# Single-Agent Dynamics in Hedonic Games

Joint work with Felix Brandt, Leo Tappe, and Anaëlle Wilczynski

**Martin Bullinger**

**Online Social Choice and Welfare Seminar Series**

Technical University of Munich

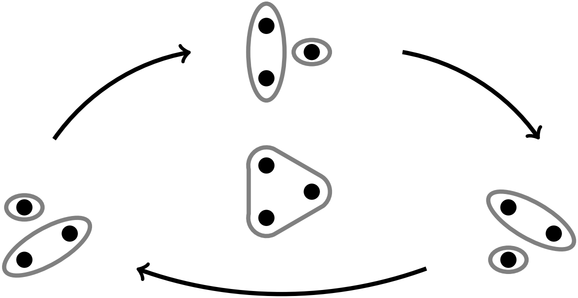
# Goals

Study dynamics in coalition formation

- Proving convergence: the art of potential functions
  - Simple potential functions
  - Modifying non-increasing potential functions
  - A global perspective on non-increasing potential functions
  - A glance on running time
- Finding and using counterexamples
  - Finding difficult counterexamples by linear programming
  - Using counterexamples in hardness reductions

# Motivation

- Convergence as proof of concept: centralized solutions through natural distributed process
- Strong proof of existence
- Complete the picture: non-reachability ‘paradox’

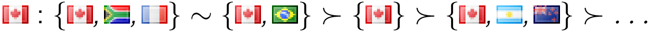


# Motivation

- Convergence as proof of concept: centralized solutions through natural distributed process
- Strong proof of existence
- Complete the picture: non-reachability ‘paradox’
- General interest in potential functions
  - Local search problems
  - Algorithmic game theory (congestion games)
  - Combinatorial algorithms (MaxFlow)
  - Local optimum search (MaxCut, TSP)
  - ...
  - Dynamics in multiagent systems (matchings, Schelling segregation, coalition formation)

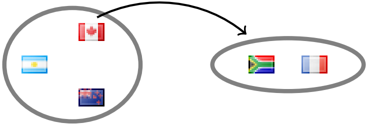
# Formal Model of Coalition Formation

- Finite set of agents $N = \{$  $, \dots \}$
- Preferences over coalitions they are part of
- Hedonic preferences: no dependencies outside coalition



# Formal Model of Coalition Formation

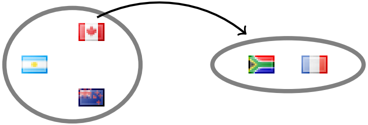
- Finite set of agents $N = \{$🇫🇷, 🇿🇦, 🇨🇦, $\dots\}$
- Preferences over coalitions they are part of
- Hedonic preferences: no dependencies outside coalition
- Dynamics of joining other coalitions



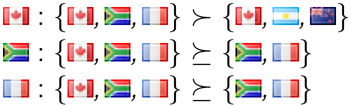
Nash deviation: 🇨🇦 : $\{$🇨🇦, 🇿🇦, 🇫🇷$\} \succ \{$🇨🇦, 🇦🇷, 🇳🇿$\}$

# Formal Model of Coalition Formation

- Finite set of agents $N = \{$  $, \dots \}$
- Preferences over coalitions they are part of
- Hedonic preferences: no dependencies outside coalition
- Dynamics of joining other coalitions

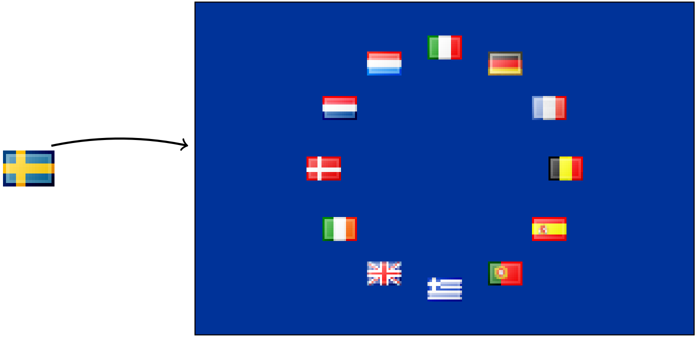


Individual deviation:



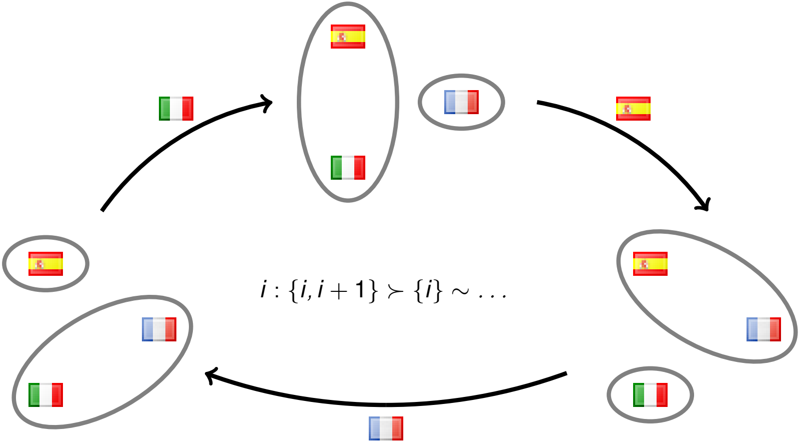
Individually stable partition: no individual deviation
(Drèze and Greenberg 1980)

# Consent in International Bodies



Art. 49, Maastricht Treaty
Address application to Council, which shall act **unanimously**

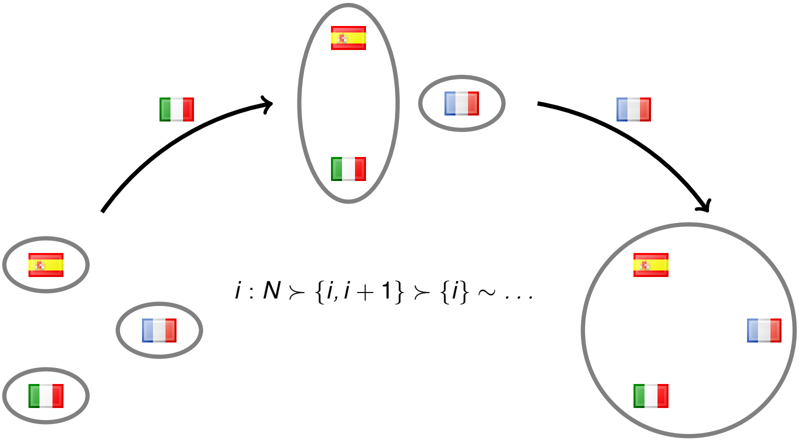
# Occurence of Cycling



- Cycling can occur
- Individually stable partitions need not exist

# Computational Problems

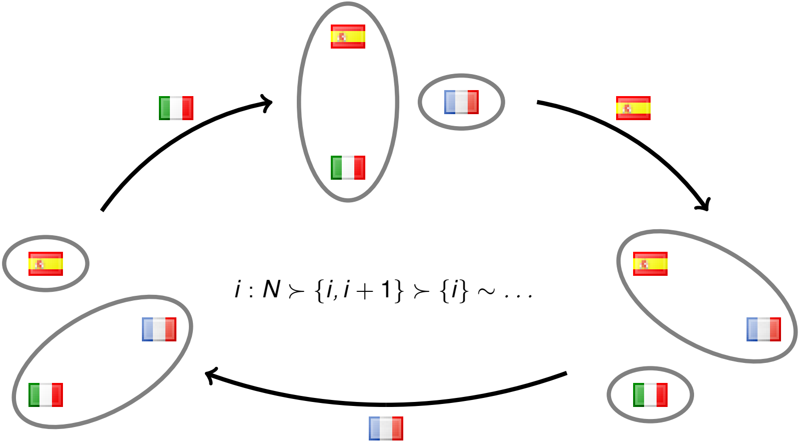
- Existence of `PATH` to stability



$$i : N \succ \{i, i+1\} \succ \{i\} \sim \ldots$$

# Computational Problems

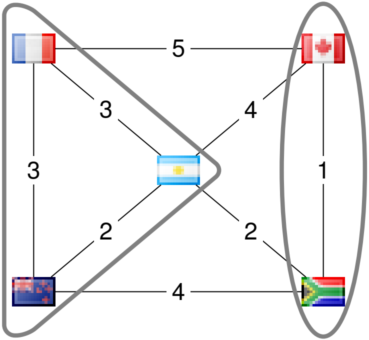
- Existence of PATH to stability
- CONVERGENCE of all sequences

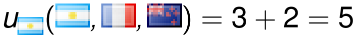


$$i : N \succ \{i, i+1\} \succ \{i\} \sim \dots$$

# Proving Convergence: The Art of Potential Functions

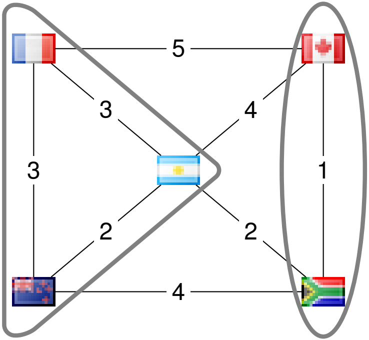
# Additively Separable Hedonic Games



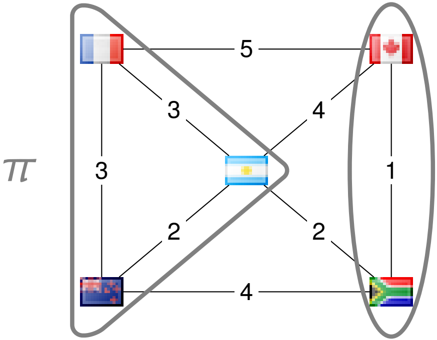

(Bogomolnaia and Jackson, 2002)

# Social Welfare: The Canonical Potential



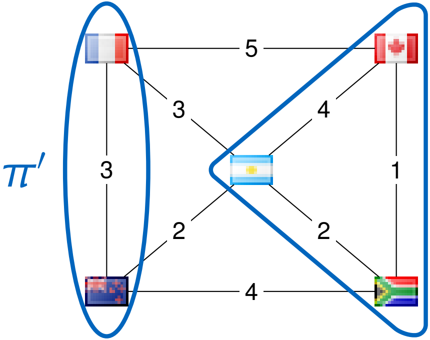

$$SW(\pi) = \sum_{i\in N} u_i(\pi)$$

# Social Welfare: The Canonical Potential




$$SW(\pi) = 6 + 5 + 5 + 1 + 1 = 18$$

# Social Welfare: The Canonical Potential

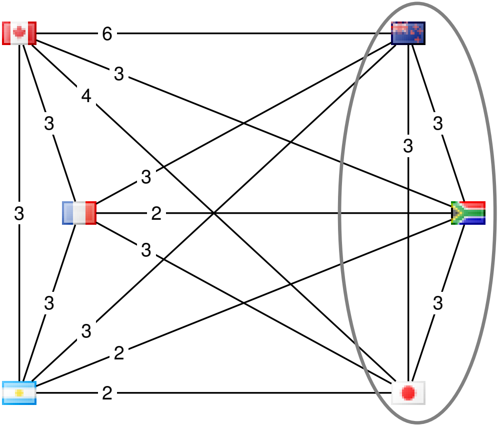



$$SW(\pi') = 20$$

Potential for Nash stability under symmetry

(Bogomolnaia and Jackson, 2002)

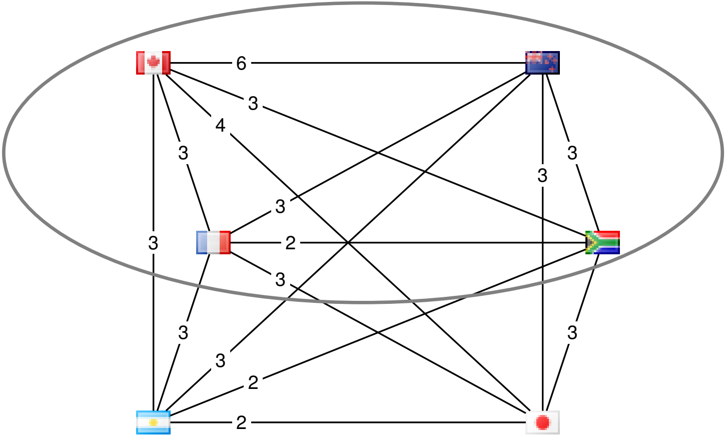
# Fractional Hedonic Games




 $= \frac{3+3}{3} = 2$

Aziz et al. (2019)

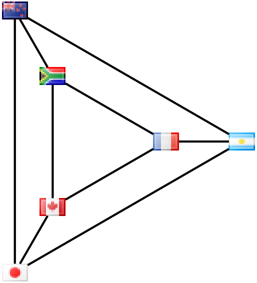
# Fractional Hedonic Games



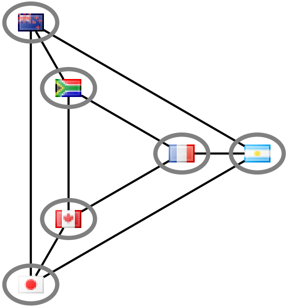

 $= \frac{6+3+3}{4} = 3$

Aziz et al. (2019)

# Binary symmetric utilities

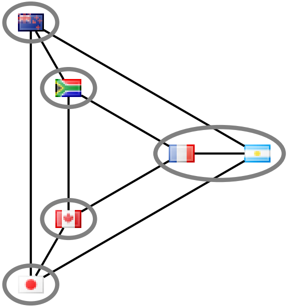


# Binary symmetric utilities

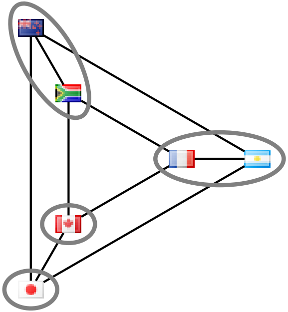


Starting point: singleton coalitions

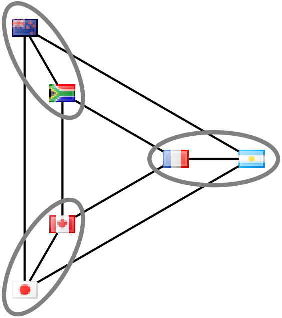
# Binary symmetric utilities



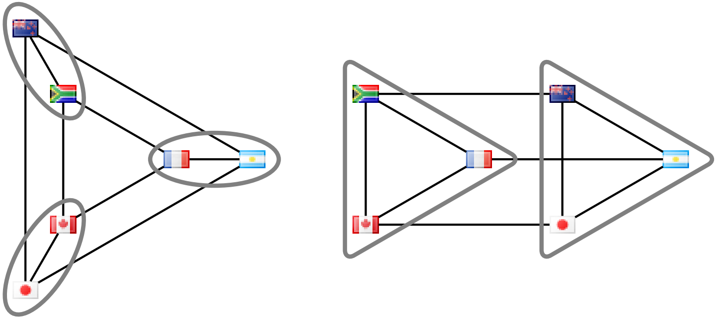
# Binary symmetric utilities



# Binary symmetric utilities



# Binary symmetric utilities



Individually stable partitions of complete partitions

# Social Welfare under Fractional Utilities

- Convergence by constructive proof
- Social welfare no increasing potential function

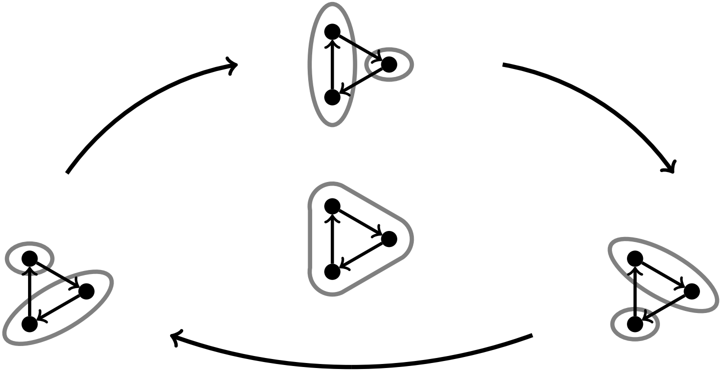
$$SW(\pi) = \sum_{C \in \pi} |C| - 1$$

- Ordered vector of coalition sizes lexicographically increasing
- Potential function via

$$\Phi(\pi) = \sum_{C \in \pi} |C|(|C| - 1)/2 = SW_{ASHG}(\pi)$$

# Discussing Assumptions

- Binary utilities needed (later)
- Non-symmetry causes non-reachability paradox



Unique stable partition non-reachable with
any non-trivial starting partition

# Non-Monotonic Potential Functions

- Under guaranteed convergence, increasing potential functions always exist
- Analysis easier / more elegant under natural potential functions
- Complex potential functions vs. complex arguments of convergence

Two techniques:

- Mapping to monotonic potential
- Global perspective of Deviation Lemma

# Hedonic Diversity Games

- Introduced by Bredereck, Elkind, and Igarashi (2019)
- Agents divided into 2 types
- Preferences over rational numbers, indicating preferred fractions of agents
- Empirical evidence for convergence (Boehmer and Elkind, 2020)
- Cycling of dynamics under any two of the following
  - Strictness
  - Single-peakedess
  - Singleton starting partition
- Convergence of dynamics starting from singleton partition under strict and single-peaked preferences

# Social Welfare Revisited

- Splitting change in potential for $\pi \xrightarrow{i} \pi'$

$$SW(\pi') - SW(\pi) = \underbrace{u_i(\pi') - u_i(\pi)}_{>0 \text{ (deviator)}} + \underbrace{Term(\pi(i), \pi'(i))}_{\text{dependent on agents involved in deviation}}$$

- Chain of deviations $\pi_0 \xrightarrow{i_1} \pi_1 \xrightarrow{i_2} \dots \xrightarrow{i_k} \pi_k$
  - Deviator part constantly improving
  - Goal: global bound for second part

**Lemma (Deviation Lemma)**

$$\sum_{j=1}^{k} |\pi_j(i_j)| - |\pi_{j-1}(i_j)| = \frac{1}{2} \sum_{i \in N} |\pi_k(i)| - |\pi_0(i)|$$

# Application: Restricted ASHGs

- Consider utilities restricted to $\{-x, y\}$
  - Friends and enemies: $\{-1, 1\}$
  - Appreciation of friends: $\{-1, n\}$ (Dimitrov et al., 2006)
  - Aversion of enemies: $\{-n, 1\}$ (Dimitrov et al., 2006)

# Application: Restricted ASHGs

- Consider utilities restricted to $\{-x, y\}$
- Single deviation:

$$\Phi(\pi_j) - \Phi(\pi_{j-1}) \geq \underbrace{\Delta}_{\text{deviator}} + y\Big(\underbrace{|\pi_j(i_j)|}_{\substack{\text{target}\\\text{coalition}}} - \underbrace{|\pi_{j-1}(i_j)|}_{\substack{\text{origin}\\\text{coalition}}}\Big)$$

- Sequence of deviations:

$$\begin{aligned}\Phi(\pi_k) - \Phi(\pi_0) &= \sum_{j=1}^{k} \Phi(\pi_j) - \Phi(\pi_{j-1}) \\ &\geq \sum_{j=1}^{k} \Delta + y\left(|\pi_j(i_j)| - |\pi_{j-1}(i_j)|\right) \\ &= k\Delta + y\sum_{j=1}^{k} |\pi_j(i_j)| - |\pi_{j-1}(i_j)|.\end{aligned}$$

# Application: Restricted ASHGs

- Sequence of deviations:

$$\begin{aligned}\Phi(\pi_k) - \Phi(\pi_0) &= \sum_{j=1}^{k} \Phi(\pi_j) - \Phi(\pi_{j-1}) \\ &\geq \sum_{j=1}^{k} \Delta + y\left(|\pi_j(i_j)| - |\pi_{j-1}(i_j)|\right) \\ &= k\Delta + y\sum_{j=1}^{k} |\pi_j(i_j)| - |\pi_{j-1}(i_j)|.\end{aligned}$$

- With Deviation Lemma:

$$\begin{aligned}\Phi(\pi_k) - \Phi(\pi_0) &\geq k\Delta + \frac{y}{2}\sum_{i\in N} |\pi_k(i)| - |\pi_0(i)| \\ &\geq k\Delta - y\frac{n(n-1)}{2}.\end{aligned}$$

# Applying the Deviation Lemma

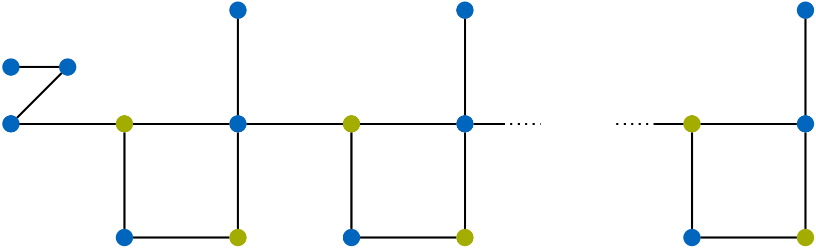
**Lemma (Deviation Lemma)**

$$\sum_{j=1}^{k} |\pi_j(i_j)| - |\pi_{j-1}(i_j)| = \frac{1}{2} \sum_{i \in N} |\pi_k(i)| - |\pi_0(i)|$$

- Purely combinatorial insight
- Not limited to utility model
  - Anonymous hedonic games
  - Dichotomous hedonic games
- Not limited to stability notion
  - Contractual Nash stability
  - Voting-based stability
- Tight result regarding
  - Number of utility values
  - Stability notion

# A Glance on Running Time

- Polynomial bounds on running time via global approach
  Upper bound $k \leq \frac{(2y+3x)n(n-1)}{2\Delta}$ for $\{-x, y\}$-ASHGs
- Potential functions do not guarantee fast convergence (based on example for MAXCUT by Monien and Tscheuschner, 2010)



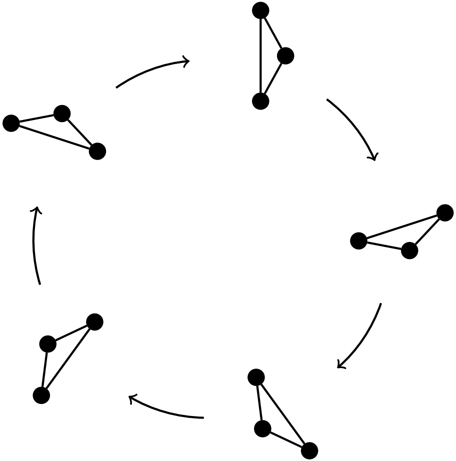
# Finding and Using Couterexamples

# Computer-Aided Counterexample Search

- Computational intractability of general existence problem
- Seek exact boundaries of (in)tractability
- Non-existence of stable states $\implies$ cycling of dynamics
- Deviations are linear inequalities for ASHGs (and FHGs!)
- Road map
  - Guess structure of game that allows cycling
  - Linear program: variables $\leftrightarrow$ edge weights of underlying graph
  - Inequalities $\leftrightarrow$ necessary deviations
  - While stable configurations exist, guess a deviation and add inequalities
- Human-made counterexamples with computer-generated weights

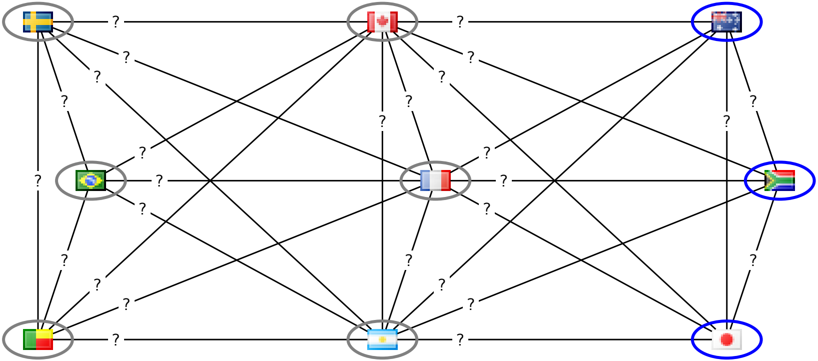
# Symmetric Fractional Hedonic Games

- Guess structure of game that allows cycling



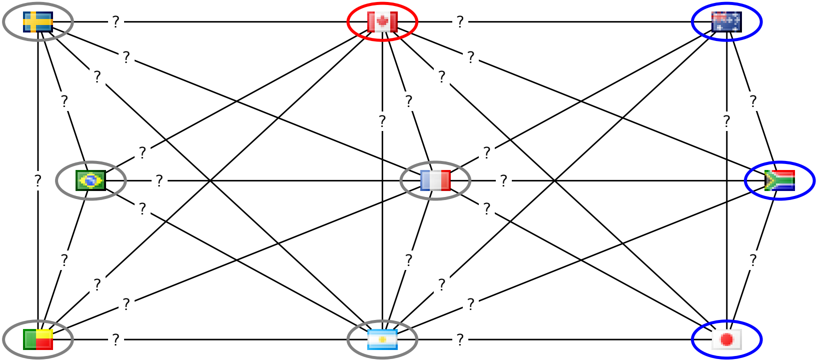
# Symmetric Fractional Hedonic Games

- Guess structure of game that allows cycling
- Defining linear program through desirable deviations



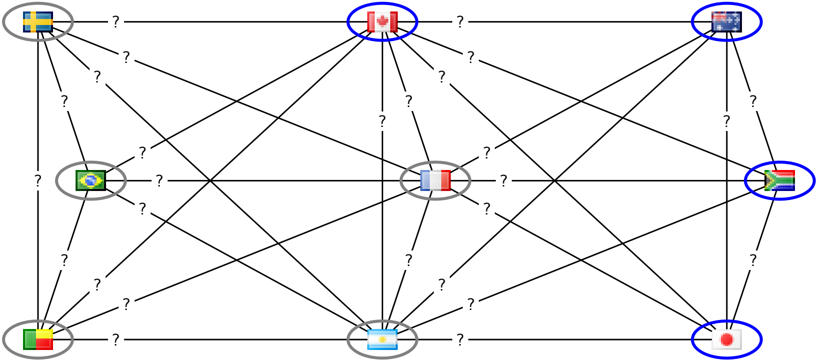
# Symmetric Fractional Hedonic Games

- Guess structure of game that allows cycling
- Defining linear program through desirable deviations



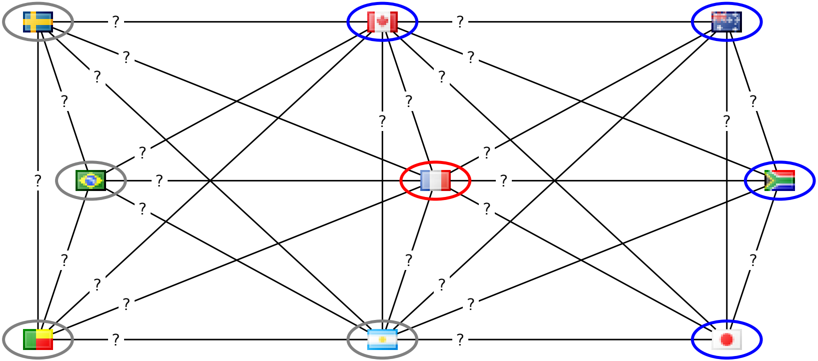
# Symmetric Fractional Hedonic Games

- Guess structure of game that allows cycling
- Defining linear program through desirable deviations



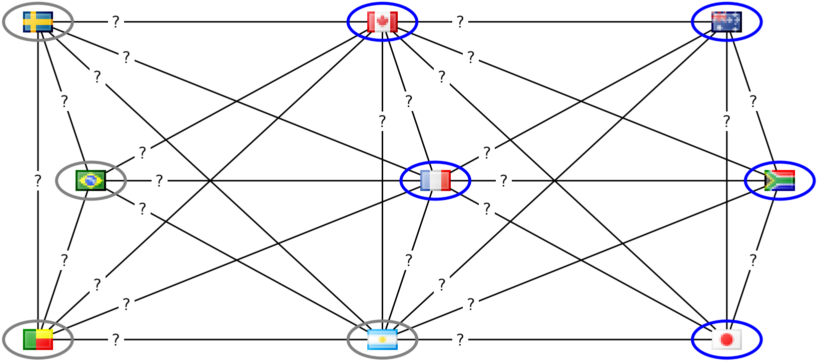
# Symmetric Fractional Hedonic Games

- Guess structure of game that allows cycling
- Defining linear program through desirable deviations



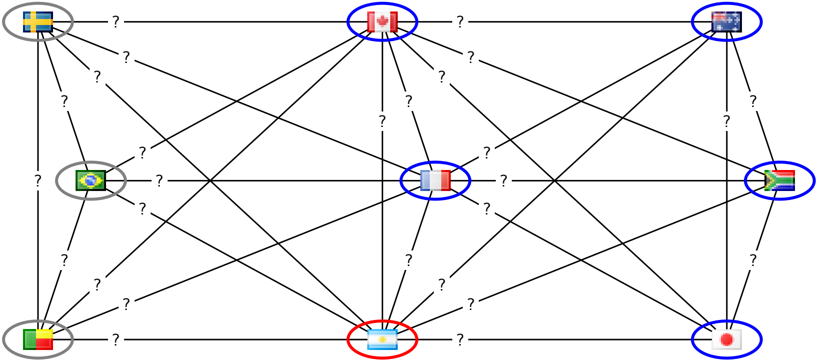
# Symmetric Fractional Hedonic Games

- Guess structure of game that allows cycling
- Defining linear program through desirable deviations



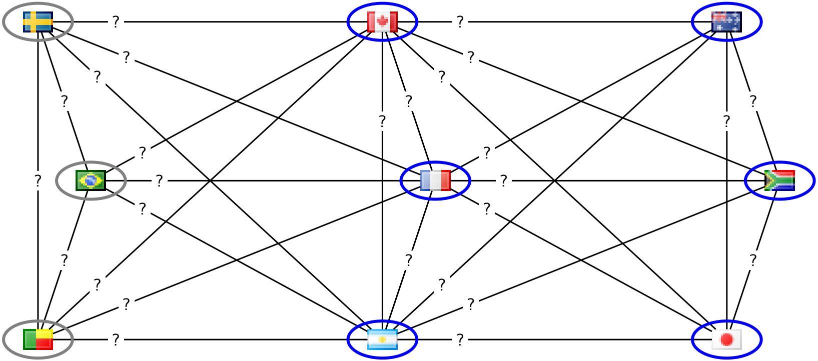
# Symmetric Fractional Hedonic Games

- Guess structure of game that allows cycling
- Defining linear program through desirable deviations



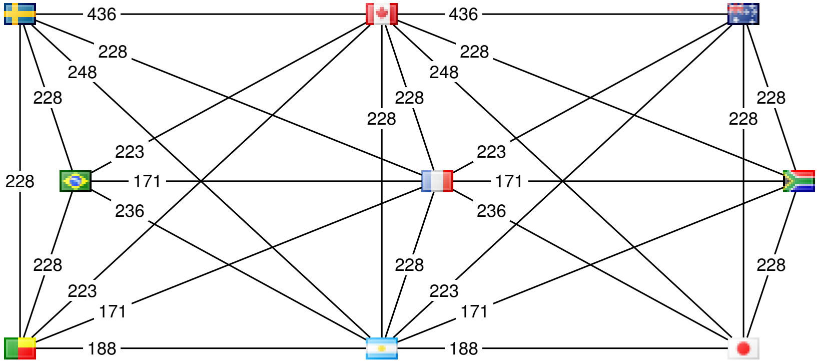
# Symmetric Fractional Hedonic Games

- Guess structure of game that allows cycling
- Defining linear program through desirable deviations



# Symmetric Fractional Hedonic Games

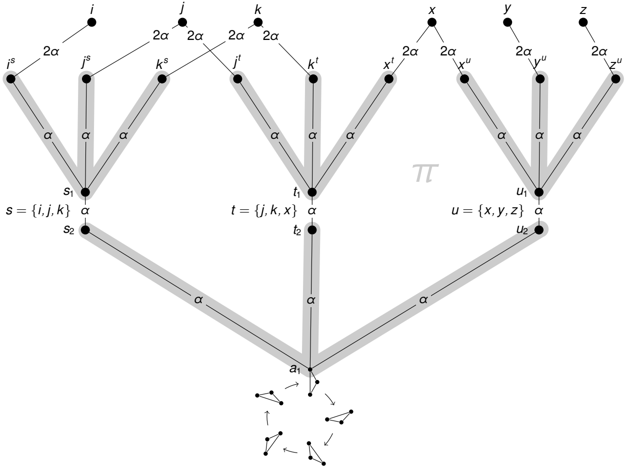
- Guess structure of game that allows cycling
- Defining linear program through desirable deviations




Weights found by linear program

# From Counterexamples to Reductions

- Model ground structure as a game
- Attach counterexamples
- Boundary of tractability through boundary to possibility



# Conclusion

- Decisive properties for convergence under individual stability
  - Additively separable hedonic games: severe weight restrictions
  - Fractional hedonic games: binary symmetric utilities
  - Hedonic diversity games: strictness (single-peaked, constraint starting partition)
- Convergence tool box
  - Potential functions: social welfare, vector of coalition-scores
  - Monotonic potential functions mean no guarantee for running time
  - Manipulating non-monotonic potential functions
  - Global bounds via Deviation Lemma
- Leverage linear programs for computer-aided counterexample search
- Counterexamples as gadgets in reductions

# Future Directions

- Explore further applications of Deviation Lemma
- Intriguing candidate for convergence
  - Binary fractional hedonic games with arbitrary starting partition
- In-depth analysis of running time of dynamics
- Combination with other properties (e.g., Pareto optimality)