Funding Public Goods via Best-Response Dynamics

Felix Brandt
(with Matthias Greger, Erel Segal-Halevi, and Warut Suksompong)

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Funding Public Projects via Best-Response Dynamics

- Protect animals
- Provide shelter
- Provide healthcare
- Prevent famines
protect animals

provide shelter

provide healthcare

prevent famines
The Model

- Each agent $i \in N$ distributes amount $C_i > 0$ of a divisible and homogeneous resource (e.g., money) among a set $A$ of public goods (e.g., charities).
- A distribution $\delta_i \in [0, C_i]^A$ is a function with $\sum_{x \in A} \delta_i(x) = C_i$.
  - The set of all distributions of $C_i$ is denoted by $\Delta(C_i)$.
- $C = \sum_{i \in N} C_i$ is called the endowment.
- $\delta = \sum_{i \in N} \delta_i \in \Delta(C)$ is the collective distribution of the endowment $C$.
- Agent $i$ receives utility $u_i(\delta) \in \mathbb{R}$ from collective distribution $\delta$. 
Related Models

- **Private provision of public goods** (e.g., Bergstrom, Blume, and Varian, 1986)
  - agents distribute their wealth between a private and a public good
  - no preferences over different public goods

- **Probabilistic social choice/ fair mixing** (e.g., Gibbard, 1977; Bogomolnaia et al., 2005)
  - ordinal, linear, or dichotomous preferences
  - exogenous fixed “endowment” of probability mass 1

- **Participatory budgeting** (e.g., Cabannes, 2004)
  - typically fixed costs for projects, which are either fully funded or not at all
  - exogenous endowment

- **Budget aggregation** (e.g., Freeman et al., 2021)
  - norm-based preferences ($\ell_1$)
  - exogenous endowment
Potential Utility Functions

- Linear
- Cobb-Douglas
- Leontief
- Dichotomous
- $\ell_1$ disutilities
- $\ell_2$ disutilities
Each agent $i \in N$ assigns a non-negative weight $v_i(x)$ to each charity $x \in A$.

Agent $i$'s utility for distribution $\delta$ is $u_i(\delta) = \prod_{x \in A} \delta(x)^{v_i(x)}$.

Equivalently, $u_i(\delta) = \sum_{x \in A} v_i(x) \cdot \log \delta(x)$.

Example: $v_i(a) = 3$, $v_i(b) = 2$, $v_i(c) = 1$.

Agent $i$’s favorite distribution of contribution 6 is (3,2,1).
Equilibrium Distributions

- Independently distributing one’s contribution while disregarding everybody else’s distributions may not be in an agent’s best interest.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
& a & b & c & C_i & u_i \\
\hline
\delta_1 & 2 & 2 & 4 & 8 \\
\delta_2 & 2 & 2 & 4 & 8 \\
\delta & 2 & 4 & 2 & 8 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
& a & b & c & C_i & u_i \\
\hline
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\delta_2 & 2 & 2 & 4 & 6 \\
\delta & 3 & 3 & 2 & 8 \\
\hline
\end{array}
\]

- \((\delta_i)_{i \in N}\) is in equilibrium if \(u_i(\delta) \geq u_i(\delta - \delta_i + \delta'_i)\) for all \(i \in N\) and \(\delta'_i \in \Delta(C_i)\).
Theorem: Every profile admits a unique equilibrium distribution.

- \((\delta_i)_{i \in N}\) is in equilibrium iff it maximizes \(\prod_{i \in N} \min_{x \in A: v_i(x) > 0} v_i(x)^{-1} \cdot \delta(x)\).

- The unique Nash equilibrium maximizes Nash welfare when all Cobb-Douglas utility functions are replaced with Leontief utility functions using the same weights.

Existence & Uniqueness

Cobb-Douglas

Leontief
Equilibrium Distributions

- **Theorem**: The equilibrium is rational-valued and can be **computed in polynomial time** via convex programming and a separation oracle.

- **Theorem**: The equilibrium distribution has nice **monotonicity properties**:
  - Agent increases contribution ⇒ utility increases, funding of no charity decreases
  - Agent increases weight for charity ⇒ funding of charity does not decrease
  - For linear utilities, both properties are violated by Nash welfare maximizing distribution.

- How do we get to the equilibrium?
  1. Implement mechanism
  2. Spending dynamics
Spending Dynamics

- Each agent has set aside a, say, monthly budget for charitable activities.
- Agents become active in round-robin order.
- Each agent observes the accumulated distribution of the last $n-1$ rounds and then distributes her own contribution **myopically optimal**.
- **Theorem**: The collective distribution of the last $n$ rounds converges to the equilibrium distribution.
- Even with occasional changes to preferences and contributions, the relative overall distribution keeps converging towards the equilibrium distribution.

<table>
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<tr>
<th>$\delta$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
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A Prisoners’ Dilemma

- The equilibrium distribution can be inefficient.
  - There is $\delta' \in \Delta(C)$ with $u_i(\delta') > u_i(\delta)$ for all $i \in N$.

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- For Leontief utility functions, equilibrium distributions are always efficient!

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Conclusion and Outlook

‣ Unique equilibrium distribution for Cobb-Douglas and Leontief utility functions that will be reached by simple best-response dynamics.

‣ Further positive results for Leontief utilities:
  • Equilibrium distribution is efficient and in the core.
  • Equilibrium mechanism is the only group-strategyproof mechanism in the core.

‣ Further results for linear, dichotomous, and $\ell_1$-based utilities.

- B., Greger, Segal-Halevi, and Suksompong. *Optimal budget aggregation with single-peaked preferences*. Presented at ACM-EC 2024

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