Proportional Multiwinner Voting with Dynamic Candidate Sets

Chris Dong[1], **Jannik Peters**[2]

[1]Technical University of Munich
[2]National University of Singapore

Abstract

Multiwinner voting is the study of electing a fixed-size committee given individual agents' preferences over candidates. Most research in this field has been limited to a static setting, with only one election over a fixed set of candidates. However, this approach overlooks the dynamic nature of real-world elections, where candidate sets are subject to change. We extend the study of proportionality in multiwinner voting to dynamic settings, allowing candidates to join or leave the election and demanding that each chosen committee satisfies proportionality without differing too much from the previously selected committee. We consider approval preferences, ranked preferences, and the proportional clustering setting. Existing voting rules turn out to be flawed and we thus give algorithms making few changes or show that such algorithms cannot exist for various proportionality axioms. In particular, we show that such algorithms cannot exist for ranked preferences and provide amortized and exact algorithms for several proportionality notions in the other two settings.

1 Dynamic Committee Selection

The *multiwinner voting* or *committee selection* problem has received considerable attention in recent years in the field of computational social choice (Faliszewski et al. 2017; Lackner and Skowron 2022). Given voters' preferences, this problem involves selecting a fixed-size subset of the candidates. In particular, the study of *proportionality* is popular—the goal here is that each cohesive group of voters should be represented sufficiently by the elected committee. As a result, several voting methods and proportionality axioms have been designed (Peters and Skowron 2020; Brill and Peters 2023), and new application domains of proportionality are being explored (e.g., participatory budgeting (Peters, Pierczyński, and Skowron 2021), clustering (Kalayci, Kempe, and Kher 2024), or sortition (Caragiannis, Micha, and Peters 2024)). However, most papers in classic multiwinner voting assume a static, one-shot model: voter and candidate sets are fixed, voters submit their preferences once, and precisely one committee is selected. This, on many occasions, fails to capture issues that arise when applying these rules. For instance, as motivated by Elkind, Obraztsova, and Teh (2024), in council elections, new candidates appear each year, while old council members drop out. As another example, in the Irish lower house (the Dáil Éireann), which is elected using the proportional single-transferable vote, members of this lower house frequently need to be replaced (in case of death or abdication) (Gallagher 1996). This replacement is often done using a single-winner voting method and can lead to disproportionality in the parliament. We take this as motivation to study *proportional multiwinner voting with dynamic candidate sets.* That is, can we select committees such that after adding or removing candidates, we only need to slightly alter our original committee to maintain proportionality? For this, we examine an online model: Inspired by the dynamic clustering (Łącki et al. 2024) and matching (Bernstein, Holm, and Rotenberg 2019) literature, we assume that we have a stream of additions and deletions of candidates. During this stream, we must maintain a proportional committee, making only minimal changes after each step.

Our Results. We study multiwinner voting rules in three dynamic settings: ranked ballots, proportional clustering, and approval ballots. Specifically, when voters submit linear orders over the candidates, we show that PSC can be satisfied by an incremental voting rule, but decremental PSC, let alone fully dynamic PSC, is not robustly achievable. Further, neither incremental nor decremental algorithms can satisfy rank-JR. When restoring either property after deletion, the number of replacements is unbounded, with our asymptotic lower bound for rank-JR being significantly steeper than for PSC. In the related setting of proportional clustering, we obtain a fully dynamic algorithm that achieves a 3.79-proportional fair outcome. We further investigate the α-q-core (Kellerhals and Peters 2023; Ebadian and Micha 2024) and show that the same algorithm for proportional fairness also gives a 5 approximation to the q-core. Surprisingly, this meets the *currently best known* upper bound on the q-core of non-dynamic algorithms. Finally, when the voters have dichotomous preferences, a fully dynamic voting rule exists that satisfies PJR+ and requires, on average, only one change per addition or deletion. For EJR, we give two kinds of approximation guarantees. To satisfy an $\mathcal{O}(\log(k))$-EJR approximation, we provide a fully dynamic rule that requires amortized one change per addition or deletion. To accommodate constant $\alpha > 1$, we provide an algorithm for α-EJR with the average number of swaps per round being constant and independent of k. For exact EJR, we further provide a rule that can handle at least one addition with a single swap, thus rendering strongly negative results like for rank-JR im-

Axiom	Incremental	Decremental	Fully Dynamic
PSC	✓	✗, $\Omega(\log(\log(k)))$ lower b.	✗, $\Omega(\log(\log(k)))$ lower b.
rank-JR	✗, $\Omega(\sqrt{k})$ lower b.	✗, $\Omega(\sqrt{k})$ lower b.	✗, $\Omega(\sqrt{k})$ lower b.
Proportional Fairness	✓3.79-approx	✓3.79-approx	✓3.79-approx
q-Core	✓5-approx	✓5-approx	✓5-approx
PJR+	✓	✓(amort.)	✓(amort.)
EJR+	✓(2,2)-approx	✓$\log(k)$-approx (amort.)	✓$\log(k)$-approx (amort.)

Table 1: An overview of our results. A ✓ indicates that a robust (amortized) dynamic algorithm exists for the respective problem and input type. An ✗ indicates that no robust fully dynamic algorithm can exist.

possible in the approval setting.

Related Work. Proportionality in the static multiwinner voting setting is well studied. In detail, we refer to Lackner and Skowron (2022) for a recent book on approval-based multiwinner voting and to the works of Aziz and Lee (2020, 2021, 2022) and Brill and Peters (2023) for proportionality with ranked preferences. For proportional clustering we refer to Chen et al. (2019) who introduced the concept of proportional fairness in clustering as well as to several follow-up works deepening the analysis of proportional fairness in clustering (Micha and Shah 2020; Li et al. 2021; Kalayci, Kempe, and Kher 2024) and relating it to multiwinner voting (Aziz et al. 2023; Kellerhals and Peters 2023).

As for related temporal or dynamic models in multiwinner voting, we refer to Elkind, Obraztsova, and Teh (2024) for a recent survey. Most notably, we are closely related to Do et al. (2022), who study an online model of approval-based multiwinner voting, in which candidates appear in an online manner and have to be irrevocably chosen or rejected at each time step. The difference is that in our model (i) the committee needs to satisfy the proportionality notion at every step of the online process and (ii) chosen candidates can be taken off the committee or even off the feasible set again. Further, Brill et al. (2023) consider a model in which they assume that the set of candidates is known in advance, but the actual availability of the candidates is unclear and needs to be requested via an invite to the committee. Similarly to Do et al. (2022), candidates are added irrevocably. Further, the preferences over all candidates are known in advance.

In a series of works Deltl, Fluschnik, and Bredereck (2023); Bredereck, Fluschnik, and Kaczmarczyk (2022); Bredereck, Kaczmarczyk, and Niedermeier (2020) considered the complexity of the *sequential committee selection problem*. In their model, the rules select one committee per time step, while constantly satisfying requirements such as optimizing egalitarian welfare or scores, subject to the committees not changing drastically over time. Most problems studied in these works are computationally intractable and thus the authors instead study the parameterized complexity of several related problems.

Finally, outside of computational social choice, there is a large body of literature on dynamic low-recourse algorithms. For instance Matuschke, Schmidt-Kraepelin, and Verschae (2019); Megow and Nölke (2020); Bernstein, Holm, and Rotenberg (2019) study online matching and Łącki et al. (2024); Bhattacharya et al. (2024a,b) study online clustering with low recourse.

2 Defining Model and Notation

We first introduce the standard settings and proportionality notions, then expand them to be dynamic. Throughout the paper, we are given a set $N = [n]$ of voters and a set $C = \{c_1, \dots, c_m\}$ of candidates. Each voter i has preferences P_i over the candidates with $P = (P_i)_{i \in N}$ being the preference profile. We deal with two different kinds of preferences for the standard multiwinner voting problem:

(i) with *approval preferences* each voter $i \in N$ has an *approval set* $A_i \subseteq C$;

(ii) with ordinal preferences each voter $i \in N$ has a strict order $\succ_i \subseteq C^2$ over the candidates.

We are further given a target size k and the goal is to select a committee $W \subseteq C$ of size k. Together $\mathcal{I} = (N, C, P, k)$ is an (approval/ordinal) instance.

Given an ordinal instance and voter $i \in N$ and candidate $c \in C$ we write $\mathrm{rank}(i, c) = |\{c' \in C : c' \succ c\}| + 1$, e.g., the candidate ranked first has rank 1.

We additionally investigate the proportional clustering problem. In this, we are instead given a (pseudo-) metric space $(N \cup C, d)$ with the distance function $d\colon (N \cup C)^2 \to \mathbb{R}_{\geq 0}$ satisfying (i) $d(i, j) = d(j, i)$ (symmetry) and (ii) $d(i, j) \leq d(i, h) + d(h, j)$ (triangle inequality) for all $i, j, h \in N \cup C$. For an agent i and subset $C' \subseteq C$ we let $d(i, C') = \min_{c \in C'} d(i, c)$ denote the distance from i to C'. Analogously, we define $d^q(i, C')$ to be the distance to the q-th furthest away candidate in C' from i.

Proportionality Notions

In the following sections, we call a group $N' \subseteq N$ of voters ℓ-large if $|N'| \geq \frac{\ell n}{k}$.

Approval Preferences. First, we assume that we are given an approval instance $\mathcal{I} = (N, C, A, k)$. For approval preferences, we call a set $N' \subseteq N$ of voters ℓ-cohesive if $|\bigcap_{i \in N'} A_i| \geq \ell$. We say that a committee satisfies

- *JR* if for every 1-cohesive and 1-large group N' there exists an $i \in N'$ with $|A_i \cap W| \geq 1$ (Aziz et al. 2017).

- *PJR* if for every $\ell \in [k]$ and ℓ-cohesive and ℓ-large group N' it holds that $|\bigcup_{i \in N'} A_i \cap W| \geq \ell$ (Sánchez-Fernández et al. 2017).
- *PJR+* if for every $\ell \in [k]$ and 1-cohesive and ℓ-large group N' it holds that $|\bigcup_{i \in N'} A_i \cap W| \geq \ell$ or $\bigcap_{i \in N'} A_i \subseteq W$ (Brill and Peters 2023).
- *EJR* if for every $\ell \in [k]$ and ℓ-cohesive and ℓ-large group N' there exists an $i \in N'$ with $|A_i \cap W| \geq \ell$ (Aziz et al. 2017).
- *EJR+* if for every $\ell \in [k]$ and 1-cohesive and ℓ-large group N' there exists an $i \in N'$ with $|A_i \cap W| \geq \ell$ or it holds that $\bigcap_{i \in N'} A_i \subseteq W$(Brill and Peters 2023).

Further, α-EJR+ with $\alpha > 1$ requires that for every $\ell \in [k]$ and 1-cohesive and $\alpha \cdot \ell$-large group N' there exists an $i \in N'$ with $|A_i \cap W| \geq \ell$ or it holds that $\bigcap_{i \in N'} A_i \subseteq W$.

Ordinal Preferences. Given ordinal preferences $(N, C, \succ, k)$, we say that a set of voters $N' \subseteq N$ is a *solid coalition* over a set $S \subseteq C$ of candidates if for any $i \in N'$ it holds that $S \succ C \setminus S$, i.e., S forms a prefix of that voters preferences. A committee W satisfies *Proportionality for Solid Coalitions (PSC)* (Dummett 1984) if for every ℓ-large group of voters N' that is a solid coalition over some $S \subseteq C$, it holds that $|W \cap S| \geq \min(\ell, |S|)$. If some solid coalition N' over S causes W to violate PSC, we say that N' (or (N', S)) is a *witness*.

For a given approval-based proportionality notion X and $r \in [k]$ we let $A_i^r = \{c \in C\colon \operatorname{rank}(i, c) \leq r\}$. Let $\mathcal{I}_r = (N, C, A^r, k)$ be the approval instance, in which every voter only approves their top r choices. Given an approval-based proportionality notion X, a committee W satisfies rank-X if W satisfies X in each instance $\mathcal{I}_r$ (Brill and Peters 2023). In the following example with $n = 4$ voters and $k = 2$, all committees satisfy PSC, but due to $r = 2$, only committees containing b satisfy rank-JR (Brill and Peters 2023).

$$
\begin{gathered}
a \succ b \succ e \succ d \succ c \\
d \succ b \succ c \succ e \succ a \\
c \succ b \succ e \succ d \succ a \\
e \succ b \succ d \succ c \succ a
\end{gathered}
$$

Proportional Clustering. In proportional clustering, the most prominent notion is that of proportional fairness (Chen et al. 2019; Kalayci, Kempe, and Kher 2024). An outcome (or clustering or committee) W is said to be γ-*proportionally fair* for some $\gamma \geq 1$ if there is no unselected candidate $c \in C \setminus W$ and 1-large group $N' \subseteq N$ of voters such that

$$\min_{c' \in W} d(i, c') > \gamma d(i, c) \text{ for all } i \in N'.$$

Proportional fairness, however, only looks at deviations to single candidates (akin to JR for approval preferences). This prompted several strengthenings of proportional fairness (Ebadian and Micha 2024; Kalayci, Kempe, and Kher 2024; Aziz et al. 2023) dealing with deviations to multiple candidates instead. As it is closest in spirit to proportional fairness, we focus here on the *q-core* as introduced by Ebadian and Micha (2024). A committee W is said to be in the α-*q*-core for some $q \in [k]$ if for all other candidate subsets $C' \subseteq C$ the following holds: there are strictly less than $\frac{|C'|}{k} n$ voters $i \in N$ for which their q-th closest candidate in W is α-times farther away than their q-th closest candidate in C'.

Online Algorithms

We consider three types of dynamic settings. First, in the *incremental setting*, we have a stream of candidates $c_1, \ldots, c_m$ appearing over time, and the voters' preferences are revealed incrementally. For this purpose, let $t \in \{k, \ldots, m\}$ denote a time step and $C_t = \{c_1, \ldots, c_t\}$ be the set of available candidates at this time step. Formally, an incremental voting rule takes as input $(\succ, t, k)$ and outputs a committee $W \subseteq C_t$ of target size k. Further, it is only allowed to depend on $\succ|_{C_t}$, i.e., we demand that $f(\succ, t, k) = f(\succ', t, k)$ whenever $\succ|_{C_t} = \succ'|_{C_t}$

Next, in the *decremental setting*, a set of candidates $C_0 = \{c_1, \ldots, c_m\}$ is given, and the voters reveal their preferences over the full set immediately. However, the candidates do not reveal when they will drop out of the election. Formally, this is modeled via a sequence $(C_t)_{t \geq 0}$, with $|C_t| = m - t$ and $C_t \subset C_{t-1}$. Then, a decremental ABC voting rule takes as input $(\succ, t, k)$ and outputs a committee $W \subseteq C_t$ of target size k. At each step t, it is allowed to use the full preference profile, but only $(C_s)_{s \leq t}$. Thus f has to be independent of the order of future deletions.

Finally, in the *fully dynamic setting*, the candidates can both join and leave as they desire. We again start with $C_0 = \{c_1, \ldots, c_m\}$, but this time only impose as constraint on the sequence of candidate sets $(C_t)_{t \geq 0}$ that for all t, we have that C_{t+1} is obtained from C_t by adding or removing one candidate. A fully dynamic rule takes as input $(\succ, t, k)$ and outputs a committee $W \subseteq C_t$ of target size k. At time t, it is only allowed to depend on $(C_s)_{s \leq t}$ and $\succ$ restricted to $\bigcup_{s \leq t} C_s$.

In all settings, when $\succ$ and k are clear from the context, we can omit these and write $f(t)$ instead of $f(\succ, t, k)$. Let X be any axiom for committees. We say that f satisfies X if $f(\succ, t, k)$ satisfies X for all k, t, and $\succ$. A rule f is *robust*, if $|f(\succ, t, k) \cap f(\succ, t+1, k)| \geq k - 1$ for all $t < m$.

For example, consider the following ranked profile:

$$
\begin{gathered}
a \succ c \succ e \succ b \succ d \\
a \succ e \succ c \succ d \succ b \\
a \succ d \succ b \succ e \succ c \\
a \succ e \succ d \succ c \succ b.
\end{gathered}
$$

Let the sequence be $C_0 = \{a, b, c, d\}$, $C_1 = \{b, c, d\}$, $C_2 = \{b, c, d, e\}$ and $k = 2$. Then, $f(0) = \{a, c\}$, $f(1) = \{c, d\}$, $f(2) = \{d, e\}$ maintains PSC at every step.

Throughout the paper, we will reference well-known rules from the static multiwinner voting setting to illustrate that the dynamic problems add a layer of depth that has yet to be accounted for. Their definitions and all missing proofs and examples are in the appendix.

3 Ordinal Preferences

In this section, we investigate ordinal preferences. In more detail, we show that PSC can be satisfied robustly in the incremental setting, while the notion of rank-JR cannot. We further show that both notions cannot be satisfied robustly in the decremental setting and give asymptotic lower bounds for the number of replacements that are required to restore proportionality after a single deletion.

As our first main result, we show that an incremental rule satisfying PSC exists. The proof entails an even stronger statement: for each committee W satisfying PSC and each newly added candidate c that causes a violation, there is always a single candidate $c' \in W$ who can be swapped with c to restore PSC.

Theorem 1. *There exists a robust incremental rule f that satisfies PSC.*

Proof. Let $\{c_1, \dots, c_t\} = C_t$ denote the candidates that joined until time step t and let $f(t)$ denote the committee selected at this time. For $t = k$, let $f(t) = C_k$. This committee trivially satisfies PSC w.r.t. the feasible set C_k.

For $t > k$, let $f(t-1)$ satisfy PSC w.r.t. C_{t-1}. Throughout the proof, we will write $W^* = f(t-1) \cup \{c_t\}$ for the previous committee together with c_t and for $c \in W^*$ we denote $W_{-c} = W^* \setminus \{c\}$ to be the committee with c removed. Our goal is to show that some W_{-c} satisfies PSC w.r.t. C_t, for some $c \in W$ for which we will need the following claims.

Claim 1: For any $c \in W^*$, let W_{-c} violate PSC on C_t due to an ℓ-large set of voters. Then, $|C \cap W^*| \leq \ell$.

To prove this, simply let N_c be ℓ-large and a solid coalition over some C that witnesses a PSC violation of W_{-c}. Then, $|C \cap W_{-c}| < \min(\ell, |C|) \leq \ell$.

Claim 2: Let $c \in W^*$, W_{-c} violate PSC and let (N_c, C) be a witness of that. Then, $c \in C$.

We prove this by contraposition. Let $N_c \subseteq N$ be an ℓ-large, solid coalition over C, but with $c \notin C$. The goal is to show that this block does not witness a violation of PSC for W_{-c}. We will do so now by examining N_c for C_{t-1}.

Case 1 is that $c_t \notin C$, which implies that N_c is also a solid coalition for C in time step $t-1$. Since $f(t-1)$ satisfies PSC by assumption, we have $\min(|C|, \ell) \leq |C \cap f(t-1)| = |C \cap W^*| = |C \cap W_{-c}|$, since both c_t and c do not occur in the block. *Case 2* is that $c_t \in C$. Then, with respect to C_{t-1}, N_c was still a solid coalition for $C \setminus \{c_t\}$ (and ℓ-large). By assumption, $f(t-1)$ satisfies PSC, and thus $\min(|C|-1, \ell) \leq |(C \setminus \{c_t\}) \cap f(t-1)| = |C \cap W^*| - 1 = |C \cap W_{-c}| - 1$, where the last equality again follows from the assumption $c \notin C$. This, however, implies $\min(s, \ell) \leq \min(s-1, \ell) + 1 \leq |C \cap W_{-c}|$, which shows that no violation can occur in Case 2 either and thus Claim 2 is proven.

Now we are ready for the main argument. Assume for contradiction that for all $c \in W^*$, the committee W_{-c} does not satisfy PSC. Then, for any $c \in W^*$, there is some ℓ_c-large solid coalition N_c over C_c witnessing a PSC violation of W_{-c}. We choose these witnesses (N_c, C_c) to be maximal with respect to $|C_c|$ first and $|N_c|$ second. We will now iteratively go over all $k+1$ candidates c in $f(t-1) \cup \{c_t\}$ and injectively assign $\frac{n}{k}$ of the voters in N_c to c. This leads to the desired contradiction, as there are only $k\frac{n}{k}$ voters in total. For this, enumerate $W^* = \{d_1, \dots, d_{k+1}\}$. We will assign each d_j to a subset of N_{d_j} as follows. For d_1, we can assign it to any $\frac{n}{k}$ voters of N_{d_1}, which must exist since N_{d_1} must be at least 1-large to be a witness. For $j \geq 2$, let injective voter assignments be made for all candidates with smaller index, i.e., d_x with $1 \leq x < j$. We claim that there are still at least $\frac{n}{k}$ candidates from N_{d_j} unassigned. If any of these voters has been assigned already, there must exist some other candidate d_x such that $N_{d_x} \cap N_{d_j} \neq \emptyset$.

Claim 3 $N_{d_x} = N_{d_j}$ and $C_{d_x} = C_{d_j}$.

Since the intersection of the two voter sets is non-empty, there exists some voter $i \in N_{d_x} \cap N_{d_j}$. Since C_{d_x} and C_{d_j} are both prefixes of the preference $\succ_i$, one must be a subset of the other. Without loss of generality, assume $C_{d_x} \subseteq C_{d_j}$. We first show that the voter set N_{d_j} is not only a witness for a PSC violation of W_{d_j}, but also for a violation of W_{d_x}. Applying Claim 2, we obtain $d_x \in C_{d_x} \subseteq C_{d_j}$ and $d_j \in C_{d_j}$. Since N_{d_j} is a witness for W_{-d_j}, it thus holds that

$$\min(|C_{d_j}|, \ell_{d_j}) > |C_{d_j} \cap W_{-d_j}| = |C_{d_j} \cap W_{-d_{jx}}|,$$

which proves that (N_{d_j}, C_{d_j}) is also a witness for the PSC violation of W_{-d_x}. Since we chose C_{d_j} to be maximal, it must be that $C_{d_j} = C_{d_x}$ and the prefixes coincide. Thus, $(N_{d_j} \cup N_{d_x}, C_{d_x})$ is also a witness for the PSC violation of W_{d_x}. Since we chose N_{d_x} to be inclusion maximal, it must be that $N_{d_j} \subseteq N_{d_x}$. Applying the same argument to W_{d_j}, we obtain $N_{d_j} \supseteq N_{d_x}$ which concludes the proof of Claim 3.

By Claim 3 and Claim 2, it follows that $d_x \in C_{d_j}$. However, by Claim 1, $\ell_{d_j} \geq |W^* \cap C_{d_j}|$, one of which is d_j itself. Hence, at most $\ell_{d_j} - 1$ different d_x with $x < j$ can have been assigned to some voters in N_{d_j}. Each of these is assigned to $\frac{n}{k}$ voters. Since $|N_{d_j}| = \ell_{d_j}\frac{n}{k}$, there are still at least $\frac{n}{k}$ voters unchosen, which we can assign to d_j. This concludes the step from $j \to j+1$, thus we can injectively assigned each $c \in W^*$ to some subset of N of size $\frac{n}{k}$, which is not possible. This is the desired contradiction and thus there is some $c \in W^*$ such that W_{-c} satisfies PSC w.r.t. C_t. This concludes the induction step $t-1 \to t$.

□

The approach of Theorem 1 does not work for deleting a candidate—if we choose poorly, we may need to replace the entire committee after a single deletion.

Example 1. *A size k committee satisfying PSC can require k additions to restore PSC after a single deletion. For this, consider the following profile with $n = k$*

$$a_1 \succ b_1 \succ a_2 \dots a_k \succ *$$
$$a_1 \succ b_2 \succ a_2 \dots a_k \succ *$$
$$\vdots$$
$$a_1 \succ b_k \succ a_2 \dots a_k \succ *$$

The committee $W = \{a_1, \ldots, a_k\}$ satisfies PSC. However, after removing a_1, all candidates b_i would need to be added.

Similarly, commonly used rules that satisfy PSC fail to distinguish between robust and non-robust committees.

Example 2. *The single transferable vote (STV) and the expanding approvals rule (EAR) can select committees that are not robust to a single deletion for PSC, even when such committees exist.*

While in Example 2 there exists a robust committee, there are instances where *any* committee satisfying PSC is not robust and requires at least $\Omega(\log(\log(k)))$ changes to restore PSC. This precludes the existence of a robust decremental—let alone fully dynamic—PSC algorithm.

Proposition 2. *There does not exist a robust decremental PSC algorithm. After a single deletion, it can be that $\Omega(\log(\log(k)))$ replacements are needed to restore PSC.*

For rank-JR, we show that neither incremental nor decremental rules can be robust.

Theorem 3. *There is no robust incremental or decremental rank-JR algorithm.*

Proof. We give the following example with $n = 12$ voters, $m = 13$ candidates $C = \{c_1, \ldots, c_6, c_1', \ldots, c_6', \bar{c}\}$, and $k = 6$. We show that the removal of c_1 from C is incompatible with decremental rank-JR and the addition of c_1, c_1' to $C \setminus \{c_1, c_1'\}$ is incompatible with incremental rank-JR:

1. $c_1 \succ c_2 \succ \ldots$	$1'$. $c_1' \succ c_2' \succ \ldots$
2. $c_1 \succ c_3 \succ \ldots$	$2'$. $c_1' \succ c_3' \succ \ldots$
3. $c_2 \succ c_5 \succ \bar{c} \ldots$	$3'$. $c_2' \succ c_5' \succ \bar{c} \ldots$
4. $c_3 \succ c_4 \succ \bar{c} \ldots$	$4'$. $c_3' \succ c_4' \succ \bar{c} \ldots$
5. $c_5 \succ c_6 \succ \bar{c} \ldots$	$5'$. $c_5' \succ c_6' \succ \bar{c} \ldots$
6. $c_4 \succ c_6 \succ \bar{c} \ldots$	$6'$. $c_4' \succ c_6' \succ \bar{c} \ldots$

Note that this instance is symmetric with regard to ranking candidates c_i and c_i'. We first show the following claim, which is crucial for both proofs:

Claim 1: If a committee W satisfies rank-JR on C, then $|W \cap \{c_2, c_3, c_2', c_3'\}| \leq 1$. To show this claim, first let a committee W be given with $\{c_2, c_3\} \subseteq W$. Since c_1, c_1' are top ranked twice, $c_1, c_1' \in W$. Further, voters $5, 6$ share candidate c_6 at position $r = 2$ and thus one of $\{c_4, c_5, c_6\}$ must be contained in W. Since $|W| = 6$, W can only contain one of the candidates $\{c_2', \ldots, c_6'\}$. Thus, there are always $\{i', j'\} \subset \{3', \ldots, 6'\}$ obtaining none of their three most preferred candidates, leading to the desired violation of rank-JR for W with $r = 3$. Now, let a committee W be given with $|W \cap \{c_2, c_3\}| = 1$ and $|W \cap \{c_2', c_3'\}| = 1$. We have to show that W violates rank-JR. Again, $c_1, c_1' \in W$. Further, voters $5, 6$ share candidate c_6 at position $r = 2$ and thus one of $\{c_4, c_5, c_6\}$ must be contained in W. Since the same argument holds for voters $5', 6'$, the committee W already contains 6 candidates and $\bar{c} \notin W$. Note however, no matter how we choose the candidates, there will be one voter $i \in \{3, 4, 5, 6\}$ not obtaining any of their top three ranked candidates: If we choose c_5 (c_4), voter 6 (voter 5) is unhappy. If we choose c_6, voter 3 or voter 4 will be unhappy as we can only choose one of $\{c_{2,3}\}$. By applying the same argument, we obtain $j \in \{3', 4', 5', 6'\}$ for which W does not contain any of their top three candidates. Since i, j both have $\bar{c}$ as their third most-preferred candidate, $\{i, j\}$ induces a rank-JR violation for W and $r = 3$. This concludes the proof of the claim.

The following is clearly true: **Claim:** If a committee W^2 on $C \setminus \{c_1, c_1'\}$ satisfies rank-JR, then $\{c_2, c_3, c_2', c_3'\} \subset W^2$.

For the incremental rank-JR violation, we see the following: Any rule f satisfying rank-JR on $C^2 = C \setminus \{c_1, c_1'\}$ must return W^2 containing $\{c_2, c_3, c_2', c_3'\}$. For $C^1 = C \setminus \{c_1'\}$, thus $|W^1 \cap \{\{c_2, c_3, c_2', c_3'\}\}| \geq 3$ and thus for $C^0 = C$ we have $|W^0 \cap \{\{c_2, c_3, c_2', c_3'\}\}| \geq 2$. This implies that f fails rank-JR by our Claim 1.

For decremental rank-JR, apply Claim 1 to obtain that wlog $W \cap \{c_2, c_3\} = \emptyset$. Deleting c_1 now requires the addition of both c_2, c_3, as desired. □

We generalize this approach to show that for any t there is an instance in which at least t changes must be made after deleting a single candidate. This provides steeper lower bounds than for PSC.

Theorem 4. *There is no incremental or decremental algorithm satisfying rank-JR and making $o(\sqrt{k})$ changes amortized per round.*

As a corollary, our results imply in the language of Brill et al. (2023) that there is no safe batch querying procedure for rank-JR or PSC.

Corollary 5. *There is no set of size k which is safe for either rank-JR or PSC.*

Theorem 4 further raises the question of whether this bound is tight.

Open Question 1. *Is there an incremental or decremental algorithm satisfying rank-JR making at most $\mathcal{O}(\sqrt{k})$ changes amortized per round?*

4 Proportional Clustering

Secondly, we turn to *proportional clustering*. Here, candidates and voters lie in a metric space $(N \cup C, d)$. Unlike the social choice settings, proportionality can only be approximated, not perfectly satisfied. Existing algorithms, such as Greedy Capture (Chen et al. 2019) or the Expanding Approvals Rule (Aziz et al. 2023), achieve a constant factor approximation to proportional fairness (Kellerhals and Peters 2023) by relying on a generalization of rank-JR to proportional clustering. We circumvent this and design a fully dynamic algorithm achieving a constant factor approximation to proportional fairness. In essence, proportional clustering is easier than proportional multiwinner voting, as voters share a metric space with the candidates. By clustering similar voters into groups, we preempt the clustering of candidates. Using this, we obtain a fully dynamic algorithm that is 3.79-proportionally fair and in the 5-q-core. Surprisingly enough, the 5-q-core bound is the same as the best-known bound obtained by Kellerhals and Peters (2023) in the offline setting.

Theorem 6. *There exists a robust fully dynamic algorithm achieving a* $\frac{3+\sqrt{21}}{2} \sim 3.79$*-proportional fair outcome and satisfying the* 5*-*q*-core for any* $q \in [k]$.

Proof Sketch of proportional fairness. We begin with pre-clustering the voters, following Aziz et al. (2023); Kalayci, Kempe, and Kher (2024), by assign each voter $i \in N$ the same budget $b_i = \frac{k}{n}$. Then we continuously increase a radius δ from 0 on until there are sets of voters $N' \subseteq N$ of diameter at most δ who have a total budget of at least 1. We create a cluster N_i for them and decrease their weights by a total of 1. Continuing this process leads to clusters $N_1, \dots, N_k$.

For a given cluster of voters N_i and candidate c we let $d(N_i, c) = \min\{d(j, c) : p_j(i) > 0\}$. Considering the clusters in order, we pick among the so far unchosen candidates some c_i minimizing $d(N_i, c_i)$. We say that a cluster N_j *envies* another cluster N_ℓ if $d(N_\ell, c_\ell) < d(N_j, c_j)$ By our enumeration, it is clear that some N_j can only envy some N_i if $i < j$. Further, the selected committee $\{c_1, \dots, c_k\}$ contains the distance minimizers for all k clusters N_i. This invariant remains true if we proceed as follows.

When we delete a candidate c we either do (i) nothing if c was not picked by any cluster or (ii) if c was picked by some cluster, we let it repick the closest current unchosen candidate d and consider it as the new N_k.

If a candidate c gets added, we check if there is any cluster N_i prefering c to its candidate c_i. Only if there are such clusters, we assign c to the smallest such i^* and in return set c_{i^*} as the current free candidate. In order, give each cluster the choice to swap the current free candidate with its own candidate.

To show that this procedure is always ρ-proportional fair, let W be any committee throughout the online process, c be any unselected candidate, and assume some large enough group of voters deviates to c. Then, we know that at least one of the agents pays for a preprocessed cluster N'' that has a diameter not larger than the deviating group. Let i be this agent and let j be the agent furthest away from c in the deviating group. Through multiple applications of the triangle inequalities and properties of the cluster N'', we obtain

$$\min\left(\frac{d(i, W)}{d(i, c)}, \frac{d(j, W)}{d(j, c)}\right) \leq \min_{x \geq 0}\left(3x, 3 + \frac{1}{x}\right),$$

implying that c improves W by at most $\frac{3+\sqrt{21}}{2}$. □

While the initial pre-clustering is not tractable, it is possible to adapt this step to obtain a polynomial time constant factor approximation with slightly larger constant.

5 Approval Preferences

Thirdly, we consider the setting of approval preferences, where each voter $i \in N$ submits an approval set $A_i \subseteq C$. We begin with two short examples, showing that a result akin to Theorem 1 is not possible, even for the weakest axiom of JR.

Example 3. *First, we give an example of a committee satisfying EJR+, for which removing any candidate from the committee, forces at least two changes, even for JR to be satisfied. For this, consider the following simple instance with* $k = 2$*, two voters, and respective approval sets* $\{a, b\}$ *and* $\{b, c\}$*. Further, there is a candidate* d *approved by no one. The committee* $\{b, d\}$ *satisfies EJR+, but after removal of* b*, only the committee* $\{a, c\}$ *satisfies JR.*

A similar example can also be created for adding a candidate. For this, consider an instance with $k = 3$ *and four voters having approval sets* $\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{f\}\{f\}$*. Consider the instance without candidate* f *and the committee* $\{b, c, d\}$*. After the addition of* f *to the instance, we would need to remove one of* b, c, d*, for whom the voter approving it, together with the voter approving* $\{a, e\}$ *would witness a JR violation.*

PJR+. Nonetheless, we can show that a good PJR+ algorithm exists. In the following, we utilize the concept of affordability and priceability and combine the ideas of Do et al. (2022) and Brill et al. (2023) to design a fully dynamic PJR+ algorithm making amortized one change per round.

Following Peters and Skowron (2020) a committee $W \subseteq C$ is *priceable* if there exists a payment system $(p_i)_{i \in N} : C \to \mathbb{R}_{\geq 0}$ satisfying the following constraints:

C1 $p_i(c) = 0$ if $c \notin A_i$ for all $c \in C$ and $i \in N$

C2 $\sum_{c \in C} p_i(c) \leq \frac{k}{n}$ for all $i \in N$

C3 $\sum_{i \in N} p_i(c) = 1$ for all $c \in W$

C4 $\sum_{i \in N} p_i(c) = 0$ for all $c \notin W$

C5 $\sum_{i \in N_c} \left(\frac{k}{n} - \sum_{d \in C} p_i(d)\right) \leq 1$ for all $c \notin W$.

A simple weakening of priceability is *affordability* (Brill and Peters 2024).

Definition 1. *A committee* $W \subseteq C$ *is* affordable *if there is a payment system for* W *satisfying constraints **C1–C4**.*

It is easy to see that any priceable committee, which satisfies **C5** with a strict inequality (**C5<**), also satisfies PJR+ (Brill and Peters 2023, Proposition 10). We call such a committee *maximally affordable*.

Corollary 7. *Every maximally affordable committee satisfies PJR+.*

For both incremental and decremental algorithms for PJR+, we show how to maintain maximally affordable committees. This approach leads to a robust incremental PJR+ algorithm. It uses a similar idea to the algorithm of Do et al. (2022), who show that PJR+ is satisfiable in their online committee selection setting.

Theorem 8 (Do et al. 2022)**.** *There exists a robust incremental PJR+ algorithm.*

The proof also shows that once a maximally affordable committee of size k has been instantiated, it will continue to satisfy PJR+ no matter how many candidates are introduced to the feasible set.

Taking the idea further, we use maximally affordable committees to construct a fully dynamic PJR+ algorithm. Our algorithm makes amortized one change per iteration.

Theorem 9. *There exists a robust fully dynamic PJR+ algorithm making amortized one change per iteration.*

Proof. Our goal is to maintain a maximally affordable committee throughout the process. Let C be the initial candidate set. We compute an affordable committee W of maximum size for C. Now, for each deleted candidate c throughout the process, we check if there are new affordable candidates. If they are, we add them to the committee, while there is still an affordable candidate. For each added candidate, we check whether this candidate is affordable. If it is, we add it to the committee.

To see that we make amortized one change per iteration, we see that (i) for each addition we only make one change and (ii) for each deletion we can at most add until we reach a maximum size affordable committee. Since each added candidate which was not added to the committee can only increase the size of the maximum affordable committee by one and since we started off with a maximum size affordable committee, we can at most add one candidate per deletion and addition. Thus, we make amortized one change per iteration. □

If one is to start with a set of $m > k$ candidates, computing the maximum size affordable committee is NP-hard. However, there is a simple fix: We can calculate any maximally affordable committee and continue with the replacements as described in the algorithm. Then, we have less than amortized $1 + \frac{m}{t}$ replacements in t steps, so if the time horizon is large enough, we can get arbitrarily close to amortized 1 replacement per step in polynomial time. This algorithm, however, only makes *amortized* one change per iteration. This leads us to our next open question.

Open Question 2. *Is there a robust decremental or fully decremental algorithm satisfying PJR+?*

EJR+. While our previous result shows that one can nearly achieve PJR+ in a fully dynamic manner, PJR+ in itself is quite a weak axiom for approval-based multiwinner voting (see for instance Peters and Skowron (2020, Example 6)). The strongest alternative to PJR+ which is still achievable in polynomial time is EJR+. Both EJR+ and its weaker version EJR, however, are not nearly as well understood as PJR+, which is often significantly easier to achieve. Consequentially, few rules are known to satisfy EJR(+).

As our first result we build upon an approximation result from Do et al. (2022) and show that a $\mathcal{O}(\log(k))$-approximation of EJR+ is possible in a fully dynamic setting making amortized one change per iteration. This in essence works similar to Theorem 9.

Theorem 10. *There exists a fully dynamic $\Theta(\log(k))$-EJR+ algorithm making amortized one change per iteration.*

While Theorem 10 only considers an approximation factor depending on k, we also can obtain bounds for general α-EJR+ at the cost of making more than amortized one change per iteration. For instance, our result gives us an incremental algorithm satisfying 2-EJR+ making amortized 2 changes per iteration.

Theorem 11. *For any $\alpha > 1$ there exists an incremental α-EJR+ algorithm making amortized $\frac{\alpha}{\alpha-1}$ changes.*

While achieving a robust algorithm satisfying exact EJR+ seems difficult, we are able to provide a committee that is robust with respect to a single added candidate. This rules out strongly negative results akin to Theorem 3 at least for incremental algorithms. We achieve this by modifying the GJCR of Brill and Peters (2023) to be "locally stable".

Theorem 12. *There exists an incremental EJR+ algorithm that is robust with respect to a single addition.*

The modification is necessary even for this single step.

Example 4. *GJCR, MES, and PAV can elect committees that are not robust with respect to a single addition.*

Naturally, this leads to the open question whether there is an incremental algorithm for EJR+ that remains robust beyond the first step.

Open Question 3. *Is there a robust (amortized) incremental rule satisfying EJR+?*

The commonly considered rules satisfying EJR+ further all fail to distinguish between decrementally robust and unrobust committees.[1]

Example 5. *Consider the profile with* $2 \times \{a_1, \dots, a_5, x\}$, $2 \times \{a_1, \dots, a_5, y\}$, $1 \times \{a_1, \dots, a_5\}$, $2 \times \{b_1, \dots, b_4, x\}$, $2 \times \{b_1, \dots, b_4, y\}$, $1 \times \{c_1, \dots, c_4, x\}$, $1 \times \{c_1, \dots, c_4, y\}$, $2 \times \{c_1, \dots, c_4\}$ *and fix* $k = n = 13$.

Then, MES, GJCR, and PAV can choose the committee $\{a_1, \dots, a_5\} \cup \{b_1, \dots, b_4\} \cup \{c_1, \dots, c_4\}$. *However, after the deletion of some a_i, say a_1, we would need to add both x and y to the committee to restore EJR+.*

It remains unknown whether EJR+ is achievable robustly in the decremental setting, even for a single deletion.

Open Question 4. *Is there a decremental rule satisfying EJR+ that is robust with respect to a single deletion?*

6 Conclusion and Open Questions

Our work leaves open several questions and possible future research directions. As a "meta" future research direction, we highlight that the understanding of EJR and EJR+ as axioms is still quite narrow. For instance, despite extensive research, it is still an open question whether there always exists a ranking satisfying EJR for every prefix of the ranking (Skowron et al. 2017; Chandak, Goel, and Peters 2024) or whether there is a safe querying procedure for EJR (Brill et al. 2023). Furthering the understanding of EJR+ and EJR, e.g., by developing different rules or characterizations of rules satisfying EJR+ might shed further light on it and help to resolve the open questions regarding EJR and EJR+.

As a further point, we restrict ourselves to dynamic *candidate* sets. A natural extension would be a setting in which not only the candidates, but also the voters are dynamic. This gives some additional difficulties. For instance, adding or deleting voters, changes the quota $\frac{n}{k}$ throughout the process. Deciding whether it is feasible to deal with this, is an interesting possibility for future work.

[1] To make the following an EJR violation, clone x and y each 5 times.

References

Amanatidis, G.; Aziz, H.; Birmpas, G.; Filos-Ratsikas, A.; Li, B.; Moulin, H.; Voudouris, A. A.; and Wu, X. 2023. Fair division of indivisible goods: Recent progress and open questions. *Artificial Intelligence*, 322: 103965.

Aziz, H.; Brill, M.; Conitzer, V.; Elkind, E.; Freeman, R.; and Walsh, T. 2017. Justified Representation in Approval-Based Committee Voting. *Social Choice and Welfare*, 48(2): 461–485.

Aziz, H.; and Lee, B. E. 2020. The expanding approvals rule: improving proportional representation and monotonicity. *Social Choice and Welfare*, 54: 1–45.

Aziz, H.; and Lee, B. E. 2021. Proportionally Representative Participatory Budgeting with Ordinal Preferences. In *Proceedings of the 35th AAAI Conference on Artificial Intelligence (AAAI)*, 5110–5118. AAAI Press.

Aziz, H.; and Lee, B. E. 2022. A characterization of proportionally representative committees. *Games and Economic Behavior*, 133: 248–255.

Aziz, H.; Lee, B. E.; Morota Chu, S.; and Vollen, J. 2023. Proportionally Representative Clustering. Technical report, arXiv:2304.13917 [cs.LG].

Bernstein, A.; Holm, J.; and Rotenberg, E. 2019. Online bipartite matching with amortized $o(log^2 n)$ replacements. *Journal of the ACM*, 66(5): 37:1–37:23.

Bhattacharya, S.; Costa, M.; Garg, N.; Lattanzi, S.; and Parotsidis, N. 2024a. Fully Dynamic k-Clustering with Fast Update Time and Small Recourse. In *Proceedings of the 65th Symposium on Foundations of Computer Science (FOCS)*. Forthcoming.

Bhattacharya, S.; Goranci, G.; Jiang, S. H.-C.; Qian, Y.; and Zhang, Y. 2024b. Dynamic Facility Location in High Dimensional Euclidean Spaces. In *Proceedings of the 41st International Conference on Machine Learning (ICML)*. Forthcoming.

Bredereck, R.; Fluschnik, T.; and Kaczmarczyk, A. 2022. When Votes Change and Committees Should (Not). In *Proceedings of the 31st International Joint Conference on Artificial Intelligence (IJCAI)*, 144–150.

Bredereck, R.; Kaczmarczyk, A.; and Niedermeier, R. 2020. Electing Successive Committees: Complexity and Algorithms. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI)*, 1846–1853.

Brill, M.; Dindar, H.; Israel, J.; Lang, J.; Peters, J.; and Schmidt-Kraepelin, U. 2023. Multiwinner Voting with Possibly Unavailable Candidates. In *Proceedings of the 37th AAAI Conference on Artificial Intelligence (AAAI)*, 5532–5539. AAAI Press.

Brill, M.; and Peters, J. 2023. Robust and Verifiable Proportionality Axioms for Multiwinner Voting. In *Proceedings of the 24th ACM Conference on Economics and Computation (ACM-EC)*, 301. ACM. Full version arXiv:2302.01989 [cs.GT].

Brill, M.; and Peters, J. 2024. Completing Priceable Committees: Utilitarian and Representation Guarantees for Proportional Multiwinner Voting. In *Proceedings of the 38th AAAI Conference on Artificial Intelligence (AAAI)*, 9528–9536. AAAI Press.

Caragiannis, I.; Micha, E.; and Peters, J. 2024. Can a Few Decide for Many? The Metric Distortion of Sortition. In *Proceedings of the 41st International Conference on Machine Learning (ICML)*. Forthcoming.

Chandak, N.; Goel, S.; and Peters, D. 2024. Proportional Aggregation of Preferences for Sequential Decision Making. In *Proceedings of the 38th AAAI Conference on Artificial Intelligence (AAAI)*, 9573–9581.

Chen, X.; Fain, B.; Lyu, L.; and Munagala, K. 2019. Proportionally Fair Clustering. In *Proceedings of the 36th International Conference on Machine Learning (ICML)*, 1032–1041.

Deltl, E. M.; Fluschnik, T.; and Bredereck, R. 2023. Algorithmics of Egalitarian versus Equitable Sequences of Committees. In *Proceedings of the 32nd International Joint Conference on Artificial Intelligence (IJCAI)*, 2651–2658.

Do, V.; Hervouin, M.; Lang, J.; and Skowron, P. 2022. Online Approval Committee Elections. In *Proceedings of the 31st International Joint Conference on Artificial Intelligence (IJCAI)*, 251–257.

Dummett, M. 1984. *Voting Procedures*. Oxford University Press.

Ebadian, S.; and Micha, E. 2024. Boosting Sortition via Proportional Representation. Technical report, arXiv:2406.00913 [cs.GT].

Elkind, E.; Obraztsova, S.; and Teh, N. 2024. Temporal Fairness in Multiwinner Voting. In *Proceedings of the 38th AAAI Conference on Artificial Intelligence (AAAI)*, 22633 – 22640.

Faliszewski, P.; Skowron, P.; Slinko, A.; and Talmon, N. 2017. Multiwinner Voting: A New Challenge for Social Choice Theory. In Endriss, U., ed., *Trends in Computational Social Choice*, chapter 2. AI Access.

Gallagher, M. 1996. By-elections to Dáil Éireann 1923–96: The anomaly that conforms. *Irish Political Studies*, 11(1): 33–60.

Kalayci, Y. H.; Kempe, D.; and Kher, V. 2024. Proportional Representation in Metric Spaces and Low-Distortion Committee Selection. In *Proceedings of the 38th AAAI Conference on Artificial Intelligence (AAAI)*, 9815–9823.

Kellerhals, L.; and Peters, J. 2023. Proportional Fairness in Clustering: A Social Choice Perspective. Technical report, arXiv:2310.18162 [cs.LG].

Łącki, J.; Haeupler, B.; Grunau, C.; Rozhoň, V.; and Jayaram, R. 2024. Fully Dynamic Consistent k-Center Clustering. In *Proceedings of the 35th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 3463–3484. SIAM.

Lackner, M.; and Skowron, P. 2022. *Multi-Winner Voting with Approval Preferences*. Springer.

Li, B.; Li, L.; Sun, A.; Wang, C.; and Wang, Y. 2021. Approximate group fairness for clustering. In *Proceedings of the 38th International Conference on Machine Learning (ICML)*, 6381–6391.

Matuschke, J.; Schmidt-Kraepelin, U.; and Verschae, J. 2019. Maintaining Perfect Matchings at Low Cost. In *Proceedings of the 46th International Colloquium on Automata, Languages, and Programming (ICALP)*.

Megow, N.; and Nölke, L. 2020. Online minimum cost matching with recourse on the line. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, 37:1–37:16.

Micha, E.; and Shah, N. 2020. Proportionally fair clustering revisited. In *Proceedings of the 47th International Colloquium on Automata, Languages, and Programming (ICALP)*, 85:1–85:16.

Peters, D.; Pierczyński, G.; and Skowron, P. 2021. Proportional Participatory Budgeting with Additive Utilities. In *Advances in Neural Information Processing Systems (NeurIPS)*, volume 34, 12726–12737.

Peters, D.; and Skowron, P. 2020. Proportionality and the Limits of Welfarism. In *Proceedings of the 21st ACM Conference on Economics and Computation (ACM-EC)*, 793–794. ACM.

Sánchez-Fernández, L.; Elkind, E.; Lackner, M.; Fernández, N.; Fisteus, J. A.; Basanta Val, P.; and Skowron, P. 2017. Proportional justified representation. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI)*, 670–676. AAAI Press.

Skowron, P.; Lackner, M.; Brill, M.; Peters, D.; and Elkind, E. 2017. Proportional Rankings. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence (IJCAI)*, 409–415. IJCAI.

This paper:

- Includes a conceptual outline and/or pseudocode description of AI methods introduced (yes)
- Clearly delineates statements that are opinions, hypothesis, and speculation from objective facts and results (yes)
- Provides well marked pedagogical references for less-familiare readers to gain background necessary to replicate the paper (yes)

Does this paper make theoretical contributions? (yes)

If yes, please complete the list below.

- All assumptions and restrictions are stated clearly and formally. (yes)
- All novel claims are stated formally (e.g., in theorem statements). (yes)
- Proofs of all novel claims are included. (yes)
- Proof sketches or intuitions are given for complex and/or novel results. (yes)
- Appropriate citations to theoretical tools used are given. (yes)
- All theoretical claims are demonstrated empirically to hold. (NA)
- All experimental code used to eliminate or disprove claims is included. (NA)

Does this paper rely on one or more datasets? (no)

Does this paper include computational experiments? (no)

ALGORITHM 1: Greedy Justified Candidate Rule (GJCR) (Brill and Peters 2023)

$W \leftarrow \emptyset$;
for ℓ *in* $k, \ldots, 1$ **do**
  while *there is* $c \notin W$*:* $|\{i \in N_c \colon |A_i \cap W| < \ell\}| \geq \frac{\ell n}{k}$ **do**
    Add candidate c maximizing $|\{i \in N_c \colon |A_i \cap W| < \ell\}|$ to W ;
  end
end
return W;

A Appendix: Multiwinner Voting Rules

In this section, we provide brief definitions of the mentioned rules.

Rules for approval ballots. *Proportional approval voting* (PAV) is the rule that assigns a score of $H(|A_i \cap W|)$ to each ballot A_i and committee W, where $H(j)$ is the j-th harmonic number. It then chooses the committees that maximize the sum of scores over the entire profile A.

The *method of equal shares* (MES) assigns each voter a budget of $\frac{k}{n}$. Each candidate can be bought into the committee for a cost of 1. MES proceeds iteratively and selects the next candidate by maximizing the minimal budget over the buyers after the purchase.

The *greedy justified candidate rule* (GJCR) also proceeds iteratively, by choosing the candidate currently causing the largest violation of EJR+. See Algorithm 1.

Rules for ranked ballots. *Single transferable vote* (STV) selects the candidates iteratively. It adds candidates that are top ranked at least $\frac{n}{k}$ times and deletes this share of supporters from the profile. In case that no candidate can be chosen, instead a candidate with the lowest amount of supporters is eliminated from the profile.

The *Expanding approvals rule* (EAR) also assembles the committee step by step. It assigns each voter a budget of 1 and proceeds rank by rank. If there is a candidate, for whom the voters giving it at most that rank have a budget of at least $\frac{n}{k}$, that candidate is added to the committee and the budget of these voters is decreased by $\frac{n}{k}$. Otherwise, the rank gets increased by 1.

B Missing Proofs for Section 3

Example 2. *The single transferable vote (STV) and the expanding approvals rule (EAR) can select committees that are not robust to a single deletion for PSC, even when such committees exist.*

Proof. For an example of STV failing this, consider the following instance

$$a \succ b \succ \cdots$$
$$a \succ c \succ \cdots$$
$$b \succ a \succ x$$
$$x \succ y \succ \cdots$$
$$c \succ a \succ x$$
$$y \succ x \succ \cdots$$

Here, with $k = 3$ STV could first select a and delete the first two voters who top-rank a from the profile. Then delete b, select x, and delete voters three and four who at this time rank x on top. Finally, delete e and choose y. Now if a withdraws, both b and c need to be added. We note that the same committee could also be selected by EAR. This instance, however, admits a robust committee, for instance $\{a, b, x\}$ is robust to a single deletion. $\square$

Proposition 2. *There does not exist a robust decremental PSC algorithm. After a single deletion, it can be that* $\Omega(\log(\log(k)))$ *replacements are needed to restore PSC.*

Proof. Let $s < t \in \mathbb{N}$ with $r = \binom{s}{t}$ Consider the following instance over the candidate set $C = \{a_1, \ldots a_r\} \cup \{b_1, \ldots, b_s\}$:

$$a_1 \succ b_1$$
$$\cdots$$
$$a_1 \succ b_t$$
$$a_2 \succ b_2$$
$$\cdots$$
$$a_2 \succ b_{t+1}$$
$$\cdots$$
$$a_r \succ b_{s-t+1} \succ \cdots$$
$$\cdots$$
$$a_r \succ b_s \succ \cdots$$
$$(t-1) \times b_1$$
$$\cdots$$
$$(t-1) \times b_s$$

To be more precise, for each subset of t candidates from B, i.e., each $B' \in \binom{B}{t}$, we add one voter ranking $a_{i(B')}$ first and $b \in B'$ second. This yields tr voters. Then, for each $i \leq s$ we add $t-1$ voters ranking b_i first. This adds further $(t-1)s$ voters. Whenever s is a multiple of t, we can set $k = r + \frac{t-1}{t}s$. Thus, a set of voters is 1-large if it has cardinality $\geq t$. PSC now implies that $a_1, \ldots, a_r$ must all be chosen. Further, if in the next time step a_i is deleted, then PSC enforces that all candidates b in the subset $B'(i) \subset B$ are chosen. Since any a_i could be deleted, this means that for the committee to guarantee PSC after one deletion, $s-1$ candidates from B must already be chosen in this round. Clearly, $s - 1 + r > \frac{t-1}{t}s + r = k$. Choose $s = t^2$. Then, $s - 1 + r - k = \frac{1}{t}s - 1 = t - 1$. So at least $t-1$ candidates b_i remain unchosen. If the corresponding a_j gets deleted, at least $t-2$ additions are necessary to restore PSC. For the asymptotic bound, observe $k \in \Theta(r) = \Theta(s^t)$. Thus, there is a constant $c \in \mathbb{R}_{>0}$ with $k \leq ct^{2t}$, which implies $\sqrt{\frac{k}{c}} \leq t^t$ and thus $\log(\sqrt{\frac{k}{c}}) \leq t\log(t)$, and finally

$W(\log(\sqrt{\frac{k}{c}})) \leq t$ for the product log function W. Thus, the number of additions necessary to restore PSC have a lower bound of $\Omega(\log(\log(k)))$. □

Theorem 4. *There is no incremental or decremental algorithm satisfying rank-JR and making $o(\sqrt{k})$ changes amortized per round.*

Proof. We now create a profile in which each committee satisfying rank-JR must add t candidates $b^1, \dots b^t$ after the deletion of some candidate a to maintain rank-JR. Consider the following instance consisting of t blocks $(B_i)_{i \leq t}$, each B_i consisting of $t(t+1)$ voters with the following $3t$ voter types $(v_i^j)_{j \leq 3t}$:

$$\begin{aligned}
1 \times v_i^1 :&\quad a_i \succ b_i^1 \succ \dots \\
&\quad \dots \\
1 \times v_i^t :&\quad a_i \succ b_i^t \succ \dots \\
(t-1) \times v_i^{t+1} :&\quad b_i^1 \succ c_i^1 \succ \dots \\
&\quad \dots \\
(t-1) \times v_i^{2t} :&\quad b_i^t \succ c_i^t \succ \dots \\
1 \times v_i^{2t+1} :&\quad c_i^1 \succ d_i^1 \succ \overline{c} \succ \dots \\
&\quad \dots \\
1 \times v_i^{3t} :&\quad c_i^t \succ d_i^t \succ \overline{c} \succ \dots
\end{aligned}$$

We set $k = t(t+1)$ and thus $\frac{n}{k} = t$. For now, consider a fixed block B_i, $i \in [t]$. First, we notice that a_i needs to be included to satisfy rank-JR, as there are t voters top ranking it. To not need t additions after the deletion of a_i, we include some $b_i^{x(i)}$ with $x(i) \leq t$ for each i. Further, we claim that $\overline{c}$ cannot be chosen: for each $i \in [t], j \in [t] \setminus \{x(i)\}$ to satisfy the $(t-1)$ voters v_i^{t+j} and the voter v_i^{2t+j} we need to include one of b_i^j, c_i^j, d_i^j to satisfy rank-JR. All additions so far considered for B_i, this enforces $t+1$ candidates. Iterating over all blocks, we overall enforce $t(t+1)$ candidates, filling the committee and leaving $\overline{c}$ unchosen. But then the set of t voters $(v_i^{2t+x(i)})_{i \leq t}$ witness a rank-JR violation, as they all rank candidate $\overline{c}$ on rank 3.

For the incremental case consider the same profile before the addition of $(a_i)_{i \leq t}$

$$\begin{aligned}
1 \times v_i^1 :&\quad b_i^1 \succ \dots \\
&\quad \dots \\
1 \times v_i^t :&\quad b_i^t \succ \dots \\
(t-1) \times v_i^{t+1} :&\quad b_i^1 \succ c_i^1 \succ \dots \\
&\quad \dots \\
(t-1) \times v_i^{2t} :&\quad b_i^t \succ c_i^t \succ \dots \\
1 \times v_i^{2t+1} :&\quad c_i^1 \succ d_i^1 \succ \overline{c} \succ \dots \\
&\quad \dots \\
1 \times v_i^{3t} :&\quad c_i^t \succ d_i^t \succ \overline{c} \succ \dots
\end{aligned}$$

Clearly, we must choose all b_j^i for $i, j \leq t$, i.e., t^2 candidates. However, after adding $a_1, \dots, a_t$, we can similarly to the decremental case prove that at most $< t$ candidates of the form b_j^i can be contained in a committee satisfying rank-JR. This constitutes $> t^2 - t$ replacements in t rounds, hence in some round there must have been $t - 1 = \Omega(\sqrt{k})$ replacements. □

C Missing Proof for Section 4

Theorem 6. *There exists a robust fully dynamic algorithm achieving a $\frac{3+\sqrt{21}}{2} \sim 3.79$-proportional fair outcome and satisfying the 5-q-core for any $q \in [k]$.*

Proof. We begin with a pre-clustering phase following the generalization of the expanding approvals rule (Aziz et al. 2023; Kalayci, Kempe, and Kher 2024). We assign each voter $i \in N$ a budget $b_i = \frac{k}{n}$. Then we continuously increase a radius δ from 0 on. If there is a set of voters $N' \subseteq N$ of diameter at most δ who have a total budget of at least 1, we create a cluster N_i for them and decrease their weights by a total of 1. Let $p_j(i)$ be the amount we decreased the budget of agent $j \in N_i$ by in this round.

This leads to clusters $N_1, \dots, N_k$. For a given cluster of voters N_i and candidate c we let $d(N_i, c) = \min\{d(j, c) \colon p_j(i) > 0\}$. We now proceed as follows: We consider the clusters $N_1, \dots, N_k$ in order. For each cluster N_i we pick among the so far unchosen candidates some c_i minimizing $d(N_i, c_i)$. We say that a cluster N_j *envies* another cluster N_ℓ if $d(N_j, c_\ell) < d(N_j, c_j)$.[2] By our enumeration, it is clear that for $i, j \leq n$, N_j can only envy N_i if $i < j$. Further, the selected committee contains the distance minimizers for all n clusters N_i. We will in each step ensure that this invariant remains true.

When we delete a candidate c we either do (i) nothing if c is not picked by any cluster or (ii) if c is picked by cluster N_i we let N_i repick the closest current unchosen candidate d. Since every cluster N_j with $j \neq i$ prefers their current candidate to d, we re-enumerate them with N_i is renamed to N_k and the former $N_{i+1}, \dots, N_k$ renamed to $N_i, \dots, N_{k-1}$. Still, for $i, j \leq n$, N_j can only envy N_i if $i < j$.

If a candidate c gets added, we first check if there is any cluster N_i for which $d(N_i, c) < d(N_i, c_i)$, i.e., c is closer than the picked candidate for this cluster. If there is not, we do not add c. If there is, however, such a cluster N_i, we assign c to such a cluster N_{i^*} with the smallest index i^* and store c_{i^*} as the *interim candidate*. As long as the set of N_j with $j > i^*$ are assigned their current choices or c_{i^*}, N_{i^*} only envies clusters with $j' < i$. Now, in ascending order, every N_j with $j > i$ gets to choose between keeping their current candidate c_j or exchanging it for the current interim candidate, thus making c_j the new interim candidate. If N_j keeps c_j, then it will not envy any N_r with $r > j$ after the process since they will be assigned to some $c_{>j}$ or the interim candidate that N_j prefers less to c_j. If N_j swaps c_j for

[2] We now perform a procedure similar to the envy-cycle elimination from fair division, see, e.g., the survey of Amanatidis et al. (2023).

the interim candidate, then it still will not envy any $N_{>j}$ after the process because each will obtain some $c_{\geq j}$. After N_k made their choice, we discard the current interim candidate and obtain a new committee containing c with just one swap on the currently selected set, but up to k re-assignments of clusters to selected candidates. Still, this time without re-enumeration, for $i, j \leq n$, N_j can only envy N_i if $i < j$.

Now we are ready to show that this procedure is always ρ-proportional fair. Let W be any committee throughout the online process, c be any unselected candidate, and N' be a group of voters deviating to c of size at least $\frac{n}{k}$. Let δ be the diameter of N'. Then, in the first step of the procedure, we know that at least one of the agents pays for a preprocessed cluster N'' of diameter at most δ. Let i be this agent and let j be the agent furthest away from c in N'. Since N'' did not pick c, c is not a (unique) minimizer of $d(N'', \cdot)$. We know that there must be an agent $h \in N''$ with $d(h, W) \leq d(N'', c) \leq d(i, c) \leq d(j, c)$. Further, by the triangle inequality we can bound the distance between any two elements of N' by their respective distances to c, so it must hold that $\delta \leq 2d(j, c)$. Thus, we get that

$$\begin{aligned}
&\min\left(\frac{d(i,W)}{d(i,c)}, \frac{d(j,W)}{d(j,c)}\right) \\
&\leq \min\left(\frac{d(i,h)+d(h,W)}{d(i,c)}, \frac{d(j,i)+d(i,W)}{d(j,c)}\right) \\
&\leq \min\left(\frac{3d(j,c)}{d(i,c)}, \frac{3d(j,c)+d(i,c)}{d(j,c)}\right) \\
&\leq \min_{x\geq 0}\left(3x, 3+\frac{1}{x}\right) = \frac{3+\sqrt{21}}{2}
\end{aligned}$$

and therefore the improvement through c is bounded by $\frac{3+\sqrt{21}}{2}$.

For the α-q-core, let $N' \subseteq N$ be an ℓ-large deviating coalition and $C' \subseteq C$ of size $|C'| = \ell$ be the set of candidates the coalition deviates to. Following Kellerhals and Peters (2023, Lemma 9) there is a candidate $c \in C'$ and a subset $N'' \subseteq N$ such that c is in the top-q choices among C' of everyone in N'' with N'' being of size at least $q\frac{n}{k}$. Let $i \in N''$ be the agent among N'' with the largest $d^q(i, C')$. Since the agents in N'' have a total budget of q there must at least exist q clusters bought partially by agents from N''. Similar to the first part of this proof, the diameter of at least q of these clusters is smaller than the diameter of N''. We know that these clusters must have chosen q of the cluster centers in W. Since the diameter of N'' is at most $d(i_1, i_2) \leq d(i_1, c) + d(c, i_2) \leq d^q(i_1, C') + d^q(i_2, C') \leq 2d^q(i, C')$ the diameter of each of these clusters is also at most $2d^q(i, C')$. Since each agent in N'' is also at most $d^q(i, C')$ away from an unselected cluster center, the cluster center selected must also be at most $d^q(i, C)$ away from someone in the group and thus at most $3d^q(i, C')$ away from the member of N''. However, since the diameter of N'' is at most $2d^q(i, C')$ this implies that agent i is at most a distance of $5d^q(i, C')$ away from q cluster centers in W, therefore showing that W is in the 5-q-core. □

D Missing Proofs for Section 5

Example 4. *GJCR, MES, and PAV can elect committees that are not robust with respect to a single addition.*

Proof. Consider the following approval profile with approval sets $1 \times \{a_1, \ldots, a_4\}, 3 \times \{a_1, \ldots, a_4, x\}, 2 \times \{b_1, \ldots, b_5, x\}, 3 \times \{b_1, \ldots, b_5\}, 1 \times \{c_1, \ldots, c_4\}, 3 \times \{c_1, \ldots, c_4, y\}, 2 \times \{d_1, \ldots, d_5, y\}, 3 \times \{d_1, \ldots, d_5\}$ with $n = 18 = k$. Here a possible MES committee is all candidates except for x and y. However, now adding a candidate approved by the three voters voting for only b candidates and the three voters only voting for d candidates, would require one other candidate to be removed. If an a or c candidate is removed that candidate witnesses an EJR+ violation. If a b or d candidate gets removed, the corresponding voters approving x or y witness an EJR+ violation, as consist of 5 voters, but only approve 4 candidates in the outcome.

To extend this to PAV consider the same instance, consisting of 6 copies of $3 \times \{a_1, \ldots, a_4\}, 1 \times \{a_1, \ldots, a_4, x\}, 4 \times \{b_1, \ldots, b_5, x\}, 1 \times \{b_1, \ldots, b_5\}$ with $n = 54 = k$ Here, a optimal PAV committee would choose the copies of the a and b candidates, as adding the x candidate for an a candidate would decrease the PAV score by $-\frac{3}{4} + \frac{4}{6} < 0$ while adding the x candidate for a b candidate would leave the PAV score unchanged. However, adding a candidate approved exactly by the $\{b_1, \ldots, b_5\}$ voters, would need this candidate to be included, leading to the same contradiction as in the first case. □

Corollary 7. *Every maximally affordable committee satisfies PJR+.*

Proof. Let $W = \{c_1, \ldots, c_\ell\}$ be maximally affordable w.r.t some $(p_i)_i$. Then clearly, $\ell \leq k$ as the voters only have budget k in total. Assume for contradiction there is a violation of $PJR+$ for W based on size-k largeness, i.e., some $c \in C \setminus W$ with some $N' \subseteq N$ such that $c \in \bigcap_{i\in N'} A_i$ and $|\bigcup_{i\in N'} A_i \cap W| < \ell$ despite $|N'| \geq \ell\frac{n}{k}$ for some $\ell \in \mathbb{N}$. The total amount of budget spent by N' is thus at most $\ell - 1$ with their starting budget being at least $\ell\frac{n}{k}\frac{k}{n} = \ell$. This is the desired contradiction as **C5<** is violated. □

Theorem 8 (Do et al. 2022)**.** *There exists a robust incremental PJR+ algorithm.*

Proof. The first k candidates that arrive we take into our committee $W^0 = C^0$. We create a partition of $W^0 = X^0 \cup Y^0$ into a maximally affordable committee X^0 and a disposable part Y^0. For this, initialize both sets as empty and assign a total budget of k equally among all voters, i.e., $\frac{k}{n}$ to every voter. They will proceed to buy candidates, each for the price of 1, into X^0 as follows: As long as there is a candidate c such that N_c has a total budget of ≥ 1 to afford it, add c to X^0, and subtract the budget of 1 off any of these voters in any way such that the budgets are not exceeded. Store this subtracted amount as $p_i(c)$. By finiteness of the budget, this process must terminate after a finite number of steps. Clearly, **C1** to **C4** are satisfied for X^0. Further, **C5<** is satisfied as else the candidate violating the inequality would be bought into the committee and the process cannot have already terminated. Clearly, $|X^0| \leq k$ as the voters only have budget k in total. Now, set $Y^0 = C^0 \setminus X^0$. This concludes the induction start.

For the induction step, let two disjoint sets $X^t, Y^t \subseteq C^t$ given such that their union is of size k and X^t is maximally affordable in C^t with cost function p^t (and thus satisfies $PJR+$ with respect to size k largeness). Let now a new candidate c^* be added, i.e., $C^{t+1} = C^t \cup \{c^*\}$. If X^t is still maximally affordable, we can set $X^{t+1} = X^t$, $Y^{t+1} = Y^t$, and $p^{t+1} = p^t$. Else, there must be $c \in C^{t+1}$ such that one of the conditions is violated. Since $C1$ to $C4$ did not change, it must be **C5<** and $c = c^*$. Thus, N_{c^*} can afford to buy c^* into the committee. Set $X^{t+1} = X^t \cup \{c^*\}$. Clearly, since there was budget left to buy c^*, we have $|X^t| < k$ and thus $|Y^t| > 0$. Thus, remove an arbitrary element $y \in Y^t$, i.e., set $Y^{t+1} = Y^t \setminus \{y\}$, Set $p_i(c^*)$ as the amount of budget that was taken from voter i to finance c^*. It is easy to check that W^{t+1} satisfies all 5 axioms and thus is maximally affordable. This concludes the induction step. ☐

Theorem 10. *There exists a fully dynamic $\Theta(\log(k))$-EJR+ algorithm making amortized one change per iteration.*

Proof. Let $H(n)$ denote the n-th harmonic number. Consider for the start a modified GJCR that considers groups of size $\geq H(k)\ell\frac{n}{k}$ instead of $\geq \ell\frac{n}{k}$. Again, this rule can be modeled via a budget of $\frac{k}{n}$ for each voter, which the voters then all uniformly spend when they can buy a candidate, which all have a unit cost. To show that on C_0 GJCR computes at most k candidates, observe that each voter can buy at most 1 candidate for violations with $\ell = 1$, 2 for violations with $\ell \leq 2$ and so on. For a violation w.r.t. ℓ, the price the voter pays it at most $\frac{k}{n\ell H(k)}$. In total, this yields $\sum_{\ell \leq k} \frac{k}{n\ell H(k)} = \frac{k}{n}$. So, no voter overspends, and since the total budget was k, we have a committee of size $\leq k$. We fill up the remaining places with placeholders. Now, if a candidate is added and creates a violation of $H(k)$-EJR+ w.r.t. some $\ell \leq k$, then each voter that is part of this violation can have spent at most $\sum_{j \leq \ell} \frac{k}{njH(k)}$. To buy this candidate into the committee, they spend at most $\leq \frac{k}{n\ell H(k)}$ and thus no one overdraws their budget. Especially, since the budget was not fully used before, there are placeholders in the committee. Replace one of them with the newly added candidate, then $H(k)$-EJR+ is restored. Conversely, if a candidate c is deleted and this creates violations of $H(k)$-EJR+, we can reimburse the voters who previously bought the candidate into the committee. With the same argument as in the instantiation and the addition of a candidate, we obtain that each voter has sufficient budget to buy the candidates causing the violations into the committee. There can be multiple of these changes to the committee after a single deletion. However, note that for $\ell \leq k$ changes to be made, there must have been ℓ total budget that was not used or freed beforehand. ☐

Theorem 11. *For any $\alpha > 1$ there exists an incremental α-EJR+ algorithm making amortized $\frac{\alpha}{\alpha-1}$ changes.*

Proof. Let $\alpha > 1$ be given. For small k with $k \leq \frac{\alpha}{\alpha-1}$, we can replace the entire committee in each step and trivially obtain the result. Else, we have $k > \frac{\alpha}{\alpha-1}$, implying $k(1-\alpha) < -\alpha$ and thus $\frac{k}{\alpha} < k-1$, which finally leads to the desired $\lceil\frac{k}{\alpha}\rceil < k$. Begin by running any EJR+ rule for the committee size $\lceil\frac{k}{\alpha}\rceil$ and select $k - \lceil\frac{k}{\alpha}\rceil$ other candidates arbitrarily. For the next $k - \lceil\frac{k}{\alpha}\rceil$ steps, while there is a candidate arriving who witnesses an α-EJR+ violation, include them in exchange for one of the arbitrarily added candidates. Afterwards, recompute a committee of size $\lceil\frac{k}{\alpha}\rceil$ satisfying EJR+ and add them into the committee by excluding arbitrary candidates. After $k - \lceil\frac{k}{\alpha}\rceil + 1$ steps, we thus have made at most k replacements. This leads to amortized $\frac{k}{k-\lceil\frac{k}{\alpha}\rceil+1} = \frac{k}{k-(\lceil\frac{k}{\alpha}\rceil-1)} < \frac{k}{k-\frac{k}{\alpha}} = \frac{1}{1-(\frac{1}{\alpha})} = \frac{\alpha}{\alpha-1}$, replacements per round which concludes the proof. ☐

ALGORITHM 2: Locally Stable GJCR

1 $W \leftarrow \emptyset$;
2 $N_{active} \leftarrow \emptyset$;
3 **for** ℓ *in* $k, \ldots, 1$ **do**
4 $\quad W_\ell = \emptyset$;
5 $\quad$ **do**
6 $\quad\quad$ **while** *there is* $c \notin W$: $|\{i \in N_c \colon |A_i \cap W| < \ell\}| \geq \frac{\ell n}{k}$ **do**
7 $\quad\quad\quad$ Choose c maximizing $|\{i \in N_c \colon |A_i \cap W| < \ell\} \setminus N_{active}|$;
8 $\quad\quad\quad$ $W_\ell \leftarrow W_\ell \cup \{c\}$;
9 $\quad\quad\quad$ $W \leftarrow W \cup \{c\}$;
10 $\quad\quad\quad$ $N_{active} \leftarrow N_{active} \cup \{i \in N_c \colon |A_i \cap W| < \ell\}$.
11 $\quad\quad$ **end**
12 $\quad$ **while** *there was a change in the last iteration*;
13 $\quad$ **for** $c \in W_\ell$, $c' \notin W$ **do**
14 $\quad\quad$ **if** $|\{\{i \in N_{c'} \colon |A_i \cap W \setminus \{c\}| < \ell\}| \geq \frac{\ell n}{k}$
15 $\quad\quad$ *and* $|i \in N_{c'} \colon |A_i \cap W \setminus \{c\}| = 0\}| > |i \in N_c \colon |\{A_i \cap W \setminus \{c\}| = 0\}|$ **then**
16 $\quad\quad\quad$ $W_\ell \leftarrow W_\ell \cup \{c'\} \setminus \{c\}$;
17 $\quad\quad\quad$ $W \leftarrow W \cup \{c'\} \setminus \{c\}$;
18 $\quad\quad\quad$ $N_{active} \leftarrow \{i \in N \colon |A_i \cap W| > 0\}$;
19 $\quad\quad$ **end**
20 $\quad$ **end**
21 **end**
22 return W;

Theorem 12. *There exists an incremental EJR+ algorithm that is robust with respect to a single addition.*

Proof. To show this theorem, we use Line 22. In essence, Line 22 runs the GJCR with an additional local swapping step at the end. This local swapping step tries to maximize the number of voters covered in each iteration. As Line 22 produces one possible outcome of GJCR, it satisfies EJR+.

If Line 22 outputs less than k candidates, the theorem follows, as we can simply include the new candidate in the committee with a single swap, swapping out an irrelevant candidate. Thus, assume it outputs k candidates and let c be the newly added candidate witnessing an EJR+ violation. Let $N' \subseteq N_c$ be the set of voters witnessing the violation with $|N'| \geq \ell\frac{n}{k}$ and $|A_i \cap W| < \ell$ for all $i \in N'$. If $|A_i \cap W| = 0$ it is easy to see that the committee could not have been of size k. Therefore, every voter in N' approves at least one candidate. Let $i \in N'$ be any such voter and let $\ell_i = |A_i \cap W|$. Let c' be any arbitrary candidate in $A_i \cap W$ and consider the committee $W' \coloneqq W \setminus \{c'\} \cup \{c\}$. Further, assume that W' does not satisfy EJR+ with its violation being witnessed by candidate c'' for threshold ℓ'' and set $N'' \subseteq N_{c''}$. We distinguish two cases:

Case 1: $\ell'' > \ell_i$. Then, in iteration ℓ'' some voter in N'' must approve at least ℓ'' candidates, one of which must be c'. Therefore, c' got bought in an iteration before ℓ_i a contradiction.restatable

Case 2: $\ell'' < \ell_i$. Then, someone in N'' must approve of c'. Since $\ell'' < \ell_i$ this voter must approve less than ℓ_i candidates in iteration ℓ_i and must therefore have contributed to buying ℓ_i. Thus, this voter approves at least $\ell_i - 1$ candidates in W' contradicting $\ell'' < \ell_i$.

Case 3: $\ell'' = \ell_i$. Let $N''_1 = \{j \in N'' \colon c' \in A_j\}$. If $N''_1 = N''$ this set must necessarily include i who still approves ℓ'' candidates in the outcome, a contradiction. Therefore, there is a $j \in N'' \setminus N''_1$. However, since for GJCR to select k candidates, every "buyer" of c' must approve exactly ℓ_i candidates. Otherwise, in the price-system constructed by GJCR, one of these "buyers" must pay less than $\frac{k}{n}$, leading to a contradiction that we selected k candidates (see Brill and Peters (2023, Proposition 8) for a full proof). Therefore, we could have swapped c'' with c' increasing the number of covered voters in iteration ℓ_i by at least 1, as there is no voter who go down to 0 approvals after the removal of c'. (The implicit assumption here is that ℓ_i is at least 2, which must be true, as otherwise j approves nothing, and we would not have selected k candidates.)

$\square$