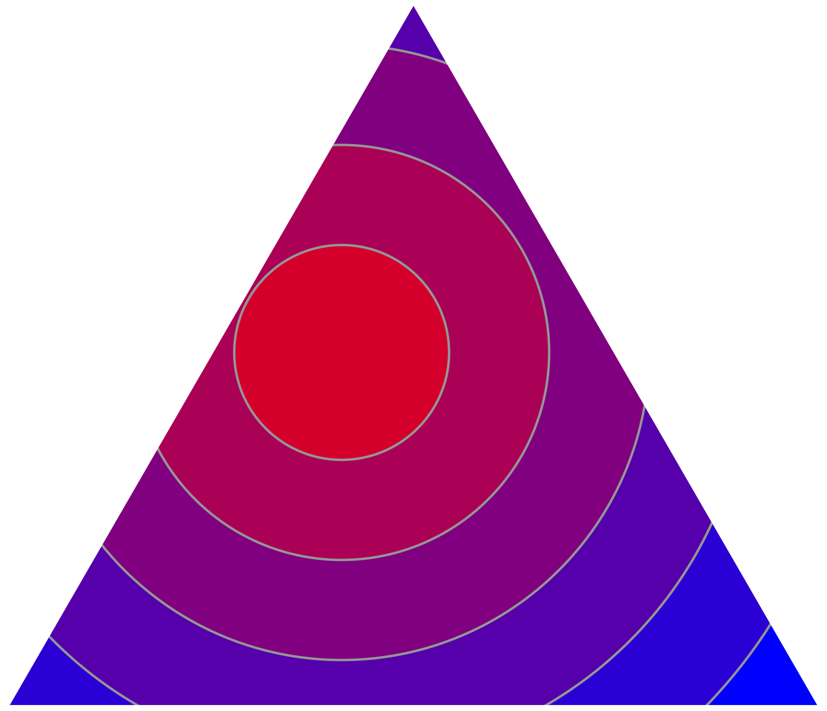

linear


Cobb-Douglas


Leontief


dichotomous


$\ell_1$ disutilities


$\ell_2$ disutilities

Charles Cobb

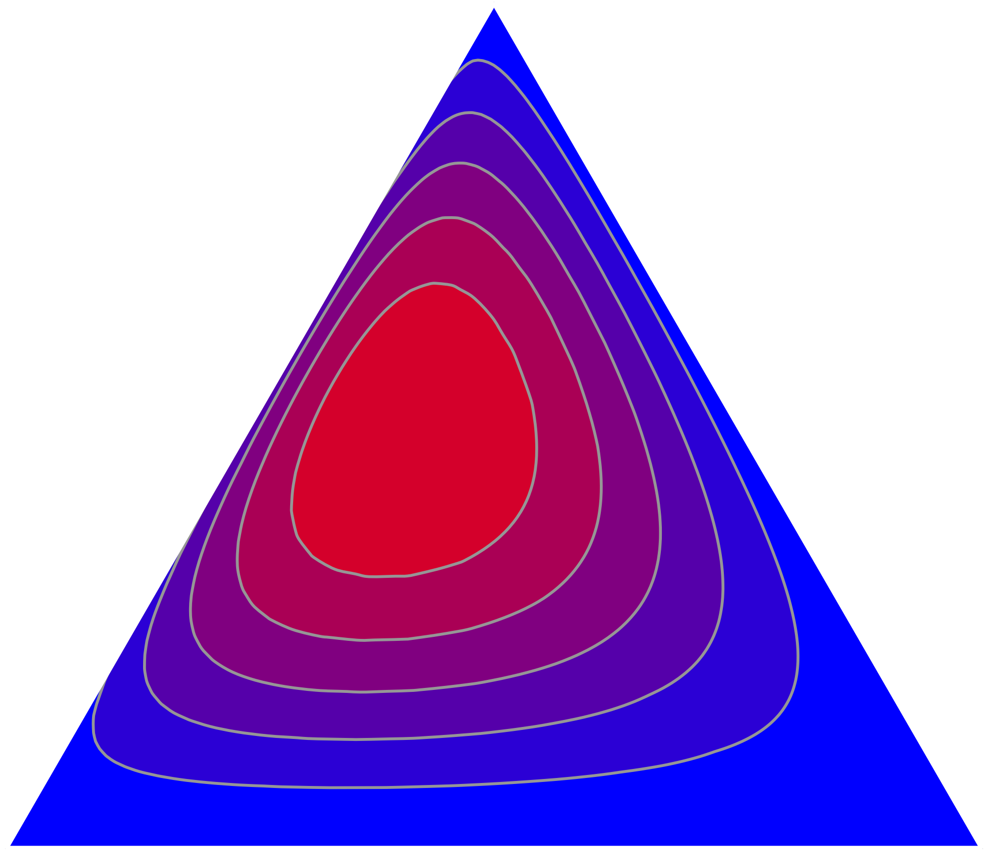
# Cobb-Douglas Utilities

Paul Douglas

- Each agent $i \in N$ assigns a non-negative weight $v_i(x)$ to each charity $x \in A$.

  Agent $i$'s utility for distribution $\delta$ is $u_i(\delta) = \prod_{x \in A} \delta(x)^{v_i(x)}$

- Equivalently, $u_i(\delta) = \sum_{x \in A} v_i(x) \cdot \log \delta(x)$.



- Example: $v_i(a) = 3$, $v_i(b) = 2$, $v_i(c) = 1$
  - Agent $i$'s favorite distribution of contribution 6 is $(3,2,1)$.

# Equilibrium Distributions

John Nash

- Independently distributing one's contribution while disregarding everybody else's distributions may not be in an agent's best interest.

$V_i(x) = 1$ $V_i(x) = 0$

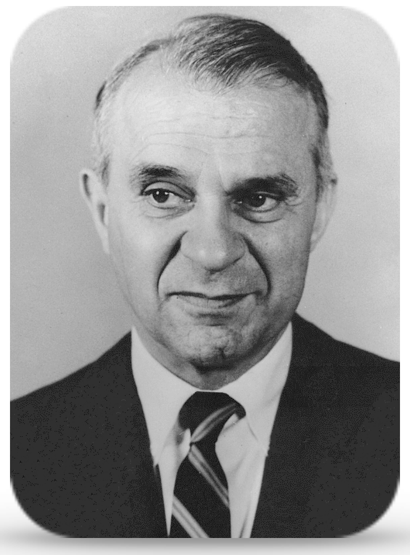
***naive spending***

| | $a$ | $b$ | $c$ | $C_i$ | $u_i$ |
|---|---|---|---|---|---|
| $\delta_1$ | 2 | 2 | | 4 | **8** |
| $\delta_2$ | | 2 | 2 | 4 | **8** |
| $\delta$ | 2 | 4 | 2 | 8 | |

***agent 1 redistributes***

| | $a$ | $b$ | $c$ | $C_i$ | $u_i$ |
|---|---|---|---|---|---|
| $\delta_1$ | 3 | 1 | | 4 | **9** |
| $\delta_2$ | | 2 | 2 | 4 | **6** |
| $\delta$ | 3 | 3 | 2 | 8 | |

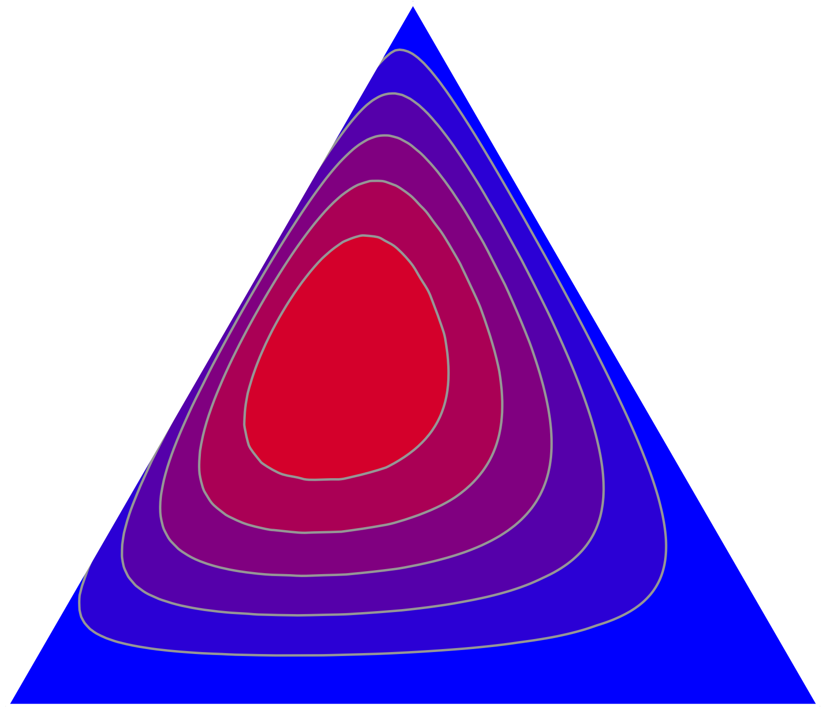
- $(\delta_i)_{i \in N}$ is in equilibrium if $u_i(\delta) \geq u_i(\delta - \delta_i + \delta_i')$ for all $i \in N$ and $\delta_i' \in \Delta(C_i)$.



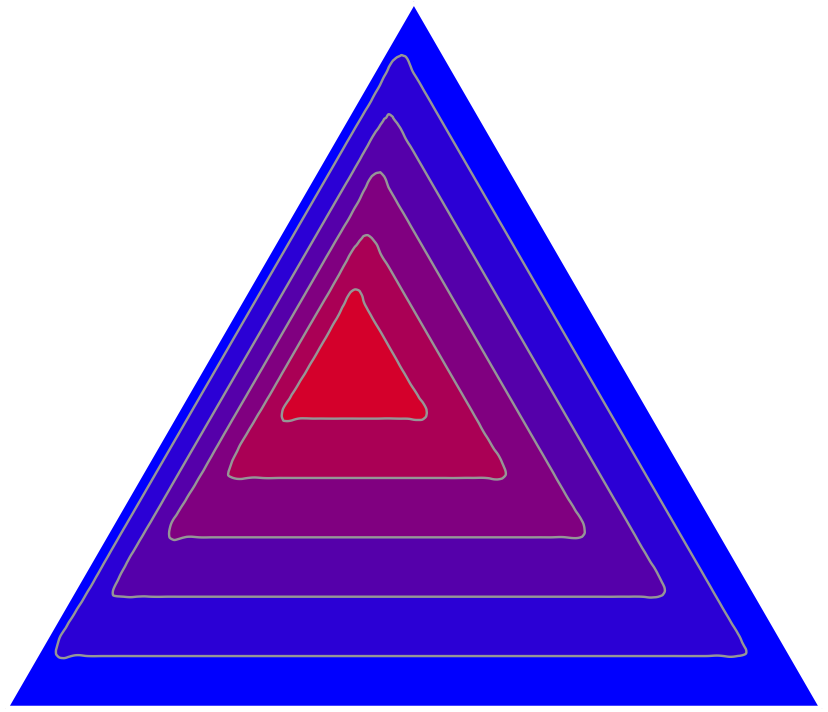
Wassily Leontief

# Existence & Uniqueness

- **Theorem**: Every profile admits a unique equilibrium distribution.
  - $(\delta_i)_{i \in N}$ is in equilibrium iff it maximizes $\prod_{i \in N} \min_{x \in A:\, v_i(x)>0} v_i(x)^{-1} \cdot \delta(x)$.
  - The unique Nash equilibrium maximizes Nash welfare when all Cobb-Douglas utility functions are replaced with Leontief utility functions using the same weights.



Cobb-Douglas



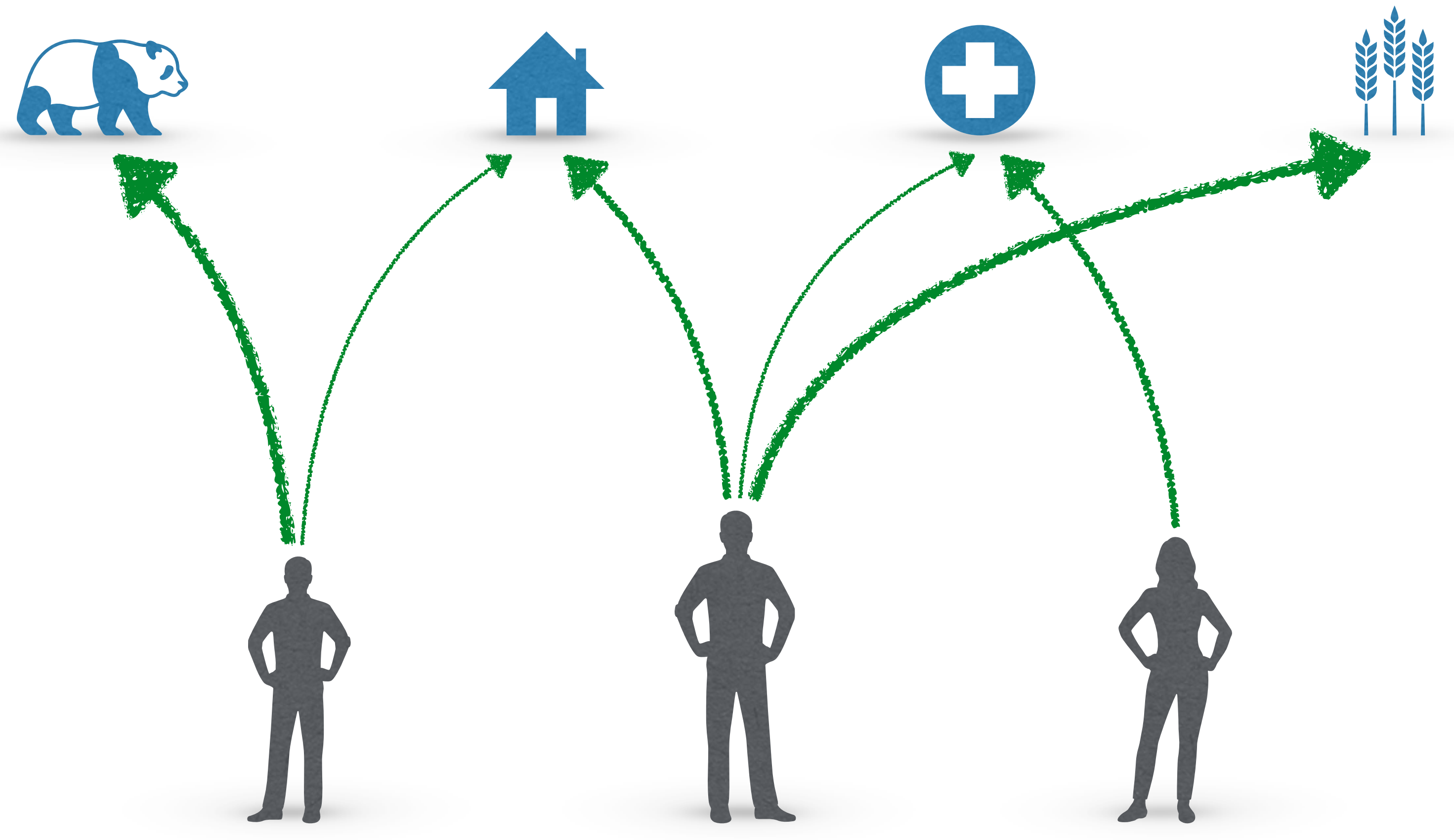
Leontief

# Equilibrium Distributions

- **Theorem**: The equilibrium is rational-valued and can be computed in polynomial time via convex programming and a separation oracle.
- **Theorem**: The equilibrium distribution has nice monotonicity properties:
  - Agent increases contribution ⇒ utility increases, funding of no charity decreases
  - Agent increases weight for charity ⇒ funding of charity does not decrease
  - For linear utilities, both properties are violated by Nash welfare maximizing distribution.
- How do we get to the equilibrium?
  1. Implement mechanism
  2. Spending dynamics

# Spending Dynamics

- Each agent has set aside a, say, *monthly* budget for charitable activities.
- Agents become active in round-robin order.
- Each agent observes the accumulated distribution of the last $n-1$ rounds and then distributes her own contribution myopically optimal.
- **Theorem**: The collective distribution of the last $n$ rounds converges to the equilibrium distribution.
- Even with occasional changes to preferences and contributions, the relative overall distribution keeps converging towards the equilibrium distribution.



| | $a$ | $b$ | $c$ | $d$ | $C_i$ | $u_i$ |
|---|---|---|---|---|---|---|
| $\delta_1$ | 18 | 18 | | | 36 | **18** |
| $\delta_2$ | | 6 | 24 | 24 | 54 | **24** |
| $\delta_3$ | | | 18 | | 18 | **42** |
| $\delta_1$ | 21 | 15 | | | 36 | **21** |
| $\delta_2$ | | 14 | 11 | 29 | 54 | **29** |
| $\delta_3$ | | | 18 | | 18 | **29** |
| $\delta_1$ | 25 | 11 | | | 36 | **25** |
| | ⋮ | ⋮ | ⋮ | ⋮ | | |
| $\delta$ | 27 | 27 | 27 | 27 | 108 | |

# A Prisoners′ Dilemma

- The equilibrium distribution can be inefficient.
  - There is $\delta' \in \Delta(C)$ with $u_i(\delta') > u_i(\delta)$ for all $i \in N$.

***equilibrium***

| | $a$ | $b$ | $c$ | $C_i$ | $u_i$ |
|---|---|---|---|---|---|
| $\delta_1$ | 4 | 2 | | 6 | **16** |
| $\delta_2$ | | 2 | 4 | 6 | **16** |
| $\delta$ | 4 | 4 | 4 | 12 | |

***Pareto improvement***

| | $a$ | $b$ | $c$ | $C_i$ | $u_i$ |
|---|---|---|---|---|---|
| $\delta_1'$ | 3 | 3 | | 6 | **18** |
| $\delta_2'$ | | 3 | 3 | 6 | **18** |
| $\delta'$ | 3 | 6 | 3 | 12 | |

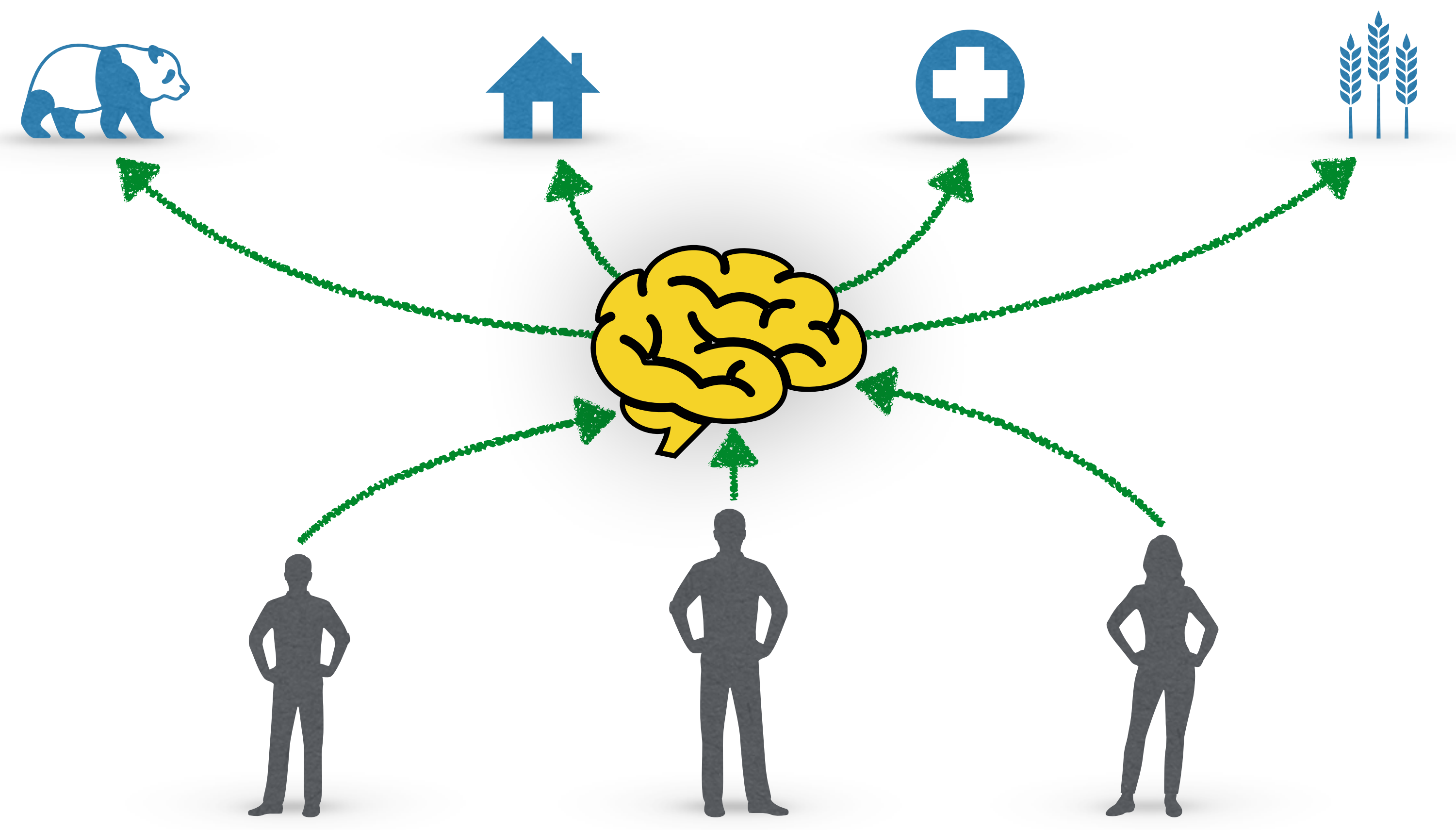
- For Leontief utility functions, equilibrium distributions are always efficient!

| | (0,3,3) | (0,2,4) |
|---|---|---|
| (3,3,0) | (3,6,3) | (3,5,4) |
| (4,2,0) | (4,5,3) | (4,4,4) |

| **Cobb-Douglas** | (0,3,3) | (0,2,4) |
|---|---|---|
| (3,3,0) | **18,18** | 15,20 |
| (4,2,0) | 20,15 | **16,16** |

| **Leontief** | (0,3,3) | (0,2,4) |
|---|---|---|
| (3,3,0) | **3,3** | 3,4 |
| (4,2,0) | 4,3 | **4,4** |

# Conclusion and Outlook



- Unique equilibrium distribution for Cobb-Douglas and Leontief utility functions that will be reached by simple best-response dynamics.
- Further positive results for Leontief utilities:
  - Equilibrium distribution is efficient and in the core.
  - Equilibrium mechanism is the only group-strategyproof mechanism in the core.
- Further results for linear, dichotomous, and $\ell_1$-based utilities.


- B., Greger, Segal-Halevi, and Suksompong. *Optimal budget aggregation with single-peaked preferences*. Presented at ACM-EC 2024
- B., Greger, Segal-Halevi, and Suksompong. *Coordinating charitable donations*. 2024. Presented at ACM-EC 2023
- Brandl, B., Greger, Peters, Stricker, and Suksompong. *Funding public projects*. Journal of Mathematical Economics, 2022.
- Brandl, B., Peters, and Stricker. *Distribution rules under dichotomous preferences: Two out of three ain't bad*. Presented at ACM-EC 2021