Single-Agent Dynamics in Hedonic Games

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Goals

Study dynamics in coalition formation

- Proving convergence: the art of potential functions
  - Simple potential functions
  - Modifying non-increasing potential functions
  - A global perspective on non-increasing potential functions
  - A glance on running time

- Finding and using counterexamples
  - Finding difficult counterexamples by linear programming
  - Using counterexamples in hardness reductions
Motivation

- Convergence as proof of concept: centralized solutions through natural distributed process
- Strong proof of existence
- Complete the picture: non-reachability ‘paradox’
Motivation

- Convergence as proof of concept: centralized solutions through natural distributed process
- Strong proof of existence
- Complete the picture: non-reachability ‘paradox’
- General interest in potential functions
  - Local search problems
  - Algorithmic game theory (congestion games)
  - Combinatorial algorithms (MaxFlow)
  - Local optimum search (MaxCut, TSP)
  - ...
  - Dynamics in multiagent systems (matchings, Schelling segregation, coalition formation)
Formal Model of Coalition Formation

- Finite set of agents $N = \{\text{[flag]}\,\}, \ldots \}$
- Preferences over coalitions they are part of
- Hedonic preferences: no dependencies outside coalition

$$\text{[flag]} : \{\text{[flag]}, \text{[flag]}, \text{[flag]}\} \sim \{\text{[flag]}, \text{[flag]}\} \succ \{\text{[flag]}\} \succ \{\text{[flag]}, \text{[flag]}, \text{[flag]}\} \succ \ldots$$
Formal Model of Coalition Formation

- Finite set of agents $N = \{\text{[flags]}\ldots\}$
- Preferences over coalitions they are part of
- Hedonic preferences: no dependencies outside coalition
- Dynamics of joining other coalitions

Nash deviation: $\text{[flag]} : \{\text{[flags]}\} \succ \{\text{[flags]}\}$
Formal Model of Coalition Formation

- Finite set of agents $N = \{\text{Agent 1, Agent 2, Agent 3, ...}\}$
- Preferences over coalitions they are part of
- Hedonic preferences: no dependencies outside coalition
- Dynamics of joining other coalitions

Individual deviation:

- $\text{Canada} : \{\text{Canada}, \text{France}, \text{South Africa}\} \succ \{\text{Canada}, \text{France}\}$
- $\text{South Africa} : \{\text{Canada}, \text{France}, \text{South Africa}\} \succeq \{\text{South Africa}\}$
- $\text{France} : \{\text{Canada}, \text{France}, \text{South Africa}\} \succeq \{\text{France}\}$

Individually stable partition: no individual deviation
(Drèze and Greenberg 1980)
Consent in International Bodies

Art. 49, Maastricht Treaty
Address application to Council, which shall act unamiously
Occurrence of Cycling

Cycling can occur
- Individually stable partitions need not exist
Computational Problems

- Existence of **PATH** to stability

\[ i : N \succ \{i, i + 1\} \succ \{i\} \sim \ldots \]
Computational Problems

- Existence of **PATH** to stability
- **CONVERGENCE** of all sequences

\[ i : N \succ \{i, i+1\} \succ \{i\} \sim \ldots \]
Proving Convergence: The Art of Potential Functions
Additively Separable Hedonic Games

$$u((\text{Argentina}, \text{France}, \text{South Africa})) = 3 + 2 = 5$$

(Bogomolnaia and Jackson, 2002)
Social Welfare: The Canonical Potential

\[ SW(\pi) = \sum_{i \in N} u_i(\pi) \]
$SW(\pi) = 6 + 5 + 5 + 1 + 1 = 18$
Social Welfare: The Canonical Potential

$SW(\pi') = 20$

Potential for Nash stability under symmetry

(Bogomolnaia and Jackson, 2002)
Fractional Hedonic Games

\[ u_{\text{frac}}(\text{Canada}, \text{Japan}, \text{South Africa}) = \frac{3 + 3}{3} = 2 \]

Aziz et al. (2019)
Fractional Hedonic Games

\[ u_{\mathcal{S}}((\text{CA}, \text{FR}, \text{GB}, \text{ZA})) = \frac{6+3+3}{4} = 3 \]

Aziz et al. (2019)
Binary symmetric utilities
Binary symmetric utilities

Starting point: singleton coalitions
Binary symmetric utilities
Binary symmetric utilities
Binary symmetric utilities
Binary symmetric utilities

Individually stable partitions of complete partitions
Social Welfare under Fractional Utilities

- Convergence by constructive proof
- Social welfare no increasing potential function

\[ SW(\pi) = \sum_{C \in \pi} |C| - 1 \]

- Ordered vector of coalition sizes lexicographically increasing
- Potential function via

\[ \Phi(\pi) = \sum_{C \in \pi} |C|(|C| - 1)/2 = SW_{ASHG}(\pi) \]
Discussing Assumptions

- Binary utilities needed (later)
- Non-symmetry causes non-reachability paradox

Unique stable partition non-reachable with any non-trivial starting partition.
Non-Monotonic Potential Functions

- Under guaranteed convergence, increasing potential functions always exist
- Analysis easier / more elegant under natural potential functions
- Complex potential functions vs. complex arguments of convergence

Two techniques:
- Mapping to monotonic potential
- Global perspective of Deviation Lemma
Hedonic Diversity Games

- Introduced by Bredereck, Elkind, and Igarashi (2019)
- Agents divided into 2 types
- Preferences over rational numbers, indicating preferred fractions of agents
- Empirical evidence for convergence (Boehmer and Elkind, 2020)
- Cycling of dynamics under any two of the following
  - Strictness
  - Single-peakedness
  - Singleton starting partition
- Convergence of dynamics starting from singleton partition under strict and single-peaked preferences
Social Welfare Revisited

- Splitting change in potential for $\pi \rightarrow \pi'$

$$SW(\pi') - SW(\pi) = u_i(\pi') - u_i(\pi) + \underbrace{\text{Term}(\pi(i), \pi'(i))}_{>0 \text{ (deviation)}}$$

dependent on agents involved in deviation

- Chain of deviations $\pi_0 \xrightarrow{i_1} \pi_1 \xrightarrow{i_2} \ldots \xrightarrow{i_k} \pi_k$
  - Deviator part constantly improving
  - Goal: global bound for second part

**Lemma (Deviation Lemma)**

$$\sum_{j=1}^{k} |\pi_j(i_j)| - |\pi_{j-1}(i_j)| = \frac{1}{2} \sum_{i \in N} |\pi_k(i)| - |\pi_0(i)|$$
Application: Restricted ASHGs

- Consider utilities restricted to \([-x, y]\)
  - Friends and enemies: \([-1, 1]\)
  - Appreciation of friends: \([-1, n]\) (Dimitrov et al., 2006)
  - Aversion of enemies: \([-n, 1]\) (Dimitrov et al., 2006)
Application: Restricted ASHGs

- Consider utilities restricted to \(\{-x, y\}\)
- Single deviation:
  \[
  \Phi(\pi_j) - \Phi(\pi_{j-1}) \geq \Delta + y \left( |\pi_j(i_j)| - |\pi_{j-1}(i_j)| \right)
  \]
- Sequence of deviations:
  \[
  \Phi(\pi_k) - \Phi(\pi_0) = \sum_{j=1}^{k} \Phi(\pi_j) - \Phi(\pi_{j-1}) \\
  \geq \sum_{j=1}^{k} \Delta + y \left( |\pi_j(i_j)| - |\pi_{j-1}(i_j)| \right) \\
  = k\Delta + y \sum_{j=1}^{k} |\pi_j(i_j)| - |\pi_{j-1}(i_j)|.
  \]
Application: Restricted ASHGs

■ Sequence of deviations:

\[ \Phi(\pi_k) - \Phi(\pi_0) = \sum_{j=1}^{k} \Phi(\pi_j) - \Phi(\pi_{j-1}) \]

\[ \geq \sum_{j=1}^{k} \Delta + y \left( |\pi_j(i_j)| - |\pi_{j-1}(i_j)| \right) \]

\[ = k\Delta + y \sum_{j=1}^{k} |\pi_j(i_j)| - |\pi_{j-1}(i_j)|. \]

■ With Deviation Lemma:

\[ \Phi(\pi_k) - \Phi(\pi_0) \geq k\Delta + \frac{y}{2} \sum_{i \in N} |\pi_k(i)| - |\pi_0(i)| \]

\[ \geq k\Delta - y \frac{n(n-1)}{2}. \]
Applying the Deviation Lemma

Lemma (Deviation Lemma)

\[
\sum_{j=1}^{k} |\pi_j(i_j)| - |\pi_{j-1}(i_j)| = \frac{1}{2} \sum_{i \in N} |\pi_k(i)| - |\pi_0(i)|
\]

- Purely combinatorial insight
- Not limited to utility model
  - Anonymous hedonic games
  - Dichotomous hedonic games
- Not limited to stability notion
  - Contractual Nash stability
  - Voting-based stability
- Tight result regarding
  - Number of utility values
  - Stability notion
A Glance on Running Time

- Polynomial bounds on running time via global approach
  - Upper bound $k \leq \frac{(2y+3x)n(n-1)}{2\Delta}$ for $\{-x, y\}$-ASHGs

- Potential functions do not guarantee fast convergence (based on example for MAXCUT by Monien and Tscheuschner, 2010)
Finding and Using Counterexamples
Computer-Aided Counterexample Search

- Computational intractability of general existence problem
- Seek exact boundaries of (in)tractability
- Non-existence of stable states $\implies$ cycling of dynamics
- Deviations are linear inequalities for ASHGs (and FHGs!)
- Road map
  - Guess structure of game that allows cycling
  - Linear program: variables $\leftrightarrow$ edge weights of underlying graph
  - Inequalities $\leftrightarrow$ necessary deviations
  - While stable configurations exist, guess a deviation and add inequalities
- Human-made counterexamples with computer-generated weights
Symmetric Fractional Hedonic Games

- Guess structure of game that allows cycling
Symmetric Fractional Hedonic Games

- Guess structure of game that allows cycling
- Defining linear program through desirable deviations
Symmetric Fractional Hedonic Games

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Weights found by linear program
From Counterexamples to Reductions

- Model ground structure as a game
- Attach counterexamples
- Boundary of tractability through boundary to possibility

\[ \alpha = \{i, j, k\} \]
\[ \alpha = \{j, k, x\} \]
\[ \alpha = \{x, y, z\} \]

\[ \Pi \]

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Conclusion

- Decisive properties for convergence under individual stability
  - Additively separable hedonic games: severe weight restrictions
  - Fractional hedonic games: binary symmetric utilities
  - Hedonic diversity games: strictness (single-peaked, constraint starting partition)

- Convergence tool box
  - Potential functions: social welfare, vector of coalition-scores
  - Monotonic potential functions mean no guarantee for running time
  - Manipulating non-monotonic potential functions
  - Global bounds via Deviation Lemma

- Leverage linear programs for computer-aided counterexample search

- Counterexamples as gadgets in reductions
Future Directions

- Explore further applications of Deviation Lemma
- Intriguing candidate for convergence
  - Binary fractional hedonic games with arbitrary starting partition
- In-depth analysis of running time of dynamics
- Combination with other properties (e.g., Pareto optimality)