Fractional Hedonic Games

HARIS AZIZ, UNSW Sydney and Data61 CSIRO
FLORIAN BRANDL, Technical University of Munich
FELIX BRANDT, Technical University of Munich
PAUL HARRENSTEIN, University of Oxford
MARTIN OLSEN, Aarhus University
DOMINIK PETERS, University of Oxford

The work we present in this paper initiated the formal study of *fractional hedonic games*, coalition formation games in which the utility of a player is the average value he ascribes to the members of his coalition. Among other settings, this covers situations in which players only distinguish between friends and non-friends and desire to be in a coalition in which the fraction of friends is maximal. Fractional hedonic games thus not only constitute a natural class of succinctly representable coalition formation games, but also provide an interesting framework for network clustering. We propose a number of conditions under which the core of fractional hedonic games is non-empty and provide algorithms for computing a core stable outcome. By contrast, we show that the core may be empty in other cases, and that it is computationally hard in general to decide non-emptiness of the core.

CCS Concepts: • **Theory of computation → Representations of games and their complexity**; *Algorithmic game theory*; *Solution concepts in game theory*; Problems, reductions and completeness; Facility location and clustering; • **Mathematics of computing** → Graph theory.

Additional Key Words and Phrases: cooperative game theory, hedonic games, core, coalition formation

1 INTRODUCTION

Hedonic games present a natural and versatile framework to study the formal aspects of coalition formation which has received much attention from both an economic and an algorithmic perspective. This work was initiated by Drèze and Greenberg [1980], Banerjee et al. [2001], Cechlárová and Romero-Medina [2001], and Bogomolnaia and Jackson [2002] and has sparked a lot of follow-up work. A recent survey was provided by Aziz and Savani [2016]. In hedonic games, coalition formation is approached from a game-theoretic angle. The outcomes are coalition structures—partitions of the players—over which the players have preferences. Moreover, the players have different individual or joint strategies at their disposal to affect the coalition structure to be formed. Various solution concepts—such as the *core*, the *strict core*, and several kinds of *individual stability*—have been proposed to analyze these games.

The characteristic feature of hedonic games is a non-externalities condition, according to which every player's preferences over the coalition structures are fully determined by the player's preferences over coalitions he belongs to, and do not depend on how the remaining players are grouped. Nevertheless, the number of coalitions a player can be a member of is exponential in the total number of players, and the development and analysis of concise representations as well as interesting subclasses of hedonic games are an ongoing concern in computer science and game theory.

Authors' addresses: Haris Aziz, UNSW Sydney and Data61 CSIRO, haris.aziz@unsw.edu.au; Florian Brandl, Technical University of Munich, brandlfl@in.tum.de; Felix Brandt, Technical University of Munich, brandtf@in.tum.de; Paul Harrenstein, University of Oxford, paul.harrenstein@cs.ox.ac.uk; Martin Olsen, Aarhus University, martino@btech.au.dk; Dominik Peters, University of Oxford, dominik.peters@cs.ox.ac.uk.

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Particularly prominent in this respect are representations in which the players are assumed to entertain preferences over the other players, which are then systematically lifted to preferences over coalitions [see, e.g., Alcalde and Revilla 2004; Aziz et al. 2013a; Banerjee et al. 2001; Bogomolnaia and Jackson 2002; Cechlárová and Romero-Medina 2001; Dimitrov et al. 2006].

The work presented in this paper pertains to what we will call *fractional hedonic games*, a subclass of hedonic games in which each player is assumed to have cardinal utilities or *valuations* over the other players. These induce preferences over coalitions by considering the *average valuation* of the members in each coalition. The higher this value, the more preferred the respective coalition is. Previously, the min, max, and sum operators have been used, respectively, for *hedonic games based on worst players* [Cechlárová and Romero-Medina 2001], *hedonic games based on best players* [Cechlárová and Hajduková 2002], and *additively separable hedonic games* [Banerjee et al. 2001]. Despite the natural appeal of taking the average value, fractional hedonic games have enjoyed surprisingly little attention prior to this work.[1] Fractional hedonic games can be represented by a weighted directed graph where the weight of the edge (i, j) denotes the value player i has for player j. However, we will be particularly interested in games that can be represented by *un*directed and *un*weighted graphs. Thus, such games have symmetric valuations that only take the values 0 and 1. With the natural graphical representation of these games, desirable outcomes for fractional hedonic games also provide an interesting angle on network clustering.

Many natural economic problems can be modeled as fractional hedonic games. A particular economic problem that we will consider is what we refer to as *Bakers and Millers*. Suppose there are two types of players, bakers and millers, where individuals of the same type are competitors, trading with players of the other type. Both types of players can freely choose the 'neighborhood' in which to set up their enterprises; in the formal model, each neighborhood forms a coalition. Millers want to be situated in a neighborhood with as many purchasing bakers relative to competing millers as possible, so as to achieve a high price for the wheat they produce. On the other hand, bakers seek a high ratio of the number of millers to the number of bakers, to keep the price of wheat low and that of bread up. We show that these problems (which belong to the class of fractional hedonic games) always admit a core stable partition. This result generalizes to situations in which there are more than two types of players who want to keep the fraction of players of their own type as low as possible. Our study of the Baker and Millers setting is inspired by Schelling's famous dynamic model of segregation [Schelling 1971, 1978].

Another example concerns the formation of political parties. The valuation of two players for each other may be interpreted as the extent to which their opinions overlap, perhaps measured by the inverse of their distance in the political spectrum. In political environments, players need to form coalitions and join parties to acquire influence. On the other hand, as parties become larger, disagreement among their members will increase, making them susceptible to split-offs. Thus, one could assume that players seek to maximize the *average* agreement with the members of their coalition.

In general, finding stable partitions for fractional hedonic games represented by social networks provides an interesting game-theoretic perspective to community detection [see, e.g., Fortunato 2010; Newman 2004] and network clustering.[2]

The contributions of the paper are as follows.

[1] Hajduková [2006] first mentioned the possibility of using the average value in hedonic games but did not further analyze it.

[2] Clauset et al. [2004] discuss how social network analysis can be used to identify clusters of like-minded buyers and sellers in Amazon's purchasing network.

- We introduce and formally define fractional hedonic games and their graphical representation. We identify the subclass of games represented by undirected and unweighted graphs (*simple* and *symmetric* fractional hedonic games) and discuss some of their properties.
- We show that fractional hedonic games may have an empty core, even in the simple symmetric case. We give an example of such a game with 40 players. We leverage this example to show that it is Σ_2^p-complete to decide whether a given simple symmetric fractional hedonic game has non-empty core. Thus, finding a partition in the core is NP-hard. It is also coNP-complete to verify whether a given partition is in the core.
- Based on the graphical representation of fractional hedonic games, we identify a number of classes of graphs which induce games that admit a non-empty core. These include graphs with degree at most 2, forests, complete multipartite graphs, bipartite graphs which admit perfect matchings, and graphs with girth at least 5. For each of these classes, we also present polynomial-time algorithms to compute a core stable partition.
- We formulate the Bakers and Millers setting as a fractional hedonic game based on complete bipartite (or, more generally, complete k-partite) graphs. We show that such games always admit a non-empty strict core, and that the grand coalition is always stable. We characterize the partitions in the strict core, and give a polynomial-time algorithm to compute a unique *finest* partition in the strict core.

2 RELATED WORK

Fractional hedonic games are related to *additively separable* hedonic games [see, e.g., Aziz et al. 2013b; Olsen 2009; Sung and Dimitrov 2010]. In both fractional hedonic games and additively separable hedonic games, each player ascribes a cardinal value to every other player. In additively separable hedonic games, utility in a coalition is derived by adding the values for the other players. By contrast, in fractional hedonic games, utility in a coalition is derived by adding the values for the other players and then dividing the sum by the total number of players in the coalition. Although conceptually additively separable and fractional hedonic games are similar, their formal properties are quite different. As neither of the two models is obviously superior, this shows how slight modeling decisions may affect the formal analysis. For example, in unweighted and undirected graphs, the grand coalition is trivially core stable for additively separable hedonic games. On the other hand, this is not the case for fractional hedonic games.[3] A fractional hedonic game approach to social networks with only non-negative weights may help to detect like-minded and densely connected communities. In comparison, when the network only has non-negative weights for the edges, any reasonable solution for the corresponding additively separable hedonic game returns the grand coalition, which is not informative.

The difference between additively separable and fractional hedonic games is reminiscent of some issues in *population ethics* (see, e.g., Arrhenius et al. 2017), which concerns the evaluation of states of the world with different numbers of individuals alive. Two prominent principles in population ethics are *total utilitarianism* and *average utilitarianism*. The former claims that a state of the world is better than another if it has a higher sum of individual utility, whereas the latter ranks states by the average utility enjoyed by the individuals. Many of the paradoxes of population ethics are analogous to properties of hedonic games. For example, average utilitarianism and fractional hedonic games both suffer from the 'Mere Addition Paradox' [Parfit 1984], according to which a state of the world (resp., a coalition) can become less attractive if we add to it another positive-utility player (but whose utility is lower than the current average). Note, however, that this paradox cannot

[3]Examples of this kind show that there are additively separable hedonic games which cannot be represented as a fractional hedonic game, and *vice versa*.

occur for simple symmetric fractional hedonic games: adding a friend always increases a player's satisfaction.

Olsen [2012] examined a variant of simple symmetric fractional hedonic games and investigated the computation and existence of Nash stable partitions. In the games he considered, however, every maximal matching is core stable and every perfect matching is a best possible outcome, even if large cliques are present in the graph. By contrast, in our setting players have an incentive to form large cliques.

Fractional hedonic games are different from, but related to, another class of hedonic games called *social distance games*, which were introduced by Brânzei and Larson [2011] and further studied by Balliu et al. [2017b,a]. In social distance games, a player's utility from another player's presence in a coalition is inversely proportional to the distance between them in the subgraph induced by the coalition. Similar ideas have been considered in other papers (see, e.g., [Nguyen et al. 2016]). In many situations, one does not derive an additional benefit from friends of friends and may in fact prefer to minimize the fraction of people one does not agree with or have direct connections with. In such scenarios, fractional hedonic games are more suitable than social distance games.

Fractional hedonic games also exhibit some similarity with the segregation and status-seeking models considered by Milchtaich and Winter [2002] and Lazarova and Dimitrov [2013]. Research on such group formation models based on types goes back to at least Schelling [1971].

Independently of our work, Feldman et al. [2012] have also considered the framework of hedonic games as an approach to graph clustering. However, their research does not relate to core and strict core stability, and they study different classes of hedonic games.

Since their inception in the conference version of this paper [Aziz et al. 2014], fractional hedonic games have already sparked some followup work. Aziz et al. [2015] took a welfare maximization approach to fractional hedonic games and considered the complexity of finding partitions that maximize utilitarian or egalitarian social welfare. Bilò et al. [2014] analyze fractional hedonic games from the viewpoint of non-cooperative game theory. They show that Nash stable partitions may not exist in the presence of negative valuations. Furthermore, they give bounds on the price of anarchy and the price of stability. Bilò et al. [2015] and Kaklamanis et al. [2016] further examine the price of (Nash) stability in simple symmetric fractional hedonic games, and Elkind et al. [2016] consider the price of Pareto optimality. Brandl et al. [2015] presented computational results for various stability concepts for fractional hedonic games. Peters and Elkind [2015] identified structural features for various classes of hedonic games for which finding stable partitions is NP-hard. Their analysis implies several hardness results for fractional hedonic games. Sliwinski and Zick [2017] have studied the PAC learnability of hedonic games, including fractional hedonic games.

Liu and Wei [2017] discuss simple symmetric fractional hedonic games (which they call *popularity games*) as a model for the formation of *socially cohesive groups*. They argue that in social networks, groups form based both on individual needs and desires, and on the group's resistance to disruption. Formally, individuals wish to maximize their *popularity* in the group (measured by the fraction of the group that they are connected to in the network), while insisting that the group is core stable. Liu and Wei [2017] identify several classes of networks in which the grand coalition is core stable and give some necessary conditions in terms of structural cohesiveness measures. They also show that it is NP-hard to decide whether the grand coalition is core stable in a given simple symmetric fractional hedonic game, and present and evaluate some heuristics for this question.

Weese et al. [2017] use fractional hedonic games as a model of the formation of jurisdictions, noting that the arrangement of political boundaries involves a tradeoff between efficiencies of scale and of geographic heterogeneity. In examining the core of their weighted symmetric fractional hedonic games, they randomly sampled such games and found that all their samples admit a non-empty core, suggesting that the problem of non-existence of stable outcomes is not a problem in practice. They also introduce a heuristic algorithm for finding a core stable outcome, which proceeds by repeatedly searching for a blocking coalition (using integer programming) and myopically implementing the corresponding coalitional deviation. They then apply this algorithm to specific games modeled using historical data from Japan about political boundary changes, finding that their algorithm always found a core solution and typically terminated within a few hours, even for games containing approximately 1,000 players. They conclude that fractional hedonic games are "an appropriate way of modeling mergers and splits of political jurisdictions" and that they "might also be used to model the formation of students into schools or classes, workers into unions, or public employees into different pension funds".

3 PRELIMINARIES

Let N be a set $\{1, \ldots, n\}$ of *agents* or *players*. A *coalition* is a subset of the players. For every player $i \in N$, we let $\mathcal{N}_i$ denote the set $\{S \subseteq N : i \in S\}$ of coalitions i is a member of. Every player i is equipped with a reflexive, complete, and transitive *preference relation* $\succsim_i$ over the set $\mathcal{N}_i$. We use $\succ_i$ and $\sim_i$ to refer to the strict and indifferent parts of $\succsim_i$, respectively. If $\succsim_i$ is also anti-symmetric we say that i's preferences are *strict*. A *hedonic game* is a pair $(N, \succsim)$, where $\succsim = (\succsim_1, \ldots, \succsim_n)$ is a profile of preference relations $\succsim_i$, modeling the preferences of the players.

A *valuation function* of a player i is a function $v_i : N \to \mathbb{R}$ assigning a real value to every player. A hedonic game $(N, \succsim)$ is said to be a *fractional hedonic game (FHG)* if, for every player i in N, there is a valuation function v_i such that for all coalitions $S, T \in \mathcal{N}_i$,

$$S \succsim_i T \text{ if and only if } v_i(S) \geq v_i(T),$$

where, abusing notation, for all $S \in \mathcal{N}_i$, we write

$$v_i(S) = \frac{\sum_{j \in S} v_i(j)}{|S|}.$$

Hence, every FHG can be compactly represented by a tuple of valuation functions $v = (v_1, \ldots, v_n)$. Throughout the paper, we assume that $v_i(i) = 0$ for all $i \in N$. It can be shown that every general FHG can be induced by valuation functions with $v_i(i) = 0$ for all $i \in N$ by shifting the valuation functions,[4] but this assumption comes with some loss of generality in restricted settings, such as simple FHGs introduced below. We will frequently associate FHGs with weighted digraphs $G = (N, N \times N, v)$ where the weight of the edge (i, j) is $v_i(j)$, that is, the valuation of player i for player j.

Two key restrictions on the valuations in an FHG will be of particular interest to us.

- An FHG is *symmetric* if $v_i(j) = v_j(i)$ for all $i, j \in N$.
- An FHG is *simple* if $v_i(j) \in \{0, 1\}$ for all $i, j \in N$.

Simple FHGs have natural appeal. Politicians may want to be in a party which maximizes the fraction of like-minded members. In general, for whatever reasons, people may want to be in a coalition with as large a fraction of people of their own social group as possible. These situations can be fruitfully modeled and understood as a simple FHG by having the players assign value 1 to like-minded or otherwise similar people, and 0 to others.

A simple FHG $(N, \succsim_i)$ can be represented by a digraph (V, A) in which $V = N$ and $(i, j) \in A$ if and only if $v_i(j) = 1$. Similarly, if $(N, \succsim_i)$ is both symmetric and simple, it can be represented by an (undirected) graph (V, E) such that $V = N$ and $\{i, j\} \in E$ if and only if $v_i(j) = v_j(i) = 1$. With this representation in mind, we will often think of graphs as simple symmetric FHGs.

[4]Let $v'_i(j) = v_i(j) - v_i(i)$ for all $i, j \in N$. Then, $v'_i(i) = 0$, for all $i \in N$ and $S \subseteq N$, $v_i(S) = \sum_{j \in S} \frac{v_i(j)}{|S|} = \sum_{j \in S} \frac{v'_i(j) + v_i(i)}{|S|} = \sum_{j \in S} \frac{v'_i(i)}{|S|} + v_i(i)$. Thus, for all $S, T \subseteq N$, $v_i(S) \geq v_i(T)$ if and only if $v'_i(S) \geq v'_i(T)$.

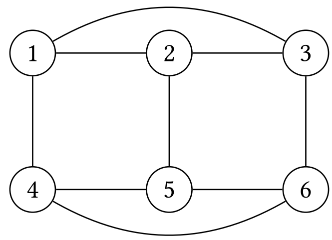


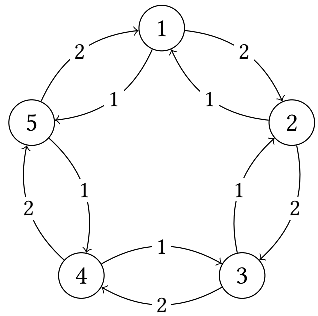
Fig. 1. Example of a simple symmetric FHG. The only core stable partition is $\{\{1, 2, 3\}, \{4, 5, 6\}\}$. In contrast, if the graph represents an additively separable hedonic game, then the partition consisting of the grand coalition is the only core stable partition.

The outcomes of hedonic games are *partitions* of the players, also known as *coalition structures*. Given a partition $\pi = \{S_1, \dots, S_m\}$ of the players, $\pi(i)$ denotes the coalition in π of which player i is a member. We also write $v_i(\pi)$ for $v_i(\pi(i))$, which is the utility that i receives in π, reflecting the hedonic nature of the games we consider. By the same token we obtain preferences over partitions from preferences over coalitions. We refer to $\{N\}$ as the *grand coalition*.

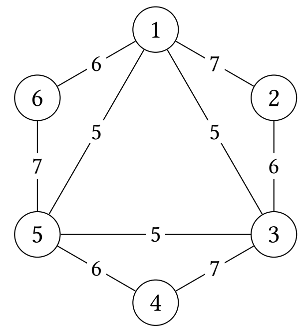
Hedonic games are analyzed using *solution concepts*, which formalize desirable ways in which the players can be partitioned (as based on the players' preferences over the coalitions). If a partition satisfies a given solution concept, it is considered to be *stable* in the sense of the solution concept. A basic requirement for partitions to be acceptable for all players is individual rationality. A partition π is *individually rational* if each player weakly prefers his coalition in π over being alone, that is, for each $i \in N$, $\pi(i) \succsim_i \{i\}$. Intuitively, if a partition is not individually rational, it cannot be stable, since one player has an incentive to leave his current coalition and be on his own instead. In this paper, we will focus on two of the most prominent solution concepts, the *core* and the *strict core*, taken from cooperative game theory. We say that a coalition $S \subseteq N$ *(strongly) blocks* a partition π if each player $i \in S$ strictly prefers S to his current coalition $\pi(i)$ in the partition π, that is, if $S \succ_i \pi(i)$ for all $i \in S$. A partition that does not admit a blocking coalition is said to be in the *core*. In a similar vein, we say that a coalition $S \subseteq N$ *weakly blocks* a partition π if each player $i \in S$ weakly prefers S to $\pi(i)$ and there exists at least one player $j \in S$ who strictly prefers S to his current coalition $\pi(j)$, that is, $S \succsim_i \pi(i)$ for all $i \in S$ and $S \succ_j \pi(j)$ for some $j \in S$. A partition that does not admit a weakly blocking coalition is in the *strict core*. Clearly, the strict core is a subset of the core. Moreover, the core is a subset of the set of individually rational coalitions, since every coalition that is not individually rational is blocked by a singleton coalition.

Example 3.1. Consider the simple symmetric FHG based on the graph depicted in Figure 1. In the grand coalition, the utility of each player is $1/2$. There is only one core stable partition: $\{\{1, 2, 3\}, \{4, 5, 6\}\}$, which yields utility $2/3$ for each player. Observe that, when interpreted as an additively separable hedonic game, this is not a stable partition, as the grand coalition would yield a higher utility—namely, 3 instead of 2—to all and thus be a blocking coalition.

Some standard graph-theoretic terminology will be useful. The complete undirected graph on n vertices is denoted by K_n. A graph (V, E) is said to be k*-partite* if V can be partitioned into k independent sets $V_1, \dots, V_k$, that is, $v, w \in V_i$ implies $\{v, w\} \notin E$. A k-partite graph is *complete* if for all $v \in V_i$ and $w \in V_j$ we have $\{v, w\} \in E$ if and only if $i \neq j$. We write $K_{n,m}$ for the complete bipartite graph where one side contains n vertices and the other side contains m vertices. A graph is *regular* if each vertex has the same degree.



(a) An FHG given by a weighted digraph whose core is empty. All missing edges have weight −10.



(b) A symmetric FHG given by a weighted graph whose core is empty. All missing edges have weight −24.

Fig. 2. Examples of (symmetric) FHGs with an empty core.

4 NEGATIVE RESULTS

For any game-theoretic solution concept, two natural questions are whether a solution is always guaranteed to exist, and whether a solution can be found efficiently. For the core of FHGs, the answer to both of these questions turns out to be negative if we do not restrict the structure of the underlying graph. In fact, for unrestricted FHGs (that is, if we allow any weighted digraph), it is easy to construct examples whose core is empty (see Figure 2a). Even if we require the game to be symmetric, it is not difficult to find examples with an empty core (see Figure 2b). Of course, the examples given are specifically constructed so as to not admit a core stable outcome, and it is plausible that "most" FHGs do admit one. Indeed, Weese et al. [2017] randomly sampled 10 million symmetric FHGs, and all of them had a non-empty core.

If we consider the *strict* core, it is also easy to construct an example of a simple symmetric FHG whose strict core is empty: take the FHG represented by a path with five vertices. If a partition includes a disconnected coalition, then a component blocks. Every connected partition is weakly blocked by a pair: adjacent singletons block; the grand coalition is blocked by any pair; a partition including a coalition S of four players is blocked by the remaining singleton and the adjacent player in S; a partition including a coalition of three players is weakly blocked by an outer player of S and the adjacent player outside S; and a partition of two pairs and a singleton is weakly blocked by the singleton player and an adjacent player in a pair.

It was open for some time whether there is a simple symmetric FHG whose core is empty. Here, we present such an example, consisting of a total of 40 players (see Figure 3). It is unclear whether smaller examples exist. Note that this result subsumes all of the non-existence results mentioned above.

Theorem 4.1. *In simple symmetric FHGs, the core can be empty.*

The proofs of this and other results can be found in the appendix.

Now that we have seen that the core of an FHG can be empty, we can move on to some computational questions. The natural problem to consider is to *find* a core stable partition. Since such a partition does not always exist, we can conveniently consider the *decision* problem of whether the core of a given FHG is non-empty. It turns out that, without imposing further restrictions,

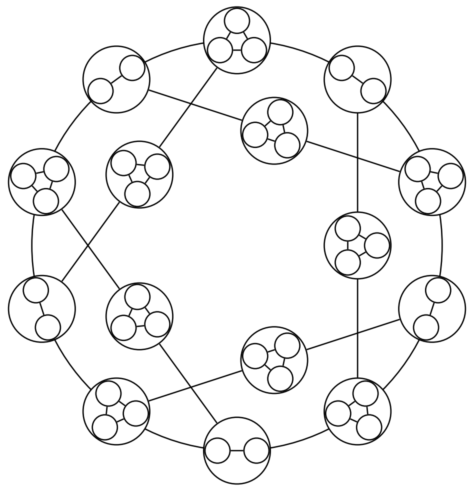


Fig. 3. A simple symmetric FHG with 40 players whose core is empty (see Theorem 4.1). An edge between two encircled cliques indicates that every vertex of one clique is connected to every vertex of the other.

answering this question is computationally difficult, and in particular NP-hard. The examples with an empty core that we have seen make for convenient gadgets in hardness reductions.

Notice that the problem of checking whether an FHG has non-empty core is not obviously contained in the class NP: the natural certificate would be a core stable partition, but it is not at all clear how to check whether a given partition is in the core; naively, this would require checking all $\Theta(2^n)$ candidate blocking coalitions. In fact, this problem of verifying whether a partition is in the core is coNP-complete (we sketch a proof in the appendix, and an alternative proof was given by Liu and Wei 2017). The natural complexity class for the non-emptiness problem is $\Sigma_2^p = \text{NP}^{\text{NP}}$ which captures the alternation of quantifiers: "does there *exist* a partition π such that *for all* coalitions S, the coalition S does not block π?". Indeed, we can show that the non-emptiness problem is complete for this class, even for simple symmetric FHGs.

Theorem 4.2. *Checking whether a simple symmetric FHG has an empty core is Σ_2^p-complete.*

The proof of this statement is a rather involved reduction from the complement of the MINMAX CLIQUE problem [Ko and Lin 1995], and uses a notion of 'subsidies' to players who are put in singleton coalitions. FHGs are not the only class of hedonic games for which checking non-emptiness of the core is Σ_2^p-complete: additively separable and Boolean hedonic games are other examples [Peters 2017; Woeginger 2013]; an open question is whether the existence of the strict core is Σ_2^p-hard for hedonic games based on enemy aversion [Ota et al. 2017; Rey et al. 2016].

As Woeginger [2013] argues, the fact that finding a core stable outcome is Σ_2^p-hard means that this solution concept is computationally much harder to handle than solution concepts like Nash stability, where the analogous decision problem is contained in NP. Indeed, recent advances in SAT and ILP solvers mean that many NP-complete problems of moderate size are now easily solvable in practice; this is not the case for Σ_2^p-complete problems.

However, Weese et al. [2017] present a heuristic algorithm that attempts to find a core stable partition by repeatedly searching for a blocking coalition using an ILP solver and implementing this deviation. They find that this approach is reasonably efficient for real-world examples with up to 1,000 players.

5 POSITIVE RESULTS

In this section, we present a number of subclasses of simple symmetric FHGs for which the core is non-empty. Since these games can be represented by unweighted and undirected graphs, we will focus on different graph classes. In particular, we show existence results for the following classes of graphs: graphs with degree at most 2, forests, complete multipartite graphs, bipartite graphs which admit perfect matchings, regular bipartite graphs, and graphs with girth at least 5. All of our proofs are constructive in the sense that we show that the core is non-empty by outlining a way to construct a partition in the core; in each case this construction can be performed in polynomial time.

For the strict core, our previous example of the path with five vertices shows that the strict core may be empty, even if the game is represented by a graph with maximum degree 2, by a forest, or a graph with girth at least 5. However, our positive results for complete multipartite graphs and bipartite graphs admitting a perfect matching also establish the existence of a partition in the strict core.

5.1 Graphs with bounded degree

If a graph is extremely sparse, then intuitively it does not admit interesting blocking coalitions. Indeed, we have the following result.

Theorem 5.1. *For simple symmetric FHGs represented by graphs of degree at most* 2*, the core is non-empty.*

The proof employs a simple greedy algorithm partitioning the players into coalitions of size at most 3. Such a strategy is successful in this case since the connected components of graphs of degree at most 2 are paths and cycles, and in this situation, players are relatively happy in a small coalition together with an immediate neighbor.

Theorem 4.1 shows that the positive result for the degree bound of 2 cannot be extended to a bound of 11 (which is the maximum degree of the example given there). It might be interesting to close this gap; but the case of degree 3 already seems difficult.

5.2 Forests

The example of an FHG with an empty core that we gave above depended crucially on an underlying cyclic structure of the game. If we do not allow such cycles, the problem disappears:

Theorem 5.2. *For simple symmetric FHGs represented by undirected forests, the core is non-empty.*

The proof proceeds by rooting each component of the forest, and exploits the fact that the preferences of a vertex are somewhat opposed to the preferences of its grandparent. Thus, blocking coalitions would need to be very 'local'. The algorithm groups the vertices in such a way that each coalition consists of one parent and a number of its descendants. This produces a partition in which vertices are locally satisfied and accordingly guarantees its stability.

The conclusion of Theorem 5.2 could be reached in an alternative way. Notice that the coalitions in any partition in the core of a simple symmetric FHG need to be *connected* in the underlying graph: otherwise, a connected component would block. (If there are coalitions consisting of isolated vertices, we can replace such coalitions by singletons.) Thus, an FHG given by a graph G can be viewed as a *hedonic game with graph structure* with communication structure given by G in the sense of Igarashi and Elkind [2016]. They showed that, by a result due to Demange [2004], the core of such games is non-empty if G is a forest. While this method works even beyond the simple symmetric case, it does not yield a polynomial-time algorithm to produce an element of the core.

Since forests are acyclic, their girth is infinite. Thus, Theorem 5.2 is also implied by Theorem 5.6 below, which shows that FHGs represented by graphs of girth at least 5 have a non-empty core. However, the proof of Theorem 5.2 is simpler and leads to a faster (linear time) algorithm for finding a stable partition.

5.3 Bakers and Millers: complete k-partite graphs

In the introduction, we referred to the *Bakers and Millers* setting, in which the players are of two different types and each of them prefers the fraction of players of the other type to be as high as possible. The setting could arise if individuals of the same type are competitors engaging in trade with individuals of the other type.

This idea can easily be extended to multiple types. Let $\Theta = \{\theta_1, \dots, \theta_t\}$ be a set of t types that forms a partition of the set N of players. Let $\theta(i)$ denote the type of player i. A hedonic game $(N, \succsim)$ is called a *Bakers and Millers game* if the preferences of each player i are such that for all coalitions $S, T \in \mathcal{N}_i$,

$$S \succsim_i T \quad \text{if and only if} \quad \frac{|S \cap \theta(i)|}{|S|} \le \frac{|T \cap \theta(i)|}{|T|}.$$

Thus, a player prefers coalitions in which a larger fraction of players are of a different type. With this formalization, we see that a Bakers and Millers game with t types is a simple symmetric FHG represented by a *complete t-partite graph* with the maximal independent sets representing the types, that is, a graph (V, E) with $V = N$ and

$$E = \{\{i, j\} : \theta(i) \neq \theta(j)\}.$$

For an example, see Figure 4, which depicts the complete bipartite graph $K_{4,10}$ with 14 vertices. There, one type is given by the vertices a, b, c, and d (“letters”), and the other type by the vertices numbered 0 through 9 (“numbers”).

In a Bakers and Millers game, the grand coalition is always in the strict core: Since the types partition the player set, observe that for every coalition S we have

$$\frac{|S \cap \theta_1|}{|S|} + \dots + \frac{|S \cap \theta_t|}{|S|} = 1.$$

Now assume for a contradiction that the grand coalition N is not in the strict core. Then there is a (weakly blocking) coalition S such that $\frac{|S\cap\theta(i)|}{|S|} < \frac{|N\cap\theta(i)|}{|N|}$ for some $i \in S$ and $\frac{|S\cap\theta(j)|}{|S|} \le \frac{|N\cap\theta(j)|}{|N|}$ for all $j \in S$. But then

$$\frac{|S \cap \theta_1|}{|S|} + \dots + \frac{|S \cap \theta_t|}{|S|} < \frac{|N \cap \theta_1|}{|N|} + \dots + \frac{|N \cap \theta_t|}{|N|},$$

that is, $1 < 1$, a contradiction. By generalizing this idea, we obtain the following theorem.

Theorem 5.3. *Let $(N, \succsim)$ be a Bakers and Millers game with type space $\Theta = \{\theta_1, \dots, \theta_t\}$ and $\pi = \{S_1, \dots, S_m\}$ a partition. Then, π is in the strict core if and only if for all types $\theta \in \Theta$ and all coalitions $S, S' \in \pi$,*

$$\frac{|S \cap \theta|}{|S|} = \frac{|S' \cap \theta|}{|S'|}.$$

Observe that the condition in Theorem 5.3 for strict core stability is trivially satisfied by the partition consisting of the grand coalition. Since every strict core stable partition is also core stable, the following result follows immediately.

Corollary 5.4. *For every Bakers and Millers game, the core and strict core are non-empty.*

Informally, Theorem 5.3 and Corollary 5.4 mean that, within each coalition in a strict core stable partition, the proportions between the types are exactly as they are in the grand coalition. Thus, every coalition is a reflection of society in this respect. Conversely, every coalition with the same proportions between the types as the grand coalition is part of some strict core-stable partition. For the Bakers and Millers game $K_{4,10}$ (see Figure 4), we find that not only the grand coalition is strict core stable, but also any bi-partition $\{X, Y\}$ in which X and Y each contain two "letters" and five "numbers". More generally, if k is a common divisor of $|\theta_1|, \ldots, |\theta_t|$, then there is a strict core stable partition consisting of k coalitions with $|\theta_i|/k$ members of each type θ_i. Furthermore, observe that merging any two coalitions with the same proportions between types, preserves these proportions.

Theorem 5.3 can now be rephrased as follows. Let d denote the greatest common divisor of $|\theta_1|, \ldots, |\theta_t|$, which we know can be computed in time linear in t [cf. Bradley 1970]. A partition π is in the strict core if and only if, for all coalitions S in π, there is a positive integer k_S such that $|S \cap \theta_i| = k_S|\theta_i|/d$ for all types θ_i. For example, for the grand coalition N we have $k_N = d$. There is also a partition π in the strict core such that $k_S = 1$ for all coalitions S in π; no finer partition is in the strict core.[5]

We say that two partitions π and π' are *identical up to renaming players of the same type* if there is a bijection $f : N \to N$ such that for all players i we have $\theta(i) = \theta(f(i))$ and $\pi' = \{\{f(i) : i \in S\} : S \in \pi\}$. Using this notion, we can state the following corollary.

Corollary 5.5. *For every Bakers and Millers game, there is a* unique *finest partition in the strict core (up to renaming players of the same type), which, moreover, can be computed in linear time.*

As the strict core is a subset of the core, the "if"-direction of Theorem 5.3 also holds for the core: every partition π such that $\frac{|S\cap\theta|}{|S|} = \frac{|S'\cap\theta|}{|S'|}$ for all types $\theta \in \Theta$ and all coalitions $S, S' \in \pi$ is in the core. The converse of this statement, however, does not generally hold: Consider three players 1, 2, and 3, with 1 belonging to type θ_1, while 2 and 3 belong to type θ_2. Then, the coalition structure $\{\{1, 2\}, \{3\}\}$ is in the core but not in the strict core: the coalition $\{1, 3\}$ would be weakly blocking. Rather, the only strict core stable partition consists of the grand coalition, as the greatest common divisor of $|\theta_1|$ and $|\theta_2|$ is 1.

5.4 Graphs with large girth

The *girth* of a graph is the length of the shortest cycle in the graph. For example, bipartite graphs have a girth of at least 4. Graphs with a girth of at least 5 do not admit triangles or cycles of length 4. FHGs described by such graphs always admit a core partition. The key idea behind this result is to pack the vertices of the graph representing a fractional game into stars while maximizing the *leximin* objective function. In the leximin objective, the goal is to maximize the utility of the agent with the least utility; then, subject to this, maximize the utility of the agent with the second least utility, and so on. The resulting partition is in the core.

Theorem 5.6. *For simple symmetric FHGs represented by graphs with girth at least* 5*, the core is non-empty. Moreover, there always exists a partition into stars that is in the core.*

The argument establishing this, while constructive, does not directly yield a polynomial-time algorithm for finding an element of the core, since it is not clear whether a star packing optimizing our leximin objective function can be found in polynomial time. In the appendix, we show that certain local maxima are also core stable, and that a stable outcome of this type can be found in polynomial time by local search. This also results in a partition into stars.

It is worth observing that there may be stable partitions which are not partitions into stars. Consider, for example, a game given by a path with four vertices. In this game, the grand coalition

[5] A partition π is *finer* than partition π', if $\pi \neq \pi'$ and for every $S \in \pi$ there is some $S' \in \pi'$ with $S \subseteq S'$.

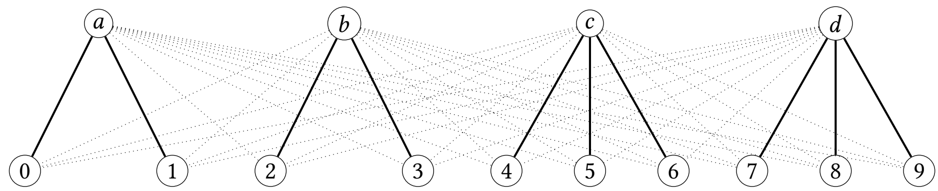


Fig. 4. The complete bipartite graph $K_{4,10}$ in which no star packing yields a stable partition. For instance, the partition indicated by the solid edges is not stable as $\{a, b, 4, 5, 6, 7, 8\}$ would deviate.

is in the core, but it is not a star packing. The leximin-optimal star packing consists of a matching into two pairs.

5.5 Bipartite graphs

For FHGs on bipartite graphs (whose girth is always at least 4), it is not always the case that there are star packings that also yield partitions in the core. For the FHG given by the complete bipartite graph $K_{4,10}$ with 14 vertices (see Figure 4), it can be checked that no partition into stars is core stable. On the other hand, recall that $K_{4,10}$ is a Bakers and Millers game. Thus, by Theorem 5.3, the grand coalition is in the core, and so the core is non-empty. By this example, to find a core element, it is not enough to search for a star packing.

We have not been able to establish whether the core is non-empty for all bipartite graphs, and this remains an interesting open problem. For certain subclasses of bipartite graphs, positive results can still be obtained. For example, perfect matchings, if they exist, are in the (strict) core.

Lemma 5.7. *For simple symmetric FHGs represented by a bipartite graph, any perfect matching is in the strict core.*

We can then obtain the following result as a corollary of Hall's Theorem.

Corollary 5.8. *For simple symmetric FHGs represented by a* regular *bipartite graph, the strict core is non-empty.*

It would be desirable to find additional examples of classes of bipartite graphs for which we can prove that the core is non-empty.

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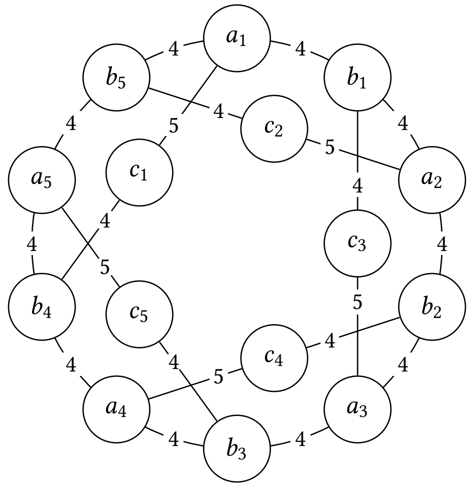

Fig. 5. A social symmetric FHG in which no core stable partition exists. The weight of an edge $\{i, j\}$ denotes $v_i(j)$. All missing edges have weight 0.

A APPENDIX: PROOFS

A.1 Empty core

To prove Theorem 4.1, we give a simple symmetric FHG that does not admit a core stable partition. Since this game is fairly large (40 players), we first illustrate the construction by giving a smaller example (15 players) from a slightly larger class of games. To this end, we say that an FHG is *social* if $v_i(j) \geq 0$ for all $i, j \in N$ [see also Peters and Elkind 2015]. Clearly, every simple FHG is also social. The FHG depicted in Figure 5 is social and symmetric, but has an empty core. We omit the proof of the latter statement, since Theorem 4.1 proves a stronger statement.

The simple symmetric FHG with an empty core depicted in Figure 6 is derived from the game given in Figure 5 by replacing all players a_i and c_i by a clique of three players and all players b_i by a clique of two players. These cliques are denoted by A_i, C_i, and B_i, respectively. Whenever two players x and y are connected in the game in Figure 5, any player from the clique X corresponding to x is connected to any player from the clique Y corresponding to y in the game depicted in Figure 6. Then, the weight of the edge between x and y is equal to the number of players in $X \cup Y$ a player in X (or Y) is connected to.

Theorem 4.1. *In simple symmetric FHGs, the core can be empty.*

Proof. We show that the core of the FHG depicted in Figure 6 is empty. For simplicity, we say that player $i \in N$ is connected to player $j \in N$ if i's valuation for j is 1 (and *vice versa*). It will be useful to keep in mind that, for all $l \in \{1, \ldots, 5\}$, $i \in A_l$ is connected to nine other players, $j \in B_l$ is connected to ten players, and $k \in C_l$ is connected to seven players. Assume for contradiction that π is a partition in the core. The first step is to show that each set A_l and C_l acts as a 'superplayer'; formally, we show that for each $l \in \{1, \ldots, 5\}$, there are $S, T \in \pi$ such that $A_l \subseteq S$ and $C_l \subseteq T$, i.e., players in each of the cliques A_l and C_l are in the same coalition. We show both statements for $l = 1$. The rest follows from the symmetry of the game.

$A_1 \subseteq S$ for some $S \in \pi$: Assume for contradiction that this is not the case. Observe that $A_1 \cup C_1$ is a 6-clique. Hence, every player in $A_1 \cup C_1$ has a valuation of $5/6$ for the coalition $A_1 \cup C_1$. Thus, at

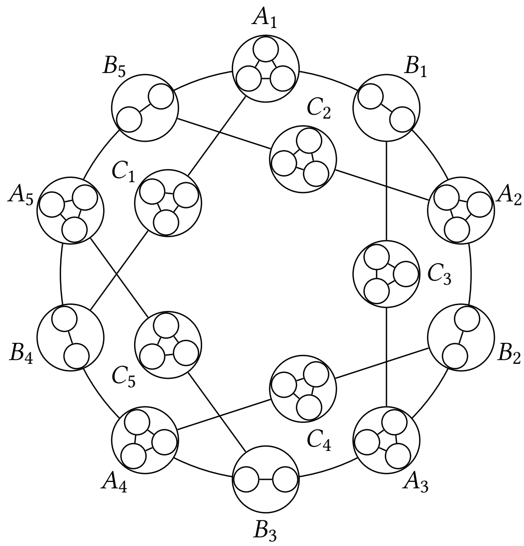

Fig. 6. A simple symmetric FHG with an empty core. For all $l \in \{1, \ldots, 5\}$, A_l and C_l denote cliques of three players and B_l denotes a clique of two players. An edge from one clique to another denotes that every player in the first clique is connected to every player in the second clique. All depicted edges have weight 1. All missing edges have weight 0.

least one player $i \in A_1 \cup C_1$ has a valuation of at least $5/6$ for his coalition in π, as otherwise $A_1 \cup C_1$ would be a blocking coalition.

- First consider the case that $i \in A_1$. We may choose i such that $v_i(\pi) \geq v_j(\pi)$ for all $j \in A_1$. The valuation of player i for any coalition that contains a player he is not connected to is at most $9/11 < 5/6$, since i is connected to nine players in total. Hence $\pi(i)$ can only contain players that i is connected to. But then $A_1 \cup \pi(i) \neq \pi(i)$ is a blocking coalition, since every player in $\pi(i)$ is connected to all players in $A_1 \setminus \pi(i)$ and, for all $j \in A_1$, $v_j(A_1 \cup \pi(i)) = v_i(A_1 \cup \pi(i)) > v_i(\pi) \geq v_j(\pi)$ by the choice of i. This is a contradiction.
- Now consider the case that $i \in C_1$. Again, we may choose i such that $v_i(\pi) \geq v_j(\pi)$ for all $j \in C_1$. If $\pi(i)$ contains a player that i is not connected to, then $v_i(\pi) \leq 7/9 < 5/6$, since i is connected to seven players in total. Hence, $\pi(i)$ can only contain players that i is connected to. With the same reasoning as in the previous case, we get $C_1 \subseteq \pi(i)$, as otherwise $C_1 \cup \pi(i)$ would be a blocking coalition. The valuation of player i for any coalition that contains no player from A_1 is at most $4/5 < 5/6$, since i is connected to four players that are not contained in A_1. Hence, $\pi(i) \cap A_1 = X \neq \emptyset$. By our initial assumption, it cannot be that $X = A_1$. Thus, $\emptyset \neq X \subsetneq A_1$, and so $\pi(i) \cap B_4 \neq \emptyset$ since $v_i(\pi) \geq 5/6$ and i would not otherwise be connected to at least five players in $\pi(i)$.
  At least one player k_1 in $A_1 \cup B_1$ and at least one player k_5 in $A_1 \cup B_5$ has a valuation of at least $4/5$ for his coalition in π, since both sets are 5-cliques and would be blocking coalitions otherwise. We will show that $k_1 \in B_1$ and $k_5 \in B_5$. To this end, observe that $v_j(\pi) \leq 4/6$ for all $j \in X$, since $\pi(i) \cap B_4 \neq \emptyset$. Hence, $k_1, k_5 \notin X$. Any player $j \in A_1 \setminus X$ is connected to at most five players that are not contained in $\pi(i)$ (two from each of B_1 and B_5 and one from $A_1 \setminus X$). Hence, $v_j(\pi) \leq 5/7 < 4/5$ if $\pi(j)$ contains a player that j is not connected to. Thus, if

$v_j(\pi) \geq 4/5$, $\pi(j)$ can only contain players that j is connected to. But then $\pi(j) \cup X$ would be a blocking coalition. This shows that $k_1, k_2 \notin A_1 \setminus X$, i.e., $k_1 \in B_1$ and $k_5 \in B_5$.

Next, we aim to show that $\pi(k_1)$ and $\pi(k_5)$ cannot contain a player that k_1 and k_5 are not connected to, respectively. Observe that k_1 is connected to at most nine players that are not in X. We exclude three cases that leave as the only remaining possibility that $\pi(k_1)$ only contains players that k_1 is connected to.

- If $\pi(k_1)$ contains at least two players that k_1 is not connected to, then $v_{k_1}(\pi) \leq 9/12 < 4/5$. This contradicts the assumption that $v_{k_1}(\pi) \geq 4/5$.
- If $\pi(k_1)$ contains one player that k_1 is not connected to and at least eight players that k_1 is connected to, then $\pi(k_1)$ contains either A_2 or C_3. In case $A_2 \subseteq \pi(k_1)$, it follows that, for all $k \in A_2$, $v_k(\pi) \leq 5/10 \leq 2/3$. Hence, A_2 is a blocking coalition, which is a contradiction. With the same reasoning, a contradiction is obtained if $C_3 \subseteq \pi(k_1)$.
- If $\pi(k_1)$ contains one player that k_1 is not connected to and at most seven players that k_1 is connected to, then $v_{k_1}(\pi) \leq 7/9 < 4/5$, which is a contradiction by the choice of k_1.

Analogously, it follows that $\pi(k_5)$ only contains players that k_5 is connected to.

To complete the proof for the case $i \in C_1$, we now distinguish two cases:

- If $B_4 \subseteq \pi(i)$, it follows that $v_j(\pi) \leq 4/7 < 2/3$ for all $j \in X$. Let $k \in A_1 \setminus X$. If $\pi(k)$ contains a player that k is not connected to, then $v_k(\pi) \leq 3/5 < 2/3$, since $\pi(k)$ cannot contain any player from $\pi(i)$ and at most two players from $B_1 \cup B_5$, since $k_1 \in B_1$, $k_5 \in B_5$, and $\pi(k_1)$ and $\pi(k_5)$ cannot contain a player that k_1 and k_5 are not connected to, respectively. Hence, $X \cup \pi(k)$ is a blocking coalition. If $\pi(k)$ only contains players that k is connected to, then $\pi(k) \cup X$ is a blocking coalition.
- If $|\pi(i) \cap B_4| = 1$, it follows that $|X| = 2$, since $v_i(\pi) \geq 5/6$. As $\pi(i)$ can only contain players that i is connected to, $\pi(i)$ has to consist of C_1, one player from B_4 and two players from A_1. Thus, $v_j(\pi) = 4/6$ for all $j \in X = A_1 \cap \pi(i)$. At least one player $k \in A_5 \cup B_4$ has a valuation of at least $4/5$ for his coalition in π, since the 5-clique $A_5 \cup B_4$ would be a blocking coalition otherwise. Observe that $v_j(\pi) = 3/6 < 4/5$ for $j \in B_4 \cap \pi(i)$. Hence, $k \notin \pi(i)$. Assume for contradiction that $k \in B_4 \setminus \pi(i)$. If $\pi(k)$ contains a player that k is not connected to, then $v_k(\pi) \leq 6/8 < 4/5$, since k is connected to only six players that are not in $\pi(i)$. Hence, $\pi(k)$ can only contain players that k is connected to. If $A_4 \subseteq \pi(k)$, then, for all $j \in A_4 \cup C_4$, $v_j(\pi) \leq 4/5 < 5/6$ and the 6-clique $A_4 \cup C_4$ would be a blocking coalition. Hence, $A_4 \nsubseteq \pi(k)$. Similarly, $A_5 \nsubseteq \pi(k)$.

  We have shown previously that either all players in A_1 are in the same coalition or there is a coalition with all players from C_1, two players from A_1, and one player from B_4. The analogous conclusion applies to A_4, C_4, and B_2 and A_5, C_5, and B_3. Hence, neither $|\pi(k) \cap A_4| = 2$ nor $|\pi(k) \cap A_5| = 2$ is possible. Thus, $|\pi(k) \cap A_4| \leq 1$ and $|\pi(k) \cap A_5| \leq 1$, which implies that $v_k(\pi) \leq 2/3 < 4/5$. This is a contradiction.

  So we get that $k \in A_5$. If $k \in \pi(k_5)$, then $\pi(k) = \pi(k_5)$ can only contain players that k_5 is connected to. Then $v_k(\pi) \geq 4/5$ requires that $\pi(k) = A_5 \cup B_5$. This implies that $v_j(\pi) \leq 4/5$ for all $j \in A_5 \cup C_5$. Thus, the 6-clique $A_5 \cup C_5$ is a blocking coalition, which is a contradiction. Hence, $k_5 \notin \pi(k)$. If then $\pi(k)$ were to contain a player that k is not connected to, then $v_k(\pi) \leq 7/9 < 4/5$, since k is connected to at most seven players that are neither in $\pi(i)$ nor in $\pi(k_5)$. This implies that $\pi(k)$ can only contain players that k is connected to. Hence $A_5 \subseteq \pi(k)$, as otherwise $A_5 \cup \pi(k)$ would be a blocking coalition. Since $v_{k_5}(\pi) \geq 4/5$ and $|X| = 2$, we get that $|\pi(k_5) \cap C_2| \geq 2$. If $\pi(k_5) \cap A_1 \neq \emptyset$, then, since $v_{k_1}(\pi) \geq 4/5$, $\pi(k_1) = B_1 \cup A_2$. This implies that $v_j(\pi) \leq 4/5$ for all $j \in A_2 \cup C_2$. Thus, the 6-clique $A_2 \cup C_2$ is a blocking coalition, which is a contradiction. If $\pi(k_5) \cap A_1 = \emptyset$, then, since $v_{k_5}(\pi) \geq 4/5$, $\pi(k_5) = B_5 \cup C_2$. Hence, $v_j(\pi) = 4/5$ for all $j \in C_2$. Since $\pi(k_1)$ cannot contain a player that k_1 is not connected to, we have that $v_j(\pi) \leq 4/5$ for all $j \in A_2$. Thus, the 6-clique $A_2 \cup C_2$ is a blocking coalition, which is a contradiction.

In both cases, we arrived at a contradiction, which contradicts the assumption that $i \in C_1$.

We have thus established that, for all $l \in \{1, \ldots, 5\}$, there is $S \in \pi$ such that $A_l \subseteq S$.

$C_1 \subseteq T$ for some $T \in \pi$: First we show that, for all $l \in \{1, \ldots, 5\}$, there is $i_l \in C_l$ such that $v_{i_l}(\pi) \geq 4/5$. Assume for contradiction that this is not the case. Without loss of generality, we may assume that $l = 1$. Observe that at least one player $i \in B_4 \cup C_1$ has a valuation of at least $4/5$ for his coalition in π, since otherwise the 5-clique $B_4 \cup C_1$ would be a blocking coalition. By assumption, we then have that $v_i(\pi) \geq 4/5$ for some $i \in B_4$ and $v_j(\pi) < 4/5$ for all $j \in C_1$. In particular, this implies that $\pi(i) \neq B_4 \cup C_1$. Thus, $\pi(i) \cap A_4 \neq \emptyset$ or $\pi(i) \cap A_5 \neq \emptyset$. From the first part of the proof, it then follows that either $A_4 \subseteq \pi(i)$ or $A_5 \subseteq \pi(i)$. If $A_4 \cup A_5 \subseteq \pi(i)$, then $v_j(\pi) \leq 9/13 < 5/6$ for all $j \in A_4$, since $\pi(j) = \pi(i)$ contains at least three players that j is not connected to (those from A_5) and j is connected to nine players in total. Now consider $k \in C_4$. If $\pi(k) = \pi(i)$, then $A_5 \subseteq \pi(k)$ and thus, $\pi(k)$ contains at least three players that k is not connected to. This implies that $v_k(\pi) \leq 7/11 < 5/6$, since k is connected to 7 players in total. If $\pi(k) \neq \pi(i)$, then $\pi(k) \cap A_4 = \emptyset$, which implies that $v_k(\pi) \leq 4/5 < 5/6$, since k is connected to 4 players that are not in A_4. In either case, we have that $v_k(\pi) < 5/6$. Hence, the 6-clique $A_4 \cup C_4$ is a blocking coalition. Thus, $\pi(i)$ cannot contain both A_4 and A_5. On the other hand, $\pi(i)$ has to contain either A_4 or A_5, since $\pi(i) \neq B_4 \cup C_5$. In combination, this implies that $\pi(i)$ contains at most seven players that i is connected to (one from B_4, three from C_1, and three from either A_4 or A_5). The valuation of player i for any coalition that contains a player that i is not connected to is thus at most $7/9 < 4/5$. This implies that $\pi(i)$ cannot contain a player that i is not connected to. Hence, $B_4 \subseteq \pi(i)$ as otherwise $B_4 \cup \pi(i)$ would be a blocking coalition. If $A_4 \subseteq \pi(i)$, we then have that $v_j(\pi) \leq 4/5 < 5/6$ for all $j \in A_4$. The valuation of $k \in C_4$ for any coalition that contains no player from A_4 is at most $4/5 < 5/6$. Hence, the 6-clique $A_4 \cup C_4$ is a blocking coalition, which is a contradiction. Similarly, a contradiction is derived if $A_5 \subseteq \pi(i)$.

Now assume for contradiction that there is $l \in \{1, \ldots, 5\}$ such that, for all $T \in \pi$, $C_l \nsubseteq T$. Without loss of generality, we may assume that $l = 1$. Let $i \in C_1$ such that $v_i(\pi) \geq v_j(\pi)$ for all $j \in C_1$. In particular, $v_i(\pi) \geq 4/5$ as argued above. The valuation of player i for any coalition that contains a player that i is not connected to is at most $7/9 < 4/5$, since i is connected to seven players in total. Hence, $\pi(i)$ cannot contain a player that i is not connected to. But then $C_1 \cup \pi(i) \neq \pi(i)$ is a blocking coalition, which is a contradiction.

We have established that, for all $l \in \{1, \ldots, 5\}$, there are $S, T \in \pi$ such that $A_l \subseteq S$ and $C_l \subseteq T$. In the remainder of the proof we successively exclude all remaining partitions.

- First consider the case that there are $i \in A_l$ and $l \in \{1, \ldots, 5\}$ such that $\pi(i)$ contains a player that i is not connected to. Without loss of generality, assume that $l = 1$. We distinguish two cases.
  - If $\pi(i) = A_1 \cup C_1 \cup Y$ with $\emptyset \neq Y \subseteq B_4$, then $v_i(\pi) \leq 5/7 < 4/5$ for all $i \in A_1$. At least one player $j \in A_1 \cup B_1$ has a valuation of at least $4/5$ for his coalition, as otherwise the 5-clique $A_1 \cup B_1$ would be a blocking coalition. By the preceding statement, we get that $j \in B_1$. Observe that $\pi(j)$ cannot contain a player that j is not connected to, as otherwise $v_j(\pi) \leq 7/9 < 4/5$, since j is connected to seven players other than those in A_1. Thus, $B_1 \subseteq \pi(j)$, as otherwise $B_1 \cup \pi(j) \neq \pi(j)$ would be a blocking coalition. If $\pi(j) = A_2 \cup B_1 \cup C_3$, $v_k(\pi) = 4/8$ for all $k \in C_3$. Then the 3-clique C_3 is a blocking coalition, which is a contradiction. If $\pi(j) = B_1 \cup A_2$, then $v_k(\pi) \leq 4/5$ for all $k \in A_2 \cup C_2$ and the 6-clique $A_2 \cup C_2$ is a blocking coalition. Hence, $\pi(j) = B_1 \cup C_3 \in \pi$. Similarly, $B_5 \cup C_2 \in \pi$. Since $j \in A_2$ is connected to four players that are not in either B_1 or C_2, this implies that $v_j(\pi) \leq 4/5$ for all $j \in A_2$. Since

$v_k(B_5 \cup C_2) = 4/5$ for all $k \in C_2$, the 6-clique $A_2 \cup C_2$ is a blocking coalition, which is a contradiction.
  - If $\pi(i)$ contains a player that i is not connected to and is not of the type excluded in the previous case, then $v_i(\pi) \leq 9/11 < 5/6$, since i is connected to nine players in total. Since $A_1 \subseteq \pi(i)$, we have that $v_j(\pi) < 5/6$ for all $j \in A_1$. Moreover, if $C_1 \subseteq \pi(i)$, then $v_k(\pi) \leq 7/9 < 5/6$ for all $k \in C_1$, since $k \in C_1$ is connected to seven players in total. If $C_1 \not\subseteq \pi(i)$, then $v_k(\pi) \leq 4/5 < 5/6$. In any case, the 6-clique $A_1 \cup C_1$ is a blocking coalition, which is a contradiction.
- Lastly, consider the complementary case that, for all $l \in \{1, \ldots, 5\}$ and $i \in A_l$, $\pi(i)$ only contains players that i is connected to. We have shown previously that, for all $l \in \{1, \ldots, 5\}$, at least one player $j_l \in C_l$ has a valuation of at least $4/5$ for his coalition. Hence, $\pi(j_l)$ cannot contain a player that j_l is not connected to, since $j_l \in C_l$ is connected to seven players in total. Since, for all $l \in \{1, \ldots, 5\}$ and $i \in A_l$, $\pi(i)$ cannot contain a player that i is not connected to, it is not possible that $\pi(j_l) = A_l \cup C_l \cup Y$ for some $\emptyset \neq Y \subseteq B_{l-2}$. Therefore, for all $l \in \{1, \ldots, 5\}$, either $\pi(j_l) = A_l \cup C_l$ or $\pi(j_l) = B_{l-2} \cup C_l$. Hence, for all $l \in \{1, \ldots, 5\}$ and $i \in A_l$, either $\pi(i) = A_l \cup C_l$ or $\pi(i) \subseteq A_l \cup B_{l-1} \cup B_l$.
  - If $A_l \cup C_l \in \pi$ for all $l \in \{1, \ldots, 5\}$, then $v_j(\pi) \leq 1/2$ for all $l \in \{1, \ldots, 5\}$ and $j \in B_l$ (which is obtained by the pair B_l). On the other hand, we have that $v_i(A_1 \cup B_1 \cup B_5) = 6/7$ for all $i \in A_1$ and $v_j(A_1 \cup B_1 \cup B_5) = 4/7 \geq 1/2$ for all $j \in B_1 \cup B_5$. Hence, $A_1 \cup B_1 \cup B_5$ is a blocking coalition, which is a contradiction.
  - If the previous case does not apply, we may assume without loss of generality that $A_1 \cup Z \in \pi$ for some $Z \subseteq B_1 \cup B_5$. If $|Z| < 3$, then $v_i(\pi) \leq 4/5 < 5/6$ for all $i \in A_1$. Since $j \in C_1$ is connected to four players that are not in A_1, we also have that $v_j(\pi) \leq 4/5 < 5/6$ for all $j \in C_1$. Hence, the 6-clique $A_1 \cup C_1$ is a blocking coalition. This implies that $|Z| \geq 3$ and we may assume without loss of generality that $B_5 \subseteq Z$. It follows that $B_4 \cup C_1 \in \pi$, since one player in C_1 has a valuation of at least $4/5$ for his coalition. This implies that $A_5 \cup C_5 \in \pi$. Also, $A_4 \cup C_4 \in \pi$, as otherwise $v_i(\pi) \leq 4/5$ for all $i \in A_4 \cup C_4$ and the 6-clique $A_4 \cup C_4$ is a blocking coalition. Furthermore $A_2 \cup C_2, A_3 \cup C_3 \in \pi$, as otherwise $B_5 \cup C_2$ and $B_1 \cup C_3$ are blocking coalitions, respectively. Hence, $v_i(\pi) = 5/6$ for all $i \in A_3$ and $v_j(\pi) \leq 1/2$ for all $j \in B_2 \cup B_3$. So $A_3 \cup B_2 \cup B_3$ is a blocking coalition, since $v_i(A_3 \cup B_2 \cup B_3) = 6/7 > 5/6$ for all $i \in A_3$ and $v_j(A_3 \cup B_2 \cup B_3) = 4/7 > 1/2$ for all $j \in B_2 \cup B_3$. Again, this is a contradiction.

In all cases, we derived a contradiction. Since the choice of π in the core was arbitrary, this shows that the core is empty. □

We remark that the game above is fragile in the sense that a game with non-empty core may be obtained by deleting a single specific player. This will be useful for the hardness proofs below.

Proposition A.1. *In the game shown in Figure 6, delete one player from B_2 from the game, so that now $|B_2| = 1$. Then the partition $\pi = \{A_1 \cup B_1 \cup B_5, A_2 \cup C_2, A_3 \cup C_3, A_4 \cup C_4, A_5 \cup C_5, B_4 \cup C_1, B_2, B_3\}$ is in the core of the resulting game.*

Proof. Suppose there was a coalition S blocking π. Note that we have $v_i(\pi) \geq 5/6$ for all $i \in A_1 \cup \cdots \cup A_5$, and $v_i(\pi) \geq 4/5$ all $i \in C_1 \cup \cdots \cup C_5$. Thus, because $9/11 < 5/6$ and $7/9 < 4/5$, such players i can only be in S if S contains only players that i is connected to. For players in A_2, the only potential such blocking coalition would be $B_1 \cup A_2 \cup B_2$ which is not better than $A_2 \cup C_2$, so $A_2 \cap S = \emptyset$. Similarly $A_3 \cap S = \emptyset$. The players in B_4 are only interested in blocking if S contains players from at least two of the sets A_4, A_5, C_1, which would contradict the property that players from the latter sets only block in coalitions containing only adjacent players. Hence $B_4 \cap S = \emptyset$. Thus also $A_4 \cap S = \emptyset$ and $A_5 \cap S = \emptyset$.

Given that S does not contain any players from $A_2 \cup \cdots \cup A_5$, it follows that also $C_l \cap S = \emptyset$ for $l \in \{2, \ldots, 5\}$. Because none of their neighbors are willing to block we further get that $B_l \cap S = \emptyset$ for $l \in \{2, 3, 4\}$. Together, it follows that $S \subseteq A_1 \cup C_1 \cup B_1 \cup B_5$.

If S contains a player from C_1, then $S \subseteq A_1 \cup C_1$, and S cannot be blocking, because for $i \in S \cap A_1$, we have $v_i(S) \leq 5/6 < 6/7 = v_i(\pi)$. So $C_1 \cap S = \emptyset$. Hence $S \subseteq A_1 \cup B_1 \cup B_5$, but no proper subset can block π. □

A.2 Hardness results

We will now show that it is computationally hard to decide whether a given FHG admits a non-empty core. This problem turns out to be Σ_2^p-complete, that is, complete for the second level of the polynomial hierarchy, even for simple symmetric FHGs. Our argument is rather involved; shorter proofs exist when aiming only for NP-hardness and without the restriction to simple symmetric games [Brandl et al. 2015; Peters and Elkind 2015].

We will start our reduction from the problem MINMAX-CLIQUE, which is Π_2^p-complete [Ko and Lin 1995]:

MINMAX-CLIQUE
Instance: An undirected graph $H = (V, E)$ whose vertex set $V = \bigcup_{i=1}^{n} \bigcup_{j=1}^{c} V_{i,j}$ is partitioned into $n \cdot c$ cells, thought of as a grid with n rows and c columns, and a target integer k.
Question: Is it the case that for every way of choosing exactly one $V_{i,j}$ for each row i, the union of the n chosen cells contains a clique of size k?

From the reduction presented by Ko and Lin [1995], it follows that this problem remains Π_2^p-complete even if $c = 2$, all the $V_{i,j}$'s contain the same number of vertices (say $|V_{i,j}| = m$), and $k = n$. From this, it is easy to see that the problem with $k = n + \frac{nm}{2}$ is also hard: just add a clique of $2nm$ (= $|V|$) new vertices to H, connect each of the new vertices to every of the old vertices, and distribute the new vertices into the grid so that each $V_{i,j}$ contains precisely m of these new vertices. To see correctness, consider a way of choosing exactly one cell per row in the old graph. By assumption, the union of those cells contains a clique of size n. In the new graph, we can add the nm new vertices to that clique, which implies existence of a clique of the required size. (Note that after this reduction, the value of "m" has doubled.)

Taking the complement of the problem we have now arrived at, we find that the following problem is Σ_2^p-complete. (Note the change from "minmax" to "maxmin".)

MAXMIN-CLIQUE
Instance: An undirected graph $H = (V, E)$ whose vertex set $V = \bigcup_{i=1}^{n} \bigcup_{j=1,2} V_{i,j}$ is partitioned into a grid with n rows and two columns, where all cells contain the same number of vertices, say $|V_{i,j}| = m$ for all i, j.
Question: Is there a way to choose exactly one of $V_{i,1}$ and $V_{i,2}$ for each row i so that the union of the n chosen cells does *not* contain a clique of size $n + \frac{nm}{2}$?

We will not give a direct reduction from MAXMIN-CLIQUE to our problem about FHGs, but will instead consider an intermediate problem first. Later, we show how to extend this to the case we are actually interested in. Our intermediate problem uses a modification of allowing so-called *supported players*, who are unusually happy in a singleton coalition. A similar device also appears in the Σ_2^p-hardness proof by Peters [2017] for additively separable hedonic games. The formal definition of our problem is as follows.

Core-non-emptiness with Supported Players
Instance: An undirected unweighted graph $G = (N, E)$, defining an FHG. This hedonic game is then modified by identifying a number of *supported* players $S \subseteq N$ who receive

a specified subsidy when they are alone, i.e., for each $i \in S$, we set $v(\{i\}) = (s_i - 1)/s_i$ for some given integer $s_i \geq 4$ (encoded in unary).
Question: Does the given hedonic game admit a non-empty core?

Later we will show a reduction from this problem to the case without supported players; there the technical assumptions that the subsidies satisfy $s_i \geq 4$ and that s_i is given in unary will become useful.

THEOREM A.2. *Core-non-emptiness with Supported Players is Σ_2^p-complete.*

PROOF. In this proof, for an integer m, we write $[m] = \{1, 2, \ldots, m\}$.

We reduce from MAXMIN-CLIQUE. So let $H = (V, E)$ be a given graph with vertex partition $V = \bigcup_{i=1}^{n} \bigcup_{j=1,2} V_{i,j}$ with $|V_{i,j}| = m$ for all i and j and with target clique size $k = n + \frac{nm}{2}$. We now construct a game $G = (N, \hat{E})$ with supported players S.

Let M be a big number; taking $M = 20m^2n$ suffices.

We produce the following players.

- For each row i, we introduce a player z_i who will eventually be responsible for choosing one of the cells $V_{i,1}$ or $V_{i,2}$.
- For each cell $V_{i,j}$, we introduce a set $X_{i,j}$ of M supported players. Each of these players receives a subsidy of $(M + 2m)/(M + 2m + 1)$.
- Each original vertex $w \in V$ is also a player $w \in N$.
- For each original vertex $w \in V$, we introduce a *mate* w'.
- For each $w \in V$, we also introduce a set C_w of $k - 3$ supported players with subsidy $(k - 2)/(k - 1)$.
- For each player z_i, we introduce a set O_{z_i} of 39 players who will form a copy of the game from Theorem 4.1 with an empty core.
- For each mate player w', we introduce a set $O_{w'}$ of 39 players who will form a copy of the game from Theorem 4.1 with an empty core.

If $X \subseteq V$ is a subset of vertices, let's write $X' = \{w' : w \in X\}$ for the collection of mates of vertices in X. Summarizing, we have produced the following set of players:

$$N = V \cup V' \cup \{z_i : i \in [n]\} \cup \bigcup_{i,j \in [n] \times [2]} X_{i,j} \cup \bigcup_{w \in V} C_w \cup \bigcup_{w' \in V'} O_{w'} \cup \bigcup_{i \in [n]} O_{z_i},$$

of which the following are supported:

$$S = \bigcup_{i,j \in [n] \times [2]} X_{i,j} \cup \bigcup_{w \in V} C_w.$$

We also need to construct the set of edges $\hat{E}$:

- All original edges from E are in $\hat{E}$.
- For each w, the set $C_w \cup \{w, w'\}$ forms a clique of size $k - 1$.
- For each cell $V_{i,j}$, the set $X_{i,j} \cup \{z_i\}$ forms a clique of size $M + 1$.
- For each $w \in V_{i,j}$, both w and w' are connected to all vertices in $X_{i,j}$.
- The sets $O_{z_i} \cup \{z_i\}$ and $O_{w'} \cup \{w'\}$ form a copy of the game from Theorem 4.1, such that the distinguished player (z_i and w'_i respectively) is one of the two players in B_2.
- There are no other edges; in particular no two mate players are adjacent.

This completes the description of the reduction.

We now show that G admits a core stable partition if and only if our instance of MAXMIN-CLIQUE is a 'yes'-instance.

$\Longrightarrow$: Suppose the game G admits a core stable partition π. We show how to choose cells $t : [n] \to \{1, 2\}$ so that $\bigcup_i V_{i,t(i)}$ contains no clique of size k.

Consider some row $i \in [n]$. Since the game restricted to the players in $O_{z_i} \cup \{z_i\}$ does not possess a core stable partition, the player z_i needs to be together with one of his neighbors outside O_{z_i}, i.e., with a neighbor from $X_{i,1} \cup X_{i,2}$. Say this neighbor x comes from $X_{i,1}$, so $x \in \pi(z_i) \cap X_{i,1}$. We show that in fact $\pi(z_i) = \{z_i\} \cup X_{i,1} \cup V'_{i,1} \cup V_{i,1}$.

- $\supseteq$: We know that $x \in \pi(z_i)$. Now x is supported with subsidy $(M + 2m)/(M + 2m + 1)$; since $\{x\}$ does not block π, it must be the case that x's utility in π is at least as high as its subsidy. Hence $|\pi(z_i)| = |\pi(x)| \geq M + 2m + 1$, and x must have at least $M + 2m$ neighbors in $\pi(z_i)$. Recalling that $m = |V_{i,j}|$ for all i and j, and looking at the reduction, we see that x only has $M + 2m$ neighbors in total, namely $V_{i,1} \cup V'_{i,1} \cup \{z_i\} \cup X_{i,1} \setminus \{x\}$, and hence this must form a subset of $\pi(z_i)$.
- $\subseteq$: If there are any additional players in $\pi(z_i)$, then x obtains utility strictly less than $(M + 2m - 1)/(M + 2m)$, and then $\{x\}$ would block π, invoking his subsidy.

We deduce that for each row i, either $\pi(z_i) = \{z_i\} \cup X_{i,1} \cup V'_{i,1} \cup V_{i,1}$ or $\pi(z_i) = \{z_i\} \cup X_{i,2} \cup V'_{i,2} \cup V_{i,2}$. This allows us to choose cell $V_{i,2}$ in the former case (setting $t(i) = 2$) and $V_{i,1}$ in the latter (setting $t(i) = 1$). Note that z_i is together with the cell that is *not* chosen.

Now let's consider the players in $X_{i,t(i)}$ corresponding to a chosen cell. Given what we know so far about π, these players only have $M + 2m - 1$ remaining neighbors (since z_i is in a different coalition). Thus, no non-singleton coalition can give such a player utility of at least the subsidy $(M + 2m)/(M + 2m + 1)$. Hence each player in $X_{i,t(i)}$ is in a singleton in π.

Now consider a vertex $w \in V_{i,t(i)}$ in a chosen cell and look at its mate w'. Since the game restricted to the players in $O_{w'} \cup \{w'\}$ does not possess a core stable partition, the player w' needs to be together with a neighbor outside $O_{w'}$, i.e., needs to be together with w and/or a player in C_w. In fact, we can see that w' needs to be together with at least one player from C_w: if not, then w' obtains utility at most $11/12$ (because w' has 10 neighbors in $O_{w'}$ plus the neighbor w), and then $C_w \cup \{w'\}$ is blocking. Thus we have shown that there is $c \in C_w$ with $c \in \pi(w')$. Since $\{c\}$ is not blocking, c's utility in π must be at least its subsidy $(k - 2)/(k - 1)$. Thus $|\pi(w')| = |\pi(c)| \geq k - 1$ and c needs to have at least $k - 2$ neighbors in $\pi(w')$. But c has exactly $k - 2$ neighbors, and so, like above, we have $\pi(w') = \{w, w'\} \cup C_w$. In particular, each $w \in V_{i,t(i)}$ in a chosen cell obtains utility $(k - 2)/(k - 1)$.

Finally, suppose for a contradiction that the union $\bigcup_i V_{i,t(i)}$ of the chosen cells contains a clique $K \subseteq V$ of size k. Then each vertex of K obtains utility $(k - 1)/k$ in K, so K blocks π, a contradiction. Hence, with our choice of $t : [n] \to \{1, 2\}$, the set $\bigcup_i V_{i,t(i)}$ contains no clique of size k.

$\Longleftarrow$: Suppose there is a way of choosing $t : [n] \to \{1, 2\}$ so that $\bigcup_i V_{i,t(i)}$ contains no clique of size k. We construct a partition π of N which is core stable. For each row $i \in [n]$:

- $\{x\} \in \pi$ for each $x \in X_{i,t(i)}$.
- $\{w, w'\} \cup C_w \in \pi$ for each $w \in V_{i,t(i)}$.
- $\{z_i\} \cup X_{i,\neg t(i)} \cup V'_{i,\neg t(i)} \cup V_{i,\neg t(i)} \in \pi$, where $\neg t(i) = 3 - t(i)$ is the not-chosen index.
- $\{c\} \in \pi$ for each $c \in C_w$ for $w \in V_{i,\neg t(i)}$.
- The players in sets O_{z_i} and $O_{w'}$ are partitioned in the way indicated in Proposition A.1.

Let us note first that each player z_i receives utility $M/(M + 2m + 1) > 11/12$, and also each mate w' either receives utility $(k - 2)/(k - 1) > 11/12$ or $(1 + M)/(M + 2m + 1) > 11/12$ since M is chosen large enough. Because z_i and w' players only have 10 neighbors in O_{z_i} and $O_{w'}$, respectively, they will not block in a coalition that is contained entirely within $O_{z_i} \cup \{z_i\}$ or $O_{w'} \cup \{w'\}$, because those only bring utility at most $10/11$.

We now show that π admits no blocking coalitions. To do so, we will go through all the players to check that they have no incentive to deviate. (We will say that a player i is *not blocking* if i is not part of any blocking coalition.) First notice that π is individually rational, and in particular every supported player receives at least its subsidy. Therefore, no singleton coalition blocks π.

- The players $x \in X_{i,\neg t(i)}$ are in a best-possible coalition: they are together with exactly their neighbors. Hence, they will never be part of a blocking coalition.
- The players $x \in X_{i,t(i)}$ (who form singletons in π and currently receive their subsidy) will not deviate, because the only coalition that gives a utility of at least their subsidy would be x's neighborhood $\{z_i\} \cup X_{i,t(i)} \cup V'_{i,t(i)} \cup V_{i,t(i)}$, yet this is not a blocking coalition since z_i is not better off in it.
- For each z_i, we have excluded all the neighbors $x \in X_{i,j}$ of z_i as possible members of a blocking coalition. This would only leave a blocking coalition contained entirely within $O_{z_i} \cup \{z_i\}$, which is not an improvement for z_i as argued above. Hence z_i will not block.
- Each $c \in C_w$ where $w \in V_{i,t(i)}$ is in a best-possible coalition because its coalition is precisely its neighborhood, and hence will not deviate.
- Each $c \in C_w$ where $w \in V_{i,\neg t(i)}$ (who forms a singleton in π and currently receives its subsidy) cannot block, because the only coalition that gives utility at least its subsidy would be its neighborhood $\{w, w'\} \cup C_w$, but this coalition is not blocking because w' is not better off (since $(1 + M)/(M + 2m + 1) > (k - 2)/(k - 1)$ by choice of M large enough).
- Consider a mate player w'. We have already shown that all of its neighbors, except possibly w and vertices in $O_{w'}$, are not part of blocking coalitions. But a blocking coalition contained in $\{w, w'\} \cup O_{w'}$ brings utility at most $11/12$ to w' (because w' only has 11 neighbors in this set), and thus w' is not better off in such a coalition. Hence no mate player is blocking.
- No player in O_{z_i} or $O_{w'}$ can be part of a blocking coalition by Proposition A.1, since z_i and w', respectively, are not blocking.
- Each $w \in V_{i,\neg t(i)}$ currently receives utility $\geq (1 + M)/(M + 2m + 1)$ which, for our choice of M large enough, is at least the utility w could receive in any coalition $S \subseteq V$ consisting entirely of original vertices (this quantity being at most $(|V| - 1)/|V| = (2nm - 1)/2nm$).
- Thus, any blocking coalition S that we have not yet excluded must consist entirely of original vertices in chosen cells, that is $S \subseteq \bigcup_{i=1}^{n} V_{i,t(i)}$. Because each $w \in S$ currently obtains utility $(k - 2)/(k - 1)$, S must give each member a utility strictly exceeding this value. We show that S is a clique in the graph H, and of size $\geq k$, which gives a contradiction.
  Let $r := |S|$. Note that $r \leq mn = |\bigcup_{i=1}^{n} V_{i,t(i)}|$. Suppose that S is not a clique. Then there exists a vertex $w \in S$ which is not connected to every other $w \in S$. Thus
  $$u_w(S) \leq \frac{r-2}{r} \leq \frac{mn-2}{mn} < \frac{k-2}{k-1},$$
  where the last inequality follows from $k > \frac{mn}{2} + 1$ by simple algebra. But because S is assumed to be blocking, we know that $(k - 2)/(k - 1) < u_w(S)$, and hence we have a contradiction. Thus, S must be a clique. Since each $w \in S$ obtains utility $> (k - 2)/(k - 1)$ in it, we must have that $|S| \geq k$, a contradiction.

Thus, no blocking coalition exists, and hence π is in the core. □

With this result in place, we can now formally state the following problem, and prove it to be hard:

Core-non-emptiness for simple symmetric FHGs
Instance: An undirected unweighted graph $G = (N, E)$, defining an FHG.
Question: Does the given FHG game admit a non-empty core?

Theorem 4.2. *Checking whether a simple symmetric FHG has an empty core is Σ_2^p-complete.*

Proof. We reduce from Core-non-emptiness with Supported Players to Core-non-emptiness for simple symmetric FHGs. So let $G = (N, E)$ be a FHG modified by having supported players $S \subseteq N$ where $i \in S$ gets subsidy $(s_i - 1)/s_i$ where $s_i \geq 4$.

We build a new FHG $H = (N', E')$ without supported players such that G possesses a core stable partition if and only if H does.

The player set N' of H subsumes every original player from N, so $N \subseteq N'$. In addition, for each supported player $i \in S$, we add a set C_i of $(s_i - 1)$ new players. Together we have $N' = N \cup \bigcup_{i \in S} C_i$.

The edge set E' of H subsumes the original edges, so $E \subseteq E'$. Also, the sets $C_i \cup \{i\}$ form a clique of s_i players for each $i \in S$. There are no other edges.

This completes the description of the reduction. Before we prove correctness, let us analyze the preferences of players $j \in C_i$. Clearly, j's unique most-preferred coalition is $C_i \cup \{i\}$, which is precisely j's neighborhood. Ranked second are all coalitions of $(s_i - 2)$ neighbors of j together with j (that is, the coalitions C_i and $C_i \setminus \{k\} \cup \{i\}$ for some $k \in C_i \setminus \{j\}$) which give j utility $(s_i - 2)/(s_i - 1)$. All other coalitions are ranked lower than these: let C be any other coalition containing j.

- If $|C| \leq s_i - 2$, then j obtains utility $\leq (s_i - 3)/(s_i - 2) < (s_i - 2)/(s_i - 1)$.
- If $|C| = s_i - 1$, then, because C is a coalition different from the ones considered above, j has at most $s_i - 3$ neighbors in C, so obtains utility $\leq (s_i - 3)/(s_i - 1) < (s_i - 2)/(s_i - 1)$.
- If $|C| = s_i$, then again, because C is assumed to be different from $C_i \cup \{i\}$, j has at most $s_i - 2$ neighbors in C, so obtains utility $\leq (s_i - 2)/s_i < (s_i - 2)/(s_i - 1)$.
- If $|C| \geq s_i + 1$, then j can obtain utility at most $(s_i - 1)/(s_i + 1)$, which is worse than $(s_i - 2)/(s_i - 1)$ for $s_i > 3$.

Suppose G has a core stable partition π. Consider the following partition π' of H:

$$\pi' = (\pi \setminus \{\{i\} : i \in S\}) \cup \{C_i \cup \{i\} : i \in S \text{ and } \{i\} \in \pi\} \cup \{C_i : i \in S \text{ and } \{i\} \notin \pi\}.$$

Thus, supported players $i \in S$ who are in a singleton coalition in π join the coalition $C_i \cup \{i\}$ in π'.

We claim that π' is core stable in H. Note first that sets of form $C_i \cup \{i\}$ are not blocking: In this coalition, player i receives utility $(s_i - 1)/s_i$ (which is equal to i's subsidy) and so if $C_i \cup \{i\}$ was blocking π', then $\{i\}$ would be blocking π. As we have seen, the coalitions C_i are ranked second-best for its members, who therefore do not block either. Hence no player from any C_i is blocking. Hence any potential blocking coalition for π' is contained entirely in N, and hence would also be a blocking coalition for π, which is a contradiction. Hence π' is core stable.

Suppose H has a core stable partition π'. First note that for each $i \in S$, either $C_i \cup \{i\} \in \pi'$ or $C_i \in \pi'$, since otherwise either C_i or $C_i \cup \{i\}$ blocks (by our observations about the preferences of players $j \in C_i$ above). Build the following partition π of N: if $C_i \cup \{i\} \in \pi'$ then put i in a singleton in π: $\{i\} \in \pi$; and for every $X \in \pi'$ with $X \subseteq N$, also put $X \in \pi$. The result is core stable in G: for suppose not, and there is a blocking coalition $Y \subseteq N$. If $Y = \{i\}$ is a singleton and i is supported, then $C_i \cup \{i\}$ would block π'. In all other cases Y would also block π'. Both cases give a contradiction, and so π is core stable. □

The reduction in the proof of Theorem 4.2 can also be used to show that it is coNP-complete to verify whether a given coalition structure is core stable in a given simple symmetric FHG. We only sketch the argument, which is by reduction from clique. Given an instance (G, k) of the clique problem (which asks whether G contains a clique of size at least k, where we may assume that

$k \geq \frac{n}{2} + 2$), we produce an FHG based on the same graph G, and make every vertex a supported player with subsidy $(k-2)/(k-1)$. In this game with supported players, the all-singletons coalition structure is in the core, unless there is a clique in G of size at least k (whence the clique would block, giving its members the payoff $(k-1)/k$). The approach of Theorem 4.2 can then be used to get rid of the subsidies.

Liu and Wei [2017] give an alternative hardness proof for the problem of verifying core membership. They show that even checking whether the *grand* coalition is core stable is coNP-complete, even for graphs of diameter 2 and which satisfy some further structural constraints. Their reduction is also from the clique problem. On the other hand, they present heuristics for solving the verification problem.

A.3 Positive results

Graphs with bounded degree

Theorem 5.1. *For simple symmetric FHGs represented by graphs of degree at most 2, the core is non-empty.*

Proof. We present a polynomial-time algorithm to compute a partition in the core. The partition is computed as follows. First keep finding K_3s until no more can be found. This takes time $O(\binom{n}{3})$. Let us call the set of vertices matched into K_3s as V_3. We remove V_3 from the graph along with E_3—the edges incident to vertices in V_3. We then repeat the procedure by deleting K_2s instead of K_3s. Let us call the set of vertices matched into pairs by V_2. The unmatched vertices $V_1 = V \setminus (V_2 \cup V_3)$ are put into singleton coalitions. The partition obtained is π.

In order to prove that π is in the core, consider the potential blocking coalitions. We know that vertices in V_3 cannot be in a blocking coalition because each vertex in V_3 is in its most favored coalition. Also there does not exist a blocking coalition consisting solely of vertices from V_1. If this were the case, then we had not deleted all K_2s from $(V \setminus V_3, E \setminus E_3)$. Now let us assume that there exists a $i_2 \in V_2$ that is in a blocking coalition. A blocking coalition has to be of size 3, since i_2 has utility $1/2$ in π and utility at most $1/2$ in any coalition of size 2 or size at least 4. Moreover, a blocking coalition cannot contain two vertices from V_1, since for this to be the case i_2 has to be connected to one vertex in V_2 and two vertices in V_1, which violates the degree constraints. Hence, the coalition is of the form $\{i_1, i_2, i_2'\}$ where $i_1 \in V_1$ and $i_2, i_2' \in V_2$. If the utility of i_2 is greater than $1/2$, then the utility of i_2' is less than $1/2$ since $\{i_1, i_2, i_2'\}$ does not form a K_3. Since i_2' obtained utility $1/2$ in π, $\{i_1, i_2, i_2'\}$ is not a blocking coalition. □

Forests

Theorem 5.2. *For simple symmetric FHGs represented by undirected forests, the core is non-empty.*

Proof. We present an algorithm to compute a partition in the core for an undirected tree. We may assume that the graph is connected—and therefore a tree—because the algorithm for a tree can be applied to each connected component separately. An example run of the following algorithm is indicated in Figure 7.

Root the tree at an arbitrary vertex $r \in V$, and run breadth-first search on the rooted tree. This partitions V into sets $L_0, \ldots, L_\ell$, where L_k consists of all vertices at distance k from r. We now construct a partition π in the core of the game. For $k = \ell - 1, \ell - 2, \ldots, 0$, run through the vertices in L_k. For each vertex $i \in L_k$ that (i) has not been assigned a coalition yet and (ii) has children in L_{k+1} that have not been assigned a coalition yet, add the coalition consisting of i and all its unassigned children to π. The player i that satisfies (i) and (ii) above is called a *boss player*. For $k = 0$, in the case that all children of r have already been assigned, add the singleton coalition $\{r\}$ to π and make r a boss player. Note that every coalition in π contains exactly one boss player.

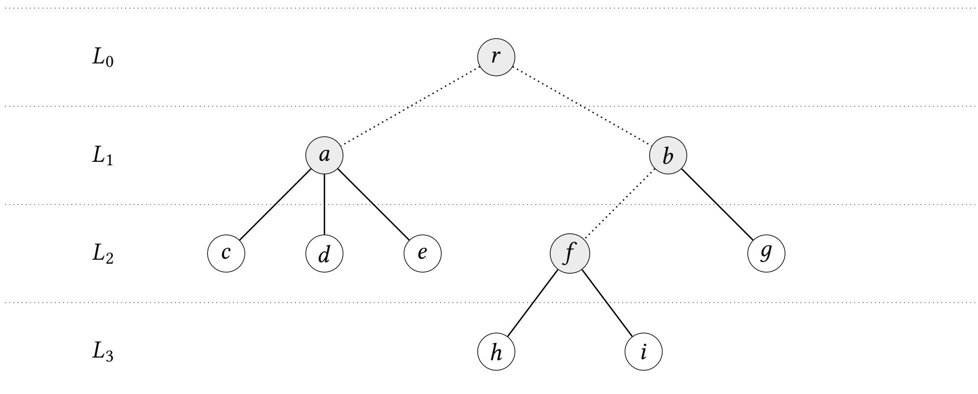

Fig. 7. Example of a tree with a core-stable partition as produced by the algorithm indicated in the proof of Theorem 5.2. Vertices connected by solid edges are in the same coalition. In the terminology of the proof, gray vertices are boss players.

We claim that π is in the core. First, we prove by induction on $k = \ell, \ell - 1, \dots, 0$ that no boss player $i \in L_k$ is part of a coalition blocking π. The base case $k = \ell$ is vacuous since L_ℓ contains no boss players. For the induction step, assume the statement holds for some $k \leq \ell$, and consider a boss player $i \in L_{k-1}$. By inductive hypothesis, no child of i who is a boss player can be part of a blocking coalition. Hence the only possible blocking coalition that i prefers to $\pi(i)$ is the coalition S consisting of i's non-boss children and i's parent, that is, $S = \pi(i) \cup \{p_i\}$, where p_i is i's parent. But S is worse than π for i's (non-boss) children. So S is not blocking, and i is not part of any blocking coalition.

We have established that no blocking coalition contains a boss player. However, no two non-boss players are adjacent in the tree, and hence no coalition containing only non-boss players can be a blocking coalition. Hence π is in the core. □

Bakers and Millers: complete k-partite graphs

Theorem 5.3. *Let $(N, \succsim)$ be a Bakers and Millers game with type space $\Theta = \{\theta_1, \dots, \theta_t\}$ and $\pi = \{S_1, \dots, S_m\}$ a partition. Then, π is in the strict core if and only if for all types $\theta \in \Theta$ and all coalitions $S, S' \in \pi$,*

$$\frac{|S \cap \theta|}{|S|} = \frac{|S' \cap \theta|}{|S'|}.$$

Proof. First assume that for all types $\theta \in \Theta$ and all coalitions S and S' in π we have $\frac{|S\cap\theta|}{|S|} = \frac{|S'\cap\theta|}{|S'|}$, but that a weakly blocking coalition T for π exists. Then, $\frac{|T\cap\theta(j)|}{|T|} \leq \frac{|\pi(j)\cap\theta(j)|}{|\pi(j)|}$ for all $j \in T$, while there is some $i \in T$ with $\frac{|T\cap\theta(i)|}{|T|} < \frac{|\pi(i)\cap\theta(i)|}{|\pi(i)|}$. Consider this i. Without loss of generality assume that $\theta_1, \dots, \theta_k$ are the types represented in T, that is, those types θ with $j \in \theta$ for some $j \in T$. By assumption we have, for all $j \in T$, $\frac{|\pi(j)\cap\theta(j)|}{|\pi(j)|} = \frac{|\pi(i)\cap\theta(j)|}{|\pi(i)|}$. Hence,

$$\frac{|T \cap \theta_1|}{|T|} + \dots + \frac{|T \cap \theta_k|}{|T|} < \frac{|\pi(i) \cap \theta_1|}{|\pi(i)|} + \dots + \frac{|\pi(i) \cap \theta_k|}{|\pi(i)|}.$$

Observe that both

$$\frac{|T \cap \theta_1|}{|T|} + \cdots + \frac{|T \cap \theta_k|}{|T|} = 1$$

$$\text{and} \quad \frac{|\pi(i) \cap \theta_1|}{|\pi(i)|} + \cdots + \frac{|\pi(i) \cap \theta_k|}{|\pi(i)|} \leq 1.$$

A contradiction follows.

For the other direction, assume that there are coalitions $S, T \in \pi$ and a type $\theta \in \Theta$ such that $\frac{|S\cap\theta|}{|S|} > \frac{|T\cap\theta|}{|T|}$. Then, $S \cap \theta \neq \emptyset$ and let $i \in S \cap \theta$. As

$$\frac{|S \cap \theta_1|}{|S|} + \cdots + \frac{|S \cap \theta_t|}{|S|} = \frac{|T \cap \theta_1|}{|T|} + \cdots + \frac{|T \cap \theta_t|}{|T|},$$

there is some type $\theta' \in \Theta$ such that $\frac{|S\cap\theta'|}{|S|} < \frac{|T\cap\theta'|}{|T|}$. Accordingly, $T \cap \theta' \neq \emptyset$.

First consider the case in which both $S \cap \theta' = \emptyset$ and $T \cap \theta = \emptyset$. Without loss of generality, we may assume that $|S| \leq |T|$. Observe that $|S| < |T \cup \{i\}|$. The coalition $T \cup \{i\}$ is weakly blocking, as

$$\frac{|(T \cup \{i\}) \cap \theta|}{|T \cup \{i\}|} = \frac{|\{i\}|}{|T \cup \{i\}|} < \frac{|\{i\}|}{|S|} \leq \frac{|S \cap \theta|}{|S|}$$

and

$$\frac{|(T \cup \{i\}) \cap \theta''|}{|T \cup \{i\}|} = \frac{|T \cap \theta''|}{|T \cup \{i\}|} \leq \frac{|T \cap \theta''|}{|T|}$$

for every type θ'' distinct from θ. (The latter inequality is not strict, as $T \cap \theta''$ may be empty.)

Finally, assume without loss of generality, that $T \cap \theta \neq \emptyset$ and let $j \in T \cap \theta$. Since S and T are distinct and both in π, also $i \neq j$. We show that the coalition $T' = (T \setminus \{j\}) \cup \{i\}$ is weakly blocking. Consider an arbitrary type $\theta'' \in \Theta$. Observe that $|T| = |T'|$ and $|T \cap \theta''| = |T' \cap \theta''|$, whether $\theta'' = \theta$ or not. Therefore, $\frac{|T\cap\theta''|}{|T|} = \frac{|T'\cap\theta''|}{|T'|}$. Accordingly, every player $k \in T \setminus \{i, j\}$ is indifferent between T and T'. To conclude the proof, observe that $\frac{|T'\cap\theta|}{|T'|} = \frac{|\pi(j)\cap\theta|}{|T|}$. Hence,

$$\frac{|\pi(i) \cap \theta(i)|}{|S|} = \frac{|S \cap \theta|}{|S|} > \frac{|T \cap \theta|}{|T|} = \frac{|\pi(j) \cap \theta|}{|T|} = \frac{|T' \cap \theta|}{|T'|},$$

that is, $T' \succ_i S$, as desired. □

Graphs with large girth

We show that in FHGs based on graphs with girth at least 5, star packings maximizing a leximin objective are core stable.

Theorem 5.6. *For simple symmetric FHGs represented by graphs with girth at least* 5*, the core is non-empty. Moreover, there always exists a partition into stars that is in the core.*

Proof. The reader is referred to Figure 8 for a graphical illustration of certain aspects of its proof.

We write S_i for a star with i vertices. Each star S_i with $i > 2$ has one center c and $i - 1$ leaves $\ell_1, \ldots, \ell_{i-1}$. We view S_2 as having two centers and no leaves. A partition π is a *star packing* if each non-singleton coalition induces a star.

We prove that a star packing that maximizes *leximin welfare* is core stable. Formally, with each star packing, denoted by π, we associate an *objective vector* $\vec{x}(\pi) = (x_1, \ldots, x_{|V|})$ such that $x_i \leq x_j$ if $1 \leq i \leq j \leq |V|$, and there is a bijection $f : V \to \{1, \ldots, |V|\}$ with $v_k(\pi) = x_{f(k)}$ for every $k \in V$. Thus, in $\vec{x}(\pi)$ the vertices/players are ordered according to their value for π in ascending order. We assume these objective vectors to be ordered lexicographically by $\geq$, e.g.,

$(1/2, 1/2, 1/2, 1/2) \geq (0, 1/3, 1/3, 2/3)$ but not vice versa. The goal is to compute a star packing that maximizes its objective vector. Intuitively, this balances the sizes of the stars in the star packing and does not leave vertices needlessly on their own. Clearly, star packings maximizing the objective are guaranteed to exist and in the remainder of the proof we argue that such star packings are in the core.

Observe first that every graph (V, E) admits a star packing such that every vertex which is not isolated (that is, which has a neighbor) is contained in some star S_i for $i \geq 2$. This can be seen by considering a spanning forest. Thus, every star packing of (V, E) that maximizes the objective vector must have this property.

Now, let π be a star packing of a graph (V, E) that maximizes the objective vector. For a contradiction, assume that there is a coalition S blocking π. Then, S contains no vertices that are isolated in S, as these obtain utility 0 and, therefore, can not be strictly better off in S than they were in π. In particular, S consists entirely of vertices that are either centers or leaves of π. Also observe that, for any two leaves ℓ, ℓ' in π we have $\{\ell, \ell'\} \notin E$. For a contradiction assume that there were such leaves ℓ, ℓ'. Then, ℓ and ℓ' must come from different centers, otherwise (V, E) would contain a triangle. Now consider partition $\pi' = \{\{\ell, \ell'\}, \pi'_1 \dots, \pi'_k\}$, where $\pi'_i = \pi_i \setminus \{\ell, \ell'\}$. Notice that π' is a star packing for which the objective vector is larger than the one for π, that is, $\vec{x}(\pi') > \vec{x}(\pi)$. To see this, observe that all leaves in π obtain at least as high a utility in π' as in π, that both ℓ and ℓ' obtain a strictly higher utility in π' than in π, and that leaves appear before centers in the ordering of the objective vectors.

Now three cases can be distinguished: (i) S contains no centers of π, (ii) S contains exactly one center of π and (iii) S contains more than one center of π.

If (i), S only contains leaves of π, between which we know there are no edges. Hence, every member of S has utility 0 and S cannot be blocking.

If (ii), we show that $\vec{x}(\pi)$ is not a maximal objective vector. Let S consist of one center c and m leaves $\ell_1, \dots, \ell_m$ of π. Since no leaves in π are neighbors, and S does not contain isolated vertices, S must be a star with c as center and $\ell_1, \dots, \ell_m$ as leaves. Observe, moreover, that $\pi(c) \neq \pi(\ell_i)$ for every leaf ℓ_i of π in S. Let ℓ denote one of the leaves and c' the center of π such that $\ell \in \pi(c')$. Then, $c' \neq c$. Consider the partition π' such that

$$\pi'(k) = \begin{cases} \pi(c) \cup \{\ell\} & \text{if } k \in \pi(c) \cup \{\ell\}, \text{ and} \\ \pi(k) \setminus \{\ell\} & \text{otherwise.} \end{cases}$$

We claim that $\vec{x}(\pi') > \vec{x}(\pi)$, contradicting our initial assumption. Observe that it suffices to prove that (a) $v_\ell(\pi') > v_\ell(\pi)$ and (b) $v_k(\pi') \geq v_\ell(\pi')$ for all k with $v_k(\pi') < v_k(\pi)$.

For (a), observe that, if $v_c(\pi) < v_c(S)$ and c is a center in both π and S, then $\frac{|\pi(c)|-1}{|\pi(c)|} < \frac{|S|-1}{|S|}$. Moreover, $v_\ell(\pi) < v_\ell(S)$, that is, $\frac{1}{|\pi(\ell)|} < \frac{1}{|S|}$. Accordingly, $|\pi(c)| < |S| < |\pi(\ell)|$. It follows that $|\pi'(\ell)| = |\pi(c) \cup \{\ell\}| \leq S < |\pi(\ell)|$ and thus $v_\ell(\pi') > v_\ell(\pi)$.

For (b), let k be such that $v_k(\pi') < v_k(\pi)$. Then either $k = c'$ or $k \in \pi(c) \setminus \{c\}$. As c' is a center and ℓ a leaf in π, c' still is a center in π'. Hence, $v_{c'}(\pi') \geq 1/2$. Moreover, ℓ is also a leaf in π' and thus $v_\ell(\pi') < 1/2$, proving the case. Now assume that $k \in \pi(c) \setminus \{c\}$. Then, with k and ℓ being both leaves in $\pi'(c)$, $v_k(\pi') = v_\ell(\pi')$.

If (iii), assume that S contains at least two centers c and c' in π. Then, $v_c(S) > v_c(\pi) \geq 1/2$ and $v_{c'}(S) > v_{c'}(\pi) \geq 1/2$. Then, both c and c' have more than half the members of S as neighbor. We distinguish two cases. If c and c' are adjacent, then there must be at least one other $k \in S$ that is adjacent to both c and c'. In that case, (V, E) contains a triangle. If, on the other hand, c and c' are not adjacent, there must be at least *two* distinct vertices k and k' in S that both c and c' are adjacent to.

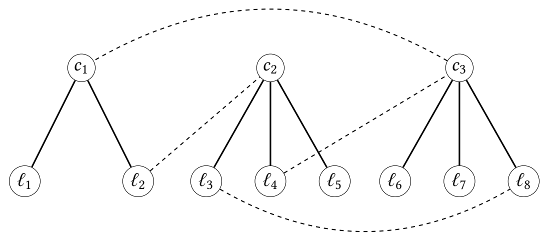

Fig. 8. A graph with girth five and a star packing indicated by the solid edges. This star packing does not have an optimal objective vector: a better one would result if ℓ_3 and ℓ_8 were to form a star. Note that $\{\ell_3, \ell_8\}$ is a blocking coalition.

Then $\{c, k, c', k'\}$ forms a cycle of length 4. Either case contradicts our initial assumption that (V, E) has girth at least 5, which concludes the proof. □

The procedure described in the proof above does not immediately yield a polynomial-time algorithm that produces a core stable partition, since it is unclear whether a leximin star packing can be found in polynomial time. However, inspecting the proof further, we see that we in fact only need a *local* optimum.

Theorem A.3. *For simple symmetric FHGs represented by graphs with girth at least* 5*, an element of the core can be found in polynomial time.*

Proof. The existence proof above showed that if a given star packing π is blocked by some coalition, then there exists a leximin-better star packing π' that could be obtained from π in one of the following two ways:

(a) two leaves ℓ, ℓ' from different stars in π with $\{\ell, \ell'\} \in E$ are removed from their respective coalitions and form the new star $\{\ell, \ell'\}$, or
(b) a leaf ℓ is moved from one star to another.

Our algorithm now proceeds as follows: start by producing some star packing of G in which every non-isolated vertex is in a star (such a star packing can be found by considering a spanning forest of G). Then improve this star packing by using operations (a) and (b) if they lead to a leximin improvement, until no more such opportunities are available. The resulting star packing is in the core by the argument in the existence proof above.

It remains to analyze the runtime of this algorithm. Clearly, the initial step and each improvement step can be executed in polynomial time, so we only need to establish that the algorithm terminates after a polynomial number of improvement steps.

Define the following potential function for each star packing π:

$$\Phi(\pi) = \sum_{i \in V\text{center}} |V| + \sum_{i \in V\text{leaf}} |V| - |\pi(i)|.$$

Note that this potential function is integral, non-negative, and bounded above by $|V|^2$. We show that every time we perform (a) or (b), the potential strictly increases. This implies that at most $|V|^2$ improvement steps will be required.

If we perform (a), then we convert the two leaves ℓ and ℓ' into centers and thereby strictly increase their contribution to Φ. We also decrease the sizes of the stars that ℓ and ℓ' were part of in π, which increases the contributions to Φ of the remaining vertices in those stars. Everyone else's contribution stays fixed.

If we perform (b), using the notation of the previous proof, we move ℓ from $\pi(c')$ to $\pi(c)$. Since by case (ii)(a) of that proof we thereby increase the utility of ℓ, the leaf ℓ has moved from a large star to a smaller star; in particular $|\pi(c')| \geq |\pi(c)| + 2$. After the move of ℓ, the contributions to Φ of the leaves of $\pi(c')$ have each increased by 1, and the contributions of leaves of $\pi(c)$ have decreased by 1. Since there are more of the former than of the latter, this is an overall strict improvement. □

Bipartite graphs

Lemma 5.7. *For simple symmetric FHGs represented by a bipartite graph, any perfect matching is in the strict core.*

Proof. Let $\{N_1, N_2\}$ be the bipartition of N. In a perfect matching π, considered as a partition, every player has utility $1/2$. Suppose there was a coalition $S \subseteq N$ that weakly blocks π. Then $v_i(S) \geq 1/2$ for every $i \in S$, and $v_j(S) > 1/2$ for some $j \in S$. Since S is weakly blocking, S must contain some player $i_1 \in N_1$ and some player $i_2 \in N_2$. Since $v_{i_1}(S) \geq 1/2$, we get that $|N_2 \cap S| \geq |N_1 \cap S|$. Since $v_{i_2}(S) \geq 1/2$, we get that $|N_1 \cap S| \geq |N_2 \cap S|$. Thus $|N_1 \cap S| = |N_2 \cap S|$. But this contradicts that $v_j(S) > 1/2$, and so S cannot exist. Hence π is in the strict core. □