Noname manuscript No.
(will be inserted by the editor)

Extending Tournament Solutions

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Received: date / Accepted: date

Abstract An important subclass of social choice functions, so-called *majoritarian (or C1) functions*, only take into account the pairwise majority relation between alternatives. In the absence of majority ties—e.g., when there is an odd number of agents with linear preferences—the majority relation is antisymmetric and complete and can thus conveniently be represented by a tournament. Tournaments have a rich mathematical theory and many formal results for majoritarian functions assume that the majority relation constitutes a tournament. Moreover, most majoritarian functions have only been defined for tournaments and allow for a variety of generalizations to unrestricted preference profiles, none of which can be seen as the unequivocal extension of the original function. In this paper, we argue that restricting attention to tournaments is justified by the existence of a *conservative extension*, which inherits most of the commonly considered properties from its underlying tournament solution.

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1 Introduction

Perhaps one of the most natural ways to aggregate binary preferences from individual agents to a group of agents is *simple majority rule*, which prescribes that one alternative is socially preferred to another whenever a majority of agents prefers the former to the latter. Majority rule intuitively appeals to democratic principles, is easy to understand and—most importantly—satisfies some attractive formal properties (May 1952). Moreover, almost all common voting rules coincide with majority rule in the two-alternative case. It would therefore seem that the existence of a majority of individuals preferring alternative a to alternative b signifies something fundamental and generic about the group's preferences over a and b.

A *majoritarian (or C1) social choice function* is a function that maps a vector of individual preference relations to a nonempty set of socially preferred alternatives while only taking into account the pairwise majority relation. When dealing with majoritarian functions, it is often assumed that there are no majority ties. This can, for example, be guaranteed by insisting on an odd number of agents with linear preferences. Under this assumption, a preference profile gives rise to a *tournament* and a majoritarian function is equivalent to a *tournament solution*, i.e., a function that associates with every complete and antisymmetric directed graph a subset of the vertices of the graph. Examples of well-studied tournament solutions are the Copeland set, the top cycle, the uncovered set, and the Slater set (see, e.g., Laslier 1997; Brandt et al. 2016).

While technically convenient, the assumption that preferences do not admit majority ties is rather artificial. Particularly if the number of agents is small, majority ties cannot be ignored. It is therefore natural to ask how a given majoritarian function can be generalized to the class of preference profiles that may admit majority ties. Mathematically speaking, we are looking for ways to apply a tournament solution to a complete, but not necessarily antisymmetric, directed graph—a so-called *weak tournament*. For many tournament solutions, generalizations or extensions to weak tournaments have been proposed (see, e.g., Peris and Subiza 1999). Often, it turns out that there are several sensible ways to generalize a tournament solution and it is unclear whether there exists a unique "correct" generalization. Even for something as elementary as the Copeland set or the top cycle, there is a variety of extensions that are regularly considered in the literature. A natural criterion for evaluating the different proposals is whether the extension satisfies appropriate generalizations of the axiomatic properties that the original tournament solution satisfies.

In this paper, we propose a generic way to extend any tournament solution to the class of weak tournaments. This so-called *conservative extension* of a tournament solution S returns all alternatives that are chosen by S in *some* orientation of the weak tournament at hand. We show that many of the most common axiomatic properties of tournament solutions are "inherited" from S to its conservative extension (see Table 1 for an overview). We argue that these results provide a justification for restricting attention to tournaments when studying majoritarian social choice functions.

Table 1 Properties that the conservative extension $[S]$ inherits from S.

Property inherited by $[S]$	Result
Condorcet-consistency	Proposition 1
monotonicity	Proposition 2
independence of unchosen alternatives	Proposition 3
set-monotonicity	Proposition 4
$\widehat{\alpha}$	Proposition 5
stability ($\widehat{\alpha} \wedge \widehat{\gamma}$)	Proposition 6
composition-consistency	Proposition 7
weak composition-consistency	Proposition 8
weak regularity	Proposition 9

The conservative extension also leads to interesting *computational* problems that have been studied as *possible winner problems* for incompletely specified tournaments (Lang et al. 2012; Aziz et al. 2015). In fact, computing the conservative extension of a tournament solution is equivalent to solving its possible winner problem when pairwise comparisons are only partially specified. Of course, there is an exponential number of orientations of a weak tournament in general. However, for many well-known tournament solutions, the corresponding conservative extensions can be computed efficiently by exploiting individual peculiarities of these concepts.

The pairwise comparisons represented by tournaments need not originate from simple majority rule. In fact, tournament solutions and variants thereof can be applied to numerous other settings such as multi-criteria decision analysis (Arrow and Raynaud 1986; Bouyssou et al. 2006), zero-sum games (Fisher and Ryan 1995; Laffond et al. 1993; Duggan and Le Breton 1996), and coalitional games (Brandt and Harrenstein 2010). The results in this paper are equally relevant for these settings than they are for social choice theory.

The paper is organized as follows. After introducing the necessary notation in Section 2, we define the conservative extension in Section 3 and show that it inherits many desirable properties in Section 4. Furthermore, we compare the conservative extension to other generalizations that have been proposed in the literature (Section 5) and study its computational complexity (Section 6) for a number of common tournament solutions.

2 Preliminaries

Let U be a universe of alternatives. For notational convenience we assume that $\mathbb{N} \subseteq U$. Every nonempty finite subset of U is called a *feasible set*. For a binary relation $\succsim$ on U and alternatives $a, b \in U$, we usually write $a \succsim b$ instead of the more cumbersome $(a, b) \in \succsim$. A *weak tournament* is a pair $W = (A, \succsim)$, where A is a feasible set and $\succsim$ is a complete binary relation on U, i.e., for all $a, b \in U$, we have $a \succsim b$ or $b \succsim a$ (or both).[1] Intuitively, $a \succsim b$ signifies that alternative a is (weakly) preferred to b. Note that completeness implies reflexivity, i.e., $a \succsim a$ for all $a \in U$. We write $a \succ b$ if $a \succsim b$ and not $b \succsim a$, and $a \sim b$ if both $a \succsim b$ and $b \succsim a$. If $a \sim b$, we say that there is *indifference* between the two alternatives. We denote the class of all weak tournaments by $\mathcal{W}$.

The relation $\succsim$ is often referred to as the *dominance relation*. One of the best-known concepts defined in terms of the dominance relation is that of a Condorcet winner. Alternative a is a *Condorcet winner* in a weak tournament $W = (A, \succsim)$ if $a \succ b$ for all alternatives $b \in A \setminus \{a\}$.

A *tournament* is a weak tournament $(A, \succsim)$ whose dominance relation $\succsim$ is antisymmetric, i.e., for all *distinct* $a, b \in U$, we have *either* $a \succsim b$ *or* $b \succsim a$ (but not both).[2] For a tournament $T = (A, \succsim)$ and distinct alternatives $a, b \in A$, $a \succsim b$ if and only if $a \succ b$. We therefore often write $T = (A, \succ)$ instead of $T = (A, \succsim)$. We denote the class of all tournaments by $\mathcal{T}$. Obviously, $\mathcal{T} \subseteq \mathcal{W}$.

For a pair of weak tournaments $W = (A, \succsim)$ and $W' = (A', \succsim')$, we say that W is *contained* in W', and write $W \subseteq W'$, if $A = A'$ and $a \succsim b$ implies $a \succsim' b$ for all $a, b \in A$. We will often deal with the set of all tournaments that are contained in a given weak tournament W.

Definition 1 For a weak tournament $W \in \mathcal{W}$, the set of *orientations* of W is given by $[W] = \{T \in \mathcal{T} : T \subseteq W\}$.

For example, the weak tournament in Figure 1 has four orientations, which are depicted in Figure 2.

Every orientation of a weak tournament $W = (A, \succsim)$ can be obtained from W by eliminating, for all distinct alternatives a and b such that $a \sim b$, one of (a, b) and (b, a) from $\succsim$.

The relation $\succsim$ can be raised to sets of alternatives and we write $A \succsim B$ to signify that $a \succsim b$ for all $a \in A$ and all $b \in B$. For a weak tournament $W = (A, \succsim)$ and a feasible set $B \subseteq A$, we will sometimes consider the *restriction* $W|_B = (B, \succsim)$ of W to B.

A *tournament solution* is a function S that maps each tournament $T = (A, \succ)$ to a nonempty subset $S(T)$ of its alternatives A called the *choice set*. It is generally assumed that choice sets only depend on $\succ|_A$ and that tournament solutions cannot distinguish between isomorphic tournaments. A tournament solution that uniquely selects the Condorcet winner whenever there is one, is said to be *Condorcet-consistent*.

Two examples of well-known tournament solutions are the top cycle and the Copeland set. The *top cycle* $TC(T)$ of a tournament $T = (A, \succ)$ is defined

[1] This definition slightly diverges from the common graph-theoretic definition where $\succsim$ is defined on A rather than on U. However, it facilitates the definition of tournament solutions and their properties.

[2] Defining tournaments with a reflexive dominance relation is non-standard. The reason we define tournaments in such a way is to ensure that every tournament is a weak tournament. Whether the dominance relation of a tournament is reflexive or not does not make a difference for any of our results.

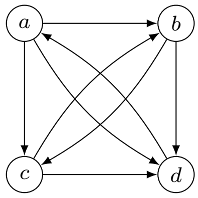

Fig. 1 Graphical representation of a weak tournament $W = (A, \succsim)$ with $A = \{a, b, c, d\}$. An edge from vertex x to vertex y represents $x \succsim y$.

as the smallest set $B \subseteq A$ such that $B \succ A \setminus B$. The *Copeland set* $CO(T)$ consists of all alternatives whose dominion is of maximal size, i.e., $CO(T) = \arg\max_{a \in A} |\{b \in A : a \succ b\}|$.

3 The Conservative Extension

In order to render tournament solutions applicable to unrestricted preference profiles, we need to generalize them to weak tournaments. A *generalized tournament solution* is a function S that maps each weak tournament $W = (A, \succsim)$ to a nonempty subset $S(W)$ of its alternatives A. A generalized tournament solution S is called an *extension* of tournament solution S' if $S(W) = S'(W)$ whenever $W \in \mathcal{T}$. For several tournament solutions, extensions have been proposed in the literature (see Section 5). Of course, there are many ways to extend any given tournament solution, and there is no definite obvious way of judging whether one proposal is better than another one.

We are interested in a *generic* way to extend any tournament solution to the class of weak tournaments. In particular, our goal is to extend tournament solutions in such a way that common axiomatic properties are "inherited" from a tournament solution to its extension. This task is not trivial, as even the arguably most cautious approach has its problems. Let the *trivial extension* of a tournament solution S be defined as the generalized tournament solution that always selects the whole feasible set A whenever the weak tournament $W = (A, \succsim) \notin \mathcal{T}$. It is easy to see that the trivial extension does not satisfy Condorcet-consistency, which also in the case of weak tournaments is the requirement that a Condorcet winner should be uniquely selected whenever it exists. The trivial extension also fails to inherit composition-consistency, which will be defined in Section 4.4.

We therefore propose to extend tournament solutions in a slightly more sophisticated way. The *conservative extension* of a tournament solution S returns all alternatives that are chosen by S in *some* orientation of the weak tournament at hand.[3]

[3] Similarly, one could define a generic extension that returns all alternatives that are chosen in *all* orientations of the weak tournament. However, this extension would not constitute a generalized tournament solution because it may return the empty set.

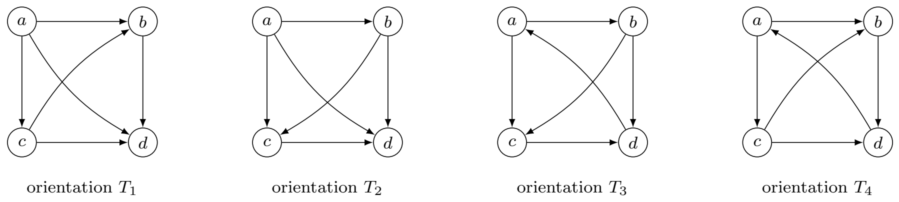

Fig. 2 The four orientations of the weak tournament W in Figure 1.

Definition 2 Let S be a tournament solution. The conservative extension $[S]$ of S is the generalized tournament solution that maps a weak tournament $W \in \mathcal{W}$ to

$$[S](W) = \bigcup_{T \in [W]} S(T).$$

This definition is reminiscent of the parallel-universes tie-breaking approach in voting theory (Conitzer et al. 2009; Brill and Fischer 2012; Freeman et al. 2015) and corresponds to selecting the set of all *possible* winners of W when indifferences are interpreted as missing edges (Lang et al. 2012; Aziz et al. 2015).

For example, consider the weak tournament W in Figure 1. Consulting Figure 2, it can be checked that:

$$\begin{array}{llll} CO(T_1) = \{a\} & CO(T_2) = \{a\} & CO(T_3) = \{a,b\} & CO(T_4) = \{a,c\} \\ TC(T_1) = \{a\} & TC(T_2) = \{a\} & TC(T_3) = \{a,b,c\} & TC(T_4) = \{a,b,c,d\} \end{array}$$

Therefore, $[CO](W) = \{a,b,c\}$ and $[TC](W) = \{a,b,c,d\}$.

4 Inheritance of Properties

The literature on (generalized) tournament solutions has identified a number of desirable properties (often called *axioms*) for these concepts. In this section, we study which properties are inherited when a tournament solution is generalized via the conservative extension. We say that a property is *inherited by the conservative extension* if, for any tournament solution S, $[S]$ satisfies the property on $\mathcal{W}$ whenever S satisfies the property on $\mathcal{T}$.

We remark that *inclusion relationships* between tournament solutions are inherited to their conservative extensions: It is not hard to see that, for any two tournament solutions S and S', if $S(T) \subseteq S'(T)$ for all tournaments T, then $[S](W) \subseteq [S'](W)$ for all weak tournaments W as well. A similar observation concerns Condorcet-consistency, which we find is also inherited by the conservative extension.

Proposition 1 *Condorcet-consistency is inherited by the conservative extension.*

Proof. Assume that S is Condorcet-consistent and consider an arbitrary weak tournament W with Condorcet winner a. Then, a will be a Condorcet winner in every $T \in [W]$. Accordingly, by Condorcet-consistency, $S(T) = \{a\}$ for every $T \in [W]$. Hence, $[S](W) = \{a\}$ as well, proving the result. □

The rest of this section is structured as follows. After stating a useful lemma (Section 4.1), we consider four classes of axiomatic properties: *dominance-based properties* that deal with changes in the dominance relation (Section 4.2), *choice-theoretic properties* that deal with varying feasible sets (Section 4.3), *composition-consistency* (Section 4.4), and *regularity* (Section 4.5).

4.1 A General Lemma

Many properties express the invariance of alternatives being chosen (or alternatives not being chosen) under a certain type of transformation of the weak tournament. That is, they have the form that if an alternative a is chosen (not chosen) from some weak tournament W, then a is also chosen (not chosen) from $f(W)$, where f is an operation that transforms weak tournaments in a particular way.[4] Formally, a *tournament operation* is a mapping $f : \mathcal{W} \to \mathcal{W}$ from the class of all weak tournaments to itself. A tournament operation f is *orientation-consistent* if applying the operation to all orientations of a weak tournament W results in the set of orientations of $f(W)$.

Definition 3 A tournament operation f is *orientation-consistent* if for all weak tournaments W,

$$f([W]) = [f(W)],$$

where $f([W]) = \{f(T) : T \in [W]\}$. Furthermore, a class F of tournament operations is *orientation-consistent* if each operation in F is orientation-consistent.

In other words, f is orientation-consistent if the diagram in Figure 3 commutes. Recalling that $\mathcal{T}$ denotes the class of all tournaments, we note that a necessary condition for f to be orientation-consistent is that $f(\mathcal{T}) \subseteq \mathcal{T}$.

Let F be a class of tournament operations, W a weak tournament, and a an alternative in U. We then say that a generalized tournament solution S is *a-inclusion invariant under F on W* if,

$$a \in S(W) \quad \text{implies} \quad a \in S(f(W)) \text{ for all } f \in F.$$

Similarly, we say that S is *a-exclusion invariant under F on W* if,

$$a \notin S(W) \quad \text{implies} \quad a \notin S(f(W)) \text{ for all } f \in F.$$

[4] For instance, the property *monotonicity* requires that a chosen alternative is still chosen if it is strengthened. In this case, the operation f would map a weak tournament W to a weak tournament W' that is identical to W except that some alternative in $S(W)$ has been strengthened with respect to another alternative. See Section 4.2 for details.

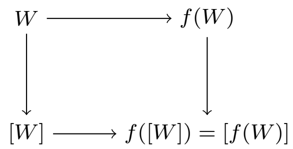

Fig. 3 Orientation-consistency.

We will see that some important tournament properties can be expressed in terms of a-inclusion invariance and a-exclusion invariance under an orientation-consistent class of tournament operations. The following useful lemma then specifies sufficient conditions for such properties to inherit from tournaments to weak tournaments.

Lemma 1 *Let S be a tournament solution, W a weak tournament, a an alternative, and F an orientation-consistent class of tournament operations.*

(i) *If S is a-inclusion invariant under F on all $T \in [W]$, so is $[S]$ on W.*
(ii) *If S is a-exclusion invariant under F on all $T \in [W]$, so is $[S]$ on W.*

Proof. For (i), assume that S is a-inclusion invariant under F on all $T \in [W]$. To prove that $[S]$ is also a-inclusion invariant under F on W, assume $a \in [S](W)$. By definition of $[S]$, we then have that $a \in S(T)$ for some $T \in [W]$. Consider this T and along with an arbitrary $f \in F$. Observe that, since $T \in [W]$, S is a-inclusion invariant under F on T. Hence, $a \in S(f(T))$. Recall that $f([W]) = \{f(T) : T \in [W]\}$ and hence $f(T) \in f([W])$. By orientation-consistency of f, we have $f([W]) = [f(W)]$ and thus $f(T) \in [f(W)]$. Thus, $a \in S(T')$ for some $T' \in [f(W)]$ and we may conclude that $a \in [S](f(W))$, as desired.

For (ii) the argument runs along analogous lines. Assume that S is a-exclusion invariant under F on all $T \in [W]$. We show that $[S]$ is also a-exclusion invariant under F on W and to this end assume $a \notin [S](W)$. Then, $a \notin S(T)$ for all $T \in [W]$. Now consider an arbitrary $f \in F$. To show that $a \notin [S](f(W))$, also consider an arbitrary $T \in [f(W)]$. Then, by orientation-consistency of f, we obtain $T \in f([W])$. Accordingly, $T = f(T')$ for some $T' \in [W]$. As $T' \in [W]$, we know that $a \notin S(T')$ and, by the same token, that S is a-exclusion invariant under F on T'. It then follows that $a \notin S(f(T'))$, that is, $a \notin S(T)$. Having chosen T arbitrarily from $[f(W)]$, it follows that $a \notin [S](f(W))$, which concludes the proof. □

4.2 Dominance-Based Properties

We first look at three properties that deal with changes in the dominance relation, namely monotonicity, independence of unchosen alternatives (IUA), and set-monotonicity. Each of these concepts convey an invariance of the choice set when some alternatives are strengthened with respect to some other alternatives. For two alternatives a and b, *strengthening* a against b refers to replacing $b \succ a$ or $b \sim a$ with $a \succ b$.[5] Formally, for a weak tournament $W = (A, \succsim)$ define $W_{a\succ b} = (A, \succsim')$, where

$$\succsim' \;=\; \succsim \setminus \{(b,a)\} \cup \{(a,b)\}.$$

Thus, $W_{a\succ b}$ is the weak tournament that is like W but with a strengthened with respect to b (unless $a \succ b$ in W, in which case $W_{a\succ b}$ is identical to W). For two alternatives a and b in U, let $f_{a\succ b}$ then be the tournament operation that maps each weak tournament W to $W_{a\succ b}$. These tournament operations are orientation-consistent, as the following lemma shows.

Lemma 2 *For all $a, b \in U$, the tournament operation $f_{a\succ b}$ is orientation-consistent.*

Proof. Let $a, b \in U$ and consider an arbitrary weak tournament $W = (A, \succ)$. If $a = b$, then trivially $f_{a\succ b}(W) = W$. Hence,

$$f_{a\succ b}([W]) = \{f_{a\succ b}(T) : T \in [W]\} = \{T : T \in [W]\} = [W] = [f_{a\succ b}(W)],$$

and for the remainder of the proof we may assume that $a \neq b$.

First consider an arbitrary tournament $T \in f_{a\succ b}([W])$. Then, $T = f_{a\succ b}(T')$ for some $T' \in [W]$. Observe that then $f_{a\succ b}(T') \in [f_{a\succ b}(W)]$, that is, $T \in [f_{a\succ b}(W)]$.

For the opposite direction, let T be an arbitrary tournament such that $T \in [f_{a\succ b}(W)]$. Either $a \succ b$ or $b \succsim a$ in W. If the former, then both $f_{a\succ b}(W) = W$ and $f_{a\succ b}(T) = T$. Hence, $f_{a\succ b}(T) \in [W]$ and, thus, $T \in f_{a\succ b}([W])$. If the latter, let $T' = f_{b\succ a}(T')$. Observe that then both $T' \in [W]$ and $f_{a\succ b}(T') = T$. It follows that $T \in f_{a\succ b}([W])$, which concludes the proof. □

A tournament solution is monotonic if a chosen alternative remains in the choice set when it is strengthened against some other alternative, while leaving everything else unchanged.

Definition 4 A generalized tournament solution S is *monotonic* if for all $W = (A, \succsim)$ and $a, b \in U$,

$$a \in S(W) \quad \text{implies} \quad a \in S(W_{a\succ b}).$$

It is easy to see that monotonicity can be phrased as an inclusion invariance condition. Invoking Lemma 1, we then obtain the following result.

[5] There is another way of strengthening a against b that is not captured by this definition, namely, replacing $b \succ a$ with $a \sim b$. Let us refer to this additional operation as a *$\sim$-strengthening.* The properties monotonicity, IUA, and set-monotonicity are usually defined in such a way that $\sim$-strengthenings are also taken into account. While we do not consider $\sim$-strengthenings, it can easily be shown that the conservative extension $[S]$ satisfies monotonicity, IUA, or set-monotonicity with respect to $\sim$-strengthenings whenever $[S]$ satisfies the respective property with respect to strengthenings as defined here.

Proposition 2 *Monotonicity is inherited by the conservative extension.*

Proof. Define, for each alternative $a \in U$,

$$F_a^{MON} = \{f_{a \succ b} : b \in U\}.$$

It can then easily be appreciated that a generalized tournament solution S is monotonic if and only if, for every alternative a, S is a-inclusion invariant under F_a^{MON} on every weak tournament W. By Lemma 2, we find that for every alternative $a \in U$, F_a^{MON} is a class of orientation-consistent tournament operations.

Now let S be a tournament solution that is monotonic on $\mathcal{T}$. Thus, for every weak tournament W and every alternative a, S is a-inclusion invariant under F_a^{MON} on every $T \in [W]$. By Lemma 1 it then follows that for every weak tournament W and every alternative a, $[S]$ is a-inclusion invariant under F_a^{MON} on W. We may conclude that $[S]$ is monotonic on $\mathcal{W}$. □

Independence of unchosen alternatives (IUA) prescribes that the choice set is invariant under any changes in the dominance relation among unchosen alternatives.

Definition 5 A generalized tournament solution S is *independent of unchosen alternatives* if for all $W = (A, \succsim)$ and $a, b \in U \setminus S(A)$,

$$S(W) = S(W_{a \succ b}).$$

Reasoning along similar lines as for monotonicity, we find that IUA is inherited from S to $[S]$.

Proposition 3 *Independence of unchosen alternatives is inherited by the conservative extension.*

Proof. For each $X \subseteq U$, let $F_X^{IUA} = \{f_{a \succ b} : a, b \in U \setminus X\}$. Observe that a generalized tournament solution S satisfies IUA if and only if, for every weak tournament W and every alternative a, S is both a-inclusion invariant and a-exclusion invariant under $F_{S(W)}^{IUA}$ on W. By virtue of Lemma 2, we find that F_X^{IUA} is a set of orientation-consistent tournament operations for each $X \subseteq U$.

Now let S be a tournament solution that satisfies IUA on $\mathcal{T}$. Consider a weak tournament W along with an arbitrary alternative a in U, and let $T \in [W]$. Consider arbitrary $b, c \in U \setminus [S](W)$. Since $S(T) \subseteq [S](W)$, we also have $b, c \in U \setminus S(T)$. And since S satisfies IUA, we get $S(T) = S(f_{b \succ c}(T))$. It follows that, for every $a \in U$, S is both a-inclusion and a-exclusion invariant under $F_{[S](W)}^{IUA}$ on every $T \in [W]$. Applying Lemma 1, we obtain that, for every $a \in U$, $[S]$ is also both a-inclusion and a-exclusion invariant under $F_{[S](W)}^{IUA}$ on W. We may therefore conclude that $[S]$ satisfies IUA on $\mathcal{W}$ as well. □

Set-monotonicity is a strengthening of both monotonicity and IUA and is the defining property in a characterization of Kelly-strategyproof tournament solutions (Brandt 2015). A tournament solution is *set-monotonic* if the choice set remains the same whenever some alternative is strengthened against some unchosen alternative.

Definition 6 A generalized tournament solution S is *set-monotonic* if for all $W = (A, \succsim)$, $a \in U$, and $b \in U \setminus S(A)$,

$$S(W) = S(W_{a \succ b}).$$

Analogously to Lemma 3, we can prove that set-monotonicity inherits from tournament solutions to their conservative extensions.

Proposition 4 *Set-monotonicity is inherited by the conservative extension.*

Proof. For each $X \subseteq U$, let $F_X^{SMON} = \{f_{a \succ b} : a \in U \text{ and } b \in U \setminus X\}$. By virtue of Lemma 2, for every X, F_X^{SMON} is a class of orientation-consistent tournament operations. Also observe that a generalized tournament solution S is set-monotonic if and only if, for every weak tournament W in $\mathcal{W}$ and every alternative a, S is both a-inclusion invariant and a-exclusion invariant under $F_{S(W)}^{SMON}$ on W.

Let S be a set-monotonic tournament solution. Consider a weak tournament W along with an arbitrary alternative $a \in U$ and let $T \in [W]$. Furthermore, let $b \in U$ and $c \in U \setminus [S](W)$. As $S(T) \subseteq [S](W)$, we also have $c \in U \setminus S(T)$. By set-monotonicity of S on $\mathcal{T}$, then $S(T) = S(f_{b \succ c}(T))$. Therefore, S is both a-inclusion and a-exclusion invariant under $F_{[S](W)}^{SMON}$ on all $T \in [W]$. By Lemma 1 it then follows that $[S]$ is both a-inclusion invariant and a-exclusion invariant under $F_{[S](W)}^{SMON}$ as well. □

4.3 Choice-Theoretic Properties

We now turn to a class of properties that relate choices from different feasible sets to each other. For all of these properties, the dominance relation $\succsim$ is fixed. We therefore write $S(A)$ for $S((A, \succsim))$ in order to simplify notation.

The central property in this section is *stability* (or self-stability) (Brandt and Harrenstein 2011), which requires that a set is chosen from two different sets of alternatives if and only if it is chosen from the union of these sets (see Figure 4).

Definition 7 A generalized tournament solution S is *stable* if for all weak tournaments $(A, \succ)$ and for all non-empty subsets $B, C, X \subseteq A$ with $X \subseteq B \cap C$,

$$X = S(B) = S(C) \quad \text{if and only if} \quad X = S(B \cup C).$$

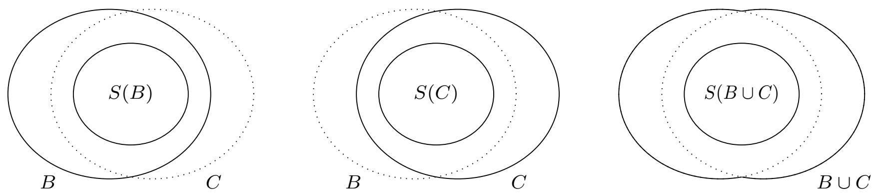

Fig. 4 A stable generalized tournament solution S chooses a set from $B \cup C$ (right) if and only if it chooses the same set from both B (left) and C (middle).

Stability is a rather demanding property that is only satisfied by few tournament solutions including the top cycle, the minimal covering set, and the bipartisan set. Stability is closely connected to rationalizability (Brandt and Harrenstein 2011) and together with monotonicity implies set-monotonicity and thereby Kelly-strategyproofness (Brandt 2015).

Stability can be factorized into conditions $\widehat{\alpha}$ and $\widehat{\gamma}$ by considering each implication in the above equivalence separately. The former is also known as Chernoff's *postulate* 5* (Chernoff 1954), the *strong superset property* (Bordes 1979), *outcast* (Aizerman and Aleskerov 1995), and the *attention filter axiom* (Masatlioglu et al. 2012).[6] A generalized tournament solution S satisfies $\widehat{\alpha}$, if for all non-empty sets of alternatives B and C,

$$S(B \cup C) \subseteq B \cap C \text{ implies } S(B \cup C) = S(B) = S(C).$$

Equivalently, S satisfies $\widehat{\alpha}$ if for all sets of alternatives B and C,

$$S(B) \subseteq C \subseteq B \text{ implies } S(B) = S(C).$$

A generalized tournament solution S satisfies $\widehat{\gamma}$, if for all sets of alternatives B and C,

$$S(B) = S(C) \text{ implies } S(B \cup C) = S(B) = S(C).$$

For a finer analysis, we split $\widehat{\alpha}$ and $\widehat{\gamma}$ into two conditions (see Brandt and Harrenstein 2011, Remark 1).

Definition 8 A generalized tournament solution S satisfies

- $\widehat{\alpha}_{\subseteq}$ if for all B, C, it holds that $S(B) \subseteq C \subseteq B$ implies $S(C) \subseteq S(B)$,
- $\widehat{\alpha}_{\supseteq}$ if for all B, C, it holds that $S(B) \subseteq C \subseteq B$ implies $S(C) \supseteq S(B)$,
- $\widehat{\gamma}_{\subseteq}$ if for all B, C, it holds that $S(B) = S(C)$ implies $S(B) \subseteq S(B \cup C)$, and
- $\widehat{\gamma}_{\supseteq}$ if for all B, C, it holds that $S(B) = S(C)$ implies $S(B) \supseteq S(B \cup C)$.

[6] We refer to Monjardet (2008) for a more thorough discussion of the origins of this condition.

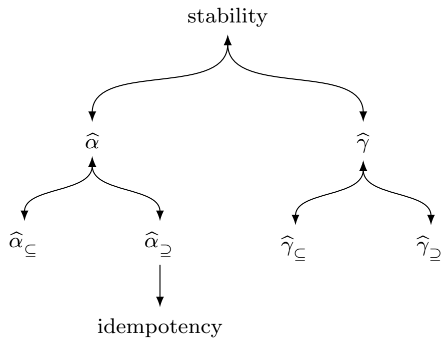

Fig. 5 Logical relationships between choice-theoretic properties.

Perhaps the most prominent among these four properties is $\widehat{\alpha}_\subseteq$, which has also been called the *weak superset property* or the *Aïzerman property* (e.g., Laslier 1997; Brandt 2009). It requires that the removal of losing alternatives cannot lead to new winning alternatives.

Obviously, for any generalized tournament solution S we have

$$\begin{aligned} S \text{ satisfies stability} &\quad \text{if and only if} \quad S \text{ satisfies } \widehat{\alpha} \text{ and } \widehat{\gamma},\\ S \text{ satisfies } \widehat{\alpha} &\quad \text{if and only if} \quad S \text{ satisfies } \widehat{\alpha}_\subseteq \text{ and } \widehat{\alpha}_\supseteq, \text{ and}\\ S \text{ satisfies } \widehat{\gamma} &\quad \text{if and only if} \quad S \text{ satisfies } \widehat{\gamma}_\subseteq \text{ and } \widehat{\gamma}_\supseteq. \end{aligned}$$

A generalized tournament solution is *idempotent* if the choice set is invariant under repeated application of the solution concept, i.e., $S(S(A)) = S(A)$ for all weak tournaments $W = (A, \succ)$. It is easily seen that $\widehat{\alpha}_\supseteq$ is stronger than idempotency since $S(W|_{S(W)}) \supseteq S(W)$ implies $S(W|_{S(W)}) = S(W)$. Figure 5 shows the logical relationships between stability and its weakenings.

For a feasible set B, we let f_B denote the tournament operation that maps a weak tournament $W = (A, \succsim)$ with $B \subseteq A$ to its restriction to B, i.e., $f_B(W) = W|_B$. Furthermore, define, for each $X \subseteq U$, the class $F_X^{\widehat{\alpha}} = \{f_B : X \subseteq B \subseteq U\}$ of tournament operations.

It is then easily seen that for every generalized tournament solution S,

(i) S satisfies $\widehat{\alpha}_\subseteq$ if and only if, for every W and every a, S is a-inclusion invariant under $F_{S(W)}^{\widehat{\alpha}}$ on W,
(ii) S satisfies $\widehat{\alpha}_\supseteq$ if and only if, for every W and every a, S is a-exclusion invariant under $F_{S(W)}^{\widehat{\alpha}}$ on W, and
(iii) S satisfies $\widehat{\alpha}$ if and only if, for every W and every a, S is both a-inclusion and a-exclusion invariant under $F_{S(W)}^{\widehat{\alpha}}$ on W.

Since for every $X \subseteq U$, the class $F_X^{\widehat{\alpha}}$ is orientation-consistent, we can apply Lemma 1 and obtain the following result.

Proposition 5 *$\widehat{\alpha}$, $\widehat{\alpha}_\subseteq$, and $\widehat{\alpha}_\supseteq$ are inherited by the conservative extension.*

Proof. We give the proof for $\phi = \widehat{\alpha}_{\subseteq}$. The argument for $\phi = \widehat{\alpha}_{\supseteq}$ runs along analogous lines. The case for $\phi = \widehat{\alpha}$ then follows as an immediate consequence.

Consider an arbitrary tournament solution S. By the above equivalences, we have that $[S]$ satisfies $\widehat{\alpha}_{\subseteq}$ if and only if, for all alternatives $a \in U$ and all weak tournaments W, $[S]$ is a-inclusion invariant under $F_{[S](W)}$ on W. Now assume that S satisfies $\widehat{\alpha}$ on $\mathcal{T}$. Also consider an arbitrary alternative $a \in U$, an arbitrary weak tournament $W = (A, \succsim)$, and an arbitrary $T \in [W]$. Furthermore, let $f \in F^{\widehat{\alpha}}_{[S](W)}$. Then, $f = f_B$ for some $S(W) \subseteq B \subseteq U$. As $S(T) \subseteq [S](W)$, also $S(T) \subseteq B \subseteq U$. Having assumed that S satisfies $\widehat{\alpha}_{\subseteq}$ on $\mathcal{T}$, we find that $a \in S(T)$ implies $a \in T|_B$, that is $a \in f_B(W)$. It follows that S is a-inclusion invariant under $F^{\widehat{\alpha}}_{[S](W)}$ on every $T \in [W]$. By Lemma 1, it follows that $[S]$ is also a-inclusion invariant under $F^{\widehat{\alpha}}_{[S](W)}$ on W. Having chosen W arbitrarily, we may conclude that $[S]$ satisfies $\widehat{\alpha}_{\subseteq}$ on $\mathcal{W}$ as well. □

For $\widehat{\gamma}$ and its descendants $\widehat{\gamma}_{\subseteq}$ and $\widehat{\gamma}_{\supseteq}$, no characterization similar in spirit to that of $\widehat{\alpha}$ is known. In fact, we were not able to prove that $\widehat{\gamma}_{\subseteq}$, $\widehat{\gamma}_{\supseteq}$, or $\widehat{\gamma}$ is inherited from a tournament solution S to its conservative extension $[S]$.[7] However, all three properties are inherited if S also satisfies $\widehat{\alpha}$.

Proposition 6 *Let S be a tournament solution that satisfies $\widehat{\alpha}$ and let $\phi \in \{\widehat{\gamma}, \widehat{\gamma}_{\subseteq}, \widehat{\gamma}_{\supseteq}\}$. If S satisfies property ϕ on $\mathcal{T}$, so does $[S]$ on $\mathcal{W}$.*

Proof. We give the proof for $\phi = \widehat{\gamma}$. The proofs for the other cases are analogous. Let S be a tournament solution satisfying $\widehat{\alpha}$ and $\widehat{\gamma}$ and let $W = (A \cup B, \succsim)$ be a weak tournament such that $[S](A) = [S](B) = X \subseteq A \cap B$. We need to show that $[S](A \cup B) = X$. By definition of $[S]$, we have

$$[S](A) = \bigcup_{T_A \in [W|_A]} S(T_A), \quad [S](B) = \bigcup_{T_B \in [W|_B]} S(T_B), \text{ and } \quad [S](A \cup B) = \bigcup_{T \in [W]} S(T).$$

We will show that for all $T \in [W]$, $S(T|_A) = S(T|_B) = S(T)$. The statement then follows from the trivial observation that every orientation of $W|_A$ can be obtained as a restriction of an orientation of W, i.e., for all $T_A \in [W|_A]$ there is a $T \in [W]$ such that $T|_A = T_A$.

Now consider an arbitrary $T \in [W]$. Obviously, $T|_A \in [W|_A]$ and $T|_B \in [W|_B]$. By assumption, we have $S(T|_A) \subseteq A \cap B$ and $S(T|_B) \subseteq A \cap B$. Applying $\widehat{\alpha}$ to A and $A \cap B$ yields

$$S(T|_A) = S(T|_{A \cap B}),$$

and applying $\widehat{\alpha}$ to B and $A \cap B$ yields

$$S(T|_B) = S(T|_{A \cap B}).$$

Therefore, $S(T|_A) = S(T|_B)$. Since S satisfies $\widehat{\gamma}$ on $\mathcal{T}$, this yields $S(T) = S(T|_A) = S(T|_B)$. □

[7] The same is true for Sen's original γ (e.g., Moulin 1986).

Since stability is equivalent to the conjunction of $\widehat{\alpha}$ and $\widehat{\gamma}$, the following statement follows as an immediate consequence of Propositions 5 and 6.

Corollary 1 *Stability is inherited by the conservative extension.*

Interestingly, requiring $\widehat{\alpha}$ so that $\widehat{\gamma}$ is inherited is less restrictive than it might seem because all common tournament solutions satisfy $\widehat{\alpha}$ *if and only if* they satisfy $\widehat{\gamma}$.[8] In general, however, it is the case that $\widehat{\alpha}$ and $\widehat{\gamma}$ are independent from each other, even though this requires the construction of rather artificial tournament solutions (see Brandt et al. 2017).

4.4 Composition-Consistency

We now consider a structural property that deals with sets of similar alternatives. A component of a tournament is a subset of alternatives that bear the same dominance relationship to all alternatives not in the set. A decomposition is a (not necessarily unique) partition of the alternatives into components. A decomposition induces a summary tournament with the components as alternatives. A tournament solution is then said to be *composition-consistent* if it selects the best alternatives from the components it selects from the summary tournament.

In order to extend the definition of composition-consistency to weak tournaments, we need to generalize the concept of a component.[9]

Definition 9 Let $W = (A, \succsim)$ be a weak tournament. A *component* of W is a feasible set $X \subseteq A$ such that X is a singleton or for all $y \in A \setminus X$, either $X \succ \{y\}$ or $\{y\} \succ X$.

The separate condition for singletons ensures that each alternative on its own constitutes a component in weak tournaments, as it is the case in tournaments as well. The following lemma establishes that components are preserved under orientation.

Lemma 3 *Let $W = (A, \succsim)$ be a weak tournament and $X \subseteq A$. Then, X is a component of W if and only if X is a component of every orientation $T \in [W]$.*

Proof. For the "only if"-direction, assume that X is a component of W and, for contradiction, that there is some orientation $T = (A, \succ')$ of W for which X is not a component. Then, there are $x, x' \in X$ and $y \in A \setminus X$ such that $x \succ' y$ and $y \succ' x'$. With X being a component of W, both $x \succ y$ and $x' \succ y$ or both $y \succ x$ and $y \succ x'$. Moreover, this has to hold in every orientation of W and a contradiction follows.

[8] For example, this statement holds for all tournament solutions considered in Section 5: TC, BP, and MC satisfy both $\widehat{\alpha}$ and $\widehat{\gamma}$, and CO, UC, BA, and TEQ satisfy neither $\widehat{\alpha}$ nor $\widehat{\gamma}$.

[9] We note that alternative definitions, such as the one discussed after Proposition 8, are conceivable.

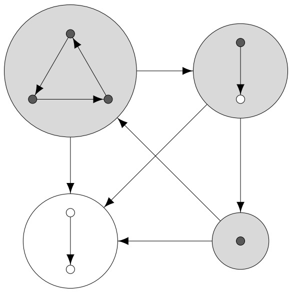

Fig. 6 Composition-consistency. The choice set of a composition-consistent tournament selects those alternatives—indicated by dark gray—that are best alternatives from the components it selects from the summary tournament—indicated by light gray.

For the "if"-direction, let X be a subset of A that is not a component of W. Then, in particular, X is not a singleton. Moreover, there are $x, x' \in X$ and $y \in A \setminus X$ such that one of the following cases obtains: both $x \succ y$ and $y \succ x$, both $x \succ y$ and $x' \sim y$, or both $x \sim y$ and $x' \sim y$. In each of these cases there is an orientation $T = (A, \succ')$ of W such that both $x \succ' y$ and $y \succ' x'$. □

Given the definition of a component, decompositions and summaries of weak tournaments, as well as composition-consistency of generalized tournament solutions, are then defined analogously to the case of tournaments.

A *decomposition* of a weak tournament $W = (A, \succsim)$ we define as a partition $\{X_1, \ldots, X_k\}$ of A such that each X_i is a component of W. Moreover, let $W_1 = (B_1, \succsim_1)$, ..., $W_k = (B_k, \succsim_k)$, and $\tilde{W} = (\{1, \ldots, k\}, \tilde{\succsim})$ be weak tournaments with $B_1, \ldots, B_k$ pairwise disjoint. Then, define the *product* $\prod(\tilde{W}, W_1, \ldots, W_k)$ of $W_1, \ldots, W_k$ with respect to $\tilde{W}$ as the weak tournament $(A, \succsim')$ such that $A = \bigcup_{i=1}^k B_i$ and, for all $1 \leq i, j \leq k$, all $b \in B_i$, and all $b' \in B_j$,

$$b \succsim' b' \text{ if and only if } i = j \text{ and } b \succsim_i b', \text{ or } i \neq j \text{ and } i \mathrel{\tilde{\succsim}} j.$$

Definition 10 A generalized tournament solution S is *composition-consistent (on $\mathcal{W}$)* if for all weak tournaments W, decompositions $\{X_1, \ldots, X_k\}$ of W, and $\tilde{W} = (\{1, \ldots, k\}, \tilde{\succsim})$ such that $W = \prod(\tilde{W}, W|_{X_1}, \ldots, W|_{X_k})$,

$$S(T) = \bigcup_{i \in S(\tilde{W})} S(W|_{X_i}).$$

Also see Figure 6 for an illustration of this concept.

Let $W = \prod(\tilde{W}, W_1, \ldots, W_k)$ where $W_i = (B_i, \succsim_i)$ for all i with $1 \leq i \leq k$. Observe that then $\{B_1, \ldots, B_k\}$ is a decomposition of W whenever $\tilde{W} \in \mathcal{T}$. If, moreover, no B_i is a singleton, the implication also holds in the opposite direction.

We find that composition-consistency is inherited to the conservative extension.

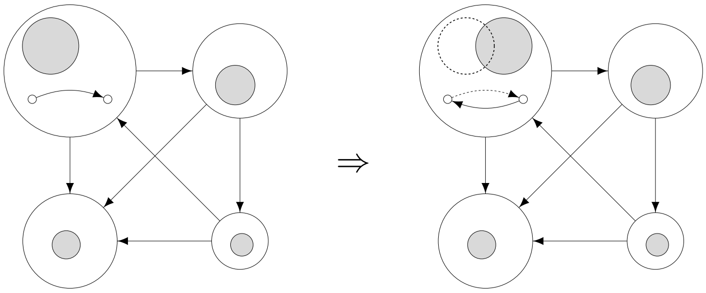

Fig. 7 Weak composition-consistency. If two tournaments only differ as to the dominance relation on some component X, then a weakly composition-consistent tournament solution selects in both tournaments the same alternatives from all components other than X and if it selects some alternative from X in one tournament, then it also selects some alternative from X in the other tournament.

Proposition 7 *Composition-consistency is inherited by the conservative extension.*

Proof. Assume S is composition-consistent. Consider an arbitrary weak tournament $W = (A, \succsim)$ along with a decomposition $\{X_1, \dots, X_k\}$ of W. Let $\tilde{W}$ be such that $W = \prod(\tilde{W}, W|_{X_1}, \dots, W|_{X_k})$. We have to show that, for all $a \in A$,

$$a \in [S](W) \quad \text{if and only if} \quad a \in [S](W|_{X_i}) \text{ for some } i \in [S](\tilde{W}).$$

First, assume that $a \in [S](W)$. Then, there is some orientation $T \in [W]$ such that $a \in S(T)$. By virtue of Lemma 3, $\{X_1, \dots, X_k\}$ is also a decomposition of T. Therefore, $T = \prod(\tilde{T}, T|_{X_1}, \dots, T|_{X_k})$, where $\tilde{T} \in [\tilde{W}]$ and $T|_{X_i} \in [W|_{X_i}]$ for all i with $1 \leq i \leq k$. Having assumed that S is composition-consistent, $a \in S(T|_{X_i})$ for some $i \in S(\tilde{T})$. As $\tilde{T} \in [\tilde{W}]$, it follows that $a \in [S](W|_{X_i})$ for some $i \in [S](\tilde{W})$.

For the opposite direction, assume $a \in [S](W|_{X_i})$ for some $i \in [S](\tilde{W})$. Then there are orientations $\tilde{T} \in [\tilde{W}]$ and $T|_{X_i} \in [W|_{X_i}]$ such that $i \in S(\tilde{T})$ and $a \in S(T|_{X_i})$. Let $T'|_{X_j} \in [W|_{X_j}]$ for all j distinct from i and define $T'' = \prod(\tilde{T}, T'_{X_1}, \dots, T_{X_i}, \dots, T'_{X_k})$. Observe that T'' is an orientation of W. By Lemma 3, moreover, $\{X_1, \dots, X_k\}$ is also a decomposition of T' and by composition-consistency of S we obtain $a \in S[T'']$. Finally, with T'' being an orientation of W, we may conclude that $a \in [S](W)$. □

The literature on tournaments also distinguishes the concept of *weak composition-consistency* (see, e.g., Moulin 1986; Laslier 1997). To show the inheritance of weak composition-consistency we first need the following definitions and notations.

For a feasible set A, we denote by $\mathcal{W}(A)$ the set of weak tournaments with A as the set of alternatives. For Y a component of a weak tournament $W = (A, \succsim)$ and $W' \in \mathcal{W}(Y)$ a weak tournament on Y, let $W^Y_{W'} = (A, \succsim'')$ denote the weak tournament that is like W except that the subtournament $W|_Y$ induced by component Y is replaced by W'.

Formally, for weak tournaments $W = \prod(\tilde{W}, W|_{X_1}, \ldots, W|_{X_k}, W|_Y)$ with components $X_1, \ldots, X_k$, Y and $W' \in \mathcal{W}(Y)$, we have $W^Y_{W'}$ denote $\prod(\tilde{W}, W|_{X_1}, \ldots, W|_{X_k}, W')$.

We are now in a position to give the definition of *weak composition-consistency* for weak tournaments. The definition extends the standard definition of weak composition-consistency for tournament solutions (Laslier 1997), which we will refer to as *weak composition-consistency on* $\mathcal{T}$. Also see Figure 7 for an illustration of this concept.

Definition 11 A generalized tournament solution S is *weakly composition-consistent (on* $\mathcal{W}$*)* if for all weak tournaments W, component Y of W, and $W' \in \mathcal{W}(Y)$,

(*i*) $S(W) \setminus Y = S(W^Y_{W'}) \setminus Y$, and
(*ii*) $S(W) \cap Y \neq \emptyset$ implies $S(W^Y_{W'}) \cap Y \neq \emptyset$.

Proposition 8 *Weak composition-consistency is inherited by the conservative extension.*

Proof. Let S be a tournament solution that is weakly composition-consistent on $\mathcal{T}$ and $W = (A, \succsim)$ a weak tournament with components $X_1, \ldots, X_k, Y$ such that $W = \prod(\tilde{W}, W|_{X_1}, \ldots, W|_{X_k}, W|_Y)$. Furthermore, let $W' \in \mathcal{W}(Y)$. We prove that

(*i*) $[S](W) \setminus Y = [S](W^Y_{W'}) \setminus Y$, and
(*ii*) $[S](W) \cap Y \neq \emptyset$ implies $[S](W^Y_{W'}) \cap Y \neq \emptyset$.

First observe that, by virtue of Lemma 3, for every orientation $T \in [W]$ we have that $\{X_1, \ldots, X_k, Y\}$ is a decomposition of T as well, i.e., $T = \prod(\tilde{T}, T|_{X_1}, \ldots, T|_{X_k}, T|_Y)$ for some $\tilde{T} \in \mathcal{T}(\{1, \ldots, k+1\})$. Moreover, it holds that $\tilde{T} \in [\tilde{W}]$, $T|_{X_1} \in [W|_{X_1}]$, $\ldots$, $T|_{X_k} \in [W|_{X_k}]$, and $T|_Y \in [W|_Y]$. Also observe that, for each $T' \in [W']$, we may assume that $T' \in \mathcal{T}(Y)$ and $T^Y_{T'} \in [W^Y_{W'}]$.

For (*i*), having assumed S to be weakly composition-consistent on $\mathcal{T}$, the following equivalences hold.

$$\begin{aligned}
&a \in [S](W) \setminus Y \\
&\text{iff } a \in S(T) \setminus Y \quad \text{for some } T \in [W] \\
&\text{iff } a \in S(T^Y_{T'}) \setminus Y \quad \text{for some } T \in [W] \text{ and some } T' \in [W'] \\
&\text{iff } a \in [S](W^Y_{W'}) \setminus Y.
\end{aligned}$$

For (*ii*), assume $[S](W) \cap Y \neq \emptyset$. Then, there is some orientation $T \in [W]$ such that $S(T) \cap Y \neq \emptyset$. Let $T \in [W']$. Then also $T \in \mathcal{T}(Y)$. Having assumed that S satisfies weak composition-consistency on $\mathcal{T}$, we obtain $S(T^Y_{T'}) \cap Y \neq \emptyset$. As $T^Y_{T'} \in [W^Y_{W'}]$, we conclude that $[S](W^Y_{W'}) \cap Y \neq \emptyset$. □

The notion of a component of a weak tournament defined here is rather strong and the associated concept of composition-consistency correspondingly weak. A natural stronger notion of composition-consistency could be based on a weaker concept of component. Thus, for weak tournaments $W = (A, \succsim)$, a component could be defined as a subset $X \subseteq A$ such that for all $y \in A \setminus X$, either $X \succ y$, $y \succ X$, or $X \sim y$. Observe that for such components Lemma 3 does no longer hold. Moreover, it can easily be seen that the conservative extension $[S]$ of *no* Condorcet-consistent tournament solution S satisfies the associated concept of composition-consistency. To appreciate this, let S be Condorcet-consistent and consider the weak tournament $W = (A, \succsim)$ with $A = \{a, b, c\}$ and $a \succ b$, $a \sim c$, and $b \sim c$ (see Figure 8). Observe that W can be oriented such that $b \succ c$ and $c \succ a$, resulting in a cyclical tournament from which every tournament solution chooses $\{a, b, c\}$. Hence, $[S](W) = \{a, b, c\}$. However, $\{\{a, b\}, \{c\}\}$ would be a decomposition under the alternate definition and, by Condorcet-consistency, alternative b would not be chosen by S from $T|_{\{a,b\}}$ for any orientation $T \in [W]$. Accordingly, if $[S]$ had been composition-consistent in the new sense, $b \notin [S](W)$, a contradiction.

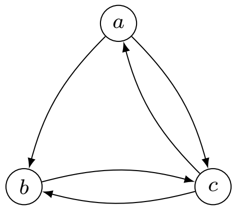

Fig. 8 Weak tournament $W = (\{a, b, c\}, \succsim)$ showing that stronger concepts of composition-consistency are not inherited by $[S]$ if S is Condorcet-consistent.

4.5 Regularity

A tournament solution is regular if it selects all alternatives from regular tournaments, i.e., tournaments in which the indegree and outdegree of every alternative are equal. Regularity extends naturally to weak tournaments.

For a weak tournament $W = (A, \succsim)$ and alternative $a \in A$, we let $d^+_W(a)$ and $d^-_W(a)$ denote the *outdegree* of a, i.e., cardinality of the dominion, and the *indegree* of a, i.e., the cardinality of the set of dominators of a, i.e.,

$$d^+_W(a) = |\{x \in A : a \succ x\}|, \text{ and}$$
$$d^-_W(a) = |\{x \in A : x \succ a\}|.$$

We omit the subscript when W is clear from the context.

A weak tournament $W = (A, \succ)$ is *regular* if $d^+(a) = d^-(a)$ for all $a \in A$. It can easily be appreciated that a tournament being regular implies its order to be odd. This, however, is not generally the case for weak tournaments, i.e., regular weak tournaments of even order exist.

A generalized tournament solution S is said to be *regular* if $S(W) = A$ for every regular weak tournament $W = (A, \succsim)$. The order of regular tournaments necessarily being odd, regularity of a tournament solution S as such does not impose any restriction on its behavior on tournaments of even order and, *ipso facto*, neither on the orientations of a weak tournament of even order. From this perspective, an arguably more natural extension of the concept of regularity for tournaments to weak tournaments takes into account the parity of weak tournaments. Thus, we say a generalized tournament solution S is *weakly regular* if $S(W) = A$ for every regular weak tournament $W = (A, \succsim)$ such that $|A|$ is odd. Observe that regularity implies weak regularity and that, on tournaments, the two notions coincide.

To prove the inheritance of weak regularity by the conservative extension we first show the following lemma, which says that in a regular weak tournament of odd order all indifferences can be eliminated without impairing regularity.

Lemma 4 *Let $W = (A, \succsim)$ be a regular weak tournament such that $|A|$ is odd. Then, there is a regular orientation $T \in [W]$.*

Proof sketch. Let $W = (A, \succsim)$ be a regular weak tournament such that $|A|$ is odd. If $|A| = 1$ the statement is trivial. So assume $|A| \geq 3$ and consider the *indifference graph* $G = (A, E)$, in which $\{x, y\} \in E$ if and only if $x \sim y$ and $x \neq y$. The degree of a vertex $a \in A$ in the graph G is given by $(|A| - 1) - d^+_W(a) - d^-_W(a)$. Since $|A| - 1$ is even and $d^+_W(a) = d^-_W(a)$, it follows that every vertex has an even degree in G. Accordingly, every connected component of G has an Eulerian cycle. Orienting each such (undirected) Eulerian cycle into a directed cycle and removing the corresponding pairs of alternatives from $\succsim$ results in a regular orientation $T \in [W]$. □

As an immediate consequence of Lemma 4 we obtain the following result.

Proposition 9 *Weak regularity is inherited by the conservative extension.*

Proof. Let S be a weakly regular tournament solution and consider an arbitrary regular weak tournament $W = (A, \succsim)$ such that $|A|$ is odd. By Lemma 4, there is a regular orientation $T \in [W]$. Accordingly, $S(T) = A$. It follows that $[S](W) = A$, as desired. □

For regularity, the situation is slightly more complicated. Although, the conservative extensions $[TC]$ and $[UC]$ of the regular tournament solutions TC and UC turn out to be regular, we find that regularity of S on tournaments does not in general extend to regularity of $[S]$ on weak tournaments.

Proposition 10 *There exists a regular tournament solution S such that $[S]$ is not regular on weak tournaments.*

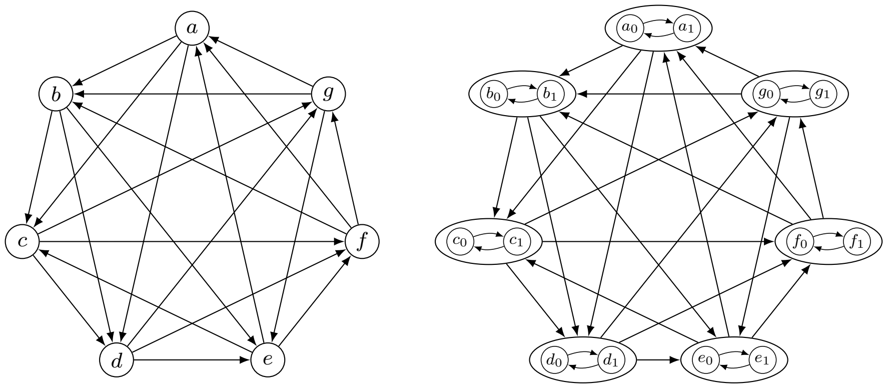

Fig. 9 The tournament T as depicted in (a) is regular but not vertex-homogeneous. For instance, there is no automorphism mapping alternative a to alternative b. The weak tournament W depicted in (b), results from T by 'replacing' *every* alternative x by a subtournament X on alternatives x_0 and x_1 such that $x_0 \sim x_1$.

Proof. A sequence $(a_1, \ldots, a_k)$ of alternatives is a *trajectory* if $i < j$ implies $a_i \succ a_j$. Let S be the tournament solution that selects the $\succ$-maximal elements of the trajectories that are of maximal length. Note that S is not regular: for the regular tournament T depicted in Figure 9(a), $S(T) = \{a, c, f\}$. Now define S^* as the tournament solution that is exactly like S apart from it choosing all alternatives from every regular tournament, i.e., for all tournaments $T = (A, \succ)$,

$$S^*(T) = \begin{cases} A & \text{if } T \text{ is regular,} \\ S(T) & \text{otherwise.} \end{cases}$$

By definition, S^* is regular.

Now consider the weak tournament $W = (A', \succsim)$ depicted in Figure 9(b). An easy check reveals that W is regular: observe that W results from the regular tournament T in Figure 9(a) by 'replacing' *every* alternative x by a subtournament X on alternatives x_0 and x_1 such that $x_0 \sim x_1$. It can also easily be verified that for every orientation $T' \in [W]$ we have that $S^*(T') \subseteq \{a_0, a_1, c_0, c_1, f_0, f_1\}$. (For an example, see the orientation in Figure 10, from which S^* selects $\{a_0, c_0, f_0\}$.) Accordingly, $[S^*](W) \subseteq \{a_0, a_1, c_0, c_1, f_0, f_1\}$. It follows that $[S^*]$ is not regular.[10] □

[10] A similar argument, involving a more complicated weak tournament can be given for the tournament solution BA_{reg} defined such that, for all tournaments $T = (A, \succ)$,

$$BA_{reg}(T) = \begin{cases} A & \text{if } T \text{ is regular,} \\ BA(T) & \text{otherwise.} \end{cases}$$

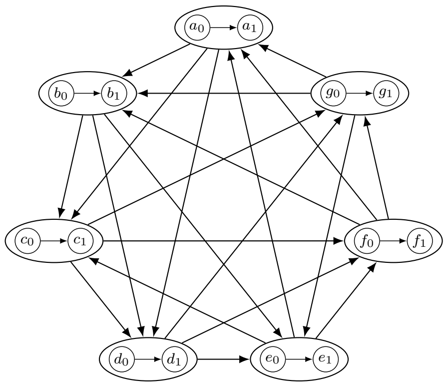

Fig. 10 An orientation of the weak tournament W.

5 Comparison to Other Generalizations

For many concrete tournament solutions, generalizations or extensions to weak tournaments have been proposed in the literature. In this section, we compare these extensions to the conservative extension (for definitions of the tournament solutions, please see Laslier 1997; Brandt et al. 2016). Note that none of these *ad hoc* extensions gives a "generic" way to extend tournament solutions to weak tournaments. For two generalized tournament solutions S and S', we write $S' \subset S$ if $S \neq S'$ and $S'(W) \subseteq S(W)$ for all weak tournaments W. In this case, we say that S' is a *refinement* of S.

All proofs and counter-examples are given in Appendix A.

Copeland Set. The Copeland set CO gives rise to a whole class of extensions that is parameterized by a number α between 0 and 1. The generalized tournament solution CO^α selects all alternatives that maximize the variant of the Copeland score in which each indifference contributes α points to an alternative's score (see, e.g., Faliszewski et al. 2009)). Henriet (1985) axiomatically characterized $CO^{\frac{1}{2}}$, arguably the most natural variant in this class. While it is easy to check that $[CO] \not\subset CO^\alpha$ for all $\alpha \in [0,1]$, the inclusion of CO^α in $[CO]$ turns out to depend on the value of α.

Proposition 11 *$CO^\alpha \subset [CO]$ if and only if $\frac{1}{2} \leq \alpha \leq 1$.*

Top Cycle. Schwartz (1972, 1986) defined two generalizations of the top cycle TC (see also Sen 1986; Brandt et al. 2009). *GETCHA* (or the *Smith set*) contains the maximal elements of the transitive closure of $\succsim$ whereas *GOCHA* (or the *Schwartz set*) contains the maximal elements of the transitive closure of $\succ$. GOCHA is always contained in GETCHA. A game-theoretical interpretation of TC gives rise to a further generalization. Duggan and Le Breton (2001) observed that the top cycle of a tournament T coincides with the unique *mixed* *saddle* $MS(T)$ of the underlying tournament game, and showed that the mixed saddle is still unique for games corresponding to weak tournaments. The solution MS is nested between GOCHA and GETCHA, and GETCHA coincides with $[TC]$.

Proposition 12 *GOCHA* $\subset MS \subset$ *GETCHA* $= [TC]$.

Bipartisan Set. Dutta and Laslier (1999) generalized the bipartisan set BP to the *essential set* ES, which is given by the set of all alternatives that are contained in the support of *some* Nash equilibrium of the underlying weak tournament game. It is easy to construct tournaments where ES is strictly smaller than $[BP]$, and there are also weak tournaments in which $[BP]$ is strictly contained in ES.

Proposition 13 $[BP] \not\subset ES$ *and* $ES \not\subset [BP]$.

Uncovered Set. Duggan (2013) surveyed several extensions of the covering relation to weak tournaments. Any such relation induces a generalization of the *uncovered set*. The so-called *deep covering* and *McKelvey covering* relations are particularly interesting extensions. Duggan (2013) showed that for all other generalizations of the covering relation he considered, the corresponding uncovered set is a refinement of the deep uncovered set UC_D. Another interesting property of UC_D is that it coincides with the conservative extension of UC.

Proposition 14 $UC_D = [UC]$.

It follows that all other UC generalizations considered by Duggan (2013) are refinements of $[UC]$.

Minimal Covering Set. The generalization of MC is only well-defined for the McKelvey covering relation and the deep covering relation. The corresponding generalized tournament solutions are known to satisfy stability. We have constructed a weak tournament in which $[MC]$ is strictly contained in both the McKelvey minimal covering set MC_M and the deep minimal covering set MC_D. There are also weak tournaments in which MC_M is strictly contained in $[MC]$.

Proposition 15 $[MC] \subset MC_D$, $[MC] \not\subset MC_M$, *and* $MC_M \not\subset [MC]$.

Corollary 1 implies that $[MC]$ satisfies the very demanding stability property. Hence, we have found a new sensible generalization of MC which is a refinement of MC_D and sometimes yields strictly smaller choice sets than MC_M.

Banks Set. Banks and Bordes (1988) discussed four different generalizations of the Banks set BA to weak tournaments, denoted by BA_1, BA_2, BA_3, and BA_4. Each of those generalizations is a refinement of the conservative extension $[BA]$.

Proposition 16 $BA_m \subset [BA]$ *for all* $m \in \{1, 2, 3, 4\}$.

Tournament Equilibrium Set. Finally, Schwartz (1990) suggested six ways to extend the *tournament equilibrium set* TEQ—and the notion of retentiveness in general—to weak tournaments. However, all of those variants can easily be shown to lead to disjoint minimal retentive sets even in very small tournaments, and none of the variants coincides with $[TEQ]$.

It is noteworthy that, in contrast to the conservative extension, some of the extensions discussed above fail to inherit properties from their corresponding tournament solutions. For instance, GOCHA violates $\widehat{\alpha}$ and BA_3 and BA_4 violate $\widehat{\alpha}_{\subseteq}$ (Banks and Bordes 1988).

Propositions 13 and 15 are surprising if one expects that every reasonable extension of a tournament solution is a refinement of its conservative extension. It is open to debate whether this assumption is unwarranted or whether these specific extensions are problematic.

6 Computational Complexity

When a tournament solution S is generalized via the conservative extension to $[S]$, it is natural to ask whether the choice set of $[S]$ can be computed efficiently. Since the number of orientations of a weak tournament can be exponential in the size of the weak tournament, tractability of the winner determination problem of S is a necessary, but not a sufficient, condition for the tractability of $[S]$. Computing the choice set of $[S]$ is mathematically equivalent to the problem of computing the set of *possible winners of S for a partially specified tournament.* The latter problem has been studied for the Copeland set CO, the top cycle TC, the uncovered set UC, and the bipartisan set BP.

Proposition 17 (Cook et al. 1998) *Computing $[CO]$ is in P.*

Proposition 18 (Lang et al. 2012) *Computing $[TC]$ is in P.*

Proposition 19 (Aziz et al. 2015) *Computing $[UC]$ is in P.*

Proposition 20 (Brill et al. 2016) *Computing $[BP]$ is NP-complete.*

Proposition 17 is shown using a polynomial-time reduction to maximum network flow; $[TC]$ and $[UC]$ can be computed by greedy algorithms. Proposition 20, which is much harder to prove, shows that tractability of S does not imply tractability of $[S]$ (assuming P $\neq$ NP). Note, however, that the *essential set*—a natural generalization of BP to weak tournaments (see Section 5)—*can* be computed in polynomial time. It is an open problem whether the conservative extension of the minimal covering set can be computed efficiently.

If computing winners is NP-complete for a tournament solution, the same is true for its conservative extension.

Lemma 5 *If winner determination for S is NP-complete, then winner determination for $[S]$ is NP-complete.*

Proof. Hardness of computing $[S]$ immediately follows from hardness of computing S, because $[S]$ and S agree whenever the weak tournament is in fact a tournament. For membership in NP, suppose that $x \in [S](W)$. Then we can guess an orientation $T \in [W]$ and an efficiently verifiable witness of the fact that $x \in S(T)$. □

Since the winner determination problem is NP-complete for the Banks set BA (Woeginger 2003), we have an immediate corollary.

Corollary 2 *Computing* $[BA]$ *is NP-complete.*

7 Conclusion

We have shown that the conservative extension inherits many desirable properties from its underlying tournament solution (see Table 1). In general, the conservative extension $[S]$ of tournament solution S is rather large and there might be more discriminating extensions of S that still satisfy its characterizing properties. However, the conservative extension may serve as "proof of concept" to show that generalizing a tournament solution in a meaningful way is possible in principle. Whether there are more discriminating solutions that are equally attractive is a different issue that can be settled for each tournament solution at hand.

Two interesting questions are whether there are other generic extensions that inherit the considered desirable properties and whether generic extensions can be characterized in terms of the properties they inherit. A challenging open computational problem is whether the conservative extension of the minimal covering set can be computed in polynomial time.

Acknowledgements

This work was supported by the Deutsche Forschungsgemeinschaft under grant BR 2312/7-2, by a Feodor Lynen research fellowship of the Alexander von Humboldt Foundation, and by the ERC under Advanced Grant 291528 ("RACE") and Starting Grant 639945 ("ACCORD"). We thank Vincent Conitzer, Christian Geist, and Hans Georg Seedig for helpful discussions and the anonymous reviewers and an associate editor for valuable feedback. Preliminary results of this paper have been presented at the 13th International Symposium on Artificial Intelligence and Mathematics (Fort Lauderdale, January 2014) and at the 28th AAAI Conference on Artificial Intelligence (Québec City, July 2014).

References

M. Aizerman and F. Aleskerov. *Theory of Choice*, volume 38 of *Studies in Mathematical and Managerial Economics*. North-Holland, 1995.

K. J. Arrow and H. Raynaud. *Social Choice and Multicriterion Decision-Making.* MIT Press, 1986.
H. Aziz, M. Brill, F. Fischer, P. Harrenstein, J. Lang, and H. G. Seedig. Possible and necessary winners of partial tournaments. *Journal of Artificial Intelligence Research*, 54:493–534, 2015.
J. S. Banks and G. A. Bordes. Voting games, indifference, and consistent sequential choice rules. *Social Choice and Welfare*, 5:31–44, 1988.
G. Bordes. Some more results on consistency, rationality and collective choice. In J. J. Laffont, editor, *Aggregation and Revelation of Preferences*, chapter 10, pages 175–197. North-Holland, 1979.
D. Bouyssou, T. Marchant, M. Pirlot, A. Tsoukiàs, and P. Vincke. *Evaluation and Decision Models: Stepping Stones for the Analyst.* Springer-Verlag, 2006.
F. Brandt. *Tournament Solutions – Extensions of Maximality and Their Applications to Decision-Making.* Habilitation Thesis, Faculty for Mathematics, Computer Science, and Statistics, University of Munich, 2009.
F. Brandt. Set-monotonicity implies Kelly-strategyproofness. *Social Choice and Welfare*, 45(4):793–804, 2015.
F. Brandt and P. Harrenstein. Characterization of dominance relations in finite coalitional games. *Theory and Decision*, 69(2):233–256, 2010.
F. Brandt and P. Harrenstein. Set-rationalizable choice and self-stability. *Journal of Economic Theory*, 146(4):1721–1731, 2011.
F. Brandt, F. Fischer, and P. Harrenstein. The computational complexity of choice sets. *Mathematical Logic Quarterly*, 55(4):444–459, 2009. Special Issue on Computational Social Choice.
F. Brandt, M. Brill, and P. Harrenstein. Tournament solutions. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 3. Cambridge University Press, 2016.
F. Brandt, M. Brill, H. G. Seedig, and W. Suksompong. On the structure of stable tournament solutions. *Economic Theory*, 2018. Forthcoming.
M. Brill and F. Fischer. The price of neutrality for the ranked pairs method. In *Proceedings of the 26th AAAI Conference on Artificial Intelligence (AAAI)*, pages 1299–1305. AAAI Press, 2012.
M. Brill, R. Freeman, and V. Conitzer. Computing possible and necessary equilibrium actions (and bipartisan set winners). In *Proceedings of the 30th AAAI Conference on Artificial Intelligence (AAAI)*, pages 418–424. AAAI Press, 2016.
H. Chernoff. Rational selection of decision functions. *Econometrica*, 22(4):422–443, 1954.
V. Conitzer, M. Rognlie, and L. Xia. Preference functions that score rankings and maximum likelihood estimation. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence (IJCAI)*, pages 109–115. AAAI Press, 2009.
W. J. Cook, W. H. Cunningham, W. R. Pulleyblank, and A. Schrijver. *Combinatorial Optimization.* Wiley and Sons, 1998.
J. Duggan. Uncovered sets. *Social Choice and Welfare*, 41(3):489–535, 2013.
J. Duggan and M. Le Breton. Dutta's minimal covering set and Shapley's saddles. *Journal of Economic Theory*, 70(1):257–265, 1996.
J. Duggan and M. Le Breton. Mixed refinements of Shapley's saddles and weak tournaments. *Social Choice and Welfare*, 18(1):65–78, 2001.
B. Dutta and J.-F. Laslier. Comparison functions and choice correspondences. *Social Choice and Welfare*, 16(4):513–532, 1999.
P. Faliszewski, E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. Llull and Copeland voting computationally resist bribery and constructive control. *Journal of Artificial Intelligence Research*, 35:275–341, 2009.
D. C. Fisher and J. Ryan. Tournament games and positive tournaments. *Journal of Graph Theory*, 19(2):217–236, 1995.
R. Freeman, M. Brill, and V. Conitzer. General tiebreaking schemes for computational social choice. In *Proceedings of the 14th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 1401–1409. IFAAMAS, 2015.
D. Henriet. The Copeland choice function: an axiomatic characterization. *Social Choice and Welfare*, 2(1):49–63, 1985.

G. Laffond, J.-F. Laslier, and M. Le Breton. The bipartisan set of a tournament game. *Games and Economic Behavior*, 5(1):182–201, 1993.

J. Lang, M. S. Pini, F. Rossi, D. Salvagnin, K. B. Venable, and T. Walsh. Winner determination in voting trees with incomplete preferences and weighted votes. *Journal of Autonomous Agents and Multi-Agent Systems*, 25(1):130–157, 2012.

J.-F. Laslier. *Tournament Solutions and Majority Voting*. Springer-Verlag, 1997.

Y. Masatlioglu, D. Nakajima, and E. Y. Ozbay. Revealed attention. *American Economic Review*, 102(5):2183–2205, 2012.

K. May. A set of independent, necessary and sufficient conditions for simple majority decisions. *Econometrica*, 20(4):680–684, 1952.

B. Monjardet. Statement of precedence and a comment on IIA terminology. *Games and Economic Behavior*, 62:736–738, 2008.

H. Moulin. Choosing from a tournament. *Social Choice and Welfare*, 3(4):271–291, 1986.

J. E. Peris and B. Subiza. Condorcet choice correspondences for weak tournaments. *Social Choice and Welfare*, 16(2):217–231, 1999.

T. Schwartz. Rationality and the myth of the maximum. *Noûs*, 6(2):97–117, 1972.

T. Schwartz. *The Logic of Collective Choice*. Columbia University Press, 1986.

T. Schwartz. Cyclic tournaments and cooperative majority voting: A solution. *Social Choice and Welfare*, 7(1):19–29, 1990.

A. K. Sen. Social choice theory. In K. J. Arrow and M. D. Intriligator, editors, *Handbook of Mathematical Economics*, volume 3, chapter 22, pages 1073–1181. Elsevier, 1986.

G. J. Woeginger. Banks winners in tournaments are difficult to recognize. *Social Choice and Welfare*, 20(3):523–528, 2003.

A Proofs for Section 5

Proposition 11 *$CO^\alpha \subset [CO]$ if and only if $\frac{1}{2} \leq \alpha \leq 1$.*

Proof. For notational convenience, define the *indifference graph* of a weak tournament $(A, \succsim)$ as the undirected graph (A, E) with $\{a, b\} \in E$ if and only if $a \sim b$ and $a \neq b$. Furthermore, $t(a)$ denotes the degree of alternative a in the indifference graph, i.e., the number of indifferences involving a. Recall that $d^+(a)$ denotes the cardinality of $\{b \in A : a \succ b\}$.

Let $0 \leq \alpha < \frac{1}{2}$. We will construct a weak tournament W_α such that $CO^\alpha(W_\alpha) \not\subseteq [CO](W_\alpha)$. Define

$$k = \left\lceil \frac{2-2\alpha}{1-2\alpha} \right\rceil .$$

The weak tournament $W_\alpha = (A, \succsim)$ has alternatives $A = \{a_i : 1 \leq i \leq k\} \cup \{x\} \cup \{b_j : 1 \leq j \leq k-1\}$. For all $i \leq k$, $a_i \succ x$, and for all $j \leq k-1$, $x \succ b_j$. Finally, $u \sim v$ for all pairs $(u, v) \in (A \setminus \{x\}) \times (A \setminus \{x\})$.

Let $s_\alpha(a)$ denote CO^α score of alternative $a \in A$, i.e., $s_\alpha(a) = d^+(a) + t(a) \cdot \alpha$. We have $s_\alpha(a_i) = 1 + (2k-2)\alpha$ for all $i \leq k$, $s_\alpha(x) = k-1$, and $s_\alpha(b_j) = (2k-2)\alpha$ for all $j \leq k-1$. The definition of k yields that $s_\alpha(x) \geq s_\alpha(a_i) > s_\alpha(b_j)$. Therefore, $x \in CO^\alpha(W_\alpha)$.

We will now show that $x \notin [CO](W_\alpha)$. Since x has no ties, we already know that its Copeland score is $k-1$ in any orientation of W_α. Let $T \in [W_\alpha]$ be such an orientation and let $\hat{T}$ be the restriction of T to $A \setminus \{x\}$. Since $\hat{T}$ has $2k-1$ alternatives, the average Copeland score in $\hat{T}$ is $k-1$. We distinguish two cases. If all alternatives in $A \setminus \{x\}$ have Copeland score $k-1$ in $\hat{T}$, then the Copeland score of alternative a_1 in T is k. If, on the other hand, not all alternatives in $A \setminus \{x\}$ have Copeland score $k-1$ in $\hat{T}$, then there exists an alternative $c \in A \setminus \{x\}$ that has a Copeland score of at least k in $\hat{T}$. The Copeland score of c in tournament T is therefore greater or equal to k. In both cases, we have found an alternative whose Copeland score in T is strictly greater than the Copeland score of x. It follows that $x \notin CO(T)$ for any orientation $T \in [W_\alpha]$ and, consequently, $x \notin [CO](W_\alpha)$.

Now let $\frac{1}{2} \leq \alpha \leq 1$. Consider a weak tournament G and an alternative $x \in CO^\alpha(G)$. We will show that $x \in [CO](G)$ by constructing an orientation $T \in [G]$ with $x \in CO(T)$.

Call an alternative y *active* if $t(y) > 0$, and *inactive* otherwise. As a first step, we make x inactive by letting x dominate all alternatives to which it was tied. Let s^* be Copeland score of x after this step and observe that all other alternatives have a CO^α score of at most s^*.

We then iteratively eliminate all remaining indifferences via the procedure described below. Throughout the procedure, the CO^α score of x will always remain maximal among the CO^α scores of all alternatives.

While there are still active alternatives, we iteratively do one of the following two operations:

- (*i*) if there is an active alternative y whose current CO^α score is less then or equal to $s^* - (1-\alpha)$, choose an arbitrary alternative z with $y \sim z$ and replace the indifference with $y \succ z$.
- (*ii*) if all active alternatives have a current CO^α score strictly greater than $s^* - (1-\alpha)$, find a cycle in the indifference graph and orient the cycle in one direction.

It is left to be shown that both operations maintain the invariant that all alternatives have a CO^α score of less than or equal to s^*. For the second operation, we also have to argue that there always exists a cycle in the indifference graph.

As for the first operation, observe that turning an indifference $a \sim b$ into $a \succ b$ increases the CO^α score of a by $1-\alpha$ and decreases the score of b by α. The first operation therefore maintains the invariant.

As for the second operation, the existence of a cycle in the indifference graph is guaranteed by the fact that every active alternative has at least two neighbors in the indifference graph. Indeed, an active alternative y with $t(y) = 1$ has a CO^α score of $d^+(y) + \alpha$, and since $d^+(y)$ is a natural number, it is impossible that

$$s^* - (1-\alpha) < d^+(y) + \alpha \leq s^* .$$

Furthermore, orienting the cycle (arbitrarily in one of the two possible directions) decreases the CO^α score of all involved alternatives by $2\alpha - 1 \geq 0$. ☐

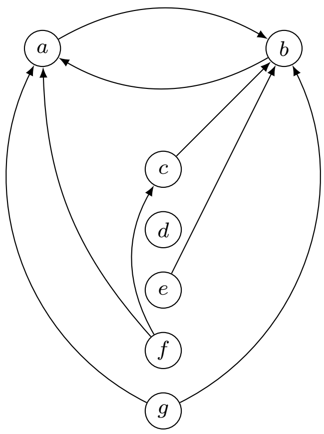

Fig. 11 Weak tournament $W_2 = (A, \succsim)$ with $A = \{a, b, c, d, e, f, g\}$. There is indifference between a and b (i.e., $a \sim b$). For all pairs for which no edge is depicted, the edge is pointing downwards. It can be verified that $ES(W_2) = \{a, b, d, e, f, g\}$ and $[BP](W_2) = \{a, b, c, d, e, f\}$. Thus, $g \in ES(W_2)$ but $g \notin [BP](W_2)$.

Proposition 12 *GOCHA* $\subset$ *MS* $\subset$ *GETCHA* $= [TC]$.

Proof. GOCHA $\subset$ GETCHA was shown by Schwartz (1972, 1986) and GOCHA $\subset$ MS $\subset$ GETCHA was shown by Duggan and Le Breton (2001). We show that GETCHA $= [TC]$. For a weak tournament $W = (A, \succsim)$, let $D^*_{\succsim}(a)$ denote the set of alternatives that can be reached by a via a $\succsim$-path.

For the inclusion GETCHA $\subseteq [TC]$, consider a weak tournament $W = (A, \succsim)$ and let $a \in$ GETCHA(W). By definition of GETCHA, $D^*_{\succsim}(a) = A$. We can construct an orientation $T_a \in [W]$ by iteratively substituting indifferences $x \sim y$ with $x \in D^*_{\succsim}(a)$ and $y \notin D^*_{\succsim}(a)$ with $x \succ y$. In T_a, alternative a can reach every other alternative via a $\succ$-path. Thus, $a \in TC(T_a) \subseteq [TC](W)$.

For the inclusion $[TC] \subseteq$ GETCHA, consider a weak tournament $W = (A, \succsim)$ and an arbitrary orientation $T \in [W]$. We show that $X = TC(T) \subseteq$ GETCHA(W). Assume for contradiction that there exists $x \in X \setminus$ GETCHA(W). Minimality of X implies that GETCHA(W) cannot be a strict subset of X. Therefore, there exists $y \in$ GETCHA$(W) \setminus X$. Since $X = TC(T)$, $x \succ y$. But this contradicts the assumption that $x \notin$ GETCHA(W). ☐

Proposition 13 $[BP] \not\subset ES$ *and* $ES \not\subset [BP]$.

Proof. For $[BP] \not\subset ES$, consider the weak tournament $W_1 = (\{a, b, c\}, \succsim)$ with $a \succ b$, $b \succ c$, and $a \sim c$. It is easily verified that $[BP](W_1) = \{a, b, c\}$ and $ES(W_1) = \{a, c\}$.

For $ES \not\subset [BP]$, consider the weak tournament $W_2 = (A, \succsim)$ depicted in Figure 11. ☐

Proposition 14 $UC_D = [UC]$.

Proof. In a tournament $T = (A, \succ)$, an alternative $y \in A$ is said to be *covered in* T if there exists an alternative $x \in A \setminus \{y\}$ such that (1) $x \succ y$ and (2) $z \succ x$ implies $z \succ y$ for all $z \in A \setminus \{x, y\}$. The uncovered set $UC(T)$ of T consists of all alternatives in A that are not covered in T.

In a weak tournament $W = (A, \succsim)$, an alternative $y \in A$ is said to be *deeply covered in* W if there exists an alternative $x \in A \setminus \{y\}$ such that (1) $x \succ y$ and (2) $z \succsim x$ implies $z \succ y$ for all $z \in A \setminus \{x, y\}$. The deep uncovered set $UC_D(W)$ of W consists of all alternatives in A that are not deeply covered in W.

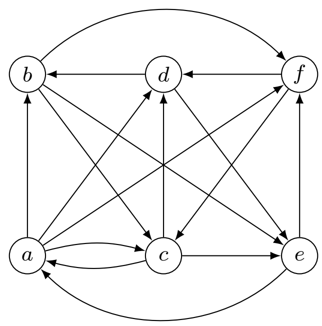

Fig. 12 A weak tournament W on $A = \{a, b, c, d, e, f\}$ with $MC_M(W) = MC_D(W) = A$ and $[MC](W) = A \setminus \{f\}$.

Let $W = (A, \succsim)$ be a weak tournament. The identity of $UC_D(W)$ and $[UC](W)$ follows from the fact that an alternative $a \in A$ is deeply covered in W if and only if a is covered in T for all orientations $T \in [W]$. □

Proposition 15 *$[MC] \subset MC_D$, $[MC] \not\subseteq MC_M$, and $MC_M \not\subseteq [MC]$.*

Proof. In a tournament $T = (A, \succ)$ the *minimal covering set* $MC(T)$ is defined as the unique smallest set $B \subseteq A$ such that $x \notin UC(T|_{B\cup\{x\}})$ for all $x \in A \setminus B$. Moreover, in a weak tournament $W = (A, \succsim)$, an alternative $y \in A$ is said to be *McKelvey-covered in* W if there exists an alternative $x \in A \setminus \{y\}$ such that (1) $x \succ y$, (2) $z \succ x$ implies $z \succ y$ for all $z \in A$, and (3) $y \succ z$ implies $x \succ z$. The *McKelvey uncovered set* $UC_M(W)$ of W consists of all alternatives in A that are not McKelvey-covered in W. In a weak tournament $W = (A, \succsim)$, the *minimal deep covering set* $MC_D(W)$ and the *minimal McKelvey covering set* $MC_M(W)$ are then defined as the (unique) smallest sets $B \subseteq A$ such that $x \notin UC_D(W|_{B\cup\{x\}})$ and $x \notin UC_M(W|_{B\cup\{x\}})$ for all $x \in A \setminus B$, respectively.

In tournaments, McKelvey-covering coincides with deep covering and is simply referred to as *covering*. Observe that if x deeply covers y in a weak tournament W, then x covers y in all tournaments $T \in [W]$.

We first show that $[MC] \subseteq MC_D$. Consider a weak tournament W and let $X = MC_D(W)$. By definition, X is externally stable w.r.t. deep covering, i.e., for all $y \in A \setminus X$ there exists $x \in A$ such that x deeply covers $y \in X \cup y$. Let T be an orientation of W. The above observation implies that X is externally stable in T. Since $MC(T)$ is contained in all externally stable sets, $MC(T) \subseteq X$.

In order to show that $[MC] \neq MC_D$ and $MC_M \not\subseteq [MC]$, consider the weak tournament W in Figure 12. It can be checked that both the McKelvey minimal covering set and the deep minimal covering set contain all alternatives, i.e., $MC_M(W) = MC_D(W) = \{a, b, c, d, e, f\}$. There are two orientations of W. Let T_1 be the orientation with $a \succ c$ and let T_2 be the orientation with $c \succ a$. Since $MC(T_1) = \{a, b, c, d, e\}$ and $MC(T_2) = \{a, b, c\}$, we have $[MC](W) = \{a, b, c, d, e\} \cup \{a, b, c\} = \{a, b, c, d, e\}$. In particular, $MC_D(W) \neq [MC](W)$ and $MC_M(W) \not\subseteq [MC](W)$.

In order to show that $[MC] \not\subseteq MC_M$, consider the tournament W' on $\{a_1, a_2, a_3, a_4, a_5, b\}$ such that $a_1 \succ a_2 \succ a_3 \succ a_4 \succ a_5 \succ a_1$, $a_i \succ b$ for $i \in \{1, 2\}$, $b \succ a_3$, and $x \sim y$ for all other pairs. It can be checked that $MC_M(W') = \{a_1, a_2, a_3, a_4, a_5\}$ and that $b \in [MC](W')$. □

Proposition 16 *$BA_m \subset [BA]$ for all $m \in \{1, 2, 3, 4\}$.*

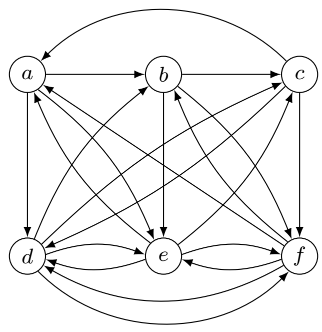

Fig. 13 A weak tournament W with $BA_1(W) = \{a, b, c\}$ and $BA_2(W) = \{d, e, f\}$.

Proof. We start by defining the four generalizations of the Banks set that were proposed by Banks and Bordes (1988). All generalizations are based on an extension of the definition of a trajectory or maximal transitive subset. Let $\mathbf{a} = (a_1, \ldots, a_k)$ be a sequence of alternatives of some weak tournament $W = (A, \succsim)$. Then, following Banks and Bordes (1988), we say

$$\mathbf{a} \text{ is } \textit{transitive}_1 \text{ if } a_i \succ a_j \text{ for all } 1 \leq i < j \leq k, \text{ and}$$
$$\mathbf{a} \text{ is } \textit{transitive}_2 \text{ if } a_i \succsim a_j \text{ for all } 1 \leq i < j \leq k$$

Furthermore, $\mathbf{a}$ is *transitive*$_3$ whenever $\mathbf{a}$ is transitive$_2$ and $a_i \succ a_j$ for some $1 \leq i < j \leq k$. Finally, $\mathbf{a}$ is *transitive*$_4$ whenever $\mathbf{a}$ is transitive$_2$ and $a_i \succ a_{i+1}$ for all $1 \leq i < k$. For $m \in \{1, 2, 3, 4\}$, we say that $\mathbf{a}$ is *maximal transitive*$_m$ in W if $(a, a_1, \ldots, a_k)$ is transitive$_m$ for no $a \in A$ and define BA_m such that, for all weak tournaments W,

$$BA_m(W) = \{a_1 : (a_1, \ldots, a_k) \text{ is maximal transitive}_m\}.$$

Banks and Bordes (1988) showed that each of their generalizations BA_m always selects a nonempty subset of alternatives. Moreover, on tournaments each BA_m coincides with the Banks set BA.

We now show that each of the generalizations BA_m is a refinement of $[BA]$.

First, let $m \in \{1, 2, 3, 4\}$ and $W = (A, \succsim)$ a weak tournament. Assume that $a \in BA_m(W)$, i.e., $a = a_1$ for some $\mathbf{a} = (a_1, \ldots, a_k)$ that is maximal transitive$_m$ in W. Observe that an orientation $T = (A, \succ')$ of W exists such that

(i) $a_i \succ' a_j$, for all a_i, a_j with $1 \leq i < j \leq k$, and
(ii) $a_i \sim x$ implies $a_i \succ' x$, for all a_i with $1 \leq i \leq k$ and $x \in A \setminus \{a_1, \ldots, a_k\}$.

Also observe that there is no $x \in A \setminus \{a_1, \ldots, a_k\}$ with $x \succ' a_i$ for all $1 \leq i \leq k$. Otherwise, also $x \succ a_i$ for all $1 \leq i \leq k$ and $\mathbf{a}$ would not be maximal transitive$_m$ in W. It thus follows that $\mathbf{a}$ is maximal transitive in T and that $a \in BA(T)$. As $T \in [W]$, we may conclude that $a \in [BA](W)$.

Second, Banks and Bordes (1988) demonstrate in their paper that for each $m \in \{2, 3, 4\}$, there is a weak tournament $W = (A, \succsim)$ with $BA_1(W) \cap BA_m(W) = \emptyset$ (see Figure 13 for the case $m = 2$). As none of the generalizations of the Banks set ever yields the empty set, there are $a, b \in A$ such that $a \in BA_1(W) \setminus BA_m(W)$ and $b \in BA_m(W) \setminus BA_1(W)$. Since both $BA_1(W) \subseteq [BA](W)$ and $BA_m(W) \subseteq [BA](W)$, it follows that $b \in [BA](W)$ whereas $b \notin BA_1(W)$ and $a \in [BA](W)$ although $a \notin BA_m(W)$. That is, $BA_m(W) \subset [BA](W)$ for each $m \in \{1, 2, 3, 4\}$, as desired. □