Computational Social Choice
From Arrow's impossibility to Fishburn's maximal lotteries

STACS 2015 Tutorial
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DSS
Decision Sciences & Systems

Motivation

‣ *What is "social choice theory"?*

   ‣ How to aggregate possibly conflicting preferences into collective choices in a fair and satisfactory way?

   ‣ Origins: mathematics, economics, and political science

   ‣ Essential ingredients

      - Autonomous agents (e.g., human or software agents)

      - A set of alternatives (depending on the application, alternatives can be political candidates, resource allocations, coalition structures, etc.)

      - Preferences over alternatives

      - Aggregation functions

‣ The axiomatic method will play a crucial role in this tutorial.

   ‣ Which formal properties should an aggregation function satisfy?

   ‣ Which of these properties can be satisfied simultaneously?

Handbook of Computational Social Choice
(Cambridge University Press, forthcoming in 2015)

Plurality

▸ Why are there different voting rules?

   ▸ What's wrong with plurality (the most widespread voting rule) where alternatives that are ranked first by most voters win?

   ▸ Consider a *preference profile* with 21 voters, who rank four alternatives as in the table on the right.

3	5	7	6
a	a	b	c
b	c	d	b
c	b	c	d
d	d	a	a

   ▸ Alternative *a* is the unique plurality winner despite

   - a majority of voters think *a* is the worst alternative,

   - *a* loses against *b*, *c*, and *d* in pairwise majority comparisons, and

   - if the preferences of all voters are reversed, *a* still wins.

   ▸ In July 2010, 22 experts on social choice theory met in France and voted on which voting rules should be used. Plurality received *no support* at all (among 18 rules).

5 Common Voting Rules

‣ ## Plurality

  ‣ Used in most democratic countries, ubiquitous

  ‣ Alternatives that are ranked first by most voters

‣ ## Borda

  ‣ Used in Slovenia, academic institutions, Eurovision song contest

  ‣ The most preferred alternative of each voter gets $m-1$ points, the second most-preferred $m-2$ points, etc. Alternatives with highest accumulated score win.

‣ ## Plurality with runoff

  ‣ Used to elect the President of France

  ‣ The two alternatives that are ranked first by most voters face off in a majority runoff.

5 Common Voting Rules (ctd.)

▸ Instant-runoff

  ‣ Used in Australia, Ireland, Malta, Academy awards

  ‣ Alternatives that are ranked first by the lowest number of voters are deleted. Repeat until no more alternatives can be deleted. The remaining alternatives win.

  ‣ In the UK 2011 alternative vote referendum, people chose plurality over instant-runoff.

▸ Sequential majority comparisons

  ‣ Used by US congress to pass laws (aka *amendment procedure*) and in many committees

  ‣ Alternatives that win a fixed sequence of pairwise comparisons (e.g., ((*a* vs. *b*) vs. *c*), etc.).

A Curious Preference Profile

33%	16%	3%	8%	18%	22%
a	b	c	c	d	e
b	d	d	e	e	c
c	c	b	b	c	b
d	e	a	d	b	d
e	a	e	a	a	a

Example due to Michel Balinski

▸ Plurality: **a wins**

▸ Borda: **b wins**

▸ Sequential majority comparisons (any order): **c wins**

▸ Instant-runoff: **d wins**

▸ Plurality with runoff: **e wins**

Rational Choice Theory

‣ A prerequisite for analyzing collective choice is to understand individual choice.

‣ Let U be a finite universe of alternatives.

‣ A *choice function* f maps a feasible set $A \subseteq U$ to a choice set $f(A) \subseteq A$.

  ‣ We require that $f(A) = \varnothing$ only if $A = \varnothing$.

‣ Not every choice function complies with our intuitive understanding of rationality.

  ‣ Certain patterns of choice from varying feasible sets may be deemed inconsistent, e.g., choosing a from $\{a,b,c\}$, but b from $\{a,b\}$.

| *A*	*f(A)*
ab	*a*
bc	*b*
ac	*a*
abc	*a*

Rationalizable Choice

▸ Binary preference relation $\succcurlyeq$ on U

  ▸ $x \succcurlyeq y$ is interpreted as *"x is at least as good as y".*

  ▸ $\succcurlyeq$ is assumed to be transitive and complete.

▸ Best alternatives

  ▸ For a binary relation $\succcurlyeq$ and a feasible set A,
    $Max(\succcurlyeq,A)= \{x{\in}A \mid \nexists y{\in}A$ such that $y > x\}$

▸ *f* is rationalizable if there exists a preference relation $\succcurlyeq$ on U such that $f(A)=Max(\succcurlyeq,A)$ for all A.

  ▸ The previously mentioned choice function *f* with $f(\{a,b,c\})=\{a\}$ and $f(\{a,b\})=\{b\}$ cannot be rationalized.

Consistent Choice

‣ It would be a nice if the non-existence of a rationalizing relation could be pointed out by finding inconsistencies.

‣ f satisfies consistency if for all A,B with $B \subseteq A$, $f(A) \cap B \neq \emptyset$ implies $f(B) = f(A) \cap B$.
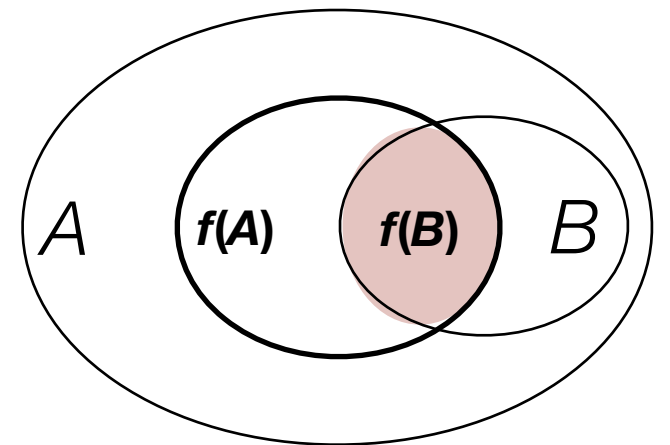
  ‣ Consequence: If x is chosen from a feasible set, then it is also chosen from all subsets that contain x.
  ‣ Example: Plurality does not satisfy consistency (when scores are computed for each feasible set).
    - $f(\{a,b,c\}) = \{a\}$ and $f(\{a,b\}) = \{b\}$

3	2	2
a	b	c
b	c	b
c	a	a

‣ Theorem (Samuelson, 1938; Arrow, 1959): A choice function is rationalizable iff it satisfies consistency.

From Choice to Social Choice

- *N* is a finite set of at least two voters.

- *R(U)* is the set of all preference relations over *U*.

- Every $R = (\succcurlyeq_1, ..., \succcurlyeq_{|N|}) \in R(U)^{|N|}$ is called a *preference profile*.

- A *social choice function (SCF)* is a function *f* that assigns a choice function to each preference profile.

  - An SCF is rationalizable (consistent) if its underlying choice functions are rationalizable (consistent) for all preference profiles.

  - We will write *f(R,A)* as a function of both *R* and *A*.

- Let $n_{xy} = |\{i \in N \mid x \succcurlyeq_i y\}|$ and define the *majority rule relation* as $(x\ R_M\ y) \Leftrightarrow n_{xy} > n_{yx}$.

# Condorcet's Paradox

Marquis de Condorcet

▸ Social choice from feasible sets of size two is easy.

  ▸ The *majority rule SCF* is defined as $f(R,\{x,y\}) = Max(R_M,\{x,y\})$.

  ▸ Majority rule can easily be characterized using uncontroversial axioms (e.g., May, 1952).

▸ Problems arise whenever there are more than two alternatives.

  ▸ Condorcet paradox (1785): R_M can be intransitive.

  ▸ Alternative x is a *Condorcet winner* in A if $x\ R_M\ y$ for all $y \in A\backslash\{x\}$.

  ▸ An SCF f is a *Condorcet extension* if $f(R,A)=\{x\}$ whenever x is a Condorcet winner in A.

| 1	1	1
a	b	c
b	c	a
c	a	b
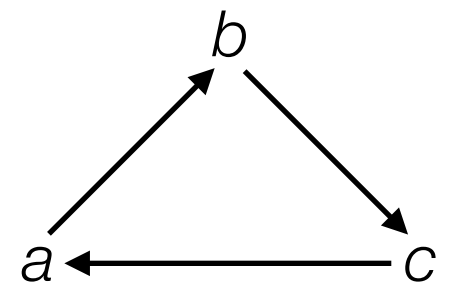
Arrow's Impossibility

Kenneth J. Arrow

▸ An SCF satisfies *independence of infeasible alternatives (IIA)* if the choice set only depends on preferences over alternatives within the feasible set.

▸ An SCF satisfies *Pareto-optimality* if an alternative will not be chosen if there exists another alternative such that all voters prefer the latter to the former.

▸ An SCF is *dictatorial* if there exists a voter whose most preferred alternative is always uniquely chosen.

▸ Theorem (Arrow, 1951): Every rationalizable SCF that satisfies IIA and Pareto-optimality is dictatorial when $|U| \geq 3$.

  ‣ Nipkow (2009) has verified a proof of Arrow's theorem using Isabelle.

  ‣ Tang & Lin (2009) reduced the statement to a finite base case that was solved by a computer.

What now?

‣ Rationalizability (or, equivalently, consistency) is incompatible with collective choice when $|U|≥3$.

  ‣ Dropping IIA offers little relief (Banks, 1995).

  ‣ Dropping Pareto-optimality offers little relief (Wilson, 1972).

  ‣ Dropping non-dictatorship is unacceptable.

‣ In this tutorial, we will consider two escape routes from Arrow's impossibility:

  ‣ SCFs that satisfy weaker notions of consistency

    - Top cycle, uncovered set, Banks set, tournament equilibrium set

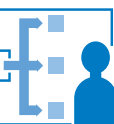 ‣ Randomized SCFs

    - Random dictatorship, maximal lotteries
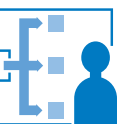
Weakly Consistent SCFs
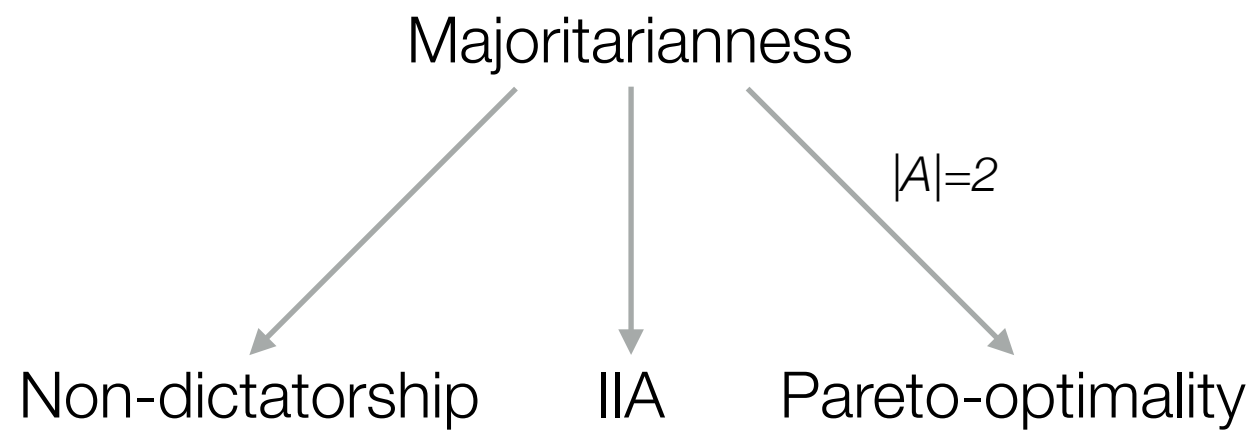
Tournaments
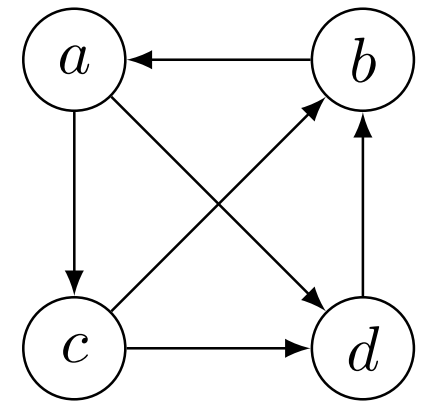
- ‣ For a given preference profile R, a feasible set A and majority rule R_M define a directed graph (A, R_M).
  - ‣ We say that b dominates a if $b\ R_M\ a$.
  - ‣ Every asymmetric directed graph is induced by some preference profile (McGarvey, 1953).

- ‣ A *majoritarian* SCF is an SCF whose output only depends on (A, R_M).
  - ‣ For simplicity, we will assume that individual preferences are anti-symmetric and that $|N|$ is odd. Hence, (A, R_M) is a tournament.
  - ‣ SCF f is said to be *finer* than SCF g if $f \subseteq g$.
  - ‣ Dominion $D(x) = \{y \in A \mid x\ R_M\ y\}$
  - ‣ Dominators $\overline{D}(x) = \{y \in A \mid y\ R_M\ x\}$

Majoritarianness

|A|=2

Non-dictatorship   IIA   Pareto-optimality

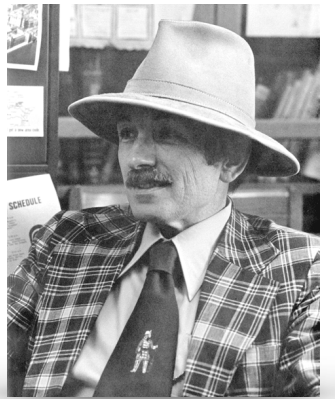Rationalizability/
Consistency

Arrow's Impossibility

Expansion

Weak expansion

Strong retentiveness

Retentiveness

The Top Cycle


John I. Good

‣ Consistency can be weakened to expansion:
$B{\subseteq}A$ and $f(A){\cap}B \neq \varnothing$ implies $f(B) \subseteq f(A)$.

‣ Theorem (Bordes, 1976): There is a unique finest majoritarian SCF satisfying expansion: the top cycle.

‣ A *dominant set* is a nonempty set of alternatives $B{\subseteq}A$ such that for all $x{\in}B$ and $y{\in}A{\setminus}B$, $x\,R_M\,y$.

  ‣ The set of dominant sets is totally ordered by set inclusion (Good, 1971).

  ‣ Hence, every tournament contains a unique minimal dominant set called the top cycle (*TC*).
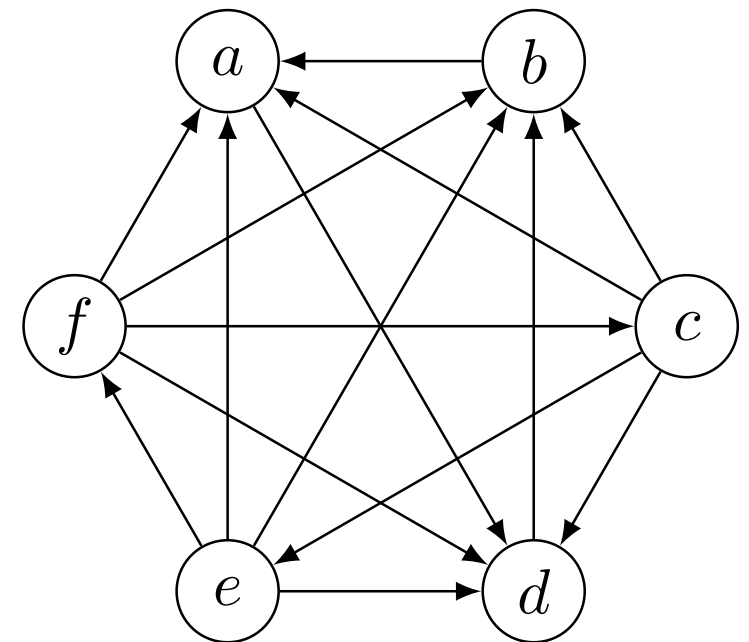
  ‣ *TC* is a Condorcet extension.

Examples
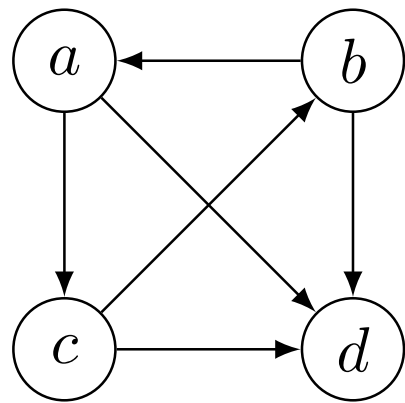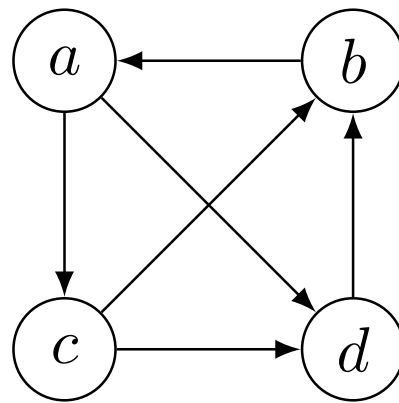
$TC(A,R_M)=\{a,b,c\}$ $TC(A,R_M)=\{a,b,c,d\}$ $TC(A,R_M)=\{c,e,f\}$

Transitive Closure

▸ The essence of Condorcet's paradox and Arrow's impossibility is that majority rule fails to be transitive.

  ‣ Why not just take the transitive (reflexive) closure $R_M{}^*$?

▸ Theorem (Deb, 1977): $TC(A, R_M) = Max(R_M{}^*, A)$.

▸ Consequences

  ‣ *TC* itself is a cycle. It is the source component in the DAG (directed acyclic graph) of strongly connected components.

  ‣ Linear-time algorithms for computing *TC* using Kosaraju's or Tarjan's algorithm for finding strongly connected components

    - Alternatively, one can initialize working set *B* with all alternatives of maximal outdegree and then iteratively add all alternatives that dominate an alternative in *B* until no more such alternatives can be found.
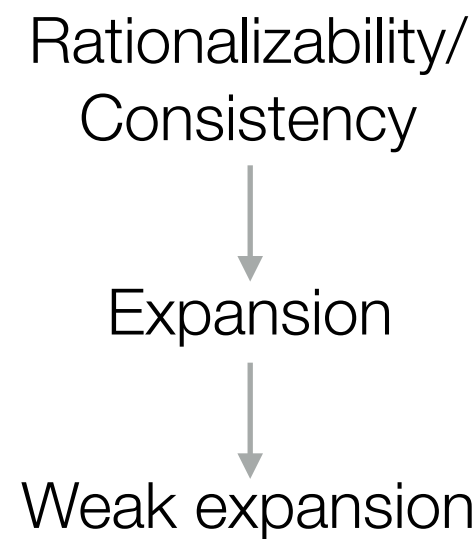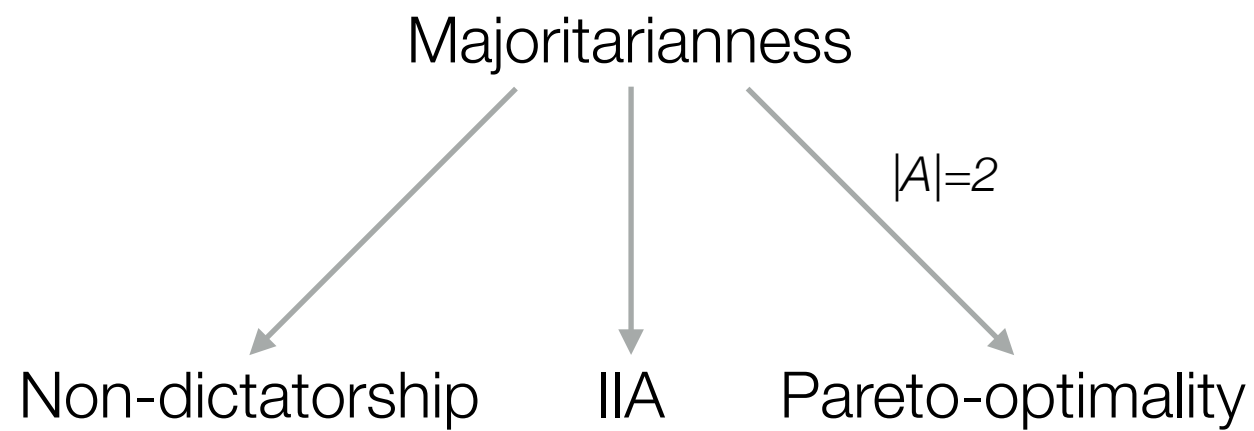
Top Cycle and Pareto-Optimality

▸ The top cycle is very large.

▸ In fact, it is so large that it fails to be Pareto-optimal when there are more than three alternatives (Ferejohn & Grether, 1977).

1	**1**	**1**
a	*b*	*d*
b	*c*	*a*
c	*d*	*b*
d	*a*	*c*

▸ Since Pareto-optimality is an essential ingredient of Arrow's impossibility, this escape route is (so far) not entirely convincing.

   ▸ Although, technically, Arrow's theorem only requires Pareto-optimality for two-element sets (which the top cycle satisfies).

Majoritarianness

Non-dictatorship   IIA   Pareto-optimality |A|=2

Rationalizability/
Consistency

Expansion

Weak expansion

Arrow's Impossibility

Top Cycle (TC)
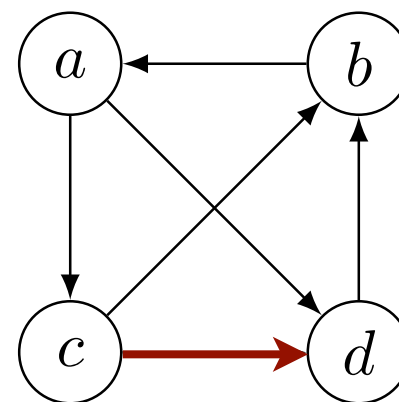
The Uncovered Set

Peter C. Fishburn

Nicholas Miller

‣ Expansion can be further weakened to weak expansion: $f(A) \cap f(B) \subseteq f(A \cup B)$.

‣ Theorem (Moulin, 1986): There is a unique finest majoritarian SCF satisfying weak expansion: the uncovered set.

‣ Given a tournament (A, R_M), x covers y (x C y), if $D(y) \subset D(x)$.

  ‣ Proposed independently by Fishburn (1977) and Miller (1980)

  ‣ Transitive subrelation of majority rule

  ‣ The uncovered set (UC) consists of all uncovered alternatives, i.e., $UC(A, P_M) = Max(C, A)$.

Examples
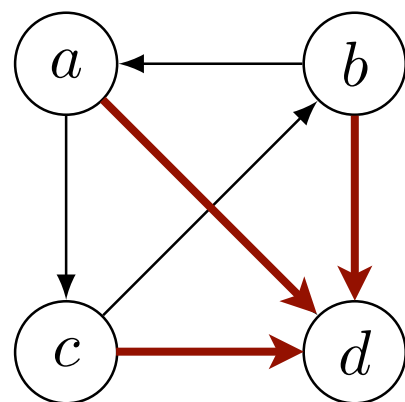
$UC(A,R_M)=\{a,b,c\}$

$UC(A,R_M)=\{a,b,c\}$

$TC(A,R_M)=\{a,b,c,d\}$

Properties of the Uncovered Set

‣ Since expansion ⇒ weak expansion, *UC⊆TC*.

  ‣ *UC* is a Condorcet extension.

‣ *UC* satisfies Pareto-optimality.

  ‣ Theorem (B. and Geist, 2014): *UC* is the largest majoritarian SCF satisfying Pareto-optimality.

‣ How can the uncovered set be efficiently computed?

  ‣ Straightforward $O(n^3)$ algorithm that computes the covering relation for every pair of alternatives

  ‣ Can we do better than that?

Uncovered Set Algorithm

▸ Equivalent characterization of *UC*

  ▸ Theorem (Shepsle & Weingast, 1984): *UC* consists precisely of all alternatives that reach every other alternative in at most two steps.

  - Such alternatives are called kings in graph theory.

▸ Hence, *UC* can be computed by squaring the tournament's adjacency matrix.

  ▸ Fastest known matrix multiplication algorithm (Le Gall, 2014): $O(n^{2.3728639})$

  ▸ Just slightly faster than Vassilevska Williams, 2011: $O(n^{2.372873})$

  ▸ Based on Coppersmith & Winograd (1990): $O(n^{2.376})$

  ▸ Matrix multiplication is believed to be feasible in linear time ($O(n^2)$).
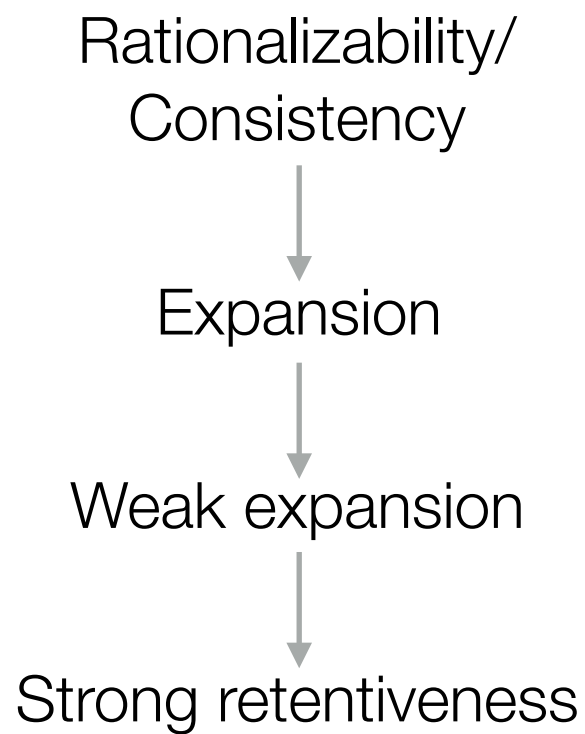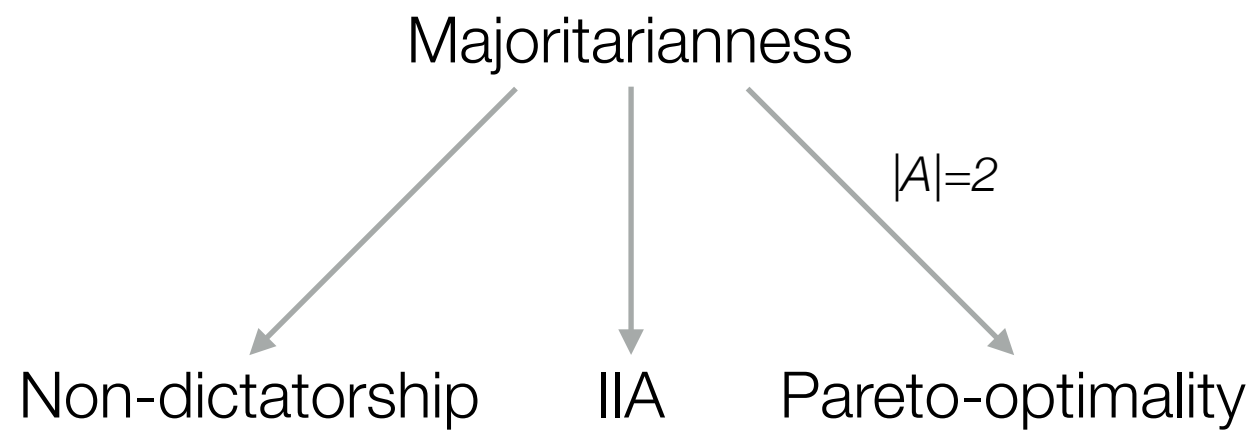
Uncovered Set Algorithm (Example)
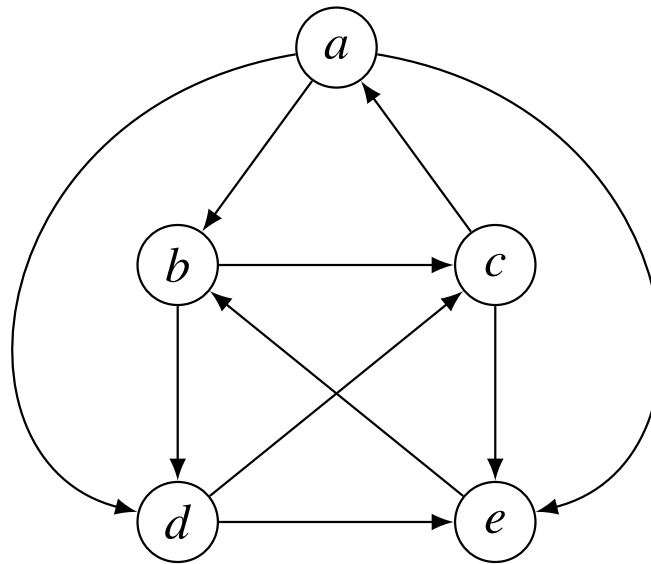
$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Majoritarianness

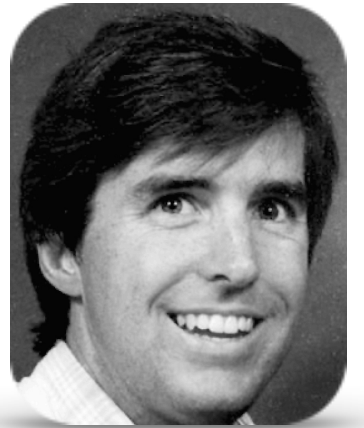Non-dictatorship   IIA   Pareto-optimality $|A|=2$

Rationalizability/
Consistency   **Arrow's Impossibility**

Expansion   **Top Cycle (TC)**

Weak expansion   **Uncovered Set (UC)**
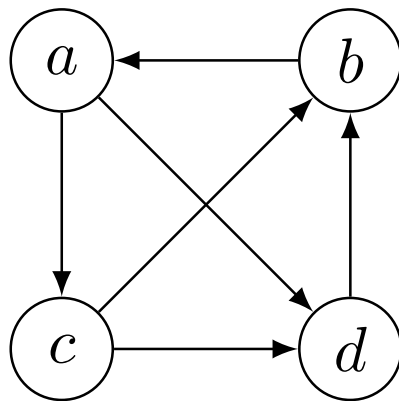
Strong retentiveness

Banks Set

Jeffrey S. Banks

▸ Weak expansion can be weakened to strong retentiveness: $f(\overline{D}(x)) \subseteq f(A)$ for all $x \in A$.

▸ Theorem (B., 2011): There is a unique finest majoritarian SCF satisfying strong retentiveness: the Banks set.

▸ A *transitive subset* of a tournament (A, R_M) is a set of alternatives $B \subseteq A$ such that R_M is transitive within B.

▸ Let $Trans(A, R_M) = \{B \subseteq A \mid B$ is transitive$\}$.

▸ The Banks set (BA) consists of the maximal elements of all inclusion-maximal transitive subsets (Banks, 1985), i.e., $BA(A, R_M) = \{Max(R_M, B) \mid B \in Max(\supseteq, Trans(A, R_M))\}$
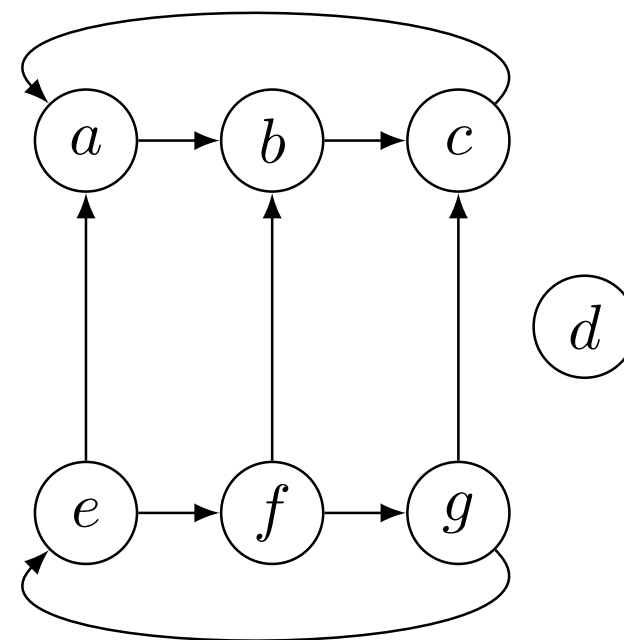
Examples

$UC(A,R_M)=\{a,b,c\}$

$BA(A,R_M)=\{a,b,c\}$

$TC(A,R_M)=\{a,b,c,d,e,f,g\}$

$UC(A,R_M)=\{a,b,c,d\}$

$BA(A,R_M)=\{a,b,c\}$

Properties of the Banks Set

‣ Since expansion ⇒ weak expansion ⇒ strong retentiveness,

  BA⊆UC⊆TC.

  ‣ As a consequence, *BA* is a Condorcet extension and satisfies Pareto-optimality.

‣ Random alternatives in *BA* can be found in linear time by iteratively constructing maximal transitive sets.

‣ Yet, computing the Banks set is NP-hard (Woeginger, 2003) and remains NP-hard even for 5 voters (B. et al., 2013).

‣ Strong retentiveness can be further weakened to retentiveness:
  $f(\overline{\overline{D}}(x)) \subseteq f(A)$ for all $x \in f(A)$.
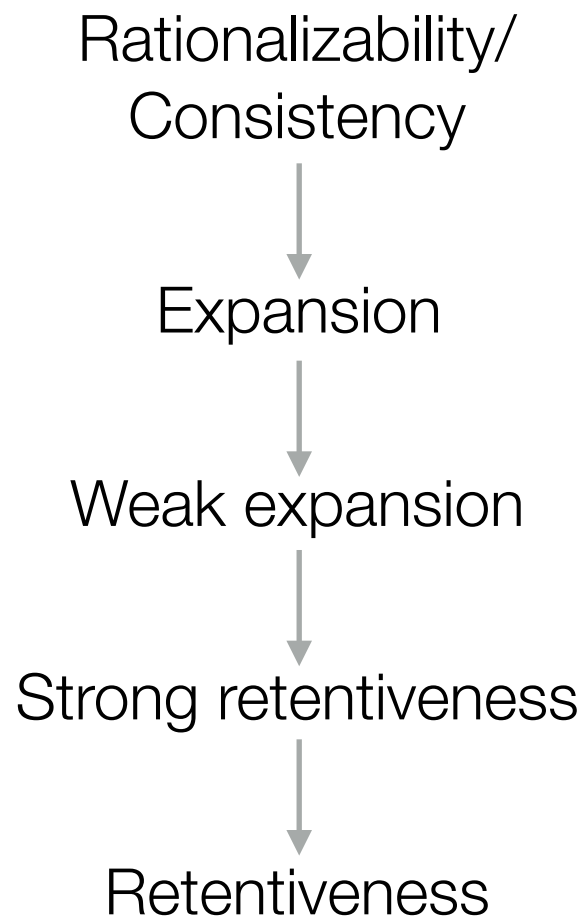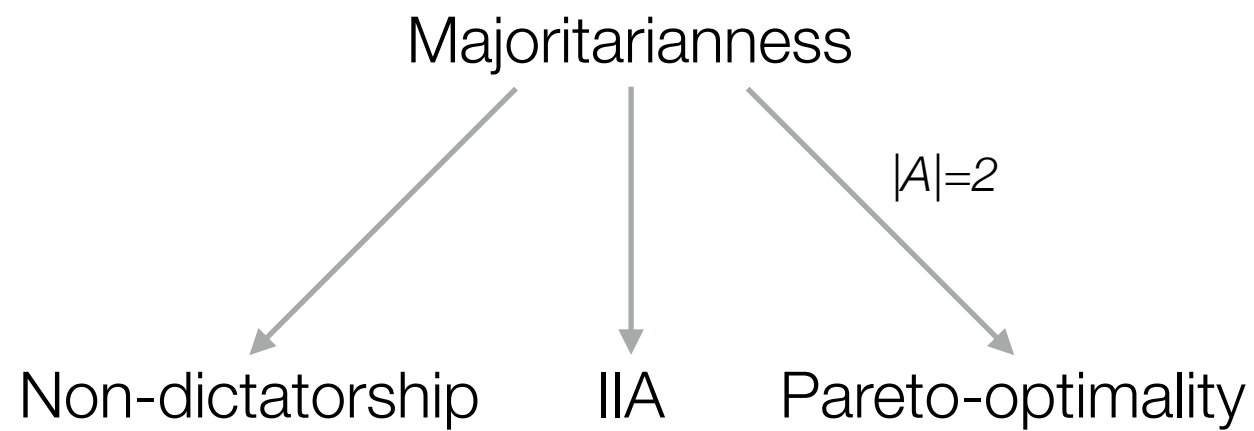
Majoritarianness

Non-dictatorship IIA Pareto-optimality

|A|=2

Rationalizability/
Consistency

Arrow's Impossibility

Expansion

Top Cycle (TC)

Weak expansion

Uncovered Set (UC)

Strong retentiveness

Banks Set (BA)

Retentiveness

Tournament Equilibrium Set

Thomas Schwartz

▸ Let *f* be an arbitrary choice function.

    ▸ A non-empty set of alternatives *B* is *f-retentive* if $f(\overline{D}(x)) \subseteq B$ for all $x \in B$.

    ▸ Idea: No alternative in the set should be "properly" dominated by an outside alternative.

▸ $\mathring{f}$ is a new choice function that yields the union of all inclusion-minimal *f*-retentive sets.

    ▸ $\mathring{f}$ satisfies retentiveness.

▸ The tournament equilibrium set (*TEQ*) of a tournament is defined as $TEQ = T\mathring{E}Q$.

    ▸ Recursive definition (unique fixed point of ring-operator)

    ▸ Theorem (Schwartz, 1990): $TEQ \subseteq BA$.
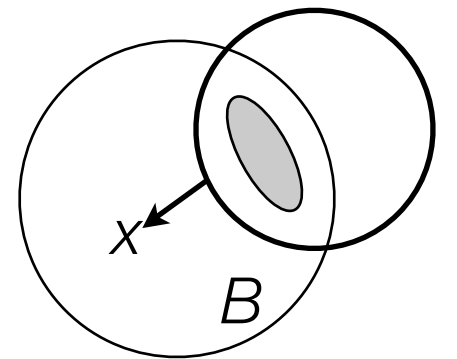
Example

- {*a,b,c*} is the unique minimal *TEQ*-retentive set.
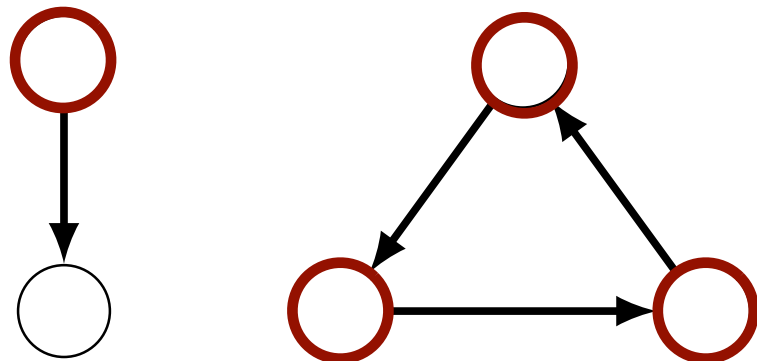 - $TEQ(\overline{D}(a)) = TEQ(\{c\}) = \{c\}$
  - $TEQ(\overline{D}(b)) = TEQ(\{a,e\}) = \{a\}$
  - $TEQ(\overline{D}(c)) = TEQ(\{b,d\}) = \{b\}$
  - $TEQ(\overline{D}(d)) = TEQ(\{a,b\}) = \{a\}$
  - $TEQ(\overline{D}(e)) = TEQ(\{a,c,d\}) = \{a,c,d\}$
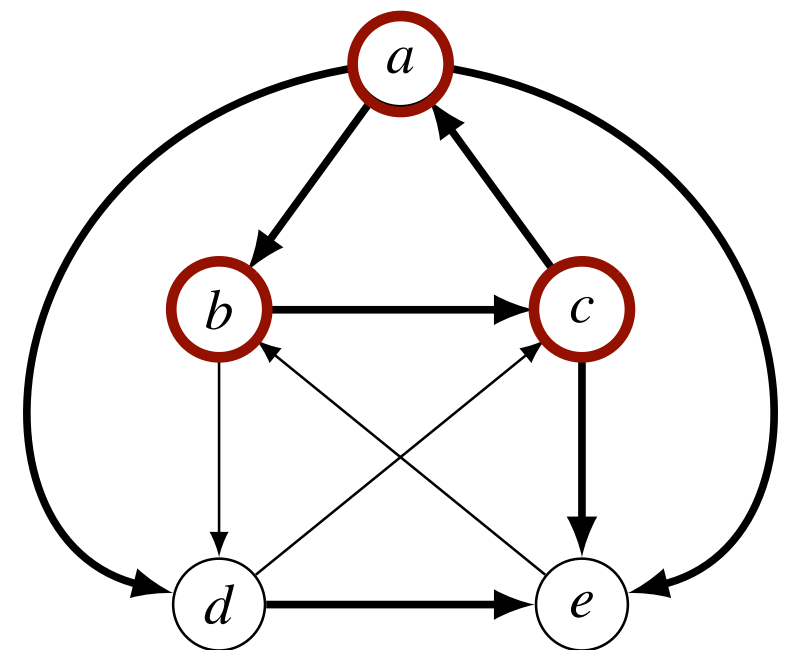
A thick edge from y to x denotes that $y \in TEQ(\overline{D}(x))$.

Properties of TEQ

‣ Computing *TEQ* is NP-hard (B. et al., 2010) and remains NP-hard even for 7 voters (Bachmeier et al., 2015).

  ‣ The best known upper bound is PSPACE!

‣ Theorem (Laffond et al., 1993; Houy 2009; B., 2011; B. and Harrenstein, 2011): The following statements are equivalent:

  ‣ Every tournament contains a unique minimal *TEQ*-retentive set. (Schwartz' Conjecture, 1990)

  ‣ *TEQ* is the unique finest majoritarian SCF satisfying retentiveness.

  ‣ *TEQ* satisfies monotonicity (and many other desirable properties).

‣ All or nothing:
Either *TEQ* is a most appealing SCF or it is severely flawed.

Schwartz's Conjecture
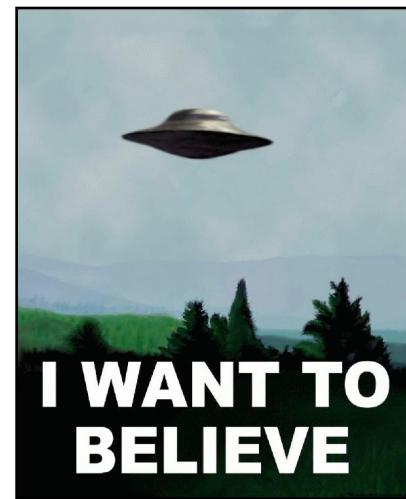
- There exists no counterexample with less than 13 alternatives (154 billion tournaments have been checked).

  - *TEQ* satisfies all nice properties if $|A|<13$.

- No counterexample was found by searching billions of random tournaments with up to 50 alternatives.

  - Checking significantly larger tournaments is intractable.

- Many non-trivial weakenings of Schwartz's conjecture are known to hold (Good, 1971; Dutta, 1988; B. et al., 2010; B., 2011).

- Theorem (B., Chudnovsky, Kim, Liu, Norin, Scott, Seymour, and Thomassé, 2012): Schwartz's conjecture is false.

Aftermath

- Non-constructive proof relying on a probabilistic argument by Erdős and Moser (1964)

  - Neither the counter-example nor its size can be deduced from proof.

  - Smallest counter-example of this type requires about 10^{136} alternatives.

- More recently, a counter-example with 24 alternatives was found with the help of a computer (B. & Seedig, 2013).

- In principle, *TEQ* is severely flawed. However, counter-examples are so extremely rare that this has no practical consequences.

  - This casts doubt on the axiomatic method.

Weakly Consistent SCFs
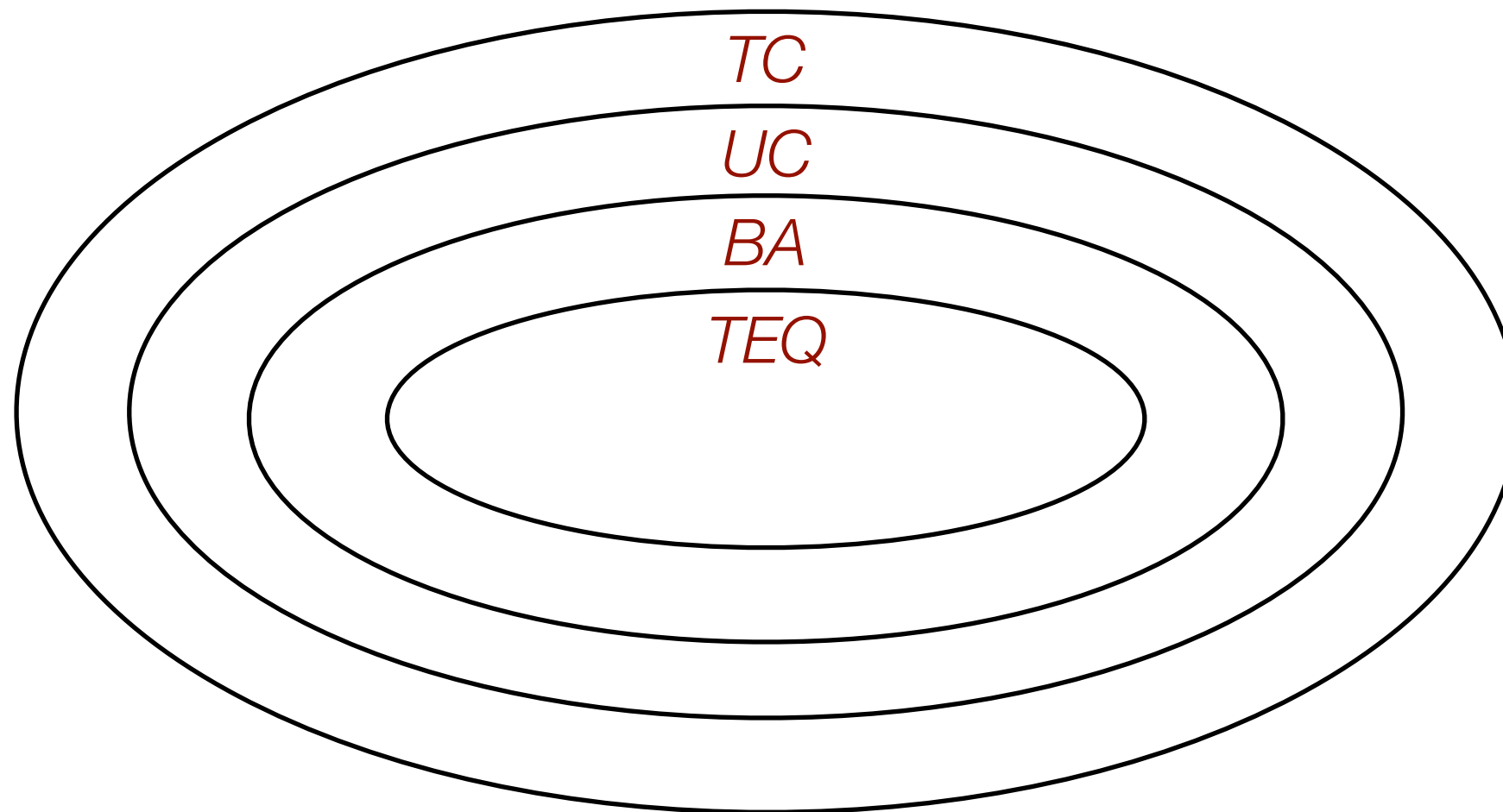
Top Cycle (1971)	TC	expansion	$O(n^2)$
Uncovered Set (1977)	UC	weak expansion	$O(n^{2.38})$
Banks Set (1985)	BA	strong retentiveness	$2^{O(n)}$
Tournament Equilibrium Set (1990)	TEQ	(retentiveness)	$2^{O(n)}$

Randomized SCFs

Random Dictatorship

Allan Gibbard

‣ A randomized SCF maps a preference profile to a lottery (probability distribution) over the alternatives.

‣ Perhaps the most notorious randomized SCF is random dictatorship.

  ‣ One agent is picked uniformly at random and his most preferred alternative is implemented as the social choice.

‣ Random dictatorship is not as bad as it may sound.

  ‣ It satisfies most of the axioms that are usually considered in social choice theory.

‣ Random dictatorship is the only Pareto-optimal randomized SCF that is strategyproof, i.e., it cannot be manipulated by lying about one's preferences (Gibbard, 1977).
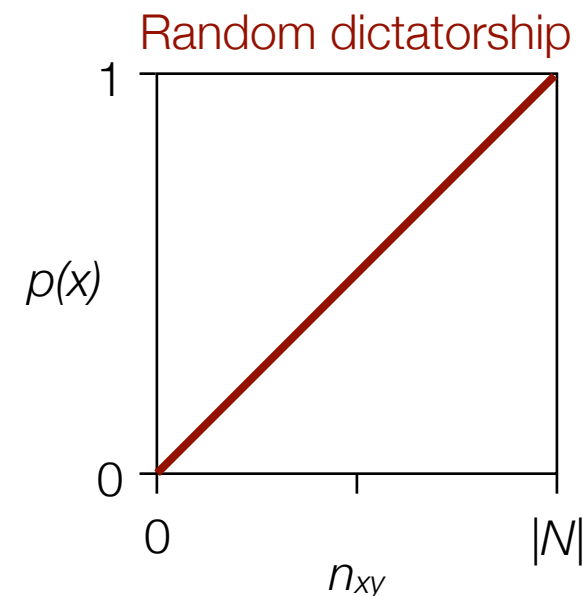
Maximal Lotteries

Germain Kreweras

Peter C. Fishburn

- Kreweras (1965) and Fishburn (1984)
  - Rediscovered by Laffond et al. (1993), Felsenthal and Machover (1992), Fisher and Ryan (1995), Rivest and Shen (2010)

- Let $g(x,y) = n_{xy} - n_{yx}$ be the *majority margin* of x and y.

- Alternative x is a (weak) Condorcet winner if $g(x,y) \geq 0$ for all y.

- Extend g to lotteries: $g(p,q) = \sum_{x,y} p(x) \cdot q(y) \cdot g(x,y)$
  - Expected majority margin

- p is a maximal lottery if $g(p,q) \geq 0$ for all q.
  - Randomized (weak) Condorcet winner
  - Always exists due to Minimax Theorem (v. Neumann, 1928)

Examples

‣ Two alternatives

Maximal lotteries

Random dictatorship
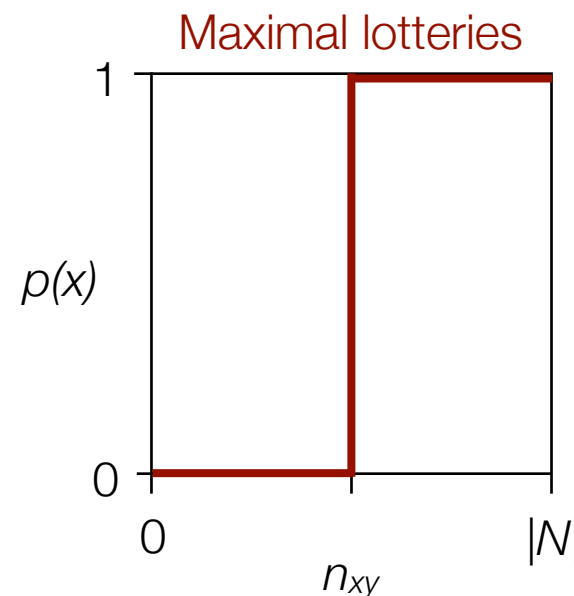
‣ *R* can be transformed into a symmetric zero-sum game.

  ‣ Maximal lotteries are mixed minimax strategies (or Nash equilibria).

2	**2**	**1**
a	b	c
b	c	a
c	a	b

	a	b	c
a	0	1	-1
b	-1	0	3
c	1	-3	0

  ‣ The unique maximal lottery is *3/5 a + 1/5 b + 1/5 c*.

Properties of Maximal Lotteries (*ML*)

‣ Maximal lotteries are almost always unique.

  ‣ Always unique for odd number of voters (Laffond et al., 1997)

‣ *ML* does not require asymmetry, completeness, or even transitivity of preferences.

  ‣ Random dictatorship requires unique maximum.

  ‣ Canonical generalization (*RSD*) requires at least one maximum.

‣ *ML* can be efficiently computed via linear programming.

  ‣ Computing *RSD* probabilities, on the other hand, is #P-complete (Aziz et al., 2013).

‣ In the assignment domain, maximal lotteries are known as popular mixed matchings (Kavitha et al., 2011).

Properties of
Maximal Lotteries (ctd.)

‣ Pareto-dominated alternatives always get zero probability in every maximal lottery.

  ‣ In fact, *ML* is even efficient with respect to stochastic dominance.

  ‣ No lottery gives more expected utility for any utility representation consistent with the voters' preferences (Aziz et al., 2012).

  ‣ Violated by *RSD* (Bogomolnaia and Moulin, 2001).

‣ *ML* is weakly strategyproof in a well-defined sense (Aziz et al., 2013).

‣ *ML* can be <u>uniquely characterized</u> using a version of consistency for randomized SCFs (Brandl et al., 2015).

Recommended Literature

- Books

  - Allingham: *Choice Theory - A very short introduction*. Oxford University Press, 2002

  - Austen-Smith and Banks: *Positive Political Theory I*, University of Michigan Press, 1999

  - Gärtner: *A Primer in Social Choice Theory*, Oxford University Press, 2009

  - Moulin: *Axioms of Cooperative Decision Making*. Cambridge University Press, 1988

  - Nitzan: *Collective Choice and Preference*. Cambridge University Press, 2010

- Introductory book chapter

  - B., Conitzer, and Endriss. *Computational Social Choice*. In "*Multiagent Systems*" (G. Weiss, ed.), MIT Press, 2013.