

A Natural Adaptive Process for Collective Decision-Making

Florian Brandl

Department for Economics
University of Bonn
53113 Bonn, Germany

Felix Brandt*

Department of Informatics
Technical University of Munich
85748 München, Germany

Consider an urn filled with balls, each labeled with one of several possible collective decisions. Now, draw two balls from the urn, let a random voter pick her more preferred as the collective decision, relabel the losing ball with the collective decision, put both balls back into the urn, and repeat. In order to prevent the permanent disappearance of some types of balls, once in a while, a randomly drawn ball is relabeled with a random collective decision. We prove that the empirical distribution of collective decisions converges towards a maximal lottery, a celebrated probabilistic voting rule proposed by Peter C. Fishburn (*Rev. Econ. Stud.*, 51(4), 1984). In fact, the probability that the collective decision in round n is made according to a maximal lottery increases exponentially in n . The proposed procedure is more flexible than traditional voting rules and bears strong similarities to natural processes studied in biology, physics, and chemistry as well as algorithms proposed in machine learning.

1. Introduction

The question of how to collectively select one of many alternatives based on the preferences of multiple agents has occupied great minds from various disciplines. Its formal study goes back to the Age of Enlightenment, in particular during the French Revolution, and the important contributions by Jean-Charles de Borda and Marie Jean Antoine Nicolas de Caritat, better known as the Marquis de Condorcet. Borda and Condorcet agreed that plurality rule—then and now the most common collective choice procedure—has serious shortcomings. This observation remains a point of consensus among social choice theorists and is largely due to the fact that plurality rule merely asks each voter

*Corresponding author: brandtf@in.tum.de, Phone +49-89-28917512, Fax +49-89-28917535

for her most-preferred alternative (see, e.g., Brams and Fishburn, 2002; Laslier, 2011).¹ When eliciting more fine-grained preferences such as complete rankings over all alternatives from the voters, much more attractive choice procedures are available. As a matter of fact, since Arrow’s (1951) seminal work, the standard assumption in social choice theory is that preferences are given in the form of binary relations that satisfy completeness, transitivity, and often anti-symmetry. Despite a number of results which prove critical limitations of choice procedures for more than two alternatives (e.g., Arrow, 1951; Gibbard, 1973; Satterthwaite, 1975), there are many encouraging results (e.g. Young, 1974; Young and Levenglick, 1978; Brams and Fishburn, 1978; Laslier, 2000). In particular, when allowing for randomization between alternatives, some of the traditional limitations can be avoided and there are appealing choice procedures that stand out (Gibbard, 1977; Brandl et al., 2016; Brandl and Brandt, 2020).

The standard framework in social choice theory rests on a number of rigid assumptions that confine its applicability: there is a fixed set of voters, a fixed set of alternatives, and a single point in time when preferences are to be aggregated; all voters are able to rank-order all alternatives; there is a central authority that has access to all these rankings, computes the outcome, and convinces voters of the outcome’s correctness, etc. On top of that, computing the outcome of many attractive choice procedures is a demanding task that requires a computer, which can render the process less transparent to voters.²

In this paper, we propose a simple urn-based procedure that implements a celebrated choice procedure called *maximal lotteries* (Fishburn, 1984). Our goal is to devise an ongoing process in which voters may arrive, leave, and change their preferences over time. Moreover, voters are never asked for their complete preference relations, but rather reveal minimal information about their preferences by choosing between two randomly drawn alternatives from time to time. No central voting authority is required. The process can be executed via a simple physical device: an urn filled with balls that allows for two primitive operations: (i) randomly sampling a ball and (ii) replacing a sampled ball of one kind with a ball of another kind.

More precisely, the process works as follows (see Figure 1). Consider an urn filled with balls that each carry the label of one alternative. The initial distribution of balls in the urn is irrelevant. In each round, a randomly selected voter will draw two balls from the urn at random. Say these two balls are labeled with alternatives 1 and 2, and the voter prefers 1 to 2. She will then change the label of the second ball to 1 and return both balls to the urn. Alternative 1 is declared the winner of this round. Once in a while, with some small probability r , which we call *mutation rate*, a randomly drawn ball is relabeled with a random alternative.

We show that if the number of balls in the urn is sufficiently large, then the empirical distribution of winners converges to a lottery close to a maximal lottery almost surely. How far the limiting distribution will be from a maximal lottery depends on r . As r

¹For example, plurality rule may select an alternative that an overwhelming majority of voters consider to be the worst of all alternatives.

²In some cases, computing the outcome was even shown to be NP-hard, i.e., the running time of all known algorithms for computing election winners increases exponentially in the number of alternatives (see, e.g., Bartholdi, III et al., 1989; Brandt et al., 2016b).

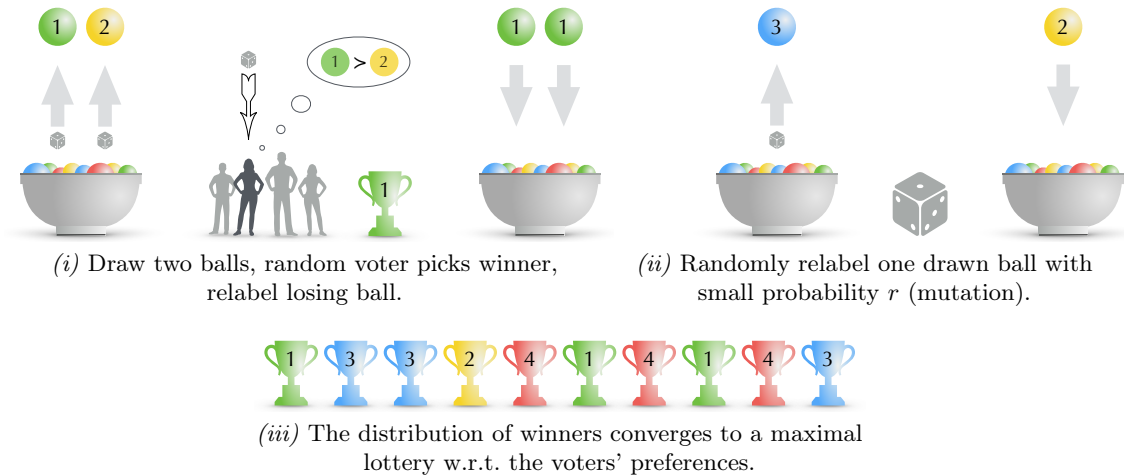


Figure 1: Illustration of one round of the urn process (*i* and *ii*) and the main result (*iii*).

goes to 0, the limiting distribution converges to a maximal lottery. We can, however, not set r to 0 as then almost surely, all alternatives except one will permanently disappear from the urn and the limiting distribution will be degenerate. Our proof not only shows convergence of the limiting distribution but also that the probability that the urn distribution is close to a maximal lottery gets arbitrarily close to 1 and increases exponentially in the number of rounds. The winners of most rounds are thus selected according to approximate maximal lotteries.

1.1. Maximal Lotteries, Dynamic Voting, and Approximate Axiomatics

Maximal lotteries are known to satisfy a number of desirable properties that are typically considered in social choice theory (see, e.g., Felsenthal and Machover, 1992; Laslier, 2000; Rivest and Shen, 2010; Hoang, 2017; Brandl et al., 2022). For example, Condorcet winners (i.e., alternatives that defeat every other alternative in a pairwise majority comparison) will be selected with probability 1, and Condorcet losers (i.e., alternatives that are defeated in all pairwise majority comparisons) will never be selected. No group of voters benefits by abstaining from an election, removing losing alternatives does not affect maximal lotteries, and each alternative's probability is unaffected by cloning other alternatives. Maximal lotteries have been axiomatically characterized using Arrow's independence of irrelevant alternatives and Pareto efficiency (Brandl and Brandt, 2020) as well as population-consistency and composition-consistency (Brandl et al., 2016). The dynamic procedure described above implements maximal lotteries while providing

- *myopic strategyproofness* within each round,
- minimal *preference elicitation* and thus increased privacy protection,
- *verifiability* realized via a simple physical procedure, and
- all-round *flexibility*.

Myopic strategyproofness: Each round’s decision is made by letting a randomly selected voter choose between two alternatives. Clearly, a voter who is only concerned with the outcome of the current round is best off by choosing the alternative that she truly prefers. If she also takes into account the outcomes of future rounds, however, she may be able to skew the distribution in the urn by choosing alternatives strategically.³

Preference elicitation: Eliciting pairwise preferences on an as-needed basis has several advantages. First, it spares the voters from the cognitive burden of having to rank-order all alternatives at once. If the number of voters is large, it may well be possible that the urn process yields satisfying results without ever querying some of the voters. Secondly, rather than submitting a complete ranking of all alternatives to a trusted authority, voters only reveal their preferences by making pairwise choices from time to time.⁴

Verifiability: Previously, the deployment of maximal lotteries required that a central authority collects the preferences of all voters, computes a maximal lottery via linear programming, and instantiates the lottery in some user-verifiable way. The urn process allows achieving these goals using a simple physical device.

Flexibility: The urn process is oblivious to changes in the voters’ preferences, the set of voters as well as the set of alternatives. Everything that has happened up to the current round is irrelevant. Since the process converges from any initial configuration, it will keep “walking in the right direction” (towards a maximal lottery of the current preference profile).

We also note some disadvantages of the urn process. The convergence of the distribution of winners to an approximate maximal lottery is an asymptotic result. In particular, for a finite number of rounds, there is a non-zero probability that the chosen alternative is subpar for a significant fraction of rounds, for example, because it is Pareto dominated. To bound this probability below an acceptable threshold, it may be necessary to run the process for an excessively large number of rounds. Second, ensuring that the limit distribution is sufficiently close to a maximal lottery could require an urn with a large number of balls. We address the first concern by showing that the probability for the distribution of winners to be far from the limit distribution converges to 0 exponentially fast in the number of rounds (Proposition 1 gives a precise statement). The speed of convergence is also evident in computational simulations we ran for various parameterizations of the process. When the preference profile admits a Condorcet winner, we can give tractable bounds on the number of balls in the urn required to achieve a good approximation in the limit. This partially mitigates the second concern since it has been

³Maximal lotteries, like any *ex post* Pareto efficient randomized choice procedure, fail to be strategyproof (Gibbard, 1977). Our simple notion of myopic strategyproofness could be strengthened by discounting future rounds.

⁴Privacy can be further increased by letting voters draw their balls privately, announce the winner, and put two balls of the same color back into the urn, without revealing the original color of the losing ball. Alternatively, the voters’ preferences can be protected completely by letting the voter publicly draw both balls, make one copy of each ball, and let her privately put back two balls of her choice. The drawing of winners can be arranged separately and at a slower pace, if desired.

observed that most real-world preference profiles admit Condorcet winners (see, e.g., Gehrlein and Lepelley, 2011).

The axiomatic characterizations of maximal lotteries not only imply that maximal lotteries satisfy several desirable axioms, but also that any deviation from maximal lotteries leads to a violation of at least one of the axioms. Hence, a process that only guarantees an approximation of a maximal lottery will not enjoy the same axiomatic properties. However, rather than insisting on stringent axioms, one can relax them by only requiring them to hold in an approximate sense. For example, a natural notion of approximate Condorcet-consistency would require that a Condorcet winner receives probability close to 1 whenever one exists. Since the empirical distribution of winners according to our process is almost surely close to a maximal lottery and maximal lotteries are Condorcet-consistent, the process is approximately Condorcet-consistent in the above sense. More generally, many of the axioms that maximal lotteries satisfy such as population-consistency, composition-consistency, agenda-consistency, and efficiency also hold for approximate maximal lotteries. This follows from the fact that the correspondence returning the set of maximal lotteries depends continuously on the underlying preference profile and we show this exemplarily for population-consistency.

1.2. Applications, Analogs in Nature, and Opinion Formation

Maximal lotteries have been repeatedly recommended for practical use (Felsenthal and Machover, 1992; Rivest and Shen, 2010; Brandl et al., 2016; Hoang, 2017). We believe that the benefits of the urn process described above extend the applicability of maximal lotteries. Rather than for traditional political elections, probabilistic rules like maximal lotteries seem more suitable for frequently repeated low-stakes elections where some degree of randomization may not only be tolerable but even desirable. Two example applications that have been suggested for maximal lotteries are to help a group of coworkers with the daily decision where to have lunch and to select music for a party or a radio station based on the preferences of the listeners (Brandl et al., 2016). The transparency and the flexibility of the urn process seem particularly effective in the music broadcasting example. Agents come and go, they only need to select from a pair of songs rather than rank-order all of them, and individual preferences, as well as the set of available songs, can be changed at any time. Our theorem shows that the sequence of simple pairwise choices results in a socially desirable distribution of songs. Loosely speaking, the more songs are being played, the less likely it becomes that another distribution of songs would have been preferred by an expected majority of listeners.

Remarkably, dynamic processes similar to the process we describe here have recently been studied in population biology, quantum physics, chemical kinetics, and plasma physics to model phenomena such as the coexistence of species, the condensation of bosons, the reactions of molecules, and the scattering of plasmons. In each of these cases, simple interactions between randomly sampled entities can be connected to equilibrium strategies in symmetric zero-sum games. In fact, maximal lotteries are precisely the mixed Nash equilibrium (or maximin) strategies of the symmetric two-player zero-sum game given by the pairwise majority margins of the voters' preferences. We discuss these

relationships, including those to evolutionary game theory, in detail in Section 6.

An alternative interpretation of our result can be used to describe the formation of opinions. In this model, there is a population of agents, each of which entertains one of many possible opinions. Agents come together in random pairwise interactions, in which they try to convince each other of their opinion. The probabilities with which one opinion beats another are given as a square matrix and, with some small probability, an agent randomly changes her opinion. In other words, the agents correspond to the balls in the urn, the opinions correspond to the alternatives, and there are neither voters nor preference profiles as transition probabilities are given explicitly. Our theorem then shows that, if the population is large enough, the distribution of opinions within the population is close to a maximal lottery of the probability matrix most of the time. Other models of opinion formation based on different processes were for example considered by DeGroot (1974), Holley and Ligget (1975), and Goel and Lee (2014).

The remainder of the paper is structured as follows. After defining our model in Section 2, we state the main result (Theorem 1) and a rough proof sketch in Section 3. The full proof is given in the Appendix. In Section 4, we analyze the instructive special case of preference profiles that admit a Condorcet winner, which allows for a more elementary proof. Section 5 is concerned with axiomatic properties of maximal lotteries and discusses in which sense these properties can be retained for approximations of maximal lotteries. In Section 6, we extensively discuss related work from various fields of study and state a continuous version of our main result (Theorem 2) that may be of independent interest.

2. The Model

Let $[d] = \{1, \dots, d\}$ be a set of alternatives and Δ the $d - 1$ -dimensional unit simplex in \mathbb{R}^d , that is, $\Delta = \{x \in \mathbb{R}_{\geq 0}^d : \sum_{i=1}^d x_i = 1\}$. We refer to elements of Δ as lotteries. Throughout the paper, for a vector $x \in \mathbb{R}^k$ for some k , $|x| = \sum_{i=1}^k |x_i|$ denotes its L^1 -norm. For a finite set S , we write $\#S$ for the number of elements of S . For $S \subset \mathbb{R}^d$, χ_S is the indicator function of S and $\bar{\chi}_S = 1 - \chi_S$ is the indicator function of the complement of S .

A *preference relation* \succ is an asymmetric binary relation over $[d]$.⁵ By \mathcal{R} we denote the set of all preference relations. Let V be a finite set of voters. A *preference profile* $R \in \mathcal{R}^V$ specifies a preference relation for each voter. With each preference profile R , we can associate a comparison matrix $M_R \in [0, 1]^{d \times d}$ that states for each ordered pair of alternatives the fraction of voters who prefer the first to the second. That is, $M_R(i, j) = \#\{v \in V : i \succ_v j\} / \#V$. This matrix induces a skew-symmetric matrix $\bar{M}_R = M_R - M_R^\top$, which we call the skew-comparison matrix.⁶

⁵Preferences need not be transitive or complete. The definition of maximal lotteries and the urn process we describe only depend on the fractions of voters who prefer one alternative to another.

⁶A matrix M is skew-symmetric if $M = -M^\top$.

2.1. Maximal Lotteries

A lottery $p \in \Delta$ is a *maximal lottery* for a profile R if $\tilde{M}_R p \leq 0$. By $ML(R)$ we denote the set of all lotteries that are maximal for R . Most profiles admit a unique maximal lottery. For example, when the number of voters is odd and voters have strict preferences, there is always a unique maximal lottery (Laffond et al., 1997)

Example 1. Consider, for example, 900 voters, three alternatives, and a preference profile R given by the following table. Each column header contains the number of voters with the corresponding preference ranking.

	300	300	300
1	1	2	
2	3	3	
3	2	1	

Then,

$$M_R = \begin{pmatrix} 0 & 2/3 & 2/3 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{M}_R = \begin{pmatrix} 0 & 1/3 & 1/3 \\ -1/3 & 0 & 1/3 \\ -1/3 & -1/3 & 0 \end{pmatrix}.$$

The set of maximal lotteries $ML(R) = \{(1, 0, 0)^\top\}$ only contains the degenerate lottery with probability 1 on the first alternative. This alternative is a *Condorcet winner*, i.e., an alternative that is preferred to every other alternative by some majority of voters.

2.2. Markov Chains

Let S be a finite set and $\{X(n) : n \in \mathbb{N}\}$ be a discrete-time, time-homogeneous *Markov chain* with state space S . The *transition probability matrix* $P \in [0, 1]^{S \times S}$ of $\{X(n) : n \in \mathbb{N}\}$ is given by

$$P(p, p') = \mathbb{P}(X(n+1) = p' \mid X(n) = p)$$

for all $p, p' \in S$. We will frequently write $X(n, p_0)$ for $X(n)$ conditioned on $X(0) = p_0 \in S$ and call p_0 the *initial state*.

The period of a state $p \in S$ is the greatest common divisor of the return times with positive probability $\{n \in \mathbb{N} : (P^n)(p, p) > 0\}$. A Markov chain is *aperiodic* if every state has period 1. Note that any Markov chain with $P(p, p) > 0$ for all $p \in S$ is aperiodic. A Markov chain is *irreducible* if every state is reached from any other state with positive probability. That is, for any two states $p, p' \in S$, there is a positive integer n so that $(P^n)(p, p') > 0$. If $\{X(n) : n \in \mathbb{N}\}$ is irreducible and aperiodic, it has a unique *stationary distribution* $\pi \in \Delta S$ so that $\pi^\top = \pi^\top P$. The *sojourn time* $s_n(S')$ of $S' \subset S$ is the fraction of the first n rounds for which $X(k)$ is in S' . That is,

$$s_n(S') = \frac{1}{n} \sum_{k=1}^n \chi_{S'}(X(k)).$$

It follows from the ergodic theorem for Markov chains (see, e.g., Douc et al., 2018, Theorem 5.2.1) that if $\{X(n): n \in \mathbb{N}\}$ is aperiodic and irreducible, $s_n(S')$ converges to $\pi(S')$ almost surely.⁷

In the Appendix, we prove a preliminary result about the asymptotic rate of convergence of sojourn times. More precisely, we show that the probability that $s_n(S')$ differs by more than ε from $\pi(S')$ converges to 0 exponentially fast.⁸

Proposition 1. *Let $\{X(n): n \in \mathbb{N}\}$ be an irreducible Markov chain with finite state space S and stationary distribution $\pi \in \Delta(S)$. Then, for all $S' \subset S$ and $\varepsilon > 0$, there is $C > 0$ such that for all $n \in \mathbb{N}$,*

$$\mathbb{P}(|s_n(S') - \pi(S')| \leq \varepsilon) \geq 1 - e^{-[Cn]}.$$

2.3. The Urn Process

Consider an urn with $N \in \mathbb{N}$ balls, each labeled with some alternative. Viewing balls with the same label as indistinguishable, we can identify each state of the urn with an element of the discrete unit simplex $\Delta^{(N)} = \{p \in \Delta: Np \in \mathbb{N}_0^d\}$. Fix a *mutation rate* $r \in [0, 1]$.

We are interested in a Markov chain with state space $\Delta^{(N)}$ that can be informally described as follows. First, we flip a coin that has probability $1 - r$ of landing heads. If the coin shows heads, we choose one voter $v \in V$ uniformly at random and ask the voter to draw two balls from the urn. Say these two balls are labeled with alternatives 1 and 2. If $1 \succ_v 2$, the label of the second ball is changed to 1. Likewise, if $2 \succ_v 1$, the first ball is relabeled with label 2. If both balls carry the same label, the labels remain unchanged. If the coin shows tails, we draw a single ball from the urn, relabel it with an alternative chosen uniformly at random, and put it back into the urn.

This description of the process assumes that two balls are drawn *without* replacement. For the formal description, we will assume that drawing is *with* replacement since it avoids a lot of clumsy notation (that is, the second ball is drawn after returning the first ball to the urn). Intuitively, it is plausible that for a large number of balls, there is no significant difference between drawing with and without replacement. In the proof, we point out why the same arguments also carry through with minor adaptations for drawing without replacement.

We define a transition probability matrix $P^{(N,r)}$ that specifies for every pair of states the probability that the distribution of the urn transitions from the first to the second. Denote by e_i the i th unit vector in \mathbb{N}^d . For $p \in \Delta^{(N)}$ and $i, j \in [d]$ with $p' = p + \frac{e_i}{N} - \frac{e_j}{N} \in \Delta^{(N)}$, let

$$P^{(N,r)}(p, p') = \begin{cases} (1-r)2p_i p_j M_R(i, j) + \frac{r}{d} p_j & \text{if } i \neq j \\ (1-r) \sum_{k=1}^d p_k^2 + \frac{r}{d} & \text{if } i = j \end{cases}$$

⁷Multiplication by the transition probability matrix P is a measure-preserving transformation on $\Delta(S)$ with invariant measure π . For an irreducible Markov chain, this transformation is ergodic.

⁸There are also alternative notions of convergence for sojourn times. For example, for $\delta > 0$, let $\varepsilon_n > 0$ be the smallest number such that $\mathbb{P}(|s_n(S') - \pi(S')| \leq \varepsilon_n) \leq \delta$. One can ask at which rate ε_n goes to 0.

be the probability of transitioning from p to p' . For the remaining pairs of states $p, p' \in \Delta^{(N)}$, let $P^{(N,r)}(p, p') = 0$. Then, $P^{(N,r)}$ has non-negative values and its rows sum to 1 so that it is a valid transition probability matrix. For an initial state $p_0 \in \Delta^{(N)}$, we consider a Markov chain $\{X^{(N,r)}(n, p_0) : n \in \mathbb{N}\}$ with transition probability matrix $P^{(N,r)}$. The distribution of $X^{(N,r)}(n, p_0)$ over $\Delta^{(N)}$ is given by the row of $(P^{(N,r)})^n$ with index p_0 . If $r > 0$, this Markov chain is irreducible and aperiodic (since it remains in the same state with positive probability). It corresponds to the urn process described above when the initial state of the urn is p_0 .

Continuing Example 1, consider an urn containing $N = 5$ balls. Then, the transition probability matrix $P^{(N,r)}$ is an $\binom{3+5-1}{5} = 21$ -dimensional square matrix. Let the mutation rate be $r = 0.1$ and the initial state $p_0 = \frac{1}{5}(1, 2, 2)^\top$. The probability that one of the balls of the second type is replaced with one of the first type is

$$P^{(5,0.1)}(p_0, \frac{1}{5}(2, 1, 2)^\top) = 0.9 \cdot \frac{4}{25} \cdot \frac{2}{3} + 0.1 \cdot \frac{1}{3} \cdot \frac{2}{5} \sim 0.109.$$

3. The Result

We show that for any initial state $p_0 \in \Delta^{(N)}$, $X^{(N,r)}(k, p_0)$ is close to a maximal lottery for all but a small fraction of rounds k for some $r > 0$ provided the number of balls N is large enough. More precisely, take any $\delta, \tau > 0$. Then we can find $N_0 \in \mathbb{N}$ and $r > 0$ so that for every $N \geq N_0$ and every initial state $p_0 \in \Delta^{(N)}$, the sojourn time $s_n^{(N,r)}(B_\delta(ML(R)))$ of the δ -ball around $ML(R)$ for the process $X^{(N,r)}$ converges almost surely to a quantity of at least $1 - \tau$.⁹ Denote by $B_\delta(ML(R)) = \{p \in \Delta : \inf\{|p - q| : q \in ML(R)\} < \delta\}$ the set of lotteries with distance less than δ to some maximal lottery.

Theorem 1. *Let $\delta, \tau > 0$. Then, there is $r_0 > 0$ such that for all $0 < r \leq r_0$, there is $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ and $p_0 \in \Delta^{(N)}$,*

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} s_n^{(N,r)}(B_\delta(ML(R))) \geq 1 - \tau\right) = 1.$$

Moreover, there is $C > 0$ such that for all $n \in \mathbb{N}$,

$$\mathbb{P}\left(s_n^{(N,r)}(B_\delta(ML(R))) \geq 1 - \tau\right) \geq 1 - e^{-[Cn]}.$$

We outline the main steps in the proof of Theorem 1. Fix any $\delta, \tau > 0$. For any $p \in \Delta^{(N)}$, we consider the expected value of $N(X^{(N,r)}(k+1, p) - X^{(N,r)}(k, p))$, which, conditional on $X^{(N,r)}(k, p)$, is independent of k since $X^{(N,r)}$ is a time-homogeneous

⁹Theorem 1 implies that the stationary distribution of $X^{(N,r)}$ assigns probability at least $1 - \tau$ to states that are in a δ -neighborhood of $ML(R)$. Conversely, this property of the stationary distribution implies Theorem 1 by the ergodic theorem for Markov chains. The proof does however not derive the above property of the stationary distribution as an intermediate step. It is only a by-product of the final result. For more than two alternatives, our result is stronger than proving that the expectation of the stationary distribution, or, equivalently, the temporal average of the urn distribution, is close to a maximal lottery.

Markov process. Moreover, it is independent of N since the probability of replacing a ball of type j by one of type i is independent of N . Hence, these expected values induce a continuous function $f^{(r)}: \Delta \rightarrow \mathbb{R}^d$. We consider $g^{(r)}: \Delta \rightarrow \mathbb{R}^d$ with $g^{(r)}(p) = p + \frac{1}{2}f^{(r)}(p)$ and show that it maps to Δ . If $r > 0$, $g^{(r)}$ has a unique fixed-point $p^{(r)}$ (a zero of $f^{(r)}$), which is close to some lottery in $ML(R)$ for any small enough r . We choose r_0 so that $p^{(r)}$ is no more than $\frac{\delta}{2}$ away from $ML(R)$ for all $0 < r \leq r_0$.

Consider now the following differential equation with $p \in \Delta$, $t \in \mathbb{R}_{\geq 0}$, and $y(\cdot, p): \mathbb{R}_{\geq 0} \rightarrow \Delta$.

$$\begin{aligned} \frac{d}{dt}y(t, p) &= f^{(r)}(y(t, p)) \\ y(0, p) &= p \end{aligned} \tag{1}$$

A solution to (1) is a *deterministic* dynamic process that can be interpreted as the *stochastic* process we consider with a continuum of balls. We show that the unique solution $y^{(r)}(\cdot, p)$ of (1) converges to $p^{(r)}$ for any initial state $p \in \Delta$ as t goes to infinity and the convergence is uniform in p . This is done by showing that the entropy of $p^{(r)}$ relative to $y^{(r)}(t, p)$ decreases monotonically at a rate proportional to the square of the distance between $p^{(r)}$ and $y^{(r)}(t, p)$.

To relate the discrete-time stochastic process $X^{(N,r)}$ to the continuous-time deterministic process $y^{(r)}$, we extend the former to the real time axis by letting $\bar{X}^{(N,r)}(t, p) = X^{(N,r)}(k, p)$ for $t \in [\frac{k-1}{N}, \frac{k}{N})$. Given any $T > 0$, one can show that with probability close to 1, $\bar{X}^{(N,r)}$ *approximately* satisfies the integral equation corresponding to (1) for t between 0 and T and uniformly in $p \in \Delta^{(N)}$ if N is large. Using Grönwall's inequality, we show that with probability close to 1, $\bar{X}^{(N,r)}(t, p)$ and $y^{(r)}(t, p)$ are close to each other for all t from 0 to T .¹⁰ However, for t larger than T , they may (and almost surely will) be arbitrarily far apart.

To deal with this, we partition the time axis into consecutive intervals of length T and synchronize the deterministic process with the probabilistic process at the beginning of each interval. More precisely, since $y^{(r)}(t, p)$ converges to $p^{(r)}$ as t goes to infinity uniformly in p , we can find $T > 0$ such that $y^{(r)}(t, p)$ is no more than $\frac{\delta}{4}$ away from $p^{(r)}$ for all but possibly a $1 - \frac{\tau}{2}$ fraction of the interval $[0, T]$ for all p . Moreover, we can choose N large enough so that with probability at least $1 - \frac{\tau}{2}$, the distance between $\bar{X}^{(N,r)}$ and $y^{(r)}$ is less than $\frac{\delta}{4}$ for all t in an interval of length T provided both processes start at the same point at the beginning of the interval. We chop up the time axis into intervals $[0, T]$, $[T, 2T]$, \dots . On the interval $[(k-1)T, kT]$, we compare $\bar{X}^{(N,r)}(t, p)$ to $y^{(r)}(t - (k-1)T, \bar{x}_{k-1})$, where $\bar{x}_{k-1} = X^{(N,r)}((k-1)T, p)$. That is, we reset $y^{(r)}$ to the position of $\bar{X}^{(N,r)}$ at the beginning of the interval. In those intervals where the distance between both processes is never more than $\frac{\delta}{4}$, $\bar{X}^{(N,r)}$ is no more than $\frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}$ away from $p^{(r)}$ for all but a $\frac{\tau}{2}$ fraction of the interval. By the choice of N , the union of those

¹⁰In the language of functional analysis, this step corresponds to an approximation of an operator semi-group. Consider the operators $\Gamma(t)$ on probability measures on Δ induced by mapping $p \in \Delta$ to $y^{(r)}(t, p)$. Then $\{\Gamma(t): t \geq 0\}$ is an operator semi-group (that is, $\Gamma(s+t) = \Gamma(s)\Gamma(t)$). On $\Delta(\Delta^{(N)})$, we approximate $\Gamma(t)$ by $(P^{(N,r)})^{Nt}$.

intervals is almost surely at least a $1 - \frac{\tau}{2}$ fraction of the time axis. Summing over all intervals, this is enough to conclude that $\bar{X}^{(N,r)}$ is no more than $\frac{\delta}{2}$ away from $p^{(r)}$ at least a $1 - \tau$ fraction of the time. That is, the sojourn time of the $\frac{\delta}{2}$ -ball around $p^{(r)}$ for the process $\bar{X}^{(N,r)}$ converges almost surely to a quantity of at least $1 - \tau$. Since $p^{(r)}$ is no more than $\frac{\delta}{2}$ away from $ML(R)$ if $0 < r \leq r_0$, we can get the same conclusion with δ in place of $\frac{\delta}{2}$ and $ML(R)$ in place of $p^{(r)}$. Translating this statement back to $X^{(N,r)}$ gives the first part of Theorem 1.

The second statement follows from applying Proposition 1 to $\{X^{(N,r)}(n) : n \in \mathbb{N}\}$ and $S' = B_\delta(ML(R))$. The stationary distribution of $\{X^{(N,r)}(n) : n \in \mathbb{N}\}$ assigns probability at least $1 - \tau$ to states in $B_\delta(ML(R))$. Hence, for any $\varepsilon > 0$, the probability that the sojourn time of $B_\delta(ML(R))$ is at least $1 - \tau - \varepsilon$ converges to 1 exponentially fast.

Theorem 1 is a statement about the distribution in the urn. Recall that the collective decision in each round is the winner of the pairwise comparison between the two drawn balls. The following argument shows that the empirical distribution of winners is also close to a maximal lottery. Suppose the distribution of balls in the urn is $p \in \Delta^{(N)}$. Then the probability that $i \in [d]$ is the collective decision is

$$w_i = p_i \left(p_i + 2 \sum_{j \neq i} M_R(i, j) p_j \right) = p_i \left(p_i + \sum_{j \neq i} (\tilde{M}_R(i, j) + 1) p_j \right) = p_i \left(1 + \tilde{M}_R p \right)$$

where we used that $2M_R(i, j) = \tilde{M}_R(i, j) + 1$, $\sum_{j \in [d]} p_j = 1$, and $M_R(i, i) = 0$. If $p \in B_\delta(ML(R))$, then $(\tilde{M}_R p)_i \leq \delta$ for all $i \in [d]$. Hence, $w_i \in [p_i - \delta, p_i + \delta]$ for all i , so that $w \in B_{\delta d}(ML(R))$. For every $\delta' > 0$, choosing $\delta = \tau = \frac{\delta'}{2d}$ in Theorem 1 thus shows that the empirical distribution of collective decisions is almost surely in a ball of radius δ' around $ML(R)$.

Another straightforward corollary of Theorem 1 is that the temporal average of the $X^{(N,r)}(k, p_0)$ is almost surely close to $ML(R)$ for $r \leq r_0$ provided that N is large enough. We define the temporal averages of the $X^{(N,r)}(k, p_0)$.

$$Z^{(N,r)}(n, p_0) = \frac{1}{n} \cdot \sum_{k=0}^{n-1} X^{(N,r)}(k, p_0)$$

By the ergodic theorem for Markov chains, $Z^{(N,r)}$ converges almost surely to the stationary distribution $\pi \in \Delta(\Delta^{(N)})$ of the Markov chain $\{X^{(N,r)}(n, p_0) : n \in \mathbb{N}_0\}$. Theorem 1 shows that the limiting sojourn time of the set of states $B_\delta(ML(R))$ is at least $1 - \tau$. Hence, $\pi(B_\delta(ML(R)))$ is at least $1 - \tau$. By choosing $\tau \leq \frac{\delta}{2}$ and using that $|\cdot|$ is bounded by 2 on $\Delta^{(N)}$, we get that almost surely, the limit $\lim_{n \rightarrow \infty} Z^{(N,r)}(n, p_0)$ exists and lies in a ball of radius 2δ around $ML(R)$. In particular, $Z^{(N,r)}(n, p_0)$ is almost surely in $B_{2\delta}(ML(R))$ for all but a finite number of n .

Corollary 1. *Let $\delta > 0$. Then, there is $r_0 > 0$ such that for all $0 < r \leq r_0$, there is $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ and $p_0 \in \Delta^{(N)}$,*

$$\mathbb{P} \left(\lim_{n \rightarrow \infty} Z^{(N,r)}(n, p_0) \in B_\delta(ML(R)) \right) = 1$$

As in Theorem 1, one can get a bound for finite n using Proposition 1.

Before illustrating these results via examples, we briefly discuss variations of the urn process.

Remark 1 (Decoupled winner determination). It is not necessary to synchronize the drawing of winners with the preference elicitation process in which balls are replaced (via comparisons and mutations). In particular, it may be more practical to draw winners less frequently than replacing balls.

Remark 2 (Non-uniform mutation rates). The results still hold if we let the probability of a random mutation from alternative i to alternative j depend on the pair (i, j) . It suffices if every alternative in the support of a maximal lottery can escape permanent depletion via some path of mutations. For example, the mutation rate for the pair (i, j) could be proportional to the number $M_R(i, j)$ of agents who prefer i to j . The proof can be adapted at the expense of more book-keeping.

Remark 3 (Mutation rate vs. urn size). Corollary 1 shows that the temporal average of the urn distribution converges to a maximal lottery if we let N go to infinity and then take r to 0 (see also Theorem 2). This is in contrast to other works on evolutionary dynamics that take limits in the reverse order.¹¹ While the frameworks are similar, these results are conceptually different. If in our model, r is too small compared to $\frac{1}{N}$, it will in general not be the case that the distribution in the urn is close to a maximal lottery for most rounds. For any long enough time interval, the distribution in the urn will for all r degenerate within the interval with high probability, that is, it will only contain balls of one type. If r is very small, it will stay in a degenerate state for a long time (compared to the chosen interval) with high probability. When the process leaves the degenerate state, the same will repeat itself (possibly with a different degenerate state), so that the process spends most rounds in a degenerate state.

Remark 4 (Majority voting). Rather than letting only a single voter decide on the pairwise comparison between the two randomly drawn balls, it is possible to ask all voters which alternative they prefer and replace the alternative which is less preferred by a *majority* of voters. This variant is equivalent to the original process for a single voter with intransitive preferences (given by the majority relation of the entire population of voters) and converges to a so-called C1 maximal lottery of the preference profile (see Brandl et al., 2022, for more information about C1 maximal lotteries). Randomized voting rules based on Markov chains defined via the pairwise majority relation have been studied in the literature on tournament solutions (see, e.g., Laslier, 1997; Brandt et al., 2016a).

¹¹For example, Fudenberg and Imhof (2008) study imitation dynamics with mutations in symmetric two-player games (not necessarily zero-sum). They consider the case when the mutation rate goes to 0 for a fixed population size N . For small but positive mutation rates, the dynamics spend most of the time in degenerate states where all but a small fraction of individuals play the same strategy. Letting the mutation rate go to 0 thus induces a distribution over actions. Their main result determines the limit of this distribution as the population size goes to infinity.

Remark 5 (Static or growing urn). When the initial distribution of balls in the urn is uniform and remains fixed (i.e., no balls are replaced over time), then the empirical distribution of winners converges to the lottery returned by the proportional Borda rule (see, e.g., Barberà, 1979; Heckelman, 2003). This rule violates Condorcet-consistency and Pareto efficiency. It can put probability $\frac{1}{d}$ on Pareto-dominated alternatives and almost as little as $\frac{1}{d}$ on Condorcet winners for large numbers of voters. When adding a new ball labeled with the winning alternative rather than replacing the losing one (i.e., the number of balls increases over time), neither the relative distribution in the urn nor the temporal average converges (see Section 6).

Figure 2 (left) shows a simulation of the urn process for the preference profile and corresponding skew comparison matrix given in Example 1. The urn process corresponds to a random walk within the shown triangle starting from the center (an almost uniform distribution). The first alternative in this profile is a Condorcet winner. From round 177 on, at least 90% of the balls (45 of the 50) are labeled with the Condorcet winner except for three rounds. At this point, only 160 of the 900 voters were asked to compare two alternatives. The path is tilted to the left because a majority of voters prefer alternative 2 to alternative 3. Note that the process only depends on the fractions of voters who prefer one alternative to another and is thus independent of the number of voters. Hence, if there are nine million—rather than nine hundred—voters whose preferences are distributed as in Example 1, the process could turn out exactly as shown in Figure 2. In particular, the overwhelming majority of voters would never be queried for their preferences.

We now give two other examples, for which the unique maximal lottery is not degenerate.

Example 2. Consider 900 voters, three alternatives, and the following preference profile R , leading to a so-called Condorcet cycle or Condorcet paradox.

300	300	300
1	2	3
2	3	1
3	1	2

Then,

$$M_R = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{M}_R = \begin{pmatrix} 0 & 1/3 & -1/3 \\ -1/3 & 0 & 1/3 \\ 1/3 & -1/3 & 0 \end{pmatrix}.$$

The set of maximal lotteries $ML(R) = \{(1/3, 1/3, 1/3)\}$ consists of the uniform lottery over the three alternatives. A simulation of an urn process for this profile is given in Figure 2 (right). This time, the initial distribution is degenerate with all balls being of type 2. It can be seen how the distribution of balls in the urn closes in on the maximal lottery and remains in its neighborhood for most of the time while the temporal average converges to the maximal lottery.

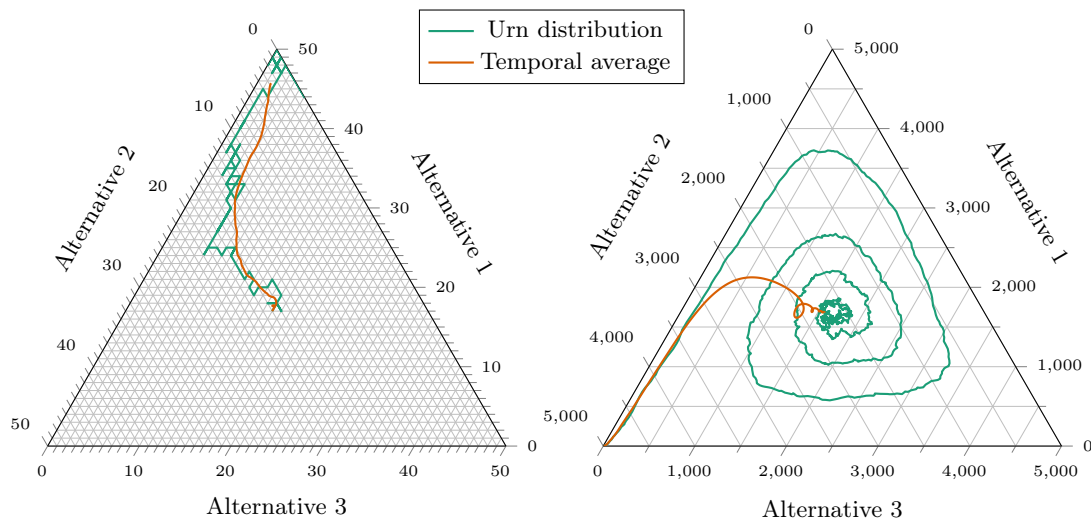


Figure 2: Simulations of the urn process.

The left diagram shows the urn process for the profile given in Example 1 using an urn with $N = 50$ balls for 1,000 rounds and mutation rate $r = 0.02$, starting from an almost uniform distribution. Each intersection of the grid lines corresponds to a configuration of the urn. The right diagram shows the urn process for the profile given in Example 2 using an urn with $N = 5,000$ balls for 500,000 rounds and mutation rate $r = 0.04$, starting from the degenerate distribution in which all balls are labeled with Alternative 2. The green lines depict the actual distribution of balls while the red lines depict the temporal average of urn distributions until the given round.

Example 3. Consider the following preference profile R with 900 voters and four alternatives.

	375	300	225
1	3	4	
2	1	2	
3	2	3	
4	4	1	

Then,

$$\tilde{M}_R = \begin{pmatrix} 0 & 1/3 & -1/9 & 1/3 \\ -1/3 & 0 & 2/9 & 1/3 \\ 1/9 & -2/9 & 0 & 1/3 \\ -1/3 & -1/3 & -1/3 & 0 \end{pmatrix}.$$

The set of maximal lotteries $ML(R) = \{(1/3, 1/6, 1/2, 0)\}$ consists of a single lottery, which is supported on the first three alternatives. A simulation of an urn process for this profile starting from the uniform distribution is given in Figure 3. The figure shows the distribution in the urn, the temporal average of urn distributions, and the difference of the urn distribution and the maximal lottery in terms of the relative entropy.

We use the relative entropy (rather than the distance $|p - q|$) to measure how much the distribution in the urn diverges from the maximal lottery since the proof of Theorem 1

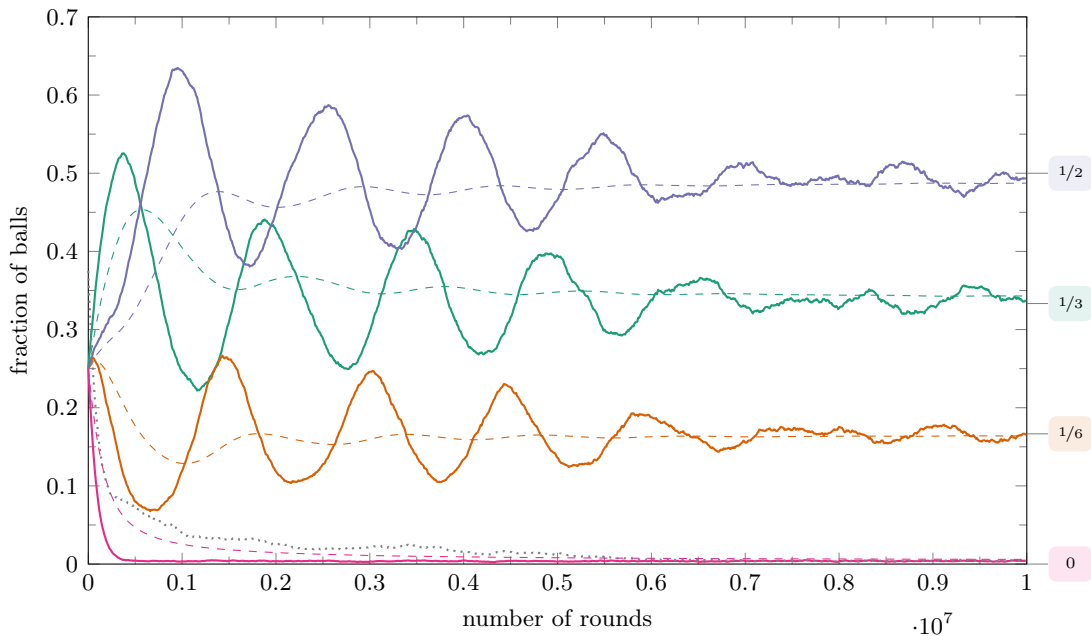


Figure 3: Simulation of the urn process for the profile in Example 3 on an urn with $N = 50,000$ balls for 10^7 rounds and mutation rate $r = 0.01$. The solid lines show the fraction of balls in the urn. The dashed lines show the temporal average of the fraction of balls in the urn until the given round. The unique maximal lottery is $p = (1/3, 1/6, 1/2, 0)$. The dotted line shows the relative entropy $D(p | q) = \sum_{i \in [d]} p_i \log(\frac{p_i}{q_i})$ of p with respect to the distribution in the urn q .

shows that the entropy of the maximal lottery relative to the continuous approximation of the discrete process converges monotonically to 0.

The literature on equilibrium learning often focusses on minimizing *regret* rather than relative entropy with respect to an equilibrium distribution (see, e.g., Foster and Vohra, 1999; Auer et al., 2002). In our context, the regret of the urn distribution at round n is $\max_{i \in [d]} (\tilde{M}_R X^{(N,r)}(n, p_0))_i$. It follows from Theorem 1 that for sufficiently large n , the regret is close to zero with high probability. It turns out that the regret of the urn distribution converges faster than its relative entropy. This is interesting insofar as in order to approximately satisfy the desirable axiomatic properties discussed in Section 5, low regret is sufficient. It can be shown that a lottery has small regret if and only if it is a maximal lottery of a nearby preference profile. In other words, even if the urn distribution is still far from a maximal lottery, the distribution can perform almost as well as a maximal lottery. We have identified preference profiles where this effect is quite noticeable.

4. The Case of a Condorcet Winner

We give an elementary proof of Theorem 1 for profiles that admit a Condorcet winner. For those profiles, the unique maximal lottery assigns probability 1 to the Condorcet winner. To analyze the stationary distribution $\pi \in \Delta(\Delta^{(N)})$ of the Markov chain induced by the urn process, it suffices to examine the fraction of balls labeled with the Condorcet winner. This allows us to relate the Markov chain to a process that is linear in the sense that each state can only transition to two different states, and is thus easy to analyze. It also enables us to give a concrete lower bound on the number of balls N required for given $\delta, \tau > 0$ for the conclusion of Theorem 1 to hold.

Let $M = M_R$ be the majority matrix of a profile R with Condorcet winner $i \in [d]$. Hence, $M_{ij} > \frac{1}{2}$ for all $j \in [d] \setminus \{i\}$. Let $\alpha = \min\{M_{ij} : j \in [d] \setminus \{i\}\} - \frac{1}{2}$. We slice up $\Delta^{(N)}$ into the level sets of i . For $k \in \{0, \dots, N\}$, let $S_k = \{p \in \Delta^{(N)} : p_i = \frac{k}{N}\}$ be the states corresponding to distributions with k of the N balls of type i . Then $\sigma_k := \sum_{p \in S_k} \pi(p)$ is the limit probability that the urn is in a state in S_k as the number of rounds goes to infinity. We want to show that if r is sufficiently small and N sufficiently large, π has most of the probability on states in S_k with k close to N .

For 4 alternatives, one can illustrate the ensuing argument as follows. The set of states $\Delta^{(N)}$ corresponds to rooms in a tetrahedral-shaped pyramid. The rooms on the k th floor correspond to S_k , so that the tip of the pyramid is the state where all balls are of type i . The urn process is a random walk through the pyramid, moving from one room to an adjacent one (which could be on the same floor, the floor below, or the floor above). With the exception of few floors close to the tip, the probability of going up is always larger than the probability of going down. It is then intuitively clear that if the pyramid is large enough, one should expect to find the random walk close to the tip of the pyramid most of the time.¹²

Recall that $P^{(N,r)}(p, q)$ is the probability of transitioning from state p to state q . Since π is a stationary distribution, we have $\pi^\top P^{(N,r)} = \pi^\top$. Consider any partition of $\Delta^{(N)}$ into two sets. For the stationary distribution, the probability of transitioning from the first set to the second is equal to the probability of transitioning from the second set to the first since the probabilities of both sets are conserved. Applying this to the sets $\bigcup_{l=0}^{k-1} S_l$ and $\bigcup_{l=k}^N S_l$ for $k \in [N]$ and noticing that the only transitions between the two sets with positive probability are from S^{k-1} to S^k and vice versa, we get

$$\sum_{p \in S_{k-1}} \pi(p) \sum_{q \in S_k} P^{(N,r)}(p, q) = \sum_{p \in S_k} \pi(p) \sum_{q \in S_{k-1}} P^{(N,r)}(p, q). \quad (2)$$

That is, the probability of being in a state in S_{k-1} and transitioning to a state in S_k equals the probability of being in a state in S_k and transitioning to a state in S_{k-1} .

¹²In the analysis of the general case, the number of balls of type i is replaced by the entropy of the urn distribution relative to a maximal lottery. The fact that the number of balls not of type i more likely than not decreases corresponds to the fact that the expected entropy relative to a maximal lottery decreases (for states close to a maximal lottery in both cases).

Now observe that for $p \in S_k$, $k \in \{0, \dots, N-1\}$, we have

$$\sum_{q \in S_{k+1}} P^{(N,r)}(p, q) \geq 2(1-r) \frac{k(N-k)}{N^2} \left(\frac{1}{2} + \alpha \right) + \frac{r}{d} \frac{N-k}{N} =: u_k$$

where the left hand side is the probability of replacing a ball of type other than i by one of type i in state $p \in S_k$ (moving up one floor in the pyramid). Similarly, we find that for $p \in S_k$, $k \in \{1, \dots, N\}$, we have

$$\sum_{q \in S_{k-1}} P^{(N,r)}(p, q) \leq 2(1-r) \frac{k(N-k)}{N^2} \left(\frac{1}{2} - \alpha \right) + r \frac{d-1}{d} \frac{k}{N} =: d_k$$

for the probability of replacing a ball of type i by one of type other than i in state $p \in S_k$ (moving down one floor in the pyramid). Plugging this into (2), we get

$$\sigma_{k-1} u_{k-1} \leq \sigma_k d_k. \tag{3}$$

All terms in (3) are strictly positive if $r > 0$.

Let N be so that $\frac{r}{Nd} \geq 2 \frac{1-r}{N^2}$ (we choose $r > 0$ later). Then,

$$\begin{aligned} u_k &\geq 2(1-r) \frac{k(N-k)}{N^2} \left(\frac{1}{2} + \alpha \right) + 2(1-r) \frac{N-k}{N^2} \\ &\geq 2(1-r) \frac{(k+1)(N-k-1)}{N^2} \left(\frac{1}{2} + \alpha \right) \end{aligned}$$

where the last inequality uses $1 \geq \frac{1}{2} + \alpha$. Similarly, we find that for $r \leq \frac{1}{d}$ and $k \leq N(1 - \frac{r}{\alpha})$,

$$d_k \leq 2(1-r) \frac{k(N-k)}{N^2} \frac{1-\alpha}{2}.$$

Hence, with this bound on k , we have

$$\frac{d_k}{u_{k-1}} \leq \frac{1-\alpha}{2(\frac{1}{2} + \alpha)} = \frac{1-\alpha}{1+2\alpha} =: \beta.$$

Thus, by (3), $\frac{\sigma_{k-1}}{\sigma_k} \leq \beta < 1$. We have shown that the cumulative probability σ_k of the states S_k decreases at least as fast the terms of the geometric series with parameter β from some k (close to N) downwards.

The maximal lottery for R is the degenerate lottery with probability 1 on i . For given $\delta, \tau > 0$, we are aiming for a lower bound on N so that the probability on states with at least $1 - \delta$ fraction of balls of type i in the stationary distribution π is at least $1 - \tau$. That is,

$$\sum_{k=\lceil N(1-\delta) \rceil}^N \sigma_k \geq 1 - \tau.$$

First observe that

$$\sum_{k \geq k_0} \beta^k = \beta^{k_0} \frac{1}{1 - \beta} \leq \tau \quad (4)$$

for $k_0 \geq \frac{\log(\tau(1-\beta))}{\log \beta}$. For our bound, N needs to be large enough so that there are at least k_0 integers in the interval $\{\lceil(1 - \delta)N\rceil, \dots, \lfloor(1 - \frac{r}{\alpha})N\rfloor\}$. The probability on states in S_k with $k < (1 - \delta)N$ will then be below τ by (4) and the choice of k_0 (since the bound on d_k assumes that $k \leq N(1 - \frac{r}{\alpha})$). Choosing $r \leq \frac{\alpha\delta}{2}$ and

$$N \geq \frac{k_0}{\delta - \frac{r}{\alpha}} \geq \frac{1}{\delta} \left\lceil \frac{\log(\tau(1-\beta))}{\log \beta} \right\rceil$$

achieves this.

In Example 1, there are three alternatives and 900 voters. Alternative 1 is a Condorcet winner as it is preferred to every other alternative by 600 of the voters ($\alpha = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$, $\beta = \frac{5}{8}$). Suppose we want that at least 90% of the balls in the urn are of type 1 in at least 90% of rounds ($\delta = 0.2$, $\tau = 0.1$). Choosing $r = \frac{\alpha\delta}{2} = \frac{1}{60}$, we need $N \geq 70$ balls in the urn. These calculations suggest that, when a Condorcet winner exists, a reasonable choice for simulations is $N \geq -\frac{1}{\delta} \log(\tau)$ and $\frac{1}{N} \leq r \leq \delta$.

5. Approximate Axiomatics

The temporal average of the urn process we describe approximates maximal lotteries. As mentioned in Section 1, maximal lotteries are renowned for satisfying a large number of desirable properties and several results have shown that no other probabilistic social choice function (PSCF) satisfies these properties. For some axioms, it is obvious that every approximation of a PSCF that satisfies the axiom satisfies an approximate version of the axiom. For example, every approximation of a Condorcet-consistent PSCF—one that returns the lottery with probability 1 on the Condorcet winner whenever one exists—is approximately Condorcet-consistent in the sense that it assigns probability close to 1 to Condorcet winners. For other axioms, this is not true in general. In this section, we discuss population-consistency as an example and show that approximations of *continuous* PSCFs that satisfy population-consistency satisfy an approximate consistency axiom. Our notion of continuity requires that a preference change by a small fraction of the voters leads to at most a small change to the chosen lottery. Since ML is continuous and population-consistent, the urn procedure is approximately population-consistent. The same holds for several other axioms satisfied by ML such as composition-consistency and agenda-consistency (see Brandl et al., 2016). We do not discuss those here since the analysis is very similar.

The notation becomes more convenient by switching to *fractional preference profiles*, which map every preference relation to the fraction of voters with these preferences. The set of all fractional preferences profiles is thus $\Delta(\mathcal{R})$. A PSCF is then a correspondence

from $\Delta(\mathcal{R})$ to Δ .¹³ Population-consistency is concerned with the consistency of PSCFs with respect to variable electorates: whenever a PSCF ϕ selects p for two profiles on disjoint electorates, then it also selects p for the union of both profiles. For fractional preference profiles, this is formalized as

$$\phi(R') \cap \phi(R'') \subset \phi(\tfrac{1}{2}R' + \tfrac{1}{2}R'') \quad (\text{population-consistency})$$

for all $R', R'' \in \Delta(\mathcal{R})$. An approximate version of population-consistency requires that $\phi(R') \cap \phi(R'')$ is contained in a small neighborhood of $\phi(\tfrac{1}{2}R' + \tfrac{1}{2}R'')$. For $\varepsilon > 0$, ϕ satisfies ε -approximate population-consistency at $R \in \Delta(\mathcal{R})$ if

$$\phi(R') \cap \phi(R'') \subset B_\varepsilon(\phi(R)) \quad (\varepsilon\text{-approximate population-consistency})$$

for all $R', R'' \in \Delta(\mathcal{R})$ with $\tfrac{1}{2}R' + \tfrac{1}{2}R'' = R$.

There are population-consistent PSCFs and $\varepsilon > 0$ such that even arbitrarily good approximations of the PSCF do not satisfy ε -approximate population-consistency for some profile. Call ψ a δ -approximation of ϕ for $\delta > 0$ if $\text{dist}(\psi(R), \phi(R)) < \delta$ for all R , where $\text{dist}(U, V) = \inf_{p \in U, q \in V} |p - q|$ for $U, V \subset \Delta$. If a PSCF is population-consistent, it may be the case that for some profile R , for every $\delta > 0$, there is a δ -approximation of the PSCF that violates ε -approximate population-consistency for R . We show that this is impossible for (upper hemi-)continuous PSCFs at profiles for which a unique lottery is returned.¹⁴

Proposition 2. *Let ϕ be a continuous PSCF that satisfies population-consistency. Then, for every $\varepsilon > 0$ and $R \in \Delta(\mathcal{R})$ with $|\phi(R)| = 1$, there is $\delta > 0$ such that every δ -approximation of ϕ satisfies ε -approximate population-consistency at R .*

Brandl et al. (2016) have shown that ML is continuous and satisfies population-consistency. Moreover, maximal lotteries are almost always unique in the sense that the set of all profiles with multiple maximal lotteries has measure zero and is nowhere dense in $\Delta(\mathcal{R})$.¹⁵ Theorem 1 implies that the number of balls in the urn and the mutation rate can be chosen so that the temporal average of the urn distribution is almost surely at most δ away from some maximal lottery. Hence, Proposition 2 implies that the urn process satisfies ε -approximate population-consistency in a well-defined sense for arbitrary $\varepsilon > 0$.

It is straightforward to check that Proposition 2 follows from the following lemma.

Lemma 1. *Let $\phi: U \rightrightarrows \mathbb{R}^d$ be an upper hemi-continuous correspondence from a compact subset $U \subset \mathbb{R}^k$ to \mathbb{R}^d . Suppose that for all $u', u'' \in U$, $\phi(u') \cap \phi(u'') \subset \phi(\tfrac{1}{2}u' + \tfrac{1}{2}u'')$. Then, for all $u \in U$ and $\varepsilon > 0$, there is $\delta > 0$ such that for all $u', u'' \in U$ with $u = \tfrac{1}{2}u' + \tfrac{1}{2}u''$,*

$$B_\delta(\phi(u')) \cap B_\delta(\phi(u'')) \subset B_\varepsilon(\phi(u)).$$

¹³This definition of a PSCF entails that it is anonymous (the identities of the voters are irrelevant) and homogeneous (replicating the entire electorate does not affect the outcome).

¹⁴A PSCF ϕ is upper hemi-continuous if for all $R \in \Delta(\mathcal{R})$ and every sequence $(R^k)_{k \in \mathbb{N}} \subset \Delta(\mathcal{R})$ converging to R , whenever a sequence $(p^k)_{k \in \mathbb{N}} \subset \Delta$ with $p^k \in \phi(R^k)$ converges to $p \in \Delta$, then $p \in \phi(R)$.

¹⁵These facts follow from considering the dimension of the kernel of a sub-matrix of the skew comparison matrix.

Proof. Let $u \in U$ and $\varepsilon > 0$. Define

$$\delta' = \inf \left\{ \text{dist} \left(\phi(u') \setminus B_{\frac{\varepsilon}{2}}(\phi(u)), \phi(u'') \setminus B_{\frac{\varepsilon}{2}}(\phi(u)) \right) : u', u'' \in U \text{ with } \frac{1}{2}u' + \frac{1}{2}u'' = u \right\}.$$

(By convention, $\text{dist}(\cdot, \cdot)$ takes the value ∞ if one of its arguments is the empty set.) Since ϕ is upper hemi-continuous, U is compact, and $\phi(u') \cap \phi(u'') \subset \phi(u)$ for u', u'' as above, $\delta' > 0$.

Now let $\delta = \frac{1}{2} \min\{\varepsilon, \delta'\}$. Let $u', u'' \in U$ with $\frac{1}{2}u' + \frac{1}{2}u'' = u$ and $p \in B_{\delta}(\phi(u')) \cap B_{\delta}(\phi(u''))$. Then there are $q' \in \phi(u')$ and $q'' \in \phi(u'')$ with $|q' - p| < \delta$ and $|q'' - p| < \delta$. Hence, $|q' - q''| < \delta'$. By definition of δ' , either $q' \in B_{\frac{\varepsilon}{2}}(\phi(u))$ or $q'' \in B_{\frac{\varepsilon}{2}}(\phi(u))$. Either way, $p \in B_{\delta + \frac{\varepsilon}{2}}(\phi(u)) \subset B_{\varepsilon}(\phi(u))$, which is what had to be shown. \square

6. Discussion of Related Work

Since the urn process described in this paper only depends on the transition probabilities induced by \tilde{M}_R and r , it is applicable to various problems unrelated to collective decision-making.

6.1. Population dynamics

The urn process is related to the *replicator equation* in population dynamics and evolutionary game theory (see, e.g., Taylor and Jonker, 1978; Schuster and Sigmund, 1983; Hofbauer and Sigmund, 1998). In its basic form, it states that the change in the relative frequency of a species equals the relative fitness of the species (that is, its fitness relative to the entire population) minus the change in the size of the entire population. When the fitness depends linearly on the relative frequencies of the species and the population size is constant, this is the differential equation (5) below with $r = 0$. The models in this stream of research are typically not discrete, which means that the distribution of species changes *continuously* with time and space.

Allesina and Levine (2011) study the competition and coexistence of species in nature via a mathematical framework that is similar to our urn process. There is a fixed finite number of individuals, each of which is assigned to some species at random. In each round, two randomly selected individuals interact. The superior species will replace the individual of the inferior. Which species is superior to which species is given in the form of a tournament graph, which can be represented by a binary comparison matrix. Interestingly, these tournaments are sampled from distributions that are obtained via multiple rankings of the species called “limiting factors”, similar to the preferences of voters. Simulations with large populations (e.g., 25,000 individuals) then show that the relative frequencies of the species oscillate around the equilibrium strategy of the skew comparison matrix. However, this phenomenon is an artifact of the population size and the limited time horizon. In the long run, as mentioned in Section 1, all species but one will almost surely become extinct.

Knebel et al. (2015) study a similar process (including random mutations) in the context of quantum physics. Here, balls in the urn model correspond to bosons and alternatives to quantum states. The distribution of quantum states determines which states are condensates and are thus observed macroscopically. Since the number of particles in such systems is typically large, they focus on a deterministic process with a continuum of particles as described in Section 3. Leveraging a classic result from evolutionary game theory (Hofbauer and Sigmund, 1998, Theorem 5.2.3), they show that the *temporal average* of this process converges to an equilibrium strategy (i.e., a maximal lottery) of the zero-sum game induced by the transition probabilities between quantum states. All states with probability zero in the equilibrium strategy are depleted; the fractions of the remaining states are bounded away from 0 for all times. Knebel et al. neglect mutations for the continuous process, which may cause the process to cycle around the equilibrium strategy without converging to it. Within our proof of Theorem 1 (see Theorem 2), we show that the continuous process with mutations does converge (and not only its temporal average). Knebel et al. (2015, Supplementary Note 1) argue that the discrete process with mutations is well-approximated by the continuous process if the number of particles is large and mutations become vanishingly unlikely. Hence, they conclude that the temporal average of the discrete process converges to an equilibrium strategy, which is in the spirit of Corollary 1. Our understanding is that their arguments are heuristic and not intended to provide a rigorous derivation of this result. In particular, the arguments do not seem to use that mutations happen with non-zero probability. Without mutations, however, the discrete process almost surely enters a state with a degenerate distribution.

In earlier work, Knebel et al. (2013) have connected the survival and extinction of states to the Pfaffian of the transition matrix. This is reminiscent of a statement by Kaplansky (1995) about the support of equilibrium strategies in symmetric zero-sum games. Reichenbach et al. (2006) study the extinction probabilities for three states with cyclical dominance (“rock-paper-scissors”) for finite populations.

Laslier and Laslier (2017) consider a discrete urn process that is similar to ours, but in which the number of balls in the urn increases over time. Two balls are drawn at random and a binary comparison matrix specifies which alternative wins against which alternative (this could be seen as a single voter with possibly intransitive preferences in our model). Rather than replacing the losing ball, a new ball of the same type as the winning ball is added to the urn. They show that the distribution in the urn does not converge unless one alternative beats all alternatives (which corresponds to the Condorcet winner case). However, the fraction of alternatives not contained in the support of the maximal lottery of the skew comparison matrix goes to zero. They then consider a modified process, in which three balls are drawn from the urn. Whenever one of three balls beats both other balls, a new ball of the same type is added to the urn. Otherwise, one of the three types is chosen at random and a ball of that type is added. Their main result is that, for this modified process, the distribution in the urn converges towards the (unique) maximal lottery of the skew comparison matrix. Laslier and Laslier

	Model	Interaction	Mutations	Pop. Size	Convergence
Allesina et al. (2011)	discrete	pairs, det.	no	fixed	— ^a
Knebel et al. (2015)	cont.	pairs, det.	no ^b	fixed	temporal average
Laslier et al. (2017)	discrete	pairs, det.	no	increasing	support of distribution
Laslier et al. (2017)	discrete	triples, det.	no	increasing	distribution
Grilli et al. (2017)	cont.	triples, stoch.	no	fixed	distribution
Theorem 1	discrete	pairs, stoch.	yes	fixed	sojourn time
Corollary 1	discrete	pairs, stoch.	yes	fixed	temporal average
Theorem 2	cont.	pairs, det.	yes	fixed	distribution

Table 1: Comparison of related models and results.

a: In simulations, Allesina and Levine (2011) observe that the temporal average of their process comes close to a maximal lottery after a finite number of rounds. However, when the process is run long enough, the distribution will almost surely degenerate since there are no mutations.

b: While Knebel et al. (2015) consider a discrete process with mutations, the continuous process they study has no mutations.

neither consider the empirical distribution of winners nor the temporal average of the distribution in the urn. For the process with two drawn balls, it can be shown that not even the temporal average converges. Since the number of balls in the urn increases, convergence is generally very slow.

Grilli et al. (2017) consider a dynamical process in population biology to explain the stable coexistence of multiple species. Based on Laslier and Laslier’s findings, Grilli et al. adapt the replicator equation to interactions of triples of individuals: in each round, a randomly chosen individual dies; it is replaced by the winner of a comparison between three randomly selected individuals, where the winner is determined as in Laslier and Laslier’s process with three drawn balls based on a comparison matrix describing stochastic interactions between pairs of species. Hence, the number of individuals remains constant. They show that with a continuum of individuals, this process converges to an equilibrium strategy of the zero-sum game corresponding to the skew comparison matrix. For a finite number of individuals, permanent coexistence of multiple species is a probability zero event. However, they argue that interactions of three or more individuals can prolong coexistence compared to pairwise interactions.

Comparison. Table 1 summarized the key differences between the above mentioned results. In comparison, the main contribution of our work is that we are able to show for a discrete (rather than continuous) process based on stochastic (rather than deterministic) interactions between pairs (rather than triples) that the actual distribution in the urn is close to a maximal lottery most of the time (rather than convergence of the temporal average). Methodologically, the approach we take to cope with the discrete process is related to that of Benaim and Weibull (2003), who study more general population

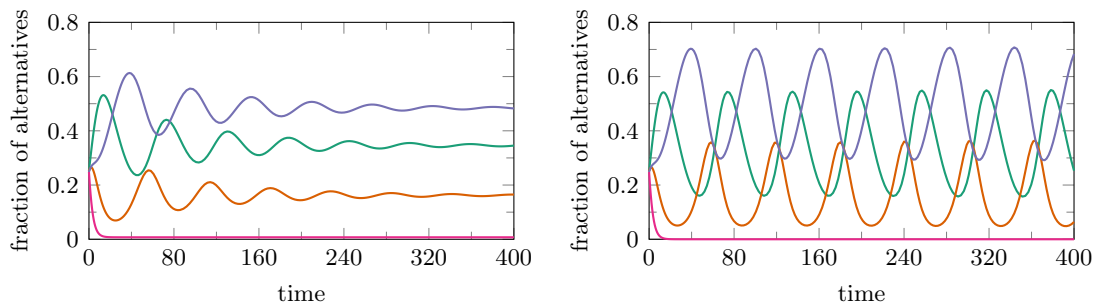


Figure 4: The continuous deterministic process $y(t)$ solving Equation (5) for the profile in Example 3 with $r = 0.01$ on the left and $r = 0$ on the right. For strictly positive r , $y(t)$ converges to a zero of $f^{(r)}$ (see Theorem 2). For $r = 0$, it approaches an orbit of constant entropy relative to a zero of $f^{(r)}$.

processes in n -player games.¹⁶

It has been observed repeatedly that without mutations (i.e., $r = 0$), the deterministic process described by the differential equation (5) does not in general converge, but only approaches an orbit of constant entropy relative to a zero of the fitness function f . When introducing mutations, the limiting behavior of the process changes qualitatively (see Figure 4). As Theorem 2 shows, it then converges to a zero of the fitness function (see Appendix B).

Theorem 2. Let $f^{(r)}: \Delta \rightarrow \mathbb{R}^d$ be defined by

$$f_i^{(r)}(p) = 2(1-r)p_i(\tilde{M}p)_i + r\left(\frac{1}{d} - p_i\right)$$

If $r > 0$, $f^{(r)}$ has a unique zero $p^{(r)}$ and the unique solution $y(t)$ of

$$\begin{aligned} \frac{d}{dt}y(t) &= f^{(r)}(y(t)) \\ y(0) &= p \end{aligned} \tag{5}$$

converges to $p^{(r)}$ as $t \rightarrow \infty$. Moreover, if r goes to 0, then $p^{(r)}$ converges to $ML(R)$.

We believe that this result as well as Theorem 1 and Corollary 1 are also of relevance for the natural sciences. In particular, a discrete model may describe the mentioned natural

¹⁶In Benaim and Weibull's model, each player has a population of N individuals who play pure strategies. In each round, one individual of one player can update their strategy based on the distributions of pure strategies of all players. An update rule induces a deterministic process described by a differential equation similar to (5) below. They show that if N is large, the distributions of strategies among the individuals of each role in this probabilistic process approximate the deterministic process described by the differential equation. Our setting corresponds to a symmetric two-player zero-sum game and an update rule based on the comparison matrix \tilde{M} . The special properties of this instance allow us to make more precise statements about the behavior of the deterministic process, and, thus, of the probabilistic process for large N . In particular, we show that the deterministic process converges and that its limit approximates a maximal lottery.

phenomena more accurately than continuous ones. As Corollary 1 shows, the expectation of the discrete process with a large number of individuals is a good approximation of the continuous process. Furthermore, the observation that convergence is only guaranteed if mutations occur with small probability and the number of individuals is large enough seems noteworthy.

6.2. Equilibrium learning

When interpreting the skew comparison matrix as a symmetric two-player zero-sum game and maximal lotteries as equilibrium strategies, our result can be phrased as a result about a learning procedure for equilibrium play. Such procedures have been extensively studied in game theory and, in particular for zero-sum games, a number of simple and attractive procedures have been proposed. The earliest of these is *fictitious play* (Brown, 1951; Robinson, 1951). More recently, the *multiplicative weights update algorithm* (e.g., Freund and Schapire, 1999; Arora et al., 2012) and *regret matching* (Hart and Mas-Colell, 2000, 2013) have been celebrated in game theory, optimization, and machine learning. When translating the multiplicative weights update algorithm to our setting, one obtains a dynamic urn process, in which voters need to compare a drawn ball to all possible alternatives and adjust the distribution in the urn accordingly. It does not suffice to replace a single ball and the total number of balls does not remain constant. Also, the multiplicative weights update algorithm only guarantees convergence of the temporal average. The actual distribution does not converge, even for self-play in symmetric zero-sum games (Bailey and Piliouras, 2018).

A very active subarea of machine learning is concerned with *multi-armed bandits*, a simple model of learning optimal sequential decisions when only very limited information is available (see, e.g., Bubeck and Cesa-Bianchi, 2012; Slivkins, 2019). The theory of adversarial bandits is closely connected to learning in repeated multi-player games and it turns out that the prototypical algorithm for adversarial bandits, Exp3 (which stands for “exponential-weight algorithm for exploration and exploitation”), bears some similarities to the urn process we describe in this paper. Exp3 can be formulated as an algorithm that learns an equilibrium strategy of a symmetric zero-sum game in self-play by iteratively updating a probability distribution merely based on the payoff associated with two actions randomly drawn from the current distribution. In contrast to multiplicative weights, Exp3 is a probabilistic algorithm like the urn process. Auer et al. (2002) prove strong bounds on the expected average regret and average regret achieved by Exp3 after a finite number of rounds, which imply that the temporal average of the distributions converges to a strategy close to an equilibrium. How close it gets to an equilibrium depends on a parameter that is roughly related to our mutation rate. Exp3 updates a probability distribution rather than the contents of a discrete urn and we are not aware of convergence results beyond the temporal average.

6.3. Population protocols

The process we describe approximately computes a mixed Nash equilibrium of a symmetric zero-sum game. This problem is known to be equivalent to linear programming. In fact, deciding whether an action is played with positive probability in an equilibrium of a symmetric zero-sum game is P-complete, even when all payoffs are -1 , 0 , or 1 (Brandt and Fischer, 2008, Theorem 5), which, loosely speaking, means that the problem is at least as hard as any problem that can be solved in polynomial time. The urn process can thus be seen as a probabilistic algorithm that approximates polynomial-time computable functions. In contrast to traditional computing devices such as Turing machines, the urn process is based on unordered elementary entities that randomly interact according to very simple replacement rules. Related decentralized models of computation with applications to sensor networks and molecular computing are studied under the name “population protocols” in computer science (e.g., Angluin et al., 2006; Aspnes and Ruppert, 2009). While the urn process has the same *modus operandi* as population protocols, the input-output behavior is different. The input of population protocols is given by the initial distribution of balls in the urn and the output has been reached if all balls belong to a certain subset of types. By contrast, the input for our urn process is encoded in the matrix describing the replacement rules and the (approximate) output is given by the distribution of balls in the urn after sufficiently many rounds.

Acknowledgments

This material is based on work supported by the Deutsche Forschungsgemeinschaft under grants BR 2312/11-1, BR 2312/12-1, and BR 5969/1-1. The authors thank Stefano Allesina, Vincent Conitzer, Javier Esparza, Erwin Frey, Drew Fudenberg, Philipp Geiger, Umberto Grandi, Matthias Greger, Sean Horan, Johannes Knebel, Jean-François Laslier, Patrick Lederer, Hervé Moulin, Robert Schapire, Jörgen Weibull, Peyton Young, and the participants of the COMSOC video seminar (March 2021), the International Conference on “New Directions in Social Choice” (St. Petersburg, July 2021), the Hausdorff Center for Mathematics Symposium (Bonn, August 2021), the seminar of the Center of Economic Research of ETH Zürich (Zurich, November 2021), and the Nobel Symposium “One Hundred Years of Game Theory” (Stockholm, December 2021) for stimulating discussions and encouraging feedback.

APPENDIX: Proofs

A. A Continuous Vector Field Induced by the Markov Chain

In this section, we define a continuous mapping from Δ to Δ based on the *expected* urn distribution in the subsequent round for each state of the Markov chain. We then show that this mapping admits a unique fixed-point corresponding to an approximate maximal lottery.

Recall that $\{X^{(N,r)}(n, p_0) : n \in \mathbb{N}_0\}$ is a discrete-time, time-homogeneous Markov chain with state space $\Delta^{(N)}$ and transition probability matrix

$$P^{(N,r)}(p, p') = \begin{cases} 2(1-r)p_i p_j M(i, j) + \frac{r}{d} p_j & \text{if } i \neq j \\ 2(1-r) \sum_{k=1}^d p_k^2 + \frac{r}{d} & \text{if } i = j \end{cases}$$

for $p \in \Delta^{(N)}$ and $p' = p + \frac{e_i}{N} - \frac{e_j}{N}$ for $i, j \in \{1, \dots, d\}$ with $p' \in \Delta^{(N)}$. All other transition probabilities are 0. If $r > 0$, it is irreducible and aperiodic and thus admits a unique stationary distribution in $\Delta(\Delta^{(N)})$, a probability distribution over urn distributions. We omit writing the initial state p_0 whenever it is convenient.

For $i \in [d]$, we calculate the expected change in the i th component of $X^{(N,r)}$ times N given that $X^{(N,r)}$ is in state $p \in \Delta^{(N)}$.

$$\begin{aligned} & N \mathbb{E} \left(X_i^{(N,r)}(n+1) - X_i^{(N,r)}(n) \mid X^{(N,r)}(n) = p \right) \\ &= N \sum_{p' \in \Delta^{(N)}} (p'_i - p_i) P^{(N,r)}(p, p') \\ &= 2(1-r) \sum_{j \neq i} p_i p_j (M(i, j) - M(j, i)) + \frac{r}{d} \sum_{j \neq i} (p_j - p_i) \\ &= 2(1-r) p_i \sum_{j \neq i} \tilde{M}(i, j) p_j + \frac{r}{d} (1 - p_i - (d-1)p_i) \\ &= 2(1-r) p_i (\tilde{M}p)_i + r \left(\frac{1}{d} - p_i \right) \end{aligned}$$

For the last equality, recall that $\tilde{M}(i, i) = 0$ since \tilde{M} is skew-symmetric.

Based on this, we define the continuous function $f^{(r)} : \Delta \rightarrow \mathbb{R}^d$ with

$$f_i^{(r)}(p) = 2(1-r) p_i (\tilde{M}p)_i + r \left(\frac{1}{d} - p_i \right). \quad (6)$$

Let $g^{(r)} : \Delta \rightarrow \Delta$ with $g^{(r)}(p) = p + \frac{1}{2} f(p)$ for $p \in \Delta$. We show that $g^{(r)}$ is well-defined (that is, indeed maps to Δ) and has a fixed-point. If $r > 0$, this fixed-point is unique and we denote it by $p^{(r)}$. As r goes to 0, $p^{(r)}$ converges to a maximal lottery for the profile R that induces \tilde{M} . We note that if $r = 0$, $g^{(r)}$ has a unique fixed-point if and only if there is a unique maximal lottery.

Lemma 2. For $r > 0$, $g^{(r)}$ has a unique fixed-point $p^{(r)}$. Moreover, for every $\delta > 0$, there is r_0 so that $p^{(r)} \in B_\delta(ML(R))$ for all $r \leq r_0$.

Proof. We verify that $g^{(r)}$ maps to Δ . For all $p \in \Delta$, $\sum_{i \in [d]} f_i^{(r)}(p) = 2(1-r)p^\top \tilde{M}p + r \left(1 - \sum_{i \in [d]} p_i\right) = 0$ since \tilde{M} is skew-symmetric and $p \in \Delta$. Moreover,

$$f_i^{(r)}(p) = 2(1-r)p_i \underbrace{(\tilde{M}p)_i}_{\geq -1} + r \left(\frac{1}{d} - p_i\right) \geq -2p_i.$$

Thus,

$$g_i^{(r)}(p) \geq p_i + \frac{1}{2}(-2p_i) \geq 0.$$

It follows that $g^{(r)}$ maps to Δ . Moreover, $g^{(r)}$ is continuous since $f^{(r)}$ is continuous. Hence, by Brouwer's Theorem, $g^{(r)}$ has a fixed point $p^{(r)}$.

Now let $r > 0$. Then, for all $p \in \Delta$ with $f^{(r)}(p) = 0$, we have for all $i \in [d]$, $p_i > 0$ since $p_i = 0$ implies $f_i^{(r)}(p) = r \frac{1}{d} > 0$. Hence, we can rewrite $f^{(r)}(p) = 0$ as follows: for all $i \in [d]$,

$$2(1-r)(\tilde{M}p)_i = r \left(1 - \frac{1}{p_i d}\right) \tag{7}$$

To show that $f^{(r)}$ has a unique zero, assume that $f^{(r)}(p) = f^{(r)}(q) = 0$ for $p, q \in \Delta$. We have

$$\begin{aligned} 0 &= 2(1-r) \left(p^\top \tilde{M}q + q^\top \tilde{M}p\right) \\ &= 2(1-r) \sum_{i \in [d]} p_i (\tilde{M}q)_i + q_i (\tilde{M}p)_i \\ &\stackrel{(7)}{=} r \sum_{i \in [d]} p_i \left(1 - \frac{1}{q_i d}\right) + q_i \left(1 - \frac{1}{p_i d}\right) \\ &= r \left(2 - \frac{1}{d} \sum_{i \in [d]} \frac{p_i}{q_i} + \frac{q_i}{p_i}\right) \\ &= -\frac{r}{d} \sum_{i \in [d]} \frac{(p_i - q_i)^2}{p_i q_i} \leq -\frac{r}{d} |p - q|_2^2 \end{aligned}$$

where the first equality uses the skew-symmetry of \tilde{M} (hence, $p^\top \tilde{M}q = -q^\top \tilde{M}p$), the third equality follows from (7) and the fact that p and q are zeros of $f^{(r)}$, and the last two are algebra. ($|\cdot|_2$ denotes the L^2 -norm.) This sequence of equalities implies that $p = q$. Hence, $p^{(r)}$ is the unique zero of $f^{(r)}$ for $r > 0$. Since every fixed-point of $g^{(r)}$ is a zero of $f^{(r)}$, $g^{(r)}$ has a unique fixed-point.

For the last statement, let $\delta > 0$. By (7), for all $r > 0$ and $i \in [d]$,

$$\left(\tilde{M}p^{(r)}\right)_i = \frac{r}{2(1-r)} \left(1 - \frac{1}{p_i^{(r)}d}\right) \leq \frac{r}{2(1-r)}. \quad (8)$$

Suppose for every $r_0 > 0$, there is $r < r_0$ so that $p^{(r)} \notin B_\delta(ML(R))$. Then we can find a sequence (r_n) going to 0 so that $p^{(r_n)} \notin B_\delta(ML(R))$ for all n . By passing to a subsequence, we may assume that $p^{(r_n)} \rightarrow p \notin B_\delta(ML(R))$. But from (8) it follows that $\tilde{M}p \leq 0$ so that $p \in ML(R)$, which is a contradiction. \square

B. Properties of the Deterministic Process

In this section, we study a deterministic version of the stochastic process described by the Markov chain. We thus have a continuum of balls and continuous time, and show that this process converges to the unique fixed-point identified in the previous section.

Function $f^{(r)}$ defined in Equation (6) gives rise to a (first-order ordinary) differential equation for continuously differentiable functions from $[0, \infty)$ to Δ , that is, functions in $\mathcal{C}^1([0, \infty), \Delta)$. For $y \in \mathcal{C}^1([0, \infty), \Delta)$ and $p_0 \in \Delta$, consider

$$\begin{aligned} \frac{d}{dt}y(t) &= f^{(r)}(y(t)) \\ y(0) &= p_0 \end{aligned} \quad (9)$$

We show that (9) has a unique global solution $y^{(r)}$ for all $r > 0$ and $p_0 \in \Delta$. Moreover, this solution converges to the zero $p^{(r)}$ of $f^{(r)}$ as t goes to infinity. Since r remains fixed throughout this section, we frequently omit the superscript (r) .

The proof that (9) has a unique local solution with values in \mathbb{R}^d is standard. Only the fact that the solution does not leave the domain Δ of f and can thus be extended to a global solution requires attention.

Lemma 3. *For every $p_0 \in \Delta$, (9) has a unique solution $y \in \mathcal{C}^1([0, \infty), \Delta)$ with $y(0) = p_0$.*

Proof. Note that f is Lipschitz-continuous in a neighborhood of Δ . It follows from the Picard-Lindelöf Theorem that for any $t_0 \in [0, \infty)$ and $p \in \Delta$, the system

$$\begin{aligned} \frac{d}{dt}y(t) &= f(y(t)) \\ y(t_0) &= p \end{aligned} \quad (10)$$

has a unique local solution, that is, a solution $y \in \mathcal{C}^1((t_0 - \varepsilon, t_0 + \varepsilon), \mathbb{R}^d)$.

We observe that y maps to Δ . First, by the same arguments as in the proof of Lemma 2, we have

$$\frac{d}{dt} \sum_{i \in [d]} y_i(t) = \sum_{i \in [d]} f_i(y(t)) = 0$$

whenever $y(t) \in \Delta$. Second, if $y_i(t) = 0$, then $\frac{d}{dt}y_i(t) = f_i(y(t)) > 0$. Hence, $y(t) \in \Delta$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. Since $t_0 \in [0, \infty)$ was arbitrary, it follows that y can be uniquely extended to a global solution in $\mathcal{C}^1([0, \infty), \Delta)$. \square

Denote by $y^{(r)}(t, p_0) \in \mathcal{C}^1([0, \infty), \Delta)$ the unique solution to (9) with $y^{(r)}(0, p_0) = p_0$. We will sometimes suppress the argument p_0 when it is clear from the context.

We want to show that if $r > 0$, $y^{(r)}(t, p_0)$ converges to the zero $p^{(r)}$ of $f^{(r)}$ as t goes to infinity. Moreover, the convergence is uniform in p_0 . The proof of this fact in Lemma 5 uses the relative entropy (aka the Kullback–Leibler Divergence) of $p, q \in \Delta$, which is defined as

$$D(p \mid q) = \sum_{i \in [d]} p_i \log \left(\frac{p_i}{q_i} \right).$$

Moreover, the following lower bound on the relative entropy will be helpful (see, e.g., Cover and Thomas, 2006, Lemma 11.6.1).

Lemma 4. *For all $p, q \in \Delta$,*

$$D(p \mid q) \geq \frac{1}{2 \log 2} |p - q|^2.$$

Lemma 5. *Let $r > 0$. Then,*

$$\limsup_{t \rightarrow \infty} \left\{ \left| y^{(r)}(t, p_0) - p^{(r)} \right| : p_0 \in \Delta \right\} = 0.$$

Proof. Fix p_0 in the interior of Δ and write $y = y(\cdot, p_0)$. We show that the entropy of $p^{(r)}$ relative to $y(t)$ decreases at a rate of at least $\frac{r}{d\sqrt{d}} |p^{(r)} - y(t)|_2^2$.

$$\begin{aligned} \frac{d}{dt} D(p^{(r)} \mid y(t)) &= \frac{d}{dt} \sum_{i \in [d]} p_i^{(r)} \log \left(\frac{p_i^{(r)}}{y_i(t)} \right) = - \sum_{i \in [d]} p_i^{(r)} \frac{\frac{d}{dt} y_i(t)}{y_i(t)} \\ &\stackrel{(i)}{=} - \sum_{i \in [d]} p_i^{(r)} \frac{f_i(y(t))}{y_i(t)} \\ &= - \sum_{i \in [d]} p_i^{(r)} \frac{2(1-r)y_i(t)(\tilde{M}y(t))_i + r \left(\frac{1}{d} - y_i(t) \right)}{y_i(t)} \\ &= -2(1-r) \sum_{i \in [d]} p_i^{(r)} (\tilde{M}y(t))_i - r \sum_{i \in [d]} p_i^{(r)} \left(\frac{1}{y_i(t)d} - 1 \right) \\ &\stackrel{(ii)}{=} 2(1-r) \sum_{i \in [d]} y_i(t) (\tilde{M}p^{(r)})_i - r \left(\sum_{i \in [d]} \frac{p_i^{(r)}}{y_i(t)d} - 1 \right) \\ &\stackrel{(iii)}{=} \sum_{i \in [d]} y_i(t) r \left(1 - \frac{1}{p_i^{(r)}d} \right) - r \left(\sum_{i \in [d]} \frac{p_i^{(r)}}{y_i(t)d} - 1 \right) \end{aligned}$$

$$\begin{aligned}
 &= r \left(2 - \frac{1}{d} \sum_{i \in [d]} \frac{y_i(t)}{p_i^{(r)}} + \frac{p_i^{(r)}}{y_i(t)} \right) \\
 &\stackrel{(iv)}{=} -\frac{r}{d} \sum_{i \in [d]} \frac{(p_i^{(r)} - y_i(t))^2}{p_i^{(r)} y_i(t)} \leq -\frac{r}{d\sqrt{d}} |p^{(r)} - y(t)|^2
 \end{aligned}$$

Here, (i) follows from the fact that y satisfies (9), (ii) uses the skew-symmetry of \tilde{M} and $\sum_{i \in [d]} p_i^{(r)} = 1$, (iii) uses (7), and (iv) uses $a^2 + b^2 = (a + b)^2 - 2ab$ for any $a, b \in \mathbb{R}$.

For $t \geq t_0 \geq 0$, we have

$$0 \leq D(p^{(r)} | y(t)) = D(p^{(r)} | y(t_0)) + \int_{t_0}^t \frac{d}{ds} D(p^{(r)} | y(s)) ds \leq D(p^{(r)} | y(t_0)).$$

Combining this with the sequence of equalities above, we see that

$$0 \leq \frac{r}{d} \sum_{i=1}^d \int_{t_0}^t \frac{(p_i^{(r)} - y_i(s))^2}{p_i^{(r)} y_i(s)} ds = - \int_{t_0}^t \frac{d}{ds} D(p^{(r)} | y(s)) ds \leq D(p^{(r)} | y(t_0)). \quad (11)$$

We want to prove that $y(t, p_0)$ converges to $p^{(r)}$ uniformly in p_0 as t goes to ∞ . That is, for all $\varepsilon > 0$, there exists $T > 0$ such that for all $t \geq T$ and all $p_0 \in \Delta$, $|y(t, p_0) - p^{(r)}| < \varepsilon$. To this end, first note that if $y_i(t, p_0) < \frac{r}{4d}$, then

$$\frac{d}{dt} y_i(t, p_0) = 2(1-r)y_i(t, p_0) \underbrace{\left(\tilde{M} y(t, p_0) \right)_i}_{\geq -1} + r \left(\frac{1}{d} - y_i(t, p_0) \right) \geq -\frac{r}{2d} + \frac{r}{d} \geq \frac{r}{2d}.$$

Hence, for all $p_0 \in \Delta$, $i \in [d]$, and $t \geq 1$, $y_i(t, p_0) \geq \frac{r}{4d}$. We can thus upper bound $D(p^{(r)} | y(t, p_0))$ for all $p_0 \in \Delta$ and $t \geq 1$ by $C = \max_{p \in \Delta^r} D(p^{(r)} | p) < \infty$, where $\Delta^r = \{p \in \Delta : p_i \geq \frac{r}{4d} \text{ for all } i \in [d]\}$.

Now we prove uniform convergence in p_0 . Let $\varepsilon > 0$. It follows from (11) with $t_0 = 1$ that given $\delta > 0$, for all $p_0 \in \Delta^r$,

$$\int_{t \geq 1} \bar{\chi}_{B_\delta(p^{(r)})}(y(t, p_0)) dt \leq \frac{Cd\sqrt{d}}{r\delta^2}.$$

Hence, for every $p_0 \in \Delta^r$, we can find $t_0(p_0, \delta) \in [1, 1 + \frac{Cd\sqrt{d}}{r\delta^2}]$ such that

$$\left| y(t_0(p_0, \delta), p_0) - p^{(r)} \right| < \delta. \quad (12)$$

Using the estimate $\log(x - \delta) \geq \log(x) - \frac{\delta}{x - \delta}$ for the last inequality, we find that

$$D(p^{(r)} | y(t_0(p_0, \delta))) \leq \sum_{i \in [d]} \log \left(\frac{p_i^{(r)}}{p_i^{(r)} - \delta} \right) = \sum_{i \in [d]} \log(p_i^{(r)}) - \log(p_i^{(r)} - \delta)$$

$$\leq \sum_{i \in [d]} \frac{\delta}{p_i^{(r)} - \delta} \leq \delta C'$$

where $C' = 2d \max\{\frac{1}{p_i^{(r)}} : i \in [d]\}$ if $\delta \in (0, \frac{1}{2} \min\{p_i^{(r)} : i \in [d]\})$.

We use this bound and the fact that the relative entropy is non-increasing in t to show that $|y(t, p_0) - p^{(r)}| < \varepsilon$ for $t \geq t_0(p_0, \delta)$ for sufficiently small δ . By Lemma 4, we have for all $p \in \Delta$, $D(p^{(r)} | p) \geq \frac{1}{2 \log 2} |p^{(r)} - p|^2$. Hence, $|p^{(r)} - y(t, p_0)| \leq \sqrt{2 \log(2) \delta C'}$ for $t \geq t_0(p_0, \delta)$. Recalling that $t_0(p_0, \delta) \leq 1 + \frac{Cd\sqrt{d}}{r\delta^2} =: T$, we have for $\delta \in (0, \frac{\varepsilon^2}{2 \log(2) C'})$ that $|p^{(r)} - y(t, p_0)| < \varepsilon$ for all $t \geq T$ and $p_0 \in \Delta$. Since ε was arbitrary, this proves uniform convergence. \square

The next lemma states that for any $\delta > 0$, if the process $y^{(r)}$ starts sufficiently close to $p^{(r)}$, it will never get further than δ away from $p^{(r)}$.

Lemma 6. *Let $r > 0$ and $\delta > 0$. Then, there is $\eta > 0$ such that*

$$\sup \left\{ \left| y^{(r)}(t, p) - p^{(r)} \right| : t \geq 0, p \in B_\eta(p^{(r)}) \right\} < \delta$$

Proof. Recall that $p_i^{(r)} > 0$ for all $i \in [d]$. By Lemma 4, if $p \notin B_\delta(p^{(r)})$, then $D(p^{(r)} | p) \geq \frac{1}{2\sqrt{2}} \delta^2 =: C$. Since $D(p^{(r)} | \cdot)$ is continuous on the interior of Δ and $D(p^{(r)} | p^{(r)}) = 0$, there is $\eta > 0$ such that $D(p^{(r)} | p) < C$ for all $p \in B_\eta(p^{(r)})$. In the proof of Lemma 5, we have seen that $D(p^{(r)} | y^{(r)}(t, p))$ is non-increasing in t . Hence, for $p \in B_\eta(p^{(r)})$, it follows that $|y^{(r)}(t, p) - p^{(r)}| < \delta$ for all $t \geq 0$. \square

Summarizing Lemma 2, Lemma 3, and Lemma 5, we get the following theorem.

Theorem 2. *Let $f^{(r)} : \Delta \rightarrow \mathbb{R}^d$ be defined by*

$$f_i^{(r)}(p) = 2(1-r)p_i(\tilde{M}p)_i + r \left(\frac{1}{d} - p_i \right)$$

If $r > 0$, $f^{(r)}$ has a unique zero $p^{(r)}$ and the unique solution $y(t)$ of

$$\begin{aligned} \frac{d}{dt} y(t) &= f^{(r)}(y(t)) \\ y(0) &= p \end{aligned} \tag{5}$$

converges to $p^{(r)}$ as $t \rightarrow \infty$. Moreover, if r goes to 0, then $p^{(r)}$ converges to $ML(R)$.

Remark 6 (Drawing without replacement). For the urn process with drawing without replacement, $f^{(r)}$ as derived in Appendix A would depend on N . The solution $y^{(r)}$ of the differential equation (9) and the unique zero $p^{(r)}$ of $f^{(r)}$ would thus also depend on N . The previous lemmas carry over to this case with the straightforward adaptations.

C. Properties of the Probabilistic Process

In this section, we study the behavior of the Markov chain $X^{(N,r)}$ by exploring its connections to the deterministic process $y^{(r)}$.

We estimate the distance between $X^{(N,r)}$ and the set of maximal lotteries in several steps. First, we choose T_0 large enough so that $y^{(r)}(\cdot, p_0)$ is close to $p^{(r)}$ for all but a small fraction of the time interval $[0, T_0]$ for all initial states p_0 . In Lemma 7, we show that if N is large enough, $X^{(N,r)}$ approximately solves (the integral equation equivalent to) the differential equation (9) with high probability on the interval $[0, T_0]$ for any initial state. From this we conclude in Lemma 8 that for large enough N , $X^{(N,r)}$ is close to $y^{(r)}$ with high probability on any interval of length T_0 , provided both processes start with the same state at the beginning of that interval. Thus, $X^{(N,r)}$ is with high probability approximately equal to $p^{(r)}$ for all but a small fraction of iterations on any interval of length T_0 . Now we chop up the time line into successive intervals of length T_0 . In expectation, $X^{(N,r)}$ stays close to $y^{(r)}$ in a large fraction of these intervals. Using an adaption of the strong law of large numbers, we show in Lemma 11 that $X^{(N,r)}$ is *almost surely* close to $p^{(r)}$ for all but a small fractions of iterations. Lastly, since by Lemma 2, $p^{(r)}$ is close to a maximal lottery if r is small enough, Theorem 1 follows.

The integral equation equivalent to (9) is

$$\begin{aligned} y(t) - y(0) &= \int_0^t f^{(r)}(y(s)) ds \\ y(0) &= p_0 \end{aligned} \tag{13}$$

We show that $X^{(N,r)}$ approximately satisfies (13) (with the integral replaced by a sum) for large N on bounded time intervals. Lemma 7 below states that for any time T and any $\delta > 0$, we can choose N large enough so that with high probability, $X^{(N,r)}(n, p_0)$ does not violate (13) by more than δ within the first NT iterations independently of the initial state $p_0 \in \Delta^{(N)}$. For the proof, we use the following proposition due to Kurtz (1970, Proposition 4.1). (The statement is adapted to our setting.)

Proposition 3 (Kurtz, 1970). *Let $(z^{(N)})_{N \in \mathbb{N}}$ be a sequence of discrete-time Markov chains with states spaces $A^{(N)}$ and probability transition matrices $Q^{(N)}$. Suppose there exist sequences of positive number (α_N) and (ε_N) ,*

$$\lim_{N \rightarrow \infty} \alpha_N = \infty \quad \text{and} \quad \lim_{N \rightarrow \infty} \varepsilon_N = 0$$

such that

$$\sup_{N \in \mathbb{N}} \sup_{p \in A^{(N)}} \alpha_N \sum_{q \in A^{(N)}} |p - q| Q^{(N)}(p, q) < \infty \tag{14}$$

and

$$\lim_{N \rightarrow \infty} \sup_{p \in A^{(N)}} \alpha_N \sum_{q \in A^{(N)}, |p-q| > \varepsilon_N} |p - q| Q^{(N)}(p, q) = 0. \tag{15}$$

Let

$$G^{(N)}(p) = \alpha_N \sum_{q \in A^{(N)}} (q - p) Q^{(N)}(p, q).$$

Then, for every $\delta > 0$ and $T > 0$,

$$\lim_{N \rightarrow \infty} \sup_{p \in A^{(N)}} \mathbb{P} \left(\sup_{n \leq \alpha_n T} \left| z^{(N)}(n) - z^{(N)}(0) - \sum_{k=0}^{n-1} \frac{1}{\alpha_N} G^{(N)}(z^{(N)}(k)) \right| > \delta \mid z^{(N)}(0) = p \right) = 0.$$

The following lemma applies this result to $(X^{(N,r)})_{N \in \mathbb{N}}$ for a fixed r .

Lemma 7. For every $T > 0$ and $\delta > 0$,

$$\lim_{N \rightarrow \infty} \sup_{p \in \Delta^{(N)}} \mathbb{P} \left(\sup_{n \leq NT} \left| X^{(N,r)}(n, p) - X^{(N,r)}(0, p) - \sum_{k=0}^{n-1} f^{(r)}(X^{(N,r)}(k, p)) \right| \geq \delta \right) = 0 \quad (16)$$

Proof. Recall that $P^{(N,r)}$ is the transition probability matrix of $X^{(N,r)}$. We apply Proposition 3 with $z^{(N)} = X^{(N,r)}$, $A^{(N)} = \Delta^{(N)}$, $Q^{(N)} = P^{(N,r)}$, $\alpha_N = N$, and $\varepsilon_N = \frac{2}{N}$ and check (14) and (15):

$$\begin{aligned} & \sup_{N \in \mathbb{N}} \sup_{p \in \Delta^{(N)}} N \sum_{q \in \Delta^{(N)}} |p - q| P^{(N,r)}(Np, Nq) \\ &= \sup_{N \in \mathbb{N}} \sup_{p \in \Delta^{(N)}} N \sum_{i,j=1}^d \frac{1}{N} |e_i - e_j| P^{(N,r)}(Np, Np - e_i + e_j) \leq 2 \end{aligned}$$

and

$$\lim_{N \rightarrow \infty} \sup_{p \in \Delta^{(N)}} N \sum_{q \in \Delta^{(N)}: |p-q| > \frac{2}{N}} |p - q| P^{(N,r)}(Np, Nq) = 0$$

Recalling the definition of $f^{(r)}$ shows that $G^{(N)} = f^{(r)}$ for all N . Hence, (16) follows. \square

Note that Lemma 7 does not use the full strength of Proposition 3 since $G^{(N)} = f^{(r)}$ is independent of N . Recall from Remark 6 that for the urn process without replacement, $f^{(r)}$ does depend on N . Hence, the additional flexibility of Proposition 3 is needed in that case.

Since we want to compare the discrete-time process $X^{(N,r)}$ to the continuous-time process $y^{(r)}$ solving (9), it is convenient to turn $X^{(N,r)}$ into a continuous-time process. To this end, let $\bar{X}^{(N,r)}(t, p) = X^{(N,r)}(\lfloor Nt \rfloor, p)$ for all $t \geq 0$ and $p \in \Delta$. (That is, time is scaled by $\frac{1}{N}$.) $\bar{X}^{(N,r)}$ is a right-continuous step function, which takes steps of length $\frac{1}{N}|e_i - e_j| = \frac{2}{N}$ and is constant on time intervals $[\frac{k}{N}, \frac{k+1}{N})$. Thus, as N grows, the steps become smaller and appear in shorter intervals. Lemma 7 shows that on any bounded

time interval, $\bar{X}^{(N,r)}$ satisfies (13) up to some arbitrary error with high probability when N is large enough. That is, for every $T > 0$ and $\delta > 0$,

$$\lim_{N \rightarrow \infty} \sup_{p \in \Delta^{(N)}} \mathbb{P} \left(\sup_{t \leq T} \left| \bar{X}^{(N,r)}(t, p) - \bar{X}^{(N,r)}(0, p) - \int_0^t f^{(r)} \left(\bar{X}^{(N,r)}(s, p) \right) ds \right| \geq \delta \right) = 0 \quad (17)$$

In Lemma 8, we show that this implies that the trajectories of $y^{(r)}(\cdot, p)$ and $\bar{X}^{(N,r)}(\cdot, p)$ stay close to each other with high probability on a given bounded time interval for any initial state p for large N . Importantly for later use, the bound on the probability is uniform in p .

Lemma 8. *For every $T > 0$ and $\delta > 0$,*

$$\lim_{N \rightarrow \infty} \sup_{p \in \Delta^{(N)}} \mathbb{P} \left(\sup_{t \leq T} \left| y^{(r)}(t, p) - \bar{X}^{(N,r)}(t, p) \right| \geq \delta \right) = 0 \quad (18)$$

Proof. First observe that since $f^{(r)}$ is continuously differentiable on the compact space Δ , there is $C \in \mathbb{R}_{\geq 0}$ such that $f^{(r)}$ is Lipschitz-continuous with constant C . Let $T > 0$, $\delta > 0$, and $p \in \Delta$. If $\sup_{t \leq T} \left| \bar{X}^{(N,r)}(t, p) - \bar{X}^{(N,r)}(0, p) - \int_0^t f^{(r)}(\bar{X}^{(N,r)}(s, p)) ds \right| < \varepsilon$, then for all $t \in [0, T]$,

$$\begin{aligned} \left| y^{(r)}(t, p) - \bar{X}^{(N,r)}(t, p) \right| &= \left| y^{(r)}(t, p) - y^{(r)}(0, p) - \bar{X}^{(N,r)}(t, p) + \bar{X}^{(N,r)}(0, p) \right| \\ &< \varepsilon + \int_0^t \left| f^{(r)} \left(y^{(r)}(s, p) \right) - f^{(r)} \left(\bar{X}^{(N,r)}(s, p) \right) \right| ds \\ &\leq \varepsilon + C \int_0^t \left| y^{(r)}(s, p) - \bar{X}^{(N,r)}(s, p) \right| ds \end{aligned}$$

The first inequality follows from the assumption about $\bar{X}^{(N,r)}$ and the fact that $y^{(r)}$ satisfies (13). The second inequality uses the Lipschitz-continuity of $f^{(r)}$. We apply Grönwall's inequality to conclude that

$$\sup_{t \leq T} \left| y^{(r)}(t, p) - \bar{X}^{(N,r)}(t, p) \right| < \varepsilon e^{CT} < \delta$$

for $\varepsilon > 0$ small enough. Note that the choice of ε does not depend on $p = \bar{X}^{(N,r)}(0, p)$.

By (17), for every $\rho > 0$, we can find $N_0 \in \mathbb{N}$ such that for every $N \geq N_0$,

$$\sup_{p \in \Delta^{(N)}} \mathbb{P} \left(\sup_{t \leq T} \left| \bar{X}^{(N,r)}(t, p) - \bar{X}^{(N,r)}(0, p) - \int_0^t f^{(r)} \left(\bar{X}^{(N,r)}(s, p) \right) ds \right| \geq \varepsilon \right) < \rho$$

Hence, for all $N \geq N_0$,

$$\sup_{p \in \Delta^{(N)}} \mathbb{P} \left(\sup_{t \leq T} \left| y^{(r)}(t, p) - \bar{X}^{(N,r)}(t, p) \right| \geq \delta \right) < \rho$$

Since ρ was arbitrary, (18) follows. \square

The last tool, Lemma 10, is in essence a one-sided strong law of large numbers for indicator random variables. Instead of the usual assumption of i.i.d. random variables, it only assumes that the probability of each variable being 1 is conditionally upper bounded. The proof uses the following auxiliary lemma about binomial distributions.

Lemma 9. *Let $\alpha \in [0, 1]$ and $\{B_n : n \in \mathbb{N}_0\}$ be binomial distributions with $B_n \sim B(n, \alpha)$. Then, for any $\varepsilon > 0$, $\sum_{n \in \mathbb{N}_0} \mathbb{P}\left(\frac{B_n}{n} > \alpha + \varepsilon\right) < \infty$.*

Proof. We use the following tail bound for binomial distributions, which can be obtained from the Chernoff bound (see, e.g., Arratia and Gordon, 1989).

$$\mathbb{P}\left(\frac{B_n}{n} \geq \alpha + \varepsilon\right) \leq e^{-nD(1-\alpha-\varepsilon|1-\alpha)}$$

where $D(\beta | \gamma) = \beta \log \frac{\beta}{\gamma} + (1 - \beta) \log \frac{1-\beta}{1-\gamma}$ is the relative entropy of indicator random variables with success probabilities β and γ . Since the lemma obviously holds if $\alpha \in \{0, 1\}$, we may assume that $0 < \alpha < \alpha + \varepsilon < 1$. Let $C(\varepsilon) = D(1 - \alpha - \varepsilon | 1 - \alpha) \in (0, \infty)$. Then,

$$\sum_{n \in \mathbb{N}_0} \mathbb{P}\left(\frac{B_n}{n} \geq \alpha + \varepsilon\right) \leq \sum_{n \in \mathbb{N}_0} e^{-nC(\varepsilon)} < \infty$$

which is what we needed to show. □

Lemma 10. *Let $\alpha \in [0, 1]$. Let $\{Z_n : n \in \mathbb{N}_0\}$ be indicator random variables and, for $n \geq 1$, $S_n = \sum_{k=1}^n Z_k$. If $\mathbb{P}(Z_1 = 1) \leq \alpha$ and for all $n \geq 2$, $\mathbb{P}(Z_n = 1 | S_{n-1}) \leq \alpha$, then*

$$\mathbb{P}\left(\limsup_{n \rightarrow \infty} \frac{S_n}{n} > \alpha\right) = 0$$

Proof. Let $\{B_n : n \in \mathbb{N}_0\}$ be binomial distributions with $B_n \sim B(n, \alpha)$. We first prove that for all $n \geq 1$ and $l \in \{0, \dots, n\}$,

$$\mathbb{P}(S_n \geq l) \leq \mathbb{P}(B_n \geq l) \tag{19}$$

We proceed by induction over n . For $n = 1$, (19) follows from the assumption that $\mathbb{P}(Z_1 = 1) \leq \alpha$. Now let $n \geq 2$ and assume the statement holds for smaller values of n . Let $l \in \{0, \dots, n\}$. The case $l = 0$ is trivial since then both probabilities are 1. So assume that $l \geq 1$. We distinguish two cases.

Case 1. If $\mathbb{P}(S_{n-1} = l - 1) \leq \mathbb{P}(B_{n-1} = l - 1)$, we get that

$$\begin{aligned} \mathbb{P}(S_n \geq l) &= \mathbb{P}(S_{n-1} \geq l) + \mathbb{P}(S_{n-1} = l - 1) \mathbb{P}(Z_n = 1 | S_{n-1} = l - 1) \\ &\leq \mathbb{P}(B_{n-1} \geq l) + \mathbb{P}(B_{n-1} = l - 1) \alpha \\ &= \mathbb{P}(B_n \geq l) \end{aligned}$$

For the inequality, the bound on the first term follows from the induction hypothesis if $l \leq n - 1$ and is trivial if $l = n$ since $\mathbb{P}(S_{n-1} \geq n) = \mathbb{P}(B_{n-1} \geq n) = 0$. For the second term, we use the assumption of the present case for the first factor and the hypothesis of the lemma for the second factor.

Case 2. If $\mathbb{P}(S_{n-1} = l - 1) \geq \mathbb{P}(B_{n-1} = l - 1)$, we get that

$$\begin{aligned} \mathbb{P}(S_n \geq l) &= \mathbb{P}(S_{n-1} \geq l - 1) - \mathbb{P}(S_{n-1} = l - 1)(1 - \mathbb{P}(Z_n = 1 \mid S_{n-1} = l - 1)) \\ &\leq \mathbb{P}(B_{n-1} \geq l - 1) - \mathbb{P}(B_{n-1} = l - 1)(1 - \alpha) \\ &= \mathbb{P}(B_n \geq l). \end{aligned}$$

The inequality holds by the same arguments as in the first case. The only difference is that we now apply the induction hypothesis at $l - 1$.

This proves (19). We use Lemma 9 to conclude that for every $\varepsilon > 0$,

$$\sum_{n \in \mathbb{N}_0} \mathbb{P}\left(\frac{S_n}{n} \geq \alpha + \varepsilon\right) \leq \sum_{n \in \mathbb{N}_0} \mathbb{P}\left(\frac{B_n}{n} \geq \alpha + \varepsilon\right) < \infty.$$

Thus, it follows from the Borel-Cantelli Lemma that $\mathbb{P}\left(\limsup_{n \rightarrow \infty} \frac{S_n}{n} \geq \alpha + \varepsilon\right) = 0$. Using that $\mathbb{P}(\cdot)$ is countably additive, this gives

$$\mathbb{P}\left(\limsup_{n \rightarrow \infty} \frac{S_n}{n} > \alpha\right) \leq \sum_{m \in \mathbb{N}_0} \mathbb{P}\left(\limsup_{n \rightarrow \infty} \frac{S_n}{n} \geq \alpha + \frac{1}{m}\right) = 0$$

as desired. □

Putting together Lemma 5, Lemma 8, and Lemma 10, we show that $\bar{X}^{(N,r)}$ is almost surely close to $p^{(r)}$ most of the time for large enough N . More precisely, for $S \subset \Delta^{(N)}$, let

$$\bar{s}_t^{(N,r)}(S) = \frac{1}{t} \int_0^t \chi_S(\bar{X}^{(N,r)}(s)) ds$$

be the sojourn time of the states S for the process $\bar{X}^{(N,r)}$. For $\delta > 0$, we consider the sojourn time of the states in $B_\delta(p^{(r)})$. If N is large enough, the limit of $\bar{s}_t^{(N,r)}(B_\delta(p^{(r)}))$ for t to infinity is almost surely close to 1.

Lemma 11. *Let $\delta, \tau > 0$ and $r > 0$. Then, there is N_0 such that for all $N \geq N_0$ and $p_0 \in \Delta^{(N)}$,*

$$\mathbb{P}\left(\lim_{t \rightarrow \infty} \bar{s}_t^{(N,r)}(B_\delta(p^{(r)})) \geq 1 - \tau\right) = 1 \tag{20}$$

Proof. By Lemma 6, we can find $\eta > 0$ such that

$$\sup \left\{ \left| y^{(r)}(t, p) - p^{(r)} \right| : t \geq 0, p \in B_\eta(p^{(r)}) \right\} < \frac{\delta}{2}$$

By Lemma 5, we can find $T_1 > 0$ such that for all $T \geq T_1$,

$$\sup \left\{ \left| y^{(r)}(T, p) - p^{(r)} \right| : p \in \Delta \right\} < \eta$$

Let $T_0 = \frac{2}{\tau}T_1$. Note that $y^{(r)}$ is time-invariant, that is, $y^{(r)}(t, p) = y^{(r)}(t - t_0, y^{(r)}(t_0, p))$ for all $t \geq t_0 \geq 0$. Combining these facts, it follows that for every $p \in \Delta$, the measure of $t \in [t_0, t_0 + T_0]$ for which $y^{(r)}(t, p)$ is in an $\frac{\delta}{2}$ -ball around $p^{(r)}$ is at least $(1 - \frac{\tau}{2})T_0$. We may assume that T_0 is integral.

By Lemma 8, there is $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$,

$$\sup_{p \in \Delta^{(N)}} \mathbb{P} \left(\sup_{0 \leq t \leq T_0} \left| y^{(r)}(t, p) - \bar{X}^{(N,r)}(t, p) \right| \geq \frac{\delta}{2} \right) < \frac{\tau}{2}.$$

Now fix $N \geq N_0$ and $p \in \Delta^{(N)}$. We upper bound the fraction of time $\bar{X}^{(N,r)}$ is further than δ away from $p^{(r)}$. To simplify notation, let $t_k = kT_0$ and $\bar{x}_k = \bar{X}^{(N,r)}(t_k, p)$. For $n \geq 1$, we calculate the expected number of intervals $[t_{k-1}, t_k]$, $1 \leq k \leq n$ so that $|\bar{x}(t, p) - p^{(r)}| \geq \delta$ for some $t \in [t_{k-1}, t_k]$. Let $Z_k^{(N,r)}$ be the indicator variable for the event that $\bar{X}^{(N,r)}(t, p)$ and $y^{(r)}(t - t_{k-1}, \bar{x}_{k-1})$ differ by at least $\frac{\delta}{2}$ on the time interval $[t_{k-1}, t_k]$ given that both start at the point \bar{x}_{k-1} at time t_{k-1} . So $Z_k^{(N,r)}$ is 1 if $\sup_{t_{k-1} \leq t \leq t_k} |\bar{X}^{(N,r)}(t, p) - y^{(r)}(t - t_{k-1}, \bar{x}_{k-1})| \geq \frac{\delta}{2}$, and 0 otherwise. Notice that $\{Z_k^{(N,r)} : k \in \mathbb{N}_0\}$ satisfies the hypothesis of Lemma 10 with $\alpha = \frac{\tau}{2}$.¹⁷

If $Z_k^{(N,r)} = 0$, then

$$\begin{aligned} \int_{t_{k-1}}^{t_k} \bar{\chi}_{B_\delta(p^{(r)})}(\bar{X}^{(N,r)}(t, p)) dt &\leq \int_{t_{k-1}}^{t_k} \bar{\chi}_{B_{\frac{\delta}{2}}(y^{(r)}(t-t_{k-1}, \bar{x}_{k-1}))}(\bar{X}^{(N,r)}(t, p)) dt \\ &\quad + \int_{t_{k-1}}^{t_k} \bar{\chi}_{B_{\frac{\delta}{2}}(p^{(r)})} y^{(r)}(t - t_{k-1}, \bar{x}_{k-1}) dt \\ &\leq \frac{\tau}{2} T_0. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{nT_0} \sum_{k \in [n]} \int_{t_{k-1}}^{t_k} \bar{\chi}_{B_\delta(p^{(r)})}(\bar{X}^{(N,r)}(t, p)) dt &= \sum_{\substack{k \in [n] \\ Z_k^{(N,r)} = 0}} \frac{1}{nT_0} \int_{t_{k-1}}^{t_k} \bar{\chi}_{B_\delta(p^{(r)})}(\bar{X}^{(N,r)}(t, p)) dt \\ &\quad + \sum_{\substack{k \in [n] \\ Z_k^{(N,r)} = 1}} \frac{1}{nT_0} \int_{t_{k-1}}^{t_k} \bar{\chi}_{B_\delta(p^{(r)})}(\bar{X}^{(N,r)}(t, p)) dt \\ &\leq \frac{\tau}{2} + \frac{1}{n} \sum_{k=1}^n Z_k^{(N,r)}. \end{aligned}$$

Applying Lemma 10 to $\{Z_k^{(N,r)} : k \in \mathbb{N}_0\}$ with $\alpha = \frac{\tau}{2}$ gives

$$\mathbb{P} \left(\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n Z_k^{(N,r)} \geq \frac{\tau}{2} \right) = 0.$$

¹⁷While the probability that $Z_n^{(N,r)}$ equals 1 may depend on $Z_k^{(N,r)}$ for $k < n$, the bound of $\frac{\tau}{2}$ holds independently of the $Z_k^{(N,r)}$ since the bound obtained in Lemma 8 is uniform in the initial state p .

Hence, with the preceding inequality we get

$$\mathbb{P} \left(\lim_{t \rightarrow \infty} \bar{s}_t^{(N,r)}(B_\delta(p^{(r)})) \geq 1 - \tau \right) = \mathbb{P} \left(\lim_{t \rightarrow \infty} \int_0^t \chi_{B_\delta(p^{(r)})}(\bar{X}^{(N,r)}) \geq 1 - \tau \right) = 1$$

which is (20). \square

Theorem 1. *Let $\delta, \tau > 0$. Then, there is $r_0 > 0$ such that for all $0 < r \leq r_0$, there is $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ and $p_0 \in \Delta^{(N)}$,*

$$\mathbb{P} \left(\lim_{n \rightarrow \infty} s_n^{(N,r)}(B_\delta(ML(R))) \geq 1 - \tau \right) = 1.$$

Moreover, there is $C > 0$ such that for all $n \in \mathbb{N}$,

$$\mathbb{P} \left(s_n^{(N,r)}(B_\delta(ML(R))) \geq 1 - \tau \right) \geq 1 - e^{-\lfloor Cn \rfloor}.$$

Proof. By definition of $\bar{X}^{(N,r)}$, we can rewrite the conclusion of the theorem as

$$\mathbb{P} \left(\lim_{t \rightarrow \infty} \bar{s}_t^{(N,r)}(B_\delta(ML(R))) \geq 1 - \tau \right) = 1 \tag{21}$$

By Lemma 2, we can choose $r_0 > 0$ so that $p^{(r)} \in B_{\frac{\delta}{2}}(ML(R))$ for all $0 < r \leq r_0$. Applying Lemma 11 to $\frac{\delta}{2}, \tau$, and r , we get N_0 such that (20) holds (with $\frac{\delta}{2}$ in place of δ). Combining these two facts gives (21). The second statement follows from applying Proposition 1 to $\{X^{(N,r)}(n) : n \in \mathbb{N}\}$ and $S' = B_\delta(ML(R))$. \square

D. Rate of Convergence of Sojourn Times

Proposition 1. *Let $\{X(n) : n \in \mathbb{N}\}$ be an irreducible Markov chain with finite state space S and stationary distribution $\pi \in \Delta(S)$. Then, for all $S' \subset S$ and $\varepsilon > 0$, there is $C > 0$ such that for all $n \in \mathbb{N}$,*

$$\mathbb{P} \left(|s_n(S') - \pi(S')| \leq \varepsilon \right) \geq 1 - e^{-\lfloor Cn \rfloor}.$$

Proof. It follows from the ergodic theorem for Markov chains that $s_n(S')$ converges almost surely to $\pi(S')$ as n goes to infinity. Hence, since S is finite, there is $n_0 = n_0(\varepsilon)$ such that

$$\mathbb{P} \left(|s_{n_0}(S') - \pi(S')| \leq \frac{\varepsilon}{3} \mid X(1) \right) \geq 1 - \frac{\varepsilon}{3} \tag{22}$$

For $m, n \in \mathbb{N}_0$, $m < n$, let

$$s_{m,n}(S') = \frac{1}{n - m} \sum_{k=m+1}^n \chi_{S'}(X(k))$$

be the sojourn time of S' in the interval $\{m + 1, \dots, n\}$. Note that $s_{nl} = s_{0,nl} = \frac{1}{l} \sum_{k=1}^l s_{n(k-1),nk}(S')$.

For $l \geq 1$, denote by Z_l the indicator random variable of the event $|s_{n_0(l-1), n_0 l}(S') - \pi(S')| \leq \frac{\varepsilon}{3}$. Moreover, let $S_l = \sum_{k=1}^l Z_k$. By (22) and the fact that $\{X(n) : n \in \mathbb{N}\}$ is time-homogeneous, $\mathbb{P}(Z_1 = 1) \geq 1 - \frac{\varepsilon}{3}$ and for all $l \geq 2$, $\mathbb{P}(Z_l = 1 \mid S_{l-1}) \geq 1 - \frac{\varepsilon}{3}$. Let $\{B_l : l \in \mathbb{N}\}$ be a binomial distribution with $B_l \sim B(l, \frac{\varepsilon}{3})$. In the proof of Lemma 10, we show that for any $k \in \{0, \dots, l\}$,

$$\mathbb{P}(S_l \geq k) \geq \mathbb{P}(B_l \geq k)$$

The estimate in Lemma 9 shows that

$$\mathbb{P}\left(\frac{B_l}{l} \geq 1 - \frac{2}{3}\varepsilon\right) \geq 1 - e^{-D(1 - \frac{2}{3}\varepsilon | 1 - \frac{\varepsilon}{3})l}$$

where $D(1 - \frac{2}{3}\varepsilon \mid 1 - \frac{\varepsilon}{3})$ is the relative entropy of indicator random variables with success probabilities $1 - \frac{2}{3}\varepsilon$ and $1 - \frac{\varepsilon}{3}$. For us it is only relevant that $C_0 = C_0(\varepsilon) = D(1 - \frac{2}{3}\varepsilon \mid 1 - \frac{\varepsilon}{3}) > 0$. Combining the previous assertions, we get

$$\mathbb{P}\left(\frac{S_l}{l} \geq 1 - \frac{2}{3}\varepsilon\right) \geq 1 - e^{-C_0 l}$$

If $\frac{S_l}{l} \geq 1 - \frac{2}{3}\varepsilon$, then

$$|s_{n_0 l}(S) - \pi(S)| = \left| \frac{1}{l} \sum_{k=1}^l (s_{n_0(k-1), n_0 k}(S') - \pi(S')) \right| \leq \frac{1}{l} \left(\frac{\varepsilon}{3} (1 - \frac{2}{3}\varepsilon)l + \frac{2}{3}\varepsilon l \right) \leq \varepsilon$$

Hence, for $n = n_0 l$,

$$\mathbb{P}(|s_n(S') - \pi(S')| \leq \varepsilon) \geq 1 - e^{-\frac{C_0}{n_0} n}.$$

Letting $C = \frac{C_0}{n_0} \varepsilon$ gives the statement for all $n \in \mathbb{N}$. □

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