Bivariate Scoring Rules: Unifying the Characterizations of Positional Scoring Rules and Kemeny's Rule

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This paper studies social preference functions (SPFs), which map the voters' ordinal preferences over a set of alternatives to a non-empty set of strict rankings over the alternatives. Maybe the most prominent SPFs are positional scoring rules and Kemeny's rule. While these two types of rules behave intuitively quite different, they are axiomatically surprisingly similar and we thus provide a joint axiomatic characterization of these SPFs. To this end, we introduce the class of bivariate scoring rules which generalize Kemeny's rule by weighting comparisons between alternatives depending on their positions in the voters' preference relations. In particular, this class contains both Kemeny's rule and all positional scoring rules. As our main result, we then characterize the set of bivariate scoring rules as the only SPFs that satisfy—aside of some standard axioms—mild consistency conditions for variable electorates and variable agendas. As corollaries of this result, we also derive variants of the well-known characterizations of positional scoring rules by Smith (1973) and of Kemeny's rule by Young (1988). Thus, our result unifies the independent streams of research on positional scoring rules and Kemeny's rule by giving a joint axiomatic basis of these SPFs.

1. Introduction

One of the most classic problems in social choice theory is ranking aggregation: given the voters' strict preferences over a set of alternatives, a collective preference relation should be found. Indeed, even Arrow's impossibility, which is commonly considered as the starting point of modern social choice theory, has originally been stated in terms of ranking aggregation (Arrow, 1951). Moreover, the problem of ranking aggregation has found its way to numerous fields, such as meta search (Dwork et al., 2001; Renda and Straccia, 2003), computational biology (Lin, 2010; Kolde et al., 2012), engineering

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(Gaertner, 2019), and machine learning (Prati, 2012; Sarkar et al., 2014). For instance, in machine learning, ranking aggregation is used to combine rankings produced by different machine learning algorithms into a final ranking to make the outcome more robust.

In social choice theory, the problem of ranking aggregation is formalized by *social preference functions (SPFs)* which compute a non-empty set of strict winning rankings based on the voters' preferences.[1] In particular, the literature mainly focuses on two types of SPFs: positional scoring rules (e.g., Smith, 1973; Young, 1975; Myerson, 1995) and Kemeny's rule (e.g., Young and Levenglick, 1978; Barthelemy and Monjardet, 1981; Can and Storcken, 2013). In a positional scoring rule, each voter assigns a fixed score to each alternative depending on its position in the voter's preference relation and the output rankings arrange the alternatives in decreasing order of their total scores. An example of a positional scoring rules is Borda's rule where voters give $m-1$ points to their favorite alternative, $m-2$ points to their second best alternative, and so on. By contrast, in Kemeny's rule, voters give points to pairs of alternatives: a voter gives 1 point to a pair (a, b) if he prefers a to b and -1 point if he prefers b to a. Kemeny's rule then views each ranking $\succ$ as a set of ordered pairs and the score of a ranking is the sum of scores assigned to the pairs $(a, b) \in \succ$. Finally, the chosen rankings are those with maximal scores.

Intuitively, Kemeny's rule and scoring rules have a rather diametrical behavior. On the one hand, scoring rules are closely related to preference intensities: for example, Borda's rule assumes that a voter's preference between his first-ranked and last-ranked alternatives is much stronger than his preference between his first-ranked and second-ranked alternatives. This interpretation follows by simply considering the scores assigned to the alternatives by Borda's rule. On the other hand, Kemeny's rule completely ignores such preference intensities as each voter gives 1 point to a pair of alternatives (a, b) if he prefers a to b and -1 if he prefers b to a, independently of how close a and b are in the voters' preference relation. Indeed, Kemeny's rule can be seen as a natural extension of the majority rule (given the choice between two alternatives, pick the one that is preferred by a majority of the voters) to ranking aggregation. This has even been formally observed by Young and Levenglick (1978) who have characterized Kemeny's rule using a variant of Condorcet-consistency for rankings. By contrast, scoring rules inherently conflict with the concept of majority decisions (Fishburn and Gehrlein, 1976).

Despite these intuitive differences, positional scoring rules and Kemeny's rule are rather similar from an axiomatic perspective. This is, e.g., demonstrated by the characterization of positional scoring rules by Smith (1973) and the characterization of Kemeny's rule by Young (1988): both of these results are driven by mild fairness axioms and a consistency condition for variable electorates. In this paper, we thus aim to unify these two classical results by giving a joint axiomatic foundation of positional scoring rules and Kemeny's rule.

[1] Another approach to study ranking aggregation are social welfare functions (SWFs) which return a single weak ranking over the alternatives (see, e.g., Smith (1973) or Campbell and Kelly (2002)). However, every SWF can be turned into an SPF by returning all strict rankings that can be derived from the weak ranking by tie-breaking. By contrast, not every SPF can be turned into an SWF.

Our contribution. To derive the joint axiomatic basis of positional scoring rules and Kemeny's rule, we will introduce and fully axiomatize a new class of SPFs called bivariate scoring rules. For these rules, voters assign—just as for Kemeny's rule—scores to each pair of alternatives (a, b), but the exact score depends—analogous to positional scoring rules—on the positions of a and b in the voters' preference relations. The score of an output ranking $\rhd$ are then the sum of the scores assigned to pairs (a, b) with $(a, b) \in \rhd$ and the rule chooses the rankings with maximal total scores. In particular, all positional scoring rules and Kemeny's rule are bivariate scoring rules, so this class allows for a joint treatment of these SPFs. Moreover, bivariate scoring rules can be seen as variants of Kemeny's rule that incorporate preference intensities as it is, e.g., possible to assign scores depending on the distance between alternatives. Such generalizations of Kemeny's rule have attracted significant attention (e.g., Cook and Kress, 1986; Kumar and Vassilvitskii, 2010; Can, 2014; Plaia et al., 2019) as it is a common observation that, e.g., the first positions in a input ranking are more important than the later ones.

For our characterization of bivariate scoring rules, we rely (aside of mild standard axioms) on consistency conditions for both variable electorates and variable agendas of alternatives. In more detail, we use the prominent notion of reinforcement as variable electorate condition, which requires that if an SPF chooses some rankings for two disjoint elections, then precisely these common winning rankings are chosen in a combined election. Variants of this axiom feature in numerous prominent characterizations (e.g., Smith, 1973; Fishburn, 1978; Young and Levenglick, 1978; Brandl et al., 2016; Lackner and Skowron, 2021). For consistency with respect to variable agendas, we use a new axiom which we call local agenda consistency. Intuitively, this axiom requires that if a ranking $\rhd$ is chosen for a feasible set Y, then it holds for every set of *consecutive* alternatives X in $\rhd$ that $\rhd$ restricted to X is a winning ranking for X. If this was not the case, there is a better ranking for the set X and we can improve the ranking $\rhd$ by reordering the alternatives in X according to this ranking because the alternatives in X appear consecutively in $\rhd$. As our main result, we then show that *an SPF is a bivariate scoring rule if and only if it satisfies reinforcement, local agenda consistency, anonymity, neutrality, continuity, and faithfulness (Theorem 1).*

Based on this result, we furthermore infer variants of the prominent characterizations of positional scoring rules by Smith (1973) and of Kemeny's rule by Young (1988), which demonstrates that our characterization of bivariate scoring rules indeed unifies these two independent lines of research. In more detail, for our characterization of positional scoring rules, we strengthen local agenda consistency to agenda consistency. This axiom requires that the winning rankings for a smaller feasible set are derived from those of a larger feasible set by deleting the unnecessary alternatives from the winning ranking. We then show that *an SPF is a positional scoring rule if and only if it satisfies reinforcement, agenda consistency, anonymity, neutrality, continuity, and faithfulness (Corollary 1).* This result is in spirit of the prominent characterization of positional scoring rules by Smith (1973) because the conjunction of agenda consistency and reinforcement has similar consequences as separability, the main axiom of Smith (see Remark 5 for details). However, while Smith (1973) proves his result for social welfare functions, our result concerns the more general setting of SPFs.

As second corollary, we characterize Kemeny's rule as the only bivariate scoring rule that satisfies independence of infeasible alternatives.[2] This axiom requires that the chosen rankings only depend on the alternatives in the feasible set (see, e.g., Arrow (1951) or Sen (2017), Chapter 3*). In more detail, we prove that *an SPF is Kemeny's rule if and only if it satisfies reinforcement, local agenda consistency, independence of infeasible alternatives, anonymity, neutrality, continuity, and faithfulness (Corollary 2).* This result is similar to the characterization of Kemeny's rule by Young (1988) who uses mostly the same axioms but another variable agenda condition called pairwise consistency (we refer to Remark 6 for details).[3,4] This axiom implies both local agenda consistency and independence of infeasible alternatives when restricting the (smaller) set to size 2, which turns out sufficient for our proof. Since pairwise consistency also encompasses a third, more technical condition, our characterization of Kemeny's rule seems more attractive than the one by Young (1988). Finally, we note that Corollaries 1 and 2 emphasize the differences between positional scoring rules and Kemeny's rule: while the former SPFs satisfy stronger agenda consistency conditions, Kemeny's rule is additionally independent of infeasible alternatives.

Related Work. Rank aggregation is one of the most classic problems in social choice theory and it is therefore not surprising that various SPFs have been suggested in the literature, with the most prominent ones including Kemeny's rule (Kemeny, 1959), positional scoring rules (Smith, 1973), various types of runoff scoring rules (Smith, 1973; Boehmer et al., 2023), Slater's rule (Slater, 1961), and the ranked pairs method (Tideman, 1987). In this paper, we will focus on positional scoring rules and Kemeny's rule; for an overview of other rules, we refer to Arrow et al. (2002) and Brandt et al. (2016).

Both Kemeny's rule and positional scoring rules have attracted significant attention and multiple characterizations are known for both types of rules. In more detail, Young and Levenglick (1978), Young (1988), and Can and Storcken (2013) prove characterizations of Kemeny's rule. For instance, Young and Levenglick (1978) characterize Kemeny's rule based on neutrality, reinforcement, and a non-standard notion of Condorcet-consistency (see Remark 7). Moreover, Bossert and Sprumont (2014) study Kemeny's rule with respect to manipulability and several authors geometrically compared Kemeny's rule to other SPFs (Saari and Merlin, 2000; Ratliff, 2001; Klamler, 2004), all of whom conclude that Kemeny's rule performs outstandingly from an axiomatic perspec-

[2] In light of Arrow's impossibility theorem, it may sound surprising that Kemeny's rule (or any other attractive SPF) satisfies independence of infeasible alternatives. The reason why this claim holds is that we use, in contrast to Arrow's impossibility, a variable agenda framework. In this case, independence of infeasible alternatives does not impose any consistency conditions between different feasible sets and is therefore easy to satisfy. Indeed, when using a variable agenda framework, it is well-known that independence of infeasible alternatives only becomes prohibitive when combined with agenda consistency notions (see, e.g., Bordes, 1976; Sen, 1977).

[3] Young (1988) calls this concept first local stability and introduces the term pairwise consistency later on (Young, 1994, Theorem 6). Furthermore, pairwise consistency is sometimes called local independence of irrelevant alternatives (e.g., Young, 1995; Boehmer et al., 2023). We choose to call this condition pairwise consistency to avoid confusion with our other axioms.

[4] Young (1988, 1994) does not provide a formal proof of this characterization but only an intuitive argument why the claim is true. So, our proof can be seen as first formal verification of this result.

tive. On the negative side, it is computationally intractable to compute the winning rankings for Kemeny's rule (Bartholdi, III et al., 1989; Hemaspaandra et al., 2005).

The picture for positional scoring rules is similar as there is a multitude of characterizations of this class (Smith, 1973; Young, 1974; Myerson, 1995) and also of specific positional scoring rules such as Borda's rule (Young, 1974; Nitzan and Rubinstein, 1981) or the plurality rule (Richelson, 1978; Ching, 1996); we refer to Chebotarev and Shamis (1998) for a more detailed overview of this extensive stream of research. Remarkably, all these characterizations crucially rely on variants of reinforcement as positional scoring rules satisfy this condition—in contrast to Kemeny's rule—also as social choice functions (which return a set of winning alternatives rather than a set of winning rankings). Moreover, there is also a vast number of results on specific aspects of scoring rules, such as their manipulability (Pritchard and Wilson, 2007; Favardin and Lepelley, 2006), their welfare guarantees (Pivato, 2016; Skowron and Elkind, 2017), or their probability to elect the Condorcet winner (Gehrlein, 1982; Cervone et al., 2005).

In this paper, we will derive a joint characterization of Kemeny's rule and positional scoring rules, thus unifying all of the above results. We note that there are already several classes of voting rules that could be used for this purpose, such as the mean neat rules of Zwicker (2008), the simple scoring ranking rules of Conitzer et al. (2009), or the distance-based voting rules studied in the context of distance rationalizability (see Elkind and Slinko, 2016). However, these classes are too general to allow for an appealing characterization. For an example, consider the simple scoring ranking rules by Conitzer et al. (2009), which are defined by a scoring function $s(\succ, \rhd)$ that states how many points a voter with preference relation $\succ$ assigns to each output ranking $\rhd$ and simply choose the rankings with maximal total score. While this class is easy to understand, it is extremely general and thus very challenging to axiomatize. By contrast, by focusing on the smaller class of bivariate scoring rules, we can find an appealing characterization.

Finally, our results are conceptually related to numerous prominent results in social choice theory because variants of reinforcement and (local) agenda consistency have frequently been studied. For instance, reinforcement has been used to characterize positional scoring rules in the context of single winner elections (Young, 1974; Myerson, 1995; Pivato, 2013), approval voting (Fishburn, 1978; Brandl and Peters, 2022), and committee voting (Skowron et al., 2019; Lackner and Skowron, 2021; Dong and Lederer, 2023). Moreover, Brandl et al. (2016) use axioms similar to reinforcement and agenda consistency to characterize a randomized voting rule called maximal lotteries. The study of agenda consistency notions in social choice theory goes back to Sen (1971, 1977) and has since then attracted significant attention (e.g., Tideman, 1987; Laffond et al., 1996; Laslier, 1997). Hence, we employ classical ideas to derive new characterizations.

2. The Model

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote an infinite set of voters and $A = \{a_1, \dots, a_m\}$ denote a set of m alternatives. We employ in this paper both a variable agenda and a variable population framework. To this end, we define $\mathcal{F}(X) = \{Y \subseteq X \colon Y \text{ is non-empty and finite}\}$ as the set of all finite and non-empty subsets of a given set X. In particular, we interpret $\mathcal{F}(\mathbb{N})$ as the set of possible electorates, whereas $\mathbb{N}$ is the set of all possible voters. Similarly, $\mathcal{F}(A)$ is the set of all possible agendas (or feasible sets) of alternatives. Intuitively, a set $X \in \mathcal{F}(A)$ contains the alternatives that need to be ranked by our voting rules.

Given an electorate $N \in \mathcal{F}(\mathbb{N})$, each voter $i \in N$ is assumed to report a *preference relation* $\succ_i$ on A, which is formally a strict total order on A. We call preference relations also *rankings* and typically write these objects as comma separated lists; e.g., $\succ_i = (a, b, c)$ means that voter i prefers a to b to c. We will omit the brackets around preference lists whenever this helps readability. The set of all rankings over a set of alternatives X is $\mathcal{R}(X)$. Moreover, $\succ|_X$ denotes the restriction of a ranking $\succ \in \mathcal{R}(Y)$ to the alternatives in $X \subseteq Y$, i.e., $\succ|_X = \succ \cap X^2$. Finally, we call a set of alternatives X *consecutive* in a ranking $\succ \in \mathcal{R}(Y)$ if $x \succ z$ if and only if $y \succ z$ for all $x, y \in X$, $z \in Y \setminus X$ and define $C(X, Y) = \{\succ \in \mathcal{R}(Y) \colon X \text{ is consecutive in } \succ\}$ as the set of rankings on Y in which the alternatives in X appear consecutively.

A *preference profile* R for an electorate $N \in \mathcal{F}(\mathbb{N})$ is a mapping from N to $\mathcal{R}(A)$, i.e., it contains the preference relation of every voter $i \in N$. Conversely, N_R denotes the set of voters that report a preference relation in R. We note that, even though we allow for a variable agendas, preference profiles are always defined on all alternatives. When writing preference profiles, the number before a preference relation indicates how many voters report a given preference relation; for instance, $3\colon a, b, c$ means that three voters prefer a to b to c. Next, we define $\mathcal{R}^*$ as the set of all possible preference profiles: $\mathcal{R}^* = \bigcup_{N \in \mathcal{F}(\mathbb{N})} \mathcal{R}^N$. Furthermore, we call two profiles $R, R' \in \mathcal{R}^*$ disjoint if $N_R \cap N_{R'} = \emptyset$ and define the profile $R'' = R + R'$ by $\succ''_i = \succ_i$ if $i \in N_R$ and $\succ''_i = \succ'_i$ if $i \in N_{R'}$.

In this paper, we aim to study *social preference functions (SPFs)* which choose a non-empty set of winning rankings $W \subseteq \mathcal{R}(X)$ for each preference profile $R \in \mathcal{R}^*$ and each feasible set of alternatives $X \in \mathcal{F}(A)$. More formally, an SPF is a function f of the type $\mathcal{R}^* \times \mathcal{F}(A) \to \bigcup_{X \in \mathcal{F}(A)} \mathcal{F}(\mathcal{R}(X))$ such that $f(R, X) \subseteq \mathcal{R}(X)$ for all profiles $R \in \mathcal{R}^*$ and feasible sets $X \in \mathcal{F}(A)$. While SPFs are also defined for feasible sets X of size 1, we will ignore this case as there is only a single ranking if $|X| = 1$. The same model has implicitly been considered by, e.g., Young (1988) and Boehmer et al. (2023). Finally, to clearly distinguish between the preference relations of voters and the winning rankings chosen by an SPF, we use $\succ$ for the former and $\rhd$ for the latter.

2.1. Bivariate Scoring Rules

In this paper, we focus on three types of SPFs: positional scoring rules, Kemeny's rule, and our new class of bivariate scoring rules. An example illustrating these different rules can be found in Figure 1.

Positional Scoring Rules. We first introduce positional scoring rules and define to this end the rank of an alternative x in a preference relation $\succ_i$ as $r(\succ_i, x) = 1 + \{y \in A \setminus \{x\} \colon y \succ_i x\}$. Less formally, $r(\succ_i, x) = k$ means that x is the k-th best alternative of voter i. Each positional scoring rules is defined by a positional scoring function $s : \{1, \dots, m\} \to \mathbb{R}$ which is non-increasing and non-constant, i.e., it holds that

$s(1) \geq s(2) \geq \cdots \geq s(m)$ and $s(1) > s(m)$.[5] Intuitively, $s(i)$ states how many points a voter assigns to his i-th best alternative. Next, we define the total score of an alternative x in a profile R as $\hat{s}(R,x) = \sum_{i \in N_R} s(r(\succ_i, x))$. Finally, an SPF f is a *positional scoring rule* if there is a positional scoring function s such that $f(R,X) = \{\rhd \in \mathcal{R}(X) \colon \forall x, y \in X \colon x \rhd y \implies \hat{s}(R,x) \geq \hat{s}(R,y)\}$ for all profiles $R \in \mathcal{R}^*$ and all feasible sets $X \in \mathcal{F}(A)$. We note that positional scoring rules are defined in a "broad" sense: we always compute the scores with respect to the full profile R on A, even if some alternative are not in the feasible set. Common examples of positional scoring rules are Borda's rule (defined by $s(x) = m - x$) or the Plurality rule (defined by $s(1) = 1$ and $s(x) = 0$ for $x > 1$).

Kemeny's Rule. Next, we turn to Kemeny's rule which interprets rankings $\rhd \in \mathcal{R}(X)$ as a set of ordered pairs of alternatives, i.e., $\rhd = \{(x,y) \in X^2 \colon x \rhd y\}$. Formally, Kemeny's rule is also defined by a scoring function s which takes two alternatives x, y and a preference relation $\succ_i$ as input: $s(x,y,\succ_i) = 1$ if $x \succ_i y$ and $s(x,y,\succ_i) = -1$ if $y \succ_i x$. The Kemeny score of a ranking $\rhd \in \mathcal{R}(X)$ in a profile R is then defined as $\hat{s}_{Kemeny}(R, \rhd) = \sum_{(x,y) \in \rhd} \sum_{i \in N_R} s(x,y,\succ_i)$ and Kemeny's rule chooses the rankings with maximal Kemeny score, i.e., $f_{Kemeny}(R,X) = \{\rhd \in \mathcal{R}(X) \colon \forall \rhd' \in \mathcal{R}(X) \colon \hat{s}_{Kemeny}(R,\rhd) \geq \hat{s}_{Kemeny}(R,\rhd')\}$ for all profiles $R \in \mathcal{R}^*$ and feasible sets of alternatives $X \in \mathcal{F}(A)$. This definition is equivalent to choosing the rankings that minimize the total swap distance to the voters' preference relations. We note that Kemeny's rule reduces to the majority rule for agendas of size 2: $f_{Kemeny}(R, \{x,y\}) = \{(x,y)\}$ if a strict majority of voters prefers x to y, $f_{Kemeny}(R, \{x,y\}) = \{(y,x)\}$ if a strict majority prefers y to x, and $f_{Kemeny}(R, \{x,y\}) = \{(x,y),(y,x)\}$ otherwise.

Bivariate Scoring Rules. Finally, we introduce the class of bivariate scoring rule. To this end, we observe that Kemeny's rule can also be defined based on a scoring function s that only takes the ranks of two alternatives as input: $s(r(\succ_i,x), r(\succ_i,y)) = 1$ if $r(\succ_i,x) < r(\succ_i,y)$ and $s(r(\succ_i,x), r(\succ_i,y)) = -1$ if $r(\succ_i,x) > r(\succ_i,y)$. The idea for bivariate scoring rules is now to introduce weights in this function. To this end, we introduce bivariate scoring functions which are functions of the type $s : \{1,\dots,m\} \times \{1,\dots,m\} \to \mathbb{R}$ that satisfy $s(\ell,k) \geq 0$ and $s(\ell,k) = -s(k,\ell)$ for all $\ell, k \in \{1,\dots,m\}$ with $\ell < k$ and $s(\ell,k) > 0$ for some $\ell, k \in \{1,\dots,m\}$. Intuitively, a bivariate scoring function quantifies how important it is for a voter that his ℓ-th ranked alternative is ranked over his k-th ranked alternative in the output ranking. We thus define the score of a pair of alternatives (x,y) in a profile R as $\hat{s}(R,(x,y)) = \sum_{i \in N_R} s(r(\succ_i,x), r(\succ_i,y))$ and the score of a ranking $\rhd \in \mathcal{R}(X)$ as $\hat{s}(R,\rhd) = \sum_{(x,y) \in \rhd} \hat{s}(R,(x,y))$. Finally, an SPF f is a *bivariate scoring rule* if there is a bivariate scoring function s such that $f(R,X) = \{\rhd \in \mathcal{R}(X) \colon \forall \rhd' \in \mathcal{R}(X) \colon \hat{s}(R,\rhd) \geq \hat{s}(R,\rhd')\}$ for all $R \in \mathcal{R}^*$ and feasible sets of alternatives $X \in \mathcal{F}(X)$. More intuitively, bivariate scoring rules allow the election

[5] Positional scoring rules can be defined without these assumptions. However, if a positional scoring function s fails to be non-increasing, the corresponding positional scoring rule violates several desirable properties. Thus, numerous authors assume that the score vector is non-decreasing (e.g., Gehrlein, 1982; Pritchard and Wilson, 2007; Favardin and Lepelley, 2006; Pivato, 2016; Skowron and Elkind, 2017). The condition that s is non-constant only entails that the corresponding positional scoring rule does not always choose every ranking.

$$R: \begin{array}{ll} 3: & a,b,c,d \\ 2: & b,c,d,a \\ 2: & c,d,a,b \end{array} \qquad \begin{array}{c} f_{Borda}(R,\{a,b,c\}) = \{(c,b,a)\} \\ f_{Kemeny}(R,\{a,b,c\} = \{(a,b,c)\} \\ f_d(R,\{a,b,c\}) = \{(c,b,a)\} \end{array}$$

Figure 1: Examples of different SPFs. The left hand side depicts the profile R which consists of 7 voters and 4 alternatives. On the right side, we display the chosen rankings by Borda's rule, Kemeny's rule, and the bivariate scoring rule f_d defined by $s(i,j) = -(i-j)^3$ for the feasible set $\{a,b,c\}$. It can be verified that Borda's rule assigns 11 points to a, 12 points to b, and 13 points to c (and 6 points to d), so it chooses the ranking c,b,a. On the other hand, 5 voters prefer a to b, 5 voters prefer b to c, and 4 voters prefer c to a. Consequently, the ranking (a,b,c) maximizes the Kemeny score with a value of $5-2+5-2+4-3=7$ and is therefore elected by Kemeny's rule. Finally, for computing f_d, we go over every voter and each pair of alternatives to compute the scores. For instance, each voter with the preference relation b,c,d,a gives 1 point to the pair (b,c), 4 points to the pair (c,a), and 27 points to the pair (b,a) (or conversely -1 point to (c,b), -4 points to (a,c), and -27 points to (a,b)). By continuing these computations, one can verify that c,b,a maximizes the total scores as the pairs (c,b) and (b,a) both get $5\cdot(-1)+2\cdot 27 = 49$ points.

designer to value each comparison between two alternatives x,y in a voters' preference relation differently depending on the position of x and y. Thus, these SPFs can be seen as weighted variants of Kemeny's rule. Finally, we note that the assumption that $s(\ell,k) = -s(k,\ell)$ is without loss of generality because the resulting bivariate scoring rule is independent of additive shifts for each pair of indices.

The class of bivariate scoring rules contains a variety of interesting SPFs, with Kemeny's rule constituting the most apparent example. Moreover, bivariate scoring rules are a very flexible concept, and one can, for instance, also define "distance-based" rules whose bivariate scoring function $s(\ell,k)$ only depends on $\ell-k$. For instance, the rule f_d defined by $s(\ell,k) = -(\ell-k)^3$ in Figure 1 is such a rule and such SPFs have been studied before (e.g., Drissi-Bakhkhat and Truchon, 2004). Finally, we will next show that also all positional scoring rules are bivariate scoring rules.

Proposition 1. *Every positional scoring rule is a bivariate scoring rule.*

Proof. Consider an arbitrary positional scoring rule f and let s denote its positional scoring function. We define a bivariate scoring function s' by $s'(\ell,k) = s(\ell) - s(k)$ for all $\ell,k \in \{1,\dots,m\}$ and let f' denote the corresponding bivariate scoring rule. First, we note that s' is indeed a bivariate scoring function since $s(1) \geq s(2) \geq \dots \geq s(m)$ and $s(1) > s(m)$, so $s(\ell,k) \geq 0$ if $\ell < k$ and $s(1,m) > 0$. Our goal is to show that $f(R,X) = f(R',X)$ for all profiles $R \in \mathcal{R}^*$ and feasible sets $X \in \mathcal{F}(A)$. To this end, we note that $\hat{s}'(R,(x,y)) = \sum_{i\in N_R} s(r(\succ_i,x)) - s(r(\succ_i,y)) = \hat{s}(R,x) - \hat{s}(R,y)$ for all profiles $R \in \mathcal{R}^*$ and alternatives $x,y \in A$. Hence, it holds for every ranking $\rhd \in f'(R,X)$ and all alternatives $x,y \in X$ that $\hat{s}(R,x) \geq \hat{s}(R,y)$ if x is placed directly above y in

$\rhd$. Otherwise, the ranking $\rhd'$ derived from $\rhd$ by swapping x and y has a higher score than $\rhd$ since $\hat{s}'(R, \rhd') = \hat{s}'(R, \rhd) - \hat{s}'(R, (x, y)) + \hat{s}'(R, (y, x)) = \hat{s}'(R, \rhd) + 2\hat{s}(R, y) - 2\hat{s}(R, x) > \hat{s}'(R, \rhd)$. This contradicts the assumption that $\rhd \in f'(R, X)$, so it holds for all rankings $\rhd \in f'(R, X)$ that $\hat{s}(R, x) \geq \hat{s}(R, y)$ if $x \rhd y$ and x, y are consecutive in $\rhd$. Even more, if there is a non-consecutive pair of alternatives $x, y \in X$ in $\rhd$ such that $\hat{s}(R, x) < \hat{s}(R, y)$ and $x \rhd y$, we can also find a consecutive pair that satisfies these conditions. Hence, we infer that $f'(R, X) \subseteq f(R, X)$. For the converse direction, we note that if $\hat{s}(R, x) = \hat{s}(R, y)$ for two alternatives $x, y \in X$ that are consecutive in a ranking $\rhd$, then $\hat{s}'(R, \rhd) = \hat{s}'(R, \rhd')$ for the ranking $\rhd'$ derived from $\rhd$ by swapping x and y. Hence, $\rhd \in f'(R, X)$ also implies that $\rhd' \in f'(R, X)$ for these rankings. Since we already know that $f'(R, X) \subseteq f(R, X)$ and the rankings in $f(R, X)$ only differ by reordering alternatives $x, y \in X$ with $\hat{s}(R, x) = \hat{s}(R, y)$, it follows that $f(R, X) = f'(R, X)$. $\square$

2.2. Basic Axioms

We next introduce five standard axioms which will form the basis of our analysis. We note that these axioms can be used to characterize scoring rules in various contexts (Young, 1975; Myerson, 1995; Skowron et al., 2019; Lackner and Skowron, 2022). By contrast, this is not the case for SPFs as, e.g., also scoring runoff rules satisfy all axioms listed in this section.

Anonymity. Anonymity is a mild fairness condition that postulates that voters are treated equally. In more detail, an SPF f is *anonymous* if $f(R, X) = f(\pi(R), X)$ for all profiles $R \in \mathcal{R}^*$, feasible sets $X \in \mathcal{F}(A)$, and permutations $\pi : \mathbb{N} \to \mathbb{N}$. Here, $R' = \pi(R)$ denotes the profile such that $N_{R'} = \{\pi(i) \colon i \in N_R\}$ and $\succ'_{\pi(i)} = \succ_i$ for all $i \in N_R$.

Neutrality. Similar to anonymity, neutrality is a standard fairness condition for alternatives. Formally, an SPF f is *neutral* if $f(\tau(R), \tau(X)) = \{\tau(\rhd) \colon \rhd \in f(R, X)\}$ for all profiles $R \in \mathcal{R}^*$, feasible sets $X \in \mathcal{F}(A)$, and permutations $\tau : A \to A$. Here, $\rhd' = \tau(\rhd)$ is the ranking defined by $\tau(x) \rhd' \tau(y)$ if and only if $x \rhd y$ for all $x, y \in A$, and $R' = \tau(R)$ is the profile with $N_{R'} = N_R$, and $\succ'_i = \tau(\succ_i)$ for all $i \in N_R$.

Faithfulness. Faithfulness requires that SPFs should respect the voters' preferences if there is only a single voter and two alternatives in the feasible set. In more detail, an SPF f is *faithful* if for all alternatives $x, y \in A$, it holds that $(x, y) \in f(\succ_i, \{x, y\})$ for all rankings $\succ_i \in \mathcal{R}(A)$ with $x \succ_i y$ and $f(\succ_i, \{x, y\}) = \{(x, y)\}$ for some ranking $\succ_i$ with $x \succ_i y$. This condition is, for instance, weaker Pareto-optimality (which requires that if $x \succ_i y$ for all voters i in some profile R, then $x \rhd y$ for all $\rhd \in f(R, X)$).

Continuity. Continuity, also known as the overwhelming majority property (Myerson, 1995) or the Archimedean property (Smith, 1973), roughly states that large groups of voters can ensure that some of their desired outcomes are chosen. More formally, an SPF f is called *continuous* if for all profiles $R, R' \in \mathcal{R}^*$ and feasible sets $X \in \mathcal{F}(A)$, there is $\lambda \in \mathbb{N}$ such that $f(\lambda R + R', X) \subseteq f(R, X)$. Here, the profile λR consists of λ copies of R; the names of the voters in λR will not matter due to anonymity. We note that all axioms defined so far are very mild and satisfied by all commonly studied SPFs.

Reinforcement. As the last axiom in this section, we introduce reinforcement. Intuitively, this axiom requires that if some rankings are chosen for two disjoint elections, then precisely the common winning rankings should be chosen in a joint election. More formally, an SPF f is *reinforcing* if $f(R, X) \cap f(R', X) \neq \emptyset$ implies that $f(R + R', X) = f(R, X) \cap f(R', X)$ for all disjoint profiles $R, R' \in \mathcal{R}^*$ and feasible sets $X \in \mathcal{F}(A)$. The idea here is that a winning ranking is as good as any other winning ranking and better than any unchosen ranking. Thus, if some rankings are chosen for two disjoint elections, these are precisely the best rankings for the combined election.

2.3. Agenda consistency

As the last ingredient for our results, we will introduce consistency conditions for variable agendas. Roughly, these axioms aim to describe how the outcome should change if we modify the feasible set. To make this more clear, let us revisit the profile R in Figure 1, where $f_{Borda}(R, \{a, b, c\}) = \{(c, b, a)\}$. The question we now ask is how the winning ranking changes when, e.g., alternative b is deleted from the feasible set. Maybe the most straightforward idea is that we should choose the ranking (c, a)—we simply remove the now unavailable alternative b from the ranking to derive the winning ranking for the smaller feasible set. We formalize this idea with *agenda consistency* which postulates of an SPF f that $f(R, X) = \{\rhd|_X : \rhd \in f(R, Y)\}$ for all profiles $R \in \mathcal{R}^*$ and feasible sets $X, Y \in \mathcal{F}(A)$ with $X \subseteq Y$. Note that the general idea of this axiom coincides with known rationality conditions such as Sen's contraction and expansion consistency for (social) choice functions (Sen, 1971, 1977). Even though agenda consistency is a restrictive axiom, all positional scoring rules satisfy this condition as these rules always order the feasible alternatives with respect to the scores computed for the full profile.

Unfortunately, it turns out that Kemeny's rule fails agenda consistency. This can be shown by considering the profile R in Figure 1: it holds that $f_{Kemeny}(R, \{a, c\}) = \{(c, a)\}$ but $f_{Kemeny}(R, \{a, b, c\}) = \{(a, b, c)\}$. On a first glance, this may seem like a flaw of Kemeny's rule because the ordering over a and c depends on the availability of b. However, the problem runs much deeper: agenda consistency entails a high degree of transitivity for agendas of size 2. In more detail, since $f_{Kemeny}(R, \{a, b\}) = \{(a, b)\}$, $f_{Kemeny}(R, \{b, c\}) = \{(b, c)\}$, and $f_{Kemeny}(R, \{a, c\}) = \{(c, a)\}$, f_{Kemeny} fails agenda consistency *regardless of which rankings it chooses for* $\{a, b, c\}$. Or, put differently, agenda consistency requires a form of transitivity for the choice on agendas of size 2: if $f(R, \{a, b\}) = \{(a, b)\}$ and $f(R, \{b, c\}) = \{(b, c)\}$, then $f(R, \{a, c\}) = \{(a, c)\}$. In concordance with a large stream of research (e.g., May, 1954; Fishburn, 1970; Bar-Hillel and Margalit, 1988; Anand, 2009), we view this transitivity requirement of agenda consistency as unduly restrictive because such transitivity notions have empirically been shown to be unrealistic and often lead to negative theoretical results.

We therefore consider a weakening of agenda consistency which is based on the idea that a winning ranking $\rhd$ for a large set Y should only inherit to a subset X if the alternatives in X appear consecutively in $\rhd$. The reason for this is that if $\rhd|_X$ is not chosen for the set X, there is a better ranking for these alternatives. Moreover, we can reorder the alternatives $x \in X$ in the ranking $\rhd$ without affecting the alternatives in $Y \setminus X$, so we can intuitively also improve this ranking. To formalize this idea, we recall that $C(X, Y)$ denotes the set of rankings $\rhd \in \mathcal{R}(Y)$ in which the alternatives in X appear consecutively. Then, *local agenda consistency* requires that $f(R, X) \supseteq \{\rhd|_X : \rhd \in f(R, Y)\} \cap C(X, Y)$ for all profiles $R \in \mathcal{R}^*$ and feasible sets $X, Y \in \mathcal{F}(A)$ with $X \subseteq Y$. Clearly, this is a rather mild condition and we will show later that all bivariate scoring rules satisfy it.

Finally, we note that all of our axioms are independent of Arrow's independence of infeasible alternatives (Arrow, 1951). This axiom demands that the winning rankings should only depend on the available alternatives. More formally, an SPF f satisfies *independence of infeasible alternatives* if $f(R, X) = f(R', X)$ for all profiles $R, R' \in \mathcal{R}^*$ and all feasible sets $X \in \mathcal{F}(A)$ such that $N_R = N_{R'}$ and $\succ_i|_X = \succ'_i|_X$ for all $i \in N_R$. We note that in our context, independence of infeasible alternatives is easy to satisfy as it suffices to apply a voting rule to the input profile restricted to the feasible set. Moreover, this axiom is typically motivated by the fact that it allows to compute the outcome for a given feasible set only depending on the voters' preferences over this set of alternatives. By contrast, just as Sen's contraction consistency, our agenda consistency notions only relate the outcomes for different feasible sets but do no restrict the information that can be used to compute the winning rankings.

3. Characterizations

We are now ready to state our results. In more detail, we show in Section 3.1 the characterization of bivariate scoring rules, and discuss in Section 3.2 and Section 3.3 the characterizations of positional scoring rules and Kemeny's rule, respectively.

3.1. Bivariate Scoring Rules

In this section, we discuss our main result, the characterization of bivariate scoring rules, and its consequences. Since the proof of this result is rather involved, we only show in the main body that every bivariate scoring rule satisfies all given axioms and give a proof sketch for the inverse direction. The full proof can be found in the appendix.

Theorem 1. *An SPF is a bivariate scoring rule if and only if it satisfies anonymity, neutrality, continuity, faithfulness, reinforcement, and local agenda consistency.*

Proof Sketch. The proof consists of two claims: we need to show that bivariate scoring rules satisfy all given axioms and that they are the only SPFs that do so.

($\implies$) We first show that every bivariate scoring rule satisfies all considered properties. Hence, let f denote a bivariate scoring rule and let s denote its corresponding bivariate scoring function. We first note that f is clearly anonymous and neutral as the scoring function s does not depend on the names of voters or alternatives. Furthermore, f is faithful because $s(\ell, k) \geq 0$ for all $\ell, k \in \{1, \ldots, m\}$ with $\ell < k$ and this inequality is strict for some ℓ, k. Next, f satisfies continuity because it maximize scores: it holds for all profiles $R \in \mathcal{R}^*$, feasible sets $X \in \mathcal{F}(A)$, and rankings $\rhd \in f(R, X)$ and $\rhd' \in$

$\mathcal{R}(X) \setminus f(R, X)$ that $\hat{s}(R, \rhd) > \hat{s}(R, \rhd')$. Thus, there is for every other profile R' a $\lambda \in \mathbb{N}$ such that $\hat{s}(\lambda R + R', \rhd) > \hat{s}(\lambda R + R', \rhd')$ for all $\rhd \in f(R, X)$, $\rhd' \in \mathcal{R}(X) \setminus f(R, X)$, which entails that $f(\lambda R + R', X) \subseteq f(R, X)$.

Next, we will prove that f is reinforcing, for which we consider two disjoint profiles $R, R' \in \mathcal{R}^*$ and a feasible set $X \in \mathcal{F}(A)$ such that $f(R, X) \cap f(R', X) \neq \emptyset$. This means for all rankings $\rhd \in f(R, X) \cap f(R', X)$ and $\rhd' \in \mathcal{R}(X)$ that $\hat{s}(R, \rhd) \geq \hat{s}(R, \rhd')$ and $\hat{s}(R', \rhd) \geq \hat{s}(R', \rhd')$, so $\hat{s}(R + R', \rhd) \geq \hat{s}(R + R', \rhd')$ and $\rhd \in f(R + R', X)$. Moreover, if a ranking $\rhd'$ is not in $f(R, X)$ or in $f(R', X)$, then $\hat{s}(R, \rhd) > \hat{s}(R, \rhd')$ or $\hat{s}(R', \rhd) > \hat{s}(R', \rhd')$ for every $\rhd \in f(R, X) \cap f(R', X)$. This implies that $\hat{s}(R + R', \rhd) > \hat{s}(R + R', \rhd')$ and $\rhd' \not\in f(R + R, X)$. Hence, $f(R + R', X) = f(R, X) \cap f(R', X)$ and f satisfies reinforcement.

Finally, we show that f satisfies local agenda consistency. To this end, consider two feasible sets $X, Y \in \mathcal{F}(A)$ with $X \subseteq Y$, a profile $R \in \mathcal{R}^*$, and a ranking $\rhd \in f(R, Y) \cap C(X, Y)$. The score of the ranking $\rhd$ in R is $\hat{s}(R, \rhd) = \hat{s}(R, \rhd|_X) + \sum_{(x,y) \in \rhd \setminus X^2} \hat{s}(R, (x, y))$. Now, if $\rhd|_X \not\in f(R, X)$, there is a ranking $\rhd' \in \mathcal{R}(X)$ such that $\hat{s}(R, \rhd') > \hat{s}(R, \rhd|_X)$. Since the alternatives in X appear consecutively in R, we can now reorder these alternatives in $\rhd$ according to $\rhd'$ to derive another ranking $\rhd''$ on Y such that $\hat{s}(R, \rhd'') = \hat{s}(R, \rhd') + \sum_{(x,y) \in \rhd \setminus X^2} \hat{s}(R, (x, y)) > \hat{s}(R, \rhd|_X) + \sum_{(x,y) \in \rhd \setminus X^2} \hat{s}(R, (x, y)) = \hat{s}(R, \rhd)$. Hence, if $\rhd|_X \not\in f(R, X)$, then $\rhd \not\in f(R, Y)$, which contradicts our assumption and therefore shows that f satisfies local agenda consistency.

($\Longleftarrow$) Let f denote an SPF that satisfies anonymity, neutrality, continuity, faithfulness, reinforcement, and local agenda consistency. The main goal of our proof is to find a bivariate scoring function s that describes f. To this end, we will use a similar approach as presented by, e.g., Young (1975) or Young and Levenglick (1978), but we will need a significant amount of new ideas to make our proof work. We will give here only a high level overview of our proof and postpone all technical details to the appendix.

As a first step, we will show that the conjunction of our axioms implies that f is non-imposing. This means that for every feasible set $X \in \mathcal{F}(A)$ and every ranking $\rhd \in \mathcal{R}(X)$, there is a profile R such that $f(R, X) = \{\rhd\}$. Next, we will change the domain of f from preference profiles to a numerical space. To this end, we represent preference profiles R by vectors $v \in \mathbb{N}^{m!}$ which state how often each possible preference relation is reported. In particular, $v(R)$ denotes the vector corresponding to the profile R. Since f is anonymous, it is straightforward that there is a function g such that $f(R, X) = g(v(R), X)$ for all profiles $R \in \mathcal{R}^*$ and feasible sets $X \in \mathcal{F}(A)$. Moreover, it is easy to see that g inherits all desirable properties of f. As the next step, we use reinforcement to extend the domain of g from $\mathbb{N}^{m!} \times \mathcal{F}(A)$ to $\mathbb{Q}^{m!} \times \mathcal{F}(A)$ while preserving the desirable properties of f (resp. g). This leads to a new function $\hat{g}$ which still satisfies that $f(R, X) = \hat{g}(v(R), X)$ for all profiles $R \in \mathcal{R}^*$ and feasible sets $X \in \mathcal{F}(A)$. So, it suffices to describe the function $\hat{g}$ by a bivariate scoring function to show that f is a bivariate scoring rule.

To this end, we define the sets $R_\rhd = \{v \in \mathbb{Q}^{m!} \colon \rhd \in \hat{g}(v, X)\}$ for all feasible sets $X \in \mathcal{F}(A)$ and rankings $\rhd \in \mathcal{R}(X)$. Moreover, we denote by $\bar{R}_\rhd$ the closure of $R_\rhd$ with respect to $\mathbb{R}^{m!}$. We observe that $\hat{g}(v, X) = \{\rhd \in \mathcal{R}(X) \colon v \in R_\rhd\} \subseteq \{\rhd \in \mathcal{R}(X) \colon v \in$

$\bar{R}_\rhd\}$, so our next intermediate goal is to describe the sets $\bar{R}_\rhd$. Here, it can be shown that the sets $\bar{R}_\rhd$ are convex as $\hat{g}$ inherits reinforcement of f, and that $\mathrm{int}\bar{R}_\rhd \cap \mathrm{int}\bar{R}_{\rhd'} = \emptyset$ for all feasible set $X \in \mathcal{F}(A)$ and rankings $\rhd, \rhd' \in \mathcal{R}(X)$. As a consequence, we can apply the separation theorem for convex sets to find a non-zero vector $u^{\rhd,\rhd'} \in \mathbb{R}^{m!}$ such that $vu^{\rhd,\rhd'} \geq 0$ for all $v \in \bar{R}_\rhd$ and $vu^{\rhd,\rhd'} \leq 0$ for all $v \in \bar{R}_{\rhd'}$ (where vu denotes the standard scalar product). These vectors describe the sets $\bar{R}_\rhd$ as we prove that $\bar{R}_\rhd = \{v \in \mathbb{R}^{m!} \colon \forall \rhd' \in \mathcal{R}(X) \setminus \{\rhd\} \colon vu^{\rhd,\rhd'} \geq 0\}$. Thus, for understanding the sets $\bar{R}_\rhd$, we only need to understand the vectors $u^{\rhd,\rhd'}$.

For doing so, we first restrict our attention to feasible sets of size 2. The reason for this is that for a feasible set $X = \{x, y\}$, there are only two possible rankings: (x, y) and (y, x). Consequently, $\bar{R}_{(x,y)} = \{v \in \mathbb{R}^{m!} \colon vu^{(x,y),(y,x)} \geq 0\}$, so it suffices to study a single vector $u^{(x,y),(y,x)}$. We thus show that there is a bivariate scoring function s such that $u_k^{(x,y),(y,x)} = s(r(\succ_k, x), r(\succ_k, y))$ for all alternatives $x, y \in A$ and preference relations $\succ_k \in \mathcal{R}(A)$. Next, we turn to larger feasible sets and note that $\hat{g}$ inherits local agenda consistency from f. This implies that $\bar{R}_\rhd \subseteq \bar{R}_{(x,y)}$ if x, y appear consecutively in $\rhd$ and $x \rhd y$. As a consequence, it holds for all rankings $\rhd, \rhd' \in \mathcal{R}(X)$ that only differ in a single pair of alternatives (x, y) that the vector $u^{(x,y),(y,x)}$ separates the set $\bar{R}_\rhd$ from $\bar{R}_{\rhd'}$ (i.e., $vu^{(x,y),(y,x)} \geq 0$ if $v \in \bar{R}_\rhd$ and $vu^{(x,y),(y,x)} \leq 0$ if $v \in \bar{R}_{\rhd'}$) as it separates $\bar{R}_{(x,y)}$ from $\bar{R}_{(y,x)}$. The reason for this is that $\rhd \setminus \rhd' = \{(x, y)\}$ implies that x and y appear consecutively in both $\rhd$ and $\rhd'$, but in different order.

Next, we aim to understand the vectors $u^{\rhd,\rhd'}$ for rankings $\rhd, \rhd'$ with $|\rhd \setminus \rhd'| > 1$. To do so, we fix a feasible set $X = \{a_1, \dots, a_\ell\}$ and introduce a neighbor relation on rankings in $\mathcal{R}(X)$. In more detail, we associate every ranking $\rhd$ with a matrix $M^\rhd$ defined by $M^\rhd_{i,j} = 1$ if $a_i \rhd a_j$, $M^\rhd_{i,j} = -1$ if $a_j \rhd a_i$, and $M^\rhd_{i,j} = 0$ if $a_i = a_j$ otherwise. Next, we define the set $\mathcal{M}$ as the convex hull of these matrices, and call two rankings $\rhd, \rhd'$ neighbors if they are neighboring extreme points on $\mathcal{M}$. This neighborhood relation has also been used by Young and Levenglick (1978) because there is a characterization of neighbors by Young (1978) (see also Gilmore and Hoffmann (1964)), which turns out helpful in the analysis. In particular, we then show for any two neighboring rankings $\rhd, \rhd' \in \mathcal{R}(X)$ that the vector $\sum_{(x,y)\in \rhd \setminus \rhd'} u^{(x,y),(y,x)}$ separates $\bar{R}_\rhd$ from $\bar{R}_{\rhd'}$. We note that this step is the technically most involved one and includes, e.g., a complete specification of the linear (in)dependence of the vectors $u^{(x,y),(y,x)}$.

We proceed then by defining the set *Neighbor*$(\rhd)$ as the neighbors of a ranking $\rhd \in \mathcal{R}(X)$, and $\bar{R}^N_\rhd = \{v \in \mathbb{R}^{m!} \colon \forall \rhd' \in \mathit{Neighbor}(\rhd) \colon vu^{\rhd,\rhd'} \geq 0\}$. It holds that $\hat{g}(v, X) \subseteq \{\rhd \in \mathcal{R}(X) \colon v \in \bar{R}^N_\rhd\}$ for all $v \in \mathbb{Q}^{m!}$ and $X \in \mathcal{F}(A)$ since $\bar{R}_\rhd \subseteq \bar{R}^N_\rhd$. Now, by the last step, we can choose $u^{\rhd,\rhd'} = \sum_{(x,y)\in \rhd \setminus \rhd'} u^{(x,y),(y,x)}$. Moreover, the analysis of feasible sets of size 2 shows that the vectors $u^{(x,y),(y,x)}$ can be described by a bivariate scoring function s. This implies for two neighboring rankings $\rhd, \rhd' \in \mathcal{R}(X)$ that

$$\begin{aligned}\hat{s}(v, \rhd) - \hat{s}(v, \rhd') &= \sum_{(x,y) \in \rhd \setminus \rhd'} \hat{s}(v, (x, y)) - \sum_{(x,y) \in \rhd' \setminus \rhd} s(v, (x, y)) \\ &= 2 \sum_{(x,y) \in \rhd \setminus \rhd'} \hat{s}(v, (x, y)) \\ &= 2v \sum_{(x,y) \in \rhd \setminus \rhd'} u^{(x,y),(y,x)},\end{aligned}$$

where $\hat{s}(v, (x, y)) = \sum_{k=1}^{m!} v_k s(r(\succ_k, x), r(\succ_k, y))$ and $\hat{s}(v, \rhd) = \sum_{(x,y) \in \rhd} \hat{s}(v, (x, y))$. Hence, we infer that $\bar{R}^N_\rhd = \{v \in \mathbb{R}^{m!} \colon \forall \rhd' \in \mathit{Neighbor}(\rhd) \colon \hat{s}(v, \rhd) \geq \hat{s}(v, \rhd')\}$.

Finally, we note that $\{v \in \mathbb{R}^{m!} \colon \forall \rhd' \in \mathit{Neighbor}(\rhd) \colon \hat{s}(v, \rhd) \geq \hat{s}(v, \rhd')\} = \{v \in \mathbb{R}^{m!} \colon \forall \rhd' \in \mathcal{R}(X) \colon \hat{s}(v, \rhd) \geq \hat{s}(v, \rhd')\}$. The reason for this is that the second set can be seen as a linear optimization problem: given a vector $v \in \mathbb{R}^{m!}$ we choose the rankings $\rhd$ that correspond to the extreme points that maximize $\sum_{a_i, a_j \in X} M_{i,j} \cdot \hat{s}(v, (a_i, a_j))$ subject to $M \in \mathcal{M}$. It is a well-known fact that, if an extreme point is not optimal, then there is a neighboring extreme point with a higher objective value. This insight then implies the set equality, so we can now infer that $f(R, X) = \hat{g}(v(R), X) \subseteq \{\rhd \in \mathcal{R}(X) \colon \forall \rhd' \in \mathcal{R}(X) \colon \hat{s}(v(R), \rhd) \geq \hat{s}(v(R), \rhd')\}$ for all profiles $R \in \mathcal{R}^*$ and all feasible sets of alternatives $X \in \mathcal{F}(A)$. Or, put differently, f always chooses a subset of the bivariate scoring rule defined by s. As last step, we use continuity to show that this must be an equality, so f is indeed a bivariate scoring rule. □

Remark 1 (Independence of the Axioms). All axioms are necessary for Theorem 1. If we only drop local agenda consistency, then every rule that maximizes the score according to a scoring function s that assigns points to every pair of input and output rankings satisfies all axioms, and this class is a strict superset of our bivariate scoring rules. If we only drop reinforcement, we can return the union of the output of two different bivariate scoring rules. Or, to be more precise, one can check that the rule $f(R, X) = f_{\mathit{Borda}}(R, X) \cup f_{\mathit{Kemeny}}(R, X)$ only fails reinforcement. When excluding continuity from our list of axioms, we can use composite bivariate scoring rules which first compute a bivariate scoring rule and then refine it by choosing the rankings with maximal score according to another bivariate scoring rules. An example for this is the rule that first computes the rankings with maximal Borda scores and then refines the output by only choosing the rankings with maximal Kemeny scores. Faithfulness is required to exclude, e.g., the trivial rule that always returns all outcomes or bivariate scoring rules that give negative score to a pair (a, b) if a voter prefers a to b. To only violate anonymity, we can treat voters differently: for instance, we can count the vote of "even" voters $i \in 2\mathbb{N}$ twice, but the votes of the "odd" voters $i \in 2\mathbb{N} + 1$ only once. Finally, for neutrality, we can, e.g., consider a biased version of Kemeny's rule that counts the points for a specific pair of alternatives twice.

Remark 2 (Faithfulness). The example in Remark 1 demonstrating that faithfulness is required for Theorem 1 generalizes the class of bivariate scoring rules by allowing for bivariate scoring functions s with $s(\ell, k) < 0$ if $\ell < k$ or that $s(\ell, k) = 0$ for all $\ell, k \in \{1, \dots, m\}$. While these rules are not particular appealing, one may wonder whether it is possible to characterize a more general class of bivariate scoring rules when simply dropping faithfulness. This is not the case because, without faithfulness, local agenda consistency becomes trivial to satisfy: we can simply define $f(R, X) = \mathcal{R}(X)$ for all profiles $R \in \mathcal{R}^*$ and all feasible sets of alternatives $X \in \mathcal{F}(A)$ with $X \neq A$ and define $f(R, A)$ by any SPF that satisfies anonymity, neutrality, consistency, and, continuity.

Remark 3 (Local Agenda Consistency). It is noteworthy that our proof of Theorem 1 does actually not use the full power of local agenda consistency. Instead, we only use local agenda consistency for situations where the smaller feasible set consists of precisely two alternatives. Analogous claims will also hold for Corollaries 1 and 2 and this observation is, e.g, also known for Arrow's impossibility (Schwartz, 1986).

Remark 4 (SSB Utility Functions). Positional scoring rules are closely connected to vNM utility functions: in a positional scoring rule, we assume that each voter has the same canonical utiliy function and we then order the alternatives according to their social welfare. A similar interpretation is possible for bivariate scoring rule by considering the class of skew-symmetric bilinear utility (SSB) functions (Fishburn, 1984), which assign a value to each pair of alternatives (and thus discard the transitivity enforced by vNM utilities): the decision maker chooses a canonical SSB utility function and the winning rankings are those with maximal social welfare (where the welfare of a ranking is simply the sum of the utilities of each pair of alternatives in a ranking). We note that this interpretation of SSB utility functions is new as SSB utilities have not been studied in the context of rankings.

3.2. Positional Scoring Rules

Next, we turn to our characterization of positional scoring rules. We derive this result as a corollary of Theorem 1 by strengthening local agenda consistency to agenda consistency. Moreover, Corollary 1 can be interpreted as a variant of the characterization of positional scoring rules by Smith (1973), so our main result intuitively generalizes this prominent result.

Corollary 1. *An SPF is a positional scoring rule if and only if satisfies anonymity, neutrality, continuity, faithfulness, reinforcement, and agenda consistency.*

Proof. First, we note that all positional scoring rules satisfy anonymity, neutrality, continuity, faithfulness, and reinforcement because they are bivariate scoring rules. Furthermore, these SPFs are also agenda consistent because they compute the scores of each alternative with respect to the full profile and simply order the available alternatives in decreasing order of their scores. Hence, if we delete some alternatives from the feasible set, we only need to delete these alternatives from the old rankings to derive the winning rankings for the smaller feasible set.

For the other direction, let f denote an SPF satisfying all given axioms. First, since agenda consistency implies local agenda consistency, we can infer that f is a bivariate scoring rule by Theorem 1. Hence, let s denote the the bivariate scoring function of f. We will next show that $s(i,k) = s(i,j) + s(j,k)$ for all $i,j,k \in \{1,\dots,m\}$ with $i<j<k$. Assume for contradiction that this is not the case for some i,j,k and consider the following profile R for three voters: voter 1 places a at position i, b at position j, and c at position k, voter 2 places b at position i, c at position j, and a at position k, and voter 3 places c at position i, a at position j, and b at position k. All remaining alternatives can be arranged arbitrarily. Next, consider the feasible set $\{a,b\}$: it holds that $\hat{s}(R,(a,b)) = s(i,j)+s(j,k)+s(k,i) \neq 0$ and $\hat{s}(R,(b,a)) = s(j,i)+s(k,j)+s(i,k) = -\hat{s}(R,(a,b))$ because of our assumption that $s(i,k) \neq s(i,j) + s(j,k)$ and the fact that $s(x,y) = -s(y,x)$. This means that $|f(R,\{a,b\})| = 1$ and we suppose without loss of generality that $s(i,j)+s(j,k)+s(k,i) > 0$, so $f(R,\{a,b\}) = \{(a,b)\}$. Based on symmetric calculations, we also get that $f(R,\{b,c\}) = \{(b,c)\}$ and $f(R,\{a,c\}) = \{(c,a)\}$. However, this means that f fails agenda consistency for $f(R,\{a,b,c\})$ since any chosen ranking $\rhd$ has to satisfy $a \rhd b$, $b \rhd c$, and $c \rhd a$. This violates the transitivity of rankings, so our initial assumption is wrong and $s(i,k) = s(i,j) + s(j,k)$ for all $i,j,k \in \{1,\dots,m\}$ with $i<j<k$.

We can now infer the positional scoring function s' of f: we define $s'(i) = s(i,m)$ for all $i \in \{1,\dots,m-1\}$ and $s'(m) = 0$. By our previous computations, it holds that $s(i,j) = s(i,m) - s(j,m)$ for all $i,j \in \{1,\dots,m\}$ with $i<j$. Moreover, if $j<i$, then $s(i,j) = -s(j,i) = -(s(j,m) - s(i,m)) = s(i,m) - s(j,m)$, so the equality holds for all $i,j \in \{1,\dots,m\}$. This means that s' is non-increasing: if $s'(i) < s'(i+1)$ for some i, then $s(i,i+1) = s(i,m) - s(i+1,m) = s'(i) - s'(i+1) < 0$, which violates the definition of a bivariate scoring function. Also, since there are $i,j \in \{1,\dots,m\}$ with $s(i,j) > 0$, s' is non-constant as $s'(i) > s'(j)$. Thus, s' is indeed a positional scoring function. Finally, we note that $\hat{s}(R,(x,y)) = \sum_{i \in N_R} s(r(\succ_i,x),m) - s(r(\succ_i,x),m) = \hat{s}'(R,x) - s'(R,y)$ for all profiles R and alternatives $x,y \in A$. This implies for all feasible sets $X \in \mathcal{F}(A)$ and all rankings $\rhd, \rhd' \in \mathcal{R}(X)$ that

$$
\begin{aligned}
\hat{s}(R,\rhd) - \hat{s}(R,\rhd') &= \sum_{(x,y)\in\rhd} \hat{s}(R,(x,y)) - \sum_{x,y\in\rhd'} \hat{s}(R,(x,y)) \\
&= 2 \sum_{(x,y)\in\rhd\setminus\rhd'} \hat{s}(R,(x,y)) \\
&= 2 \sum_{(x,y)\in\rhd\setminus\rhd'} \hat{s}'(R,x) - \hat{s}'(R,y).
\end{aligned}
$$

Now, let f' denote the positional scoring rule defined by s'. Our goal is to show that $f(R,X) = f'(R,X)$ for all profiles $R \in \mathcal{R}^*$ and all feasible sets $X \in \mathcal{F}(A)$. To this end, consider two rankings $\rhd \in f(R,X)$ and $\rhd' \in \mathcal{R}(X) \setminus f(R,X)$ for some profile $R \in \mathcal{R}^*$ and agenda $X \in \mathcal{F}(A)$. By definition of f, we have that $\hat{s}(R,\rhd) > \hat{s}(R,\rhd')$, which means that $\sum_{(x,y)\in\rhd\setminus\rhd'} \hat{s}'(R,x) - \hat{s}'(R,y) > 0$. Hence, there is a pair of alternatives $x,y \in X$ such that $\hat{s}'(R,x) > \hat{s}'(R,y)$ and $y \rhd' x$. The definition of positional scoring rules then requires that $\rhd' \notin f'(R,X)$, and thus $f'(R,X) \subseteq f(R,X)$. For the other direction, consider a ranking $\rhd \in \mathcal{R}(X) \setminus f'(R,X)$. This means that there are alternatives $x,y \in X$ such that $x \rhd y$ but $\hat{s}'(R,x) < \hat{s}'(R,y)$. By iterating through $\rhd$, we can also find two consecutive alternatives $x', y' \in X$ in $\rhd$ that satisfy $x' \rhd y'$ but $\hat{s}'(R, x') < \hat{s}'(R, y')$. Now, consider the ranking $\rhd'$ derived from $\rhd$ by swapping x' and y'. It holds that $\hat{s}(R, \rhd') - \hat{s}(R, \rhd) = 2\hat{s}'(R, y') - 2\hat{s}'(R, x') > 0$, which shows that $\rhd \notin f(R, X)$. Hence, we also get that $f(R, X) \subseteq f'(R, X)$, so f is the positional scoring rule induced by s'. $\square$

Remark 5 (Smith's Characterization of Positional Scoring Rules). Corollary 1 is closely related to the characterization of positional scoring rules due to Smith (1973).[6] In more detail, Smith (1973) characterizes positional scoring rules in the context of social welfare functions (SWFs), which return a single weak order over the alternatives rather than a set of strict orders, and shows that an SWF is a positional scoring rule if and only if it satisfies anonymity, neutrality, continuity, and separability. The last axiom, separability, is a variant of reinforcement for SWFs which requires that if $f(R)|_{\{a,b\}} \cap f(R)|_{\{a,b\}} \neq \emptyset$, then $f(R + R')|_{\{a,b\}} = f(R)|_{\{a,b\}} \cap f(R')|_{\{a,b\}}$ for all disjoint profiles $R, R' \in \mathcal{R}^*$ and all alternatives $a, b \in A$ (here, $f(R)$, $f(R')$, and $f(R + R')$ denote the weak rankings on A returned by the SWF f for R, R', and $R + R'$, respectively).

While our result is logically independent to the one by Smith since we work in a different setting, reinforcement and agenda consistency transfer the idea of separability to SPFs. To make this more formal, we write $a \succsim_{f(R,X)} b$ if there is a ranking $\rhd \in f(R, X)$ with $a \rhd b$ and $a \succ_{f(R,X)} b$ if $a \rhd b$ for all rankings $\rhd \in f(R, X)$. Then, reinforcement and agenda consistency together require for all disjoint profiles $R, R' \in \mathcal{R}^*$, feasible sets $X \in \mathcal{F}(A)$, and alternatives $a, b \in X$ that if $\succsim_{f(R,X)}|_{\{a,b\}} \cap \succsim_{f(R',X)}|_{\{a,b\}} \neq \emptyset$, then $\succsim_{f(R+R',X)}|_{\{a,b\}} = \succsim_{f(R,X)}|_{\{a,b\}} \cap \succsim_{f(R',X)}|_{\{a,b\}}$. The reason for this is that $\succsim_{f(\bar{R},X)}|_{\{a,b\}} = f(\bar{R}, \{a, b\})$ for every profile $\bar{R}$ because of agenda consistency. Hence, if $\succsim_{f(R,X)}|_{\{a,b\}} \cap \succsim_{f(R',X)}|_{\{a,b\}} \neq \emptyset$, then $f(R, \{a, b\}) \cap f(R', \{a, b\}) \neq \emptyset$ and reinforcement shows that $f(R + R', \{a, b\}) = f(R, \{a, b\}) \cap f(R', \{a, b\})$. Applying again agenda consistency then shows that $\succsim_{f(R+R',X)}|_{\{(a,b)\}} = f(R + R', \{a, b\}) = \succsim_{f(R,X)}|_{\{a,b\}} \cap \succsim_{f(R',X)}|_{\{a,b\}}$, which proves our claim.

3.3. Kemeny's rule

As our last result, we discuss our characterization of Kemeny's rule for which we additionally use independence of infeasible alternatives. Just as Corollary 1, this result follows as simple corollary from Theorem 1. Since this corollary can be seen as a variant of the characterization of Kemeny's rule by Young (1988), Theorem 1 indeed combines the prominent characterizations of Smith (1973) and Young (1988).

Corollary 2. *An SPF is Kemeny's rule if and only if it satisfies anonymity, neutrality, continuity, faithfulness, reinforcement, local agenda consistency, and independence of infeasible alternatives.*

Proof. Since Kemeny's rule is a bivariate scoring rule, it satisfies all axioms but independence of infeasible alternatives due to Theorem 1. To show that Kemeny's rule also

[6] It should be noted that the result by Smith (1973) considers positional scoring rules defined by arbitrary scoring rules s, i.e., this author drops the assumption that s is non-increasing and non-constant. When using agenda consistency instead of local agenda consistency and faithfulness in the proof of Theorem 1, we can also infer this stronger result.

satisfies independence of infeasible alternatives, we consider two profiles R and R' and a feasible set $X \in \mathcal{F}(A)$ such that $N_R = N_{R'}$ and $\succ_i|_X = \succ'_i|_X$ for all $i \in N_R$. This means for all alternatives $x, y \in X$ and all voters $i \in N_R$ that $x \succ_i y$ if and only if $x \succ'_i y$. Furthermore, by the definition of Kemeny's rule, every voter gives 1 point to a ranking $\rhd \in \mathcal{R}(X)$ for every pair of alternatives $x, y \in X$ with $x \succ_i y$ and $x \rhd y$ and -1 points for every pair of alternatives $x, y \in X$ with $x \succ_i y$ and $y \rhd x$. Combining these two insights implies that $\hat{s}_{Kemeny}(R, \rhd) = \hat{s}_{Kemeny}(R, \rhd')$ for every ranking $\rhd \in \mathcal{R}(X)$ and thus $f_{Kemeny}(R, X) = f_{Kemeny}(R', X)$.

For the other direction, let f denote an SPF that satisfies all given axioms. By Theorem 1, f is a bivariate scoring rule that satisfies independence of infeasible alternatives and we let s denote its bivariate scoring function. By the definition of bivariate scoring functions, there are indices $i, j \in \{1, \dots, m\}$ with $i < j$ such that $s(i, j) > 0$. Our goal is now to show that $s(i', j') = s(i, j)$ for all indices i', j' with $i' < j'$. To this end, we consider two profiles R and R', which are both defined by 2 voters. In more detail, in R, a is the i-th best alternative of voter 1 and the j-th best alternative of voter 2, and b is the j-th best alternative of voter 1 and the i-th best alternative of voter 2. All other alternatives can be ranked arbitrarily. Since $s(i, j) = -s(j, i)$, it is easy to check that $f(R, \{a, b\}) = \{(a, b), (b, a)\}$. In the second profile R', voter 2 has the same preference as in R, but voter 1 places a now at position i' and b at position j'. We note that $\succ_k|_{\{a,b\}} = \succ'_k|_{\{a,b\}}$ for $k \in \{1, 2\}$, so independence of infeasible alternatives entails that $f(R', \{a, b\}) = f(R, \{a, b\}) = \{(a, b), (b, a)\}$. This means that $\hat{s}(R', (a, b)) = \hat{s}(R', (b, a))$. Moreover, it is easy to see that $\hat{s}(R', (a, b)) = s(i', j') + s(j, i) = -(s(j', i') + s(i, j)) = -\hat{s}(R', (b, a))$. We therefore infer that $s(i', j') + s(j, i) = s(i', j') - s(i, j) = 0$. Hence, $s(i, j) = s(i', j')$ for all $i', j' \in \{1, \dots, m\}$ with $i' < j'$, which means that f is Kemeny's rule as s is skew-symmetric and f is invariant under scaling s. □

Remark 6 (Young's Characterization of Kemeny's Rule). Corollary 2 can be seen as a variant of a prominent characterization in the literature: Young (1988) has shown that Kemeny's rule is the only SPF satisfying anonymity, neutrality, reinforcement, faithfulness, and pairwise consistency (see also Young, 1994, Theorem 6). The last condition in this list, pairwise consistency, requires for all profiles $R \in \mathcal{R}^*$, feasible sets of alternatives $X \in \mathcal{F}(A)$, rankings $\rhd \in f(R, X)$, and alternatives $a, b \in X$ that are consecutive in $\rhd$ that *(i)* if $a \rhd b$, then $(a, b) \in f(R, \{a, b\})$, *(ii)* if $a \rhd b$ and $(b, a) \in f(R, \{a, b\})$, then $(\rhd \setminus \{(a, b)\}) \cup \{(b, a)\} \in f(R, X)$, and *(iii)* $f(R, \{a, b\}) = f(R', \{a, b\})$ for all $R' \in \mathcal{R}^*$ with $N_{R'} = N_R$ and $\succ'_i|_{\{a,b\}} = \succ_i|_{\{a,b\}}$ for all $i \in N_R$. Clearly, the first condition corresponds to local agenda consistency for agendas of size 2, and the third condition to independence of infeasible alternatives for agendas of size 2. By contrast, none of our axioms relates to the second condition. Since our proofs also work with this restricted notions of local agenda consistency and independence of infeasible alternatives, Corollary 2 gives another, intuitively slightly stronger variant of this characterization. In particular, our result replaces the technical second condition of pairwise consistency with continuity, thus resulting in a more natural characterization of Kemeny's rule.

Remark 7 (Condorcet-consistency). There is also another characterization of Kemeny's rule by Young and Levenglick (1978) that relies on a variant of Condorcet-consistency. In more detail, this axiom postulates for all profiles $R \in \mathcal{R}^*$, feasible sets $X \in \mathcal{F}(A)$, rankings $\rhd \in f(R, X)$, and alternatives $a, b \in X$ that are consecutive in $\rhd$ that *(i)* if $a \rhd b$, then a (weak) majority of voters prefer a to b, and *(ii)* if as many voters prefer a to b than vice versa and $a \rhd b$, then $(\rhd \setminus \{(a, b)\}) \cup \{(b, a)\} \in f(R, X)$. This definition of Condorcet-consistency differs vastly from classical definitions of Condorcet-consistency and consequently, the result of Young and Levenglick (1978) is often erroneously stated with a weaker variant of Condorcet-consistency.[7] Our main result allows for another variant of this characterization. Firstly, it is easy to see that Kemeny's rule is the only bivariate scoring rule that always ranks the Condorcet winner first whenever such an alternative exists. Even more, Theorem 1 also holds when replacing local agenda consistency and IIA with another variant of Condorcet-consistency, which requires that if a is ranked directly over b in a winning ranking, then a (weak) majority of the voters must prefer a to b. This axiom implies local agenda consistency and IIA for feasible sets of size 2, so this statement follows directly from our proof.

Remark 8 (Arrow's Impossibility). It is an immediate consequence of Corollary 1 and Corollary 2 that no bivariate scoring rule satisfies both agenda consistency and independence of infeasible alternatives. Indeed, when imposing both axioms at the same time, it is easy to see that Arrow's impossibility theorem applies: only dictatorships satisfy Pareto-optimality, agenda consistency, and independence of infeasible alternatives. This follows analogous to the proofs of Arrow's impossibility in, e.g., Campbell and Kelly (2002). Hence, Corollaries 1 and 2 can be seen as attractive escape routes to this impossibility.

4. Conclusion

In this paper, we study social preference functions (SPFs) which, given the voters' strict preferences over some set of alternatives, compute a non-empty set of winning rankings over a feasible subset of alternatives. Two of the most prominent classes of SPFs are positional scoring rules and Kemeny's rule, both of which have repeatedly been characterized (e.g., Smith, 1973; Young, 1974; Young and Levenglick, 1978; Young, 1988). In this paper, we unify these two independent streams of research by characterizing the class of bivariate scoring rules, which contains both Kemeny's rule and all positional scoring rules. Roughly, bivariate scoring rules can be seen as variants of Kemeny's rule that weight comparisons between alternatives differently depending on their positions in the voters' preference relations. We characterize this class of rules by mainly relying on two axioms called reinforcement and local agenda consistency, which formalize consistency notions with respect to variable electorates and variable agendas, respectively. Based on this result, we also infer characterizations of the class of positional scoring rules and of Kemeny's rule as corollaries. These characterizations can be seen as variants of the characterizations by Smith (1973) and Young (1988), which demonstrates that our main

[7] Maybe the most prominent occurrence of such a mistake is in the chapter by Zwicker (2016), where Condorcet-consistency only requires that the first-ranked alternative in a winning ranking must be the Condorcet winner if one exists.

result indeed combines these two classical theorems. Or, put differently, our main result tries to unify the axiomatic research on positional scoring rules and Kemeny's rule.

Moreover, we note that our characterizations of positional scoring rules and Kemeny's rule also highlight the differences of these SPFs: while the latter SPF satisfy independence of infeasible alternatives, the former ones satisfy a stronger notion of agenda consistency. Since these axioms are jointly incompatible under mild additional conditions due to Arrow's impossibility (Arrow, 1951), these results thus draw a sharp boundary between Kemeny's rule and positional scoring rules.

Finally, we note that our paper can also be seen as progress to the open problem of better understanding SPFs from an axiomatic perspective. Indeed, while there are very general classes of SPFs (see, e.g., Elkind and Slinko, 2016), there is little progress in understanding these large classes from an axiomatic perspective. While our result focuses on a subset of these classes, it still is the most general characterization of SPFs so far and can thus be seen as a first step in the analysis of more general classes of SPFs.

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A. Appendix: Proof of Theorem 1

In this appendix, we provide a complete proof of Theorem 1: an SPF is a bivariate scoring rule if and only if it satisfies anonymity, neutrality, continuity, faithfulness, reinforcement, and local agenda consistency. Since we have shown in the main body that every bivariate scoring rule satisfies the given axioms, we focus here on the converse and thus assume throughout the appendix that f is an SPF that satisfies all six axioms. Our goal is to find the underlying bivariate scoring function of f to show that it is indeed a bivariate scoring rule. For this, we follow the proof sketch inj the main body. For a better readability, we place the lemmas in groups indicated by subsections: in Appendix A.1 we show that f is non-imposing, in Appendix A.2 we apply the separation theorem for convex sets to infer the vectors $u^{\rhd,\rhd'}$, in Appendix A.3 we study feasible sets of size 2, and in Appendix A.4, we extend our reasoning to larger feasible sets and proof Theorem 1.

A.1. Non-imposition

Our first goal is to show that the SPF f is non-imposing. To this end, we first construct an auxiliary profile R^a, in which all rankings over a feasible X are chosen that top-rank a specific alternative $x \in X$.

Lemma 1. *For all feasible sets $X \in \mathcal{F}(A)$ and alternatives $a \in X$, there is a profile R^a such that $f(R^a, X) = \{\rhd \in \mathcal{R}(X) \colon \forall x \in X \setminus \{a\} \colon a \rhd x\}$.*

Proof. Fix an arbitrary feasible set $X \in \mathcal{F}(A)$ and an alternative $a \in X$. If $|X| = 1$, the claim is trivial and we thus suppose that $|X| \geq 2$. For constructing the profile R^a, we will first derive a profile $R^{a,b}$ such that $f(R^{a,b}, \{a,b\}) = \{(a,b)\}$ for some alternative $b \in X \setminus \{a\}$, $(a,x) \in f(R^{a,b}, \{a,x\})$ for all $x \in X \setminus \{a,b\}$, and $f(R^{a,b}, \{x,y\}) = \{(x,y),(y,x)\}$ for all $x, y \in X \setminus \{a,b\}$. To this end, we recall that there is a preference relation $\succ^* \in \mathcal{R}(A)$ such that $a \succ^* b$ and $f(\succ^*, \{a,b\}) = \{(a,b)\}$ by faithfulness.

Now, the profile $R^{a,b}$ simply contains all preference relations $\succ \in \mathcal{R}(A)$ with $a \succ b$ once. By faithfulness, it holds for all these preference relations that $(a,b) \in f(\succ, \{a,b\})$. Since $\succ^*$ also appears in $R^{a,b}$, we can infer from reinforcement that $f(R^{a,b}, \{a,b\}) = \{(a,b)\}$. Furthermore, we note that in $R^{a,b}$, all alternatives $x, y \in X \setminus \{a,b\}$ are completely symmetric. So, anonymity and neutrality require that $f(R^{a,b}, \{x,y\}) = \{(x,y),(y,x)\}$. Finally, we will show that $(a,x) \in f(R^{a,b}, \{a,x\})$ for all $x \in X \setminus \{a,b\}$. To this end, fix such an alternative $x \in A \setminus \{a,b\}$ and let τ denote the permutation that only exchanges a and x. The key insight now is that that for every ranking $\succ$ in $R^{a,b}$ with $x \succ a$, the ranking $\tau(\succ)$ (i.e., the ranking derived from $\succ$ by exchanging a and x) is also in $R^{a,b}$ as because every alternative x that is preferred to a is also preferred to b. Now, let R^1 denote the profile that contains each ballots $\succ$ with $x \succ a \succ b$ once, R^2 is the profile that contains each ballot with $a \succ x \succ b$ once, and R^3 is the profile that consists all ballots with $a \succ b \succ x$ once. First, we note that $R^1 + R^2 + R^3 = R^{a,b}$. Secondly, faithfulness implies for R^3 that $(a,x) \in f(R^3, \{a,x\})$ because every voter prefers a to x. As the last point, note that τ maps R^1 to R^2 and R^2 to R^1. As a consequence, anonymity and neutrality require for every ranking $\succ$ that $f(R^1 + R^2, \{a, x\}) = \{(a, x), (x, a)\}$. Finally, reinforcement then shows that $(a, x) \in f(R^{a,b}, \{a, x\})$.

The profile R^a consists now of a copy of $R^{a,b}$ for every alternative $b \in X \setminus \{a\}$. By our previous analysis, we have that $f(R^{a,b}, \{a, b\}) = \{(a, b)\}$ and $(a, x) \in f(R^{a,b}, \{a, x\})$ for all $b \in X \setminus \{a\}$, $x \in X \setminus \{a, b\}$. Hence, reinforcement shows that $f(R^a, \{a, x\}) = \{(a, x)\}$ for all alternatives $x \in X \setminus \{a\}$. By local agenda consistency, this means that $a \rhd x$ for all rankings $\rhd \in f(R^a, X)$ and alternatives $x \in X \setminus \{a\}$; otherwise, there is a ranking $\rhd' \in f(R^a, X)$ and an alternative $x \in X$ such that $x \rhd' a$ and a, x are consecutive in $\rhd'$. However, local agenda consistency now implies that $(x, a) \in f(R^a, \{a, x\})$, which contradicts our previous analysis. So, $f(R^a, X) \subseteq \{\rhd \in \mathcal{R}(X) \colon \forall x \in X \setminus \{a\} \colon a \rhd x\}$. Finally, it is easy to see that all alternatives $x \in X \setminus \{a\}$ are symmetric to each other. As a consequence, anonymity and neutrality turn this subset relation into an equality, which proves this lemma. □

Based on Lemma 1, we next show that f is non-imposing, i.e., for every feasible set $X \in \mathcal{F}(A)$ and every ranking $\rhd \in \mathcal{R}(X)$, there is a profile $R \in \mathcal{R}^*$ such that $f(R, X) = \{\rhd\}$.

Lemma 2. *The SPF f is non-imposing.*

Proof. Fix an arbitrary feasible set of alternatives $X = \{a_1, \dots, a_k\} \in \mathcal{F}(X)$ and ranking $\rhd \in \mathcal{F}(X)$. We will construct the profile $R^{\rhd}$ in which f uniquely chooses $\rhd = a_1, a_2, \dots, a_k$ inductively. In more detail, we will use an induction on $\ell \in \{1, \dots, |X|-1\}$ to construct a profile R^ℓ such that $f(R^\ell, X) = \{\rhd \in \mathcal{R}(X) \colon \forall x \in X \setminus \{a_1, \dots, a_\ell\} \colon a_1 \rhd \dots \rhd a_\ell \rhd x\}$. Less formally, f should return for R^ℓ all preference relations on X which rank the first ℓ alternatives like $\rhd$ and the remaining alternatives can be placed in every possible way. Furthermore, all alternatives $x \in X \setminus \{a_1, \dots, a_\ell\}$ will be completely symmetric in the profile R^ℓ. We note that this induction proves our lemma because at $\ell = |X| - 1$, $f(R^{|X|-1}, X) = \{\rhd\}$ is the only possible outcome.

Now, the induction hypothesis $\ell = 1$ follows directly from Lemma 1 as $f(R^{a_1}, X) = \{\rhd \in \mathcal{R}(X) \colon \forall x \in X \setminus \{a_1\} \colon a_1 \rhd x\}$. Next, suppose that there is a profile R^ℓ for $\ell < |X| - 1$ such that $f(R^\ell, X) = \{\rhd \in \mathcal{R}(X) \colon \forall x \in X \setminus \{a_1, \dots, a_\ell\} \colon a_1 \rhd \dots \rhd a_\ell \rhd x\}$. Moreover, we suppose that all alternatives in $X \setminus \{a_1, \dots, a_\ell\}$ are completely symmetric to each other in R^ℓ. As first point, we observe that $f(R^\ell, \{a_{\ell+1}, \dots, a_{|X|}\}) = \mathcal{R}(\{a_{\ell+1}, \dots, a_{|X|}\})$ because of local agenda consistency. Furthermore, consider the profile $R^{a_{\ell+1}}$ of Lemma 1 for which $f(R^{a_{\ell+1}}, \{a_{\ell+1}, \dots, a_{|X|}\}) = \{\rhd \in \mathcal{R}(\{a_{\ell+1}, \dots, a_{|X|}\}) \colon \forall x \in \{a_{\ell+2}, \dots, a_{|X|}\} \colon a_{\ell+1} \rhd x\}$. By reinforcement, we derive that for every $\lambda \in \mathbb{N}$ that $f(\lambda R^\ell + R^{a_{\ell+1}}, \{a_{\ell+1}, \dots, a_{|X|}\}) = \{\rhd \in \mathcal{R}(\{a_{\ell+1}, \dots, a_{|X|}\}) \colon \forall x \in \{a_{\ell+2}, \dots, a_{|X|}\} \colon a_{\ell+1} \rhd x\}$. Moreover, continuity implies that there is a $\lambda \in \mathbb{N}$ such that $f(\lambda R^\ell + R^{a_{\ell+1}}, X) \subseteq f(R^\ell)$. Now, by using again local agenda consistency, it follows that $f(\lambda R^\ell + R^{a_{\ell+1}}, X) \subseteq \{\rhd \in \mathcal{R}(X) \colon \forall x \in X \setminus \{a_1, \dots, a_{\ell+1}\} \colon a_1 \rhd \dots \rhd a_\ell \rhd a_{\ell+1} \rhd x\}$. Finally, we note that the alternatives in $\{a_{\ell+2}, \dots, a_{|X|}\}$ are completely symmetric in both R^ℓ and $R^{a_{\ell+1}}$, so anonymity and neutrality turn this subset relation into an equality. This proves the induction step and thus concludes the proof of this lemma. □

A.2. Separating Hypeplanes

After establishing that f is non-imposing, we will work towards deriving the bivariate scoring function of f. For doing so, we will use the separating hyperplane theorem for convex sets as, e.g., showcased by Young (1975); Young and Levenglick (1978). For this, we first will change the representation of f from preference profiles to a numerical space. Hence, let $b : \{1, \dots, m!\} \to \mathcal{R}(A)$ denote an enumeration of all possible preference relations. Based on b, we can present preference profiles simply as a vector $v \in \mathbb{N}^{m!}$ (we suppose $0 \in \mathbb{N}^{m!}$: the entry v_k states how often the ballot $b(k)$ appears. To indicate the vector corresponding to a profile R, we will simply write $v(R)$, so $v(R)_k$ states how many voters report $b(k)$ in R. Now, since f is anonymous, there is a (unique) function $g : \mathbb{N}^{m!} \times \mathcal{F}(A) \to \bigcup_{X\in\mathcal{F}(A)} \mathcal{F}(\mathcal{R}(X))$ such that $f(R, X) = g(v(R), X)$ for all profiles $R \in \mathcal{R}^*$ and feasible sets $X \in \mathcal{F}(A)$. Moreover, it is easy to see that g inherits the desirable properties from f:

- Neutrality: $g(\tau(v), \tau(X)) = \{\tau(\rhd) \colon \rhd \in g(v, X)\}$ for all vectors $v \in \mathbb{N}^{m!}$, feasible sets $X \in \mathcal{F}(A)$, and permutations $\tau : A \to A$. Here, $\tau(v)$ denotes the vector defined by $\tau(v)_\ell = v_k$ for all indices ℓ, k with $\tau(b(k)) = b(\ell)$. This ensures that $v(\tau(R)) = \tau(v(R))$.

- Reinforcement: $g(v + v', X) = g(v, X) \cap g(v', X)$ for all vectors $v, v' \in \mathbb{N}^{m!}$ and feasible sets $X \in \mathcal{F}(A)$ with $g(v, X) \cap g(v', X) \neq \emptyset$.

- Local Agenda Consistency: $g(v, X) \supseteq g(v, Y) \cap C(X, Y)$ for all $v \in \mathbb{N}^{m!}$ and all feasible sets $X, Y \in \mathcal{F}(A)$ with $X \subseteq Y$.

We next extend the domain of g to $\mathbb{Q}^{m!} \times \mathcal{F}(A)$ while preserving the desirable properties of f.

Lemma 3. *There is a neutral, reinforcing, and local agenda consistent function $\hat{g}$: $\mathbb{Q}^{m!} \times \mathcal{F}(A) \to \bigcup_{X\in\mathcal{F}(A)} \mathcal{F}(\mathcal{R}(X))$ such that $g(v, X) \subseteq \mathcal{R}(X)$ and $f(R, X) = g(v(R), X)$ for all profiles $R \in \mathcal{R}^*$ and feasible sets $X \in \mathcal{F}(A)$.*

Proof. First, since f is anonymous, we know that there is a function $g : \mathbb{N}^{m!} \times \mathcal{F}(A) \to \bigcup_{X\in\mathcal{F}(A)} \mathcal{F}(\mathcal{R}(X))$ that satisfies the requirements of the lemma. For extending the definition of this function to $\mathbb{Q}^{m!}$, we will heavily rely on the profile R^* in which every preference relation $\succ \in \mathcal{R}(A)$ is reported once. Moreover, let $v^* = v(R^*)$ and note that $f(R, X) = g(v(R, X)) = \mathcal{R}(X)$ for every feasible set $X \in \mathcal{F}(A)$ because of anonymity and neutrality. We will now proceed in two steps and first generalize g to $\mathbb{Z}^{m!}$ and then to $\mathbb{Q}^{m!}$.

Step 1: Extension to $\mathbb{Z}^{m!}$

For this step, we define the function $\bar{g}(v, X) = g(v + \lambda v^*, X)$ for all $v \in \mathbb{Z}^{m!}$ and $X \in \mathcal{F}(A)$, where $\lambda \in \mathbb{N}$ is an arbitrary scalar such that $v + \lambda v^* \in \mathbb{N}^{m!}$. Now, first note that $\bar{g}$ is indeed defined for all $v \in \mathbb{Z}^{m!}$ as for all such vectors, there is $\lambda \in \mathbb{N}$ such that $v + \lambda v^* \in \mathbb{N}^{m!}$. Next, we show that $\bar{g}$ is well-defined despite the fact that we do not fully specify λ. To this end, consider a feasible set $X \in \mathcal{F}(A)$, a vector $v \in \mathbb{Z}^{m!}$, and two values $\lambda_1, \lambda_2 \in \mathbb{N}$ such that $v + \lambda_1 v^* \in \mathbb{N}^{m!}$ and $v + \lambda_2 v^* \in \mathbb{N}^{m!}$. We will show that $g(v + \lambda_1 v^*, X) = g(v + \lambda_2 v^*, X)$, which entails that $\bar{g}$ is well-defined. Now, if $\lambda_1 = \lambda_2$, this is obvious. Hence, suppose without loss of generality that $\lambda_1 > \lambda_2$, which implies that $v + \lambda_1 v^* = v + \lambda_2 v^* + (\lambda_1 - \lambda_2)v^*$. Since $g((\lambda_1 - \lambda_2)v^*, X) = \mathcal{R}(X)$ due to neutrality, we infer that $g(v + \lambda_1 v^*, X) = g(v + \lambda_1 v^*, X) \cap g((\lambda_1 - \lambda_2)v^*, X) = g(v + \lambda_1 v^*, X)$, which proves our claim.

Furthermore, we note that $f(R, X) = g(v(R) + 0v^*, X) = \bar{g}(v(R), X)$ for all profiles $R \in \mathcal{R}^*$ and feasible sets $X \in \mathcal{F}(A)$, so $\bar{g}$ indeed represents f.

Next, we show that $\bar{g}$ satisfies our four properties. To this end, consider an arbitrary vector $v_1 \in \mathbb{Z}^{m!}$ and a scalar $\lambda_1 \in \mathbb{N}$ such that $v_1 + \lambda_1 v^* \in \mathbb{N}^{m!}$. First, for neutrality, we note that $\tau(v_1 + \lambda_1 v^*) = \tau(v_1) + \lambda_1 v^*$ for every permutation $\tau : A \to A$. This implies that $\bar{g}(\tau(v_1), \tau(X)) = g(\tau(v_1) + \lambda_1 v^*, \tau(X)) = g(\tau(v_1 + \lambda_1 v^*, \tau(X)) = \{\tau(\rhd)\colon \rhd \in g(v_1 + \lambda_1 v^*, X)\} = \{\tau(\rhd)\colon \rhd \in \bar{g}(v_1, X)\}$ because of the neutrality of g, so $\bar{g}$ also satisfies this axiom.

Next, we consider local agenda consistency and consider to this end two feasible sets $X, Y \in \mathcal{F}(A)$ with $X \subseteq Y$. Now, by the local agenda consistency of f, it is straightforward that $\bar{g}(v_1, X) = g(v_1 + \lambda_1 v^*, X) \supseteq g(v_1 + \lambda_1 v^*, Y) \cap C(X, Y) = \bar{g}(v_1, Y) \cap C(X, Y)$, which demonstrates that $\bar{g}$ satisfies this axiom, too.

Finally, for reinforcement, let $v_2 \in \mathbb{Z}^{m!}$ denote another vector and $\lambda_2 \in \mathbb{N}$ such that $v_2 + \lambda_2 v^* \in \mathbb{N}^{m!}$ and suppose that $\bar{g}(v_1, X) \cap \bar{g}(v_2, X) \neq \emptyset$ for some $X \in \mathcal{F}(A)$. Now, since g is reinforcing, it holds that $\bar{g}(v_1 + v_2, X) = g(v_1 + v_2 + (\lambda_1 + \lambda_2)v^*, X) = g(v_1 + \lambda_1 v^*, X) \cap g(v_2 + \lambda_2 v^*, X) = \bar{g}(v_1, X) \cap \bar{g}(v_2, X)$. This shows that $\bar{g}$ satisfies reinforcement.

Step 2: Extension to $\mathbb{Q}^{m!}$

In the second step, we will extend $\bar{g}$ to $\mathbb{Q}^{m!}$. To this end, we define $\hat{g}(v, X) = \bar{g}(\lambda v, X)$ for all vectors $v \in \mathbb{Q}^{m!}$ and feasible sets $X \in \mathcal{F}(A)$, where $\lambda \in \mathbb{N}$ is an arbitrary scalar such that $\lambda v \in \mathbb{Z}^{m!}$. First, we note that $\bar{g}$ is indeed defined for all $v \in \mathbb{Q}^{m!}$ because each such vector can be represented as $\frac{v'}{\lambda}$ for $v' \in \mathbb{Z}^{m!}$ and $\lambda \in \mathbb{N}$. Next, we will show that $\hat{g}$ is well-defined. For this, we consider a vector $v \in \mathbb{Q}^{m!}$ and two scalars $\lambda_1, \lambda_2 \in \mathbb{N}$ such that $v\lambda_1 \in \mathbb{Z}^{m!}$ and $v\lambda_2 \in \mathbb{Z}^{m!}$. Since $\bar{g}$ is reinforcing, it is now easy to verify that $\bar{g}(\lambda_1 v, X) = \bar{g}(\lambda_1 \lambda_2 v, X) = \bar{g}(\lambda_2 v, X)$, so $\hat{g}$ is well-defined.

Moreover, we note again that $\hat{g}$ represents f since $f(R, X) = \bar{g}(1 \cdot v(R), X) = \hat{g}(v(R), X)$ for all profiles $R \in \mathcal{R}^*$ and feasible sets $X \in \mathcal{F}(A)$.

As the last point of this lemma, we again verify that $\hat{g}$ satisfies all required axioms. To this end, we let $v_1 \in \mathbb{Q}^{m!}$ denote an arbitrary vector and $\lambda_1 \in \mathbb{N}$ a scalar such that $\lambda_1 v_1 \in \mathbb{Z}^{m!}$. Now, for showing that $\hat{g}$ is neutral, we note that $\lambda_1 v_1 \in \mathbb{Z}^{m!}$ implies that $\lambda_1 \tau(v_1) \in \mathbb{Z}^{m!}$ for every permutation $\tau : A \to A$. Hence, it holds that $\hat{g}(\tau(v_1), \tau(X_) = \bar{g}(\lambda_1 \tau(v_1), \tau(X)) = \bar{g}(\tau(\lambda_1 v_1), \tau(X)) = \{\tau(\rhd)\colon \bar{g}(\lambda_1 v_1, X)\} = \{\tau(\rhd)\colon \hat{g}(v_1, X)\}$ for every $X \in \mathcal{F}(A)$, so $\hat{g}$ is neutral.

Next, we show that $\hat{g}$ is locally agenda consistent, for which we consider two feasible sets $X, Y \in \mathcal{F}(A)$ with $X \subseteq Y$. It is easy to verify that $\hat{g}(v_1, X) = \bar{g}(\lambda_1 v_1, X) \supseteq \bar{g}(\lambda_1 v_1, Y) \cap C(X, Y) = \bar{g}(v_1, Y) \cap C(X, Y)$ because $\bar{g}$ is locally agenda consistent. This proves that $\hat{g}$ satisfies this condition, too.

For reinforcement, we consider again a second vector $v_2 \in \mathbb{Q}^{m!}$ and a scalar $\lambda_2 \in \mathbb{N}$ such that $\lambda_2 v_2 \in \mathbb{Z}^{m!}$. Moreover, suppose that $\hat{g}(v_1, X) \cap \hat{g}(v_2, X) \neq \emptyset$ for some $X \in \mathcal{F}(A)$. Since $\lambda_1\lambda_2(v_1 + v_2) \in \mathbb{Z}^{m!}$, $\bar{g}(\lambda_1\lambda_2 v_1, X) = \bar{g}(\lambda_1 v_1, X)$, and $\bar{g}(\lambda_1\lambda_2 v_2, X) = \bar{g}(\lambda_2 v_2, X)$, we derive that $\hat{g}(v_1 + v_2, X) = \bar{g}(\lambda_1\lambda_2(v_1 + v_2), X) = \bar{g}(\lambda_1\lambda_2 v_1, X) \cap \bar{g}(\lambda_1\lambda_2 v_2, X) = \bar{g}(\lambda_1 v_1, X) \cap \bar{g}(\lambda_2 v_2, X) = \hat{g}(v_1, X) \cap \hat{g}(v_2, X)$. Thus, $\hat{g}$ satisfies all our required axioms. □

Since $f(R, X) = \hat{g}(v(R), X)$ for all profiles $R \in \mathcal{R}^*$, we will from now on analyze the function $\hat{g}$. To this end, we define the sets $R_\rhd = \{v \in \mathbb{Q}^{m!} \colon \rhd \in \hat{g}(v, R)\}$ for all feasible sets $X \in \mathcal{F}(A)$ and rankings $\rhd \in \mathcal{R}(X)$. Moreover, we define $\bar{R}_\rhd$ as the closure of $R_\rhd$ with respect to $\mathbb{R}^{m!}$. First note that the sets $\bar{R}_\rhd$ are symmetric. In more detail, it holds that if $v \in \bar{R}_\rhd$, then $\tau(v) \in \bar{R}_{\tau(\rhd)}$ for every permutation $\tau : A \to A$. The reason for this is that $\hat{g}$ is neutral and thus, if $\rhd \in \hat{g}(v, X)$, then $\tau(\rhd) \in \hat{g}(\tau(v), \tau(X))$. Moreover, we note that $\bigcup_{\rhd \in \mathcal{R}(X)} R_\rhd = \mathbb{Q}^{m!}$ for every feasible set $X \in \mathcal{F}(A)$ as the domain of $\hat{g}$ is $\mathbb{Q}^{m!}$ for every feasible set X. Since there are only finitely many rankings in $\mathcal{R}(X)$, this also implies that $\bigcup_{\rhd \in \mathcal{R}(X)} \bar{R}_\rhd = \mathbb{R}^{m!}$ for all feasible sets $X \in \mathcal{F}(A)$. Finally, we note that the sets $R_\rhd$ are $\mathbb{Q}$-convex: it holds for all $v, v' \in R_\rhd$ and all $\lambda \in [0, 1] \cap \mathbb{Q}$ that $v\lambda + (1 - \lambda)v' \in \mathbb{R}_\rhd$. This follows from the reinforcement of $\hat{g}$ because $v, v' \in R_\rhd$ imply that $\rhd \in \hat{g}(v, X) = \hat{g}(\lambda v, X)$ and $\rhd \in \hat{g}(v', X) = \hat{g}((1 - \lambda)v', X)$. Moreover, if $R_\rhd$ is $\mathbb{Q}$-convex, then $\bar{R}_\rhd$ is known to the convex (see Young (1975)). In fact these sets are even convex cones because $v \in R_\rhd$ entails that $\lambda v \in R_\rhd$ for every $\lambda > 0$. Finally, since $\mathcal{R}(X)$ is finite and $\bigcup_{\rhd \in \mathcal{R}(X)} \bar{R}_\rhd = \mathbb{R}^{m!}$, we infer that the sets $\bar{R}_\rhd$ are fully dimensional.

We next show that these insights imply that for every feasible set $X \in \mathcal{F}(A)$ and rankings $\rhd, \rhd' \in \mathcal{R}(X)$, there is a non-zero vector that *separates* $\bar{R}_\rhd$ from $\bar{R}_{\rhd'}$. More formally, this means that there is a vector $u^{\rhd,\rhd'}$ such that $vu^{\rhd_i,\rhd_j} \geq 0$ if $v \in \bar{R}_{\rhd_i}$ and $vu^{\rhd_i,\rhd_j} \leq 0$ if $v \in \bar{R}_{\rhd_j}$ (recall that vu denotes the standard scalar product).

Lemma 4. *For every feasible set $X \in \mathcal{F}(A)$ and rankings $\rhd, \rhd' \in \mathcal{R}(X)$, there is a non-zero vector $u^{\rhd,\rhd'} \in \mathbb{R}^{m!}$ such that $vu^{\rhd,\rhd'} \geq 0$ if $v \in \bar{R}_{\rhd'}$ and $vu^{\rhd,\rhd'} \leq 0$ if $v \in \bar{R}_{\rhd'}$.*

Proof. Consider an arbitrary feasible set $X \in \mathcal{F}(A)$ and two distinct rankings $\rhd$, $\rhd' \in \mathcal{R}(X)$. Moreover, let $\bar{R}_\rhd$ and $\bar{R}_{\rhd'}$ be defined as explained before the lemma and recall that these sets are convex cones. For finding the non-zero vector $u^{\rhd,\rhd'}$ that separates $\bar{R}_\rhd$ from $\bar{R}_{\rhd'}$, we aim to apply the separation theorem for convex sets (see, e.g., Rockafeller, 1970). This requires us to show that $\mathrm{int}\bar{R}_\rhd \cap \mathrm{int}\bar{R}_{\rhd'} = \emptyset$.

Assume for contradiction that this is not the case. Since both $\bar{R}_\rhd$ and $\bar{R}_{\rhd'}$ are fully dimension, this means that there is a point $v \in \mathrm{int}\bar{R}_\rhd \cap \mathrm{int}\bar{R}_{\rhd'} \cap \mathbb{Q}^{m!}$. As first point, we note that $\mathrm{int}\bar{R}_\rhd$ is a subset of the convex hull of $R_\rhd$. Because $R_\rhd$ is $\mathbb{Q}$-convex and $v \in \mathbb{Q}^{m!}$, this entails that $v \in R_\rhd$. An analogous argument also shows that $v \in R_{\rhd'}$, so $\rhd, \rhd' \in \hat{g}(v, R)$. Next, we note that there is a profile R such that $f(R, X) = \{\rhd\}$ because of Lemma 2. By Lemma 3, it also holds that $\hat{g}(v', X) = \{\rhd\}$ for the vector $v' = v(R) \in \mathbb{N}^{m!}$. Finally, since $v \in \mathrm{int}\bar{R}_{\rhd'}$, there is $\lambda \in (0, 1) \cap \mathbb{Q}^{m!}$ such that $v + \lambda v' \in \mathrm{int}\bar{R}_{\rhd'}$. Now, using the same reasoning as for v, it follows that $v + \lambda v' \in R_{\rhd'}$. However, this means that $\rhd' \in g(v + \lambda v', X)$ but $\rhd' \notin g(v, X) \cap g(\lambda v', X) = \{\rhd\}$. This contradicts that $\hat{g}$ is reinforcing, so the assumption that $\mathrm{int}\bar{R}_\rhd \cap \mathrm{int}\bar{R}_{\rhd'} \neq \emptyset$ must be wrong.

Next, we note that both $\text{int}\bar{R}_{\triangleright}$ and $\text{int}\bar{R}_{\triangleright'}$ are non-empty because these sets are fully dimensional. Hence, we can now apply the separation theorem for convex sets to infer that there is a non-zero vector $u^{\triangleright,\triangleright'} \in \mathbb{R}^{m!}$ such that $vu^{\triangleright,\triangleright'} \geq 0$ if $v \in \bar{R}_{\triangleright}$ and $vu^{\triangleright,\triangleright'} \leq 0$ if $v \in \bar{R}_{\triangleright'}$. This completes the proof of this lemma. □

As the last lemma for this section, we show that the separating vectors $u^{\triangleright,\triangleright'}$ for $\triangleright' \in \mathcal{R}(X) \setminus \{\triangleright\}$ fully describe the set $\bar{R}_{\triangleright}$.

Lemma 5. *Consider a feasible set $X \in \mathcal{F}(A)$ and a ranking $\triangleright \in \mathcal{R}(X)$. It holds that $\bar{R}_{\triangleright} = \{v \in \mathbb{R}^{m!} \colon \forall \triangleright' \in \mathcal{R}(X) \setminus \{\triangleright\} \colon vu^{\triangleright,\triangleright'} \geq 0\}$, where $u^{\triangleright_i,\triangleright_j}$ are arbitrary non-zero vectors that separate $\bar{R}_{\triangleright}$ from $\bar{R}_{\triangleright'}$.*

Proof. Fix an arbitrary feasible set X and a ranking $\triangleright \in \mathcal{R}(X)$. Moreover, we define $S_{\triangleright} = \{v' \in \mathbb{R}^{m!} \colon \forall \triangleright' \in \mathcal{R}(X) \setminus \{\triangleright\} \colon v'u^{\triangleright,\triangleright'} \geq 0\}$ for a simpler notation. First, we note that, by definition, $vu^{\triangleright,\triangleright'} \geq 0$ for all $v \in \bar{R}_{\triangleright}$ and $\triangleright' \in \mathcal{R}(X) \setminus \{\triangleright\}$. Hence, $v \in \bar{R}_{\triangleright}$ implies $v \in S_{\triangleright}$, so $\bar{R}_{\triangleright} \subseteq S_{\triangleright}$.

For the other direction, we first note that $\text{int}S_{\triangleright} \neq \emptyset$ since $\text{int}\bar{R}_{\triangleright} \neq \emptyset$ and $\bar{R}_{\triangleright} \subseteq S_{\triangleright}$. Thus, consider a point $v \in \text{int}S_{\triangleright}$. This implies that $vu^{\triangleright,\triangleright'} > 0$ for all $\triangleright' \in \mathcal{R}(X) \setminus \{\triangleright\}$, which in turn means that $v \notin \bar{R}_{\triangleright'}$. Otherwise, we would have $vu^{\triangleright,\triangleright'} \leq 0$. Since $\bigcup_{\triangleright'' \in \mathcal{R}(X)} \bar{R}_{\triangleright''} = \mathbb{R}^{m!}$, this means that $v \in \bar{R}_{\triangleright}$. Hence, $\text{int}S_{\triangleright} \subseteq \bar{R}_{\triangleright}$ and, since $\bar{R}_{\triangleright}$ is closed, it also holds that $S_{\triangleright} \subseteq \bar{R}_{\triangleright}$. This completes the proof of this lemma. □

A.3. Feasible Sets of Size 2

Since the vectors $u^{\triangleright,\triangleright'}$ completely specify the sets $\bar{R}_{\triangleright}$, which in turn specify $\hat{g}$ and f, we will show that these vectors can be represented by a bivariate scoring function. For this, we will first prove that these vectors can be described by a bivariate scoring function if the feasible set has size 2.

Lemma 6. *There are non-zero vectors $\hat{u}^{(x,y),(y,x)}$ for all distinct $x, y \in A$ and a bivariate scoring function s such that $\hat{u}^{(x,y),(y,x)}$ separates $\bar{R}_{(x,y)}$ from $\bar{R}_{(y,x)}$ and $\hat{u}_k^{(x,y),(y,x)} = s(r(b(k), x), r(b(k), y))$ for all alternatives $x, y \in A$ and preference relations $b(k) \in \mathcal{R}(A)$.*

Proof. Before we define the bivariate scoring function s, we first need to show some insights on the vectors $u^{(x,y),(y,x)}$ that separate $\bar{R}_{(x,y),(y,x)}$. To this end, we consider an arbitrary pair of alternatives $x, y \in A$ and let $u^{(x,y),(y,x)}$ be the vector that separates $\bar{R}_{(x,y)}$ from $\bar{R}_{(x,y)}$ given by Lemma 4. We will first show that, up to scaling with a positive scalar, $u^{(x,y),(y,x)}$ is the only vector that separates $\bar{R}_{(x,y)}$ from $\bar{R}_{(y,x)}$. Consider for this an arbitrary second non-zero vector u that also separates $\bar{R}_{(x,y)}$ from $\bar{R}_{(y,x)}$. By Lemma 5, we get that $\bar{R}_{(x,y)} = \{v \in \mathbb{R}^{m!} \colon vu^{(x,y),(y,x)} \geq 0\} = \{v \in \mathbb{R}^{m!} \colon vu \geq 0\}$ and $\bar{R}_{(x,y)} = \{v \in \mathbb{R}^{m!} \colon -vu^{(x,y),(y,x)} \geq 0\} = \{v \in \mathbb{R}^{m!} \colon -vu \geq 0\}$. The second equation follows that since $-u^{(x,y),(y,x)}$ (resp. $-u$) separates $\bar{R}_{(y,x)}$ from $\bar{R}_{(x,y)}$. This means that $\bar{R}_{(x,y)} \cap \bar{R}_{(y,x)} = \{v \in \mathbb{R}^{m!} \colon vu^{(x,y),(y,x)} = 0\} = \{v \in \mathbb{R}^{m!} \colon vu = 0\}$. Hence, the vectors $u^{(x,y),(y,x)}$ and u are linearly dependent, i.e., there is a non-zero constant $\alpha \in \mathbb{R}$ such that $\alpha u^{(x,y),(y,x)} = u$. It is furthermore straightforward that $\alpha \geq 0$ as otherwise, we have

$vu^{(x,y),(y,x)} > 0$ for all $v \in \text{int}\bar{R}_{(x,y)}$ but $vu < 0$. Hence, u is indeed only a scaling of $u^{(x,y),(y,x)}$ by a positive scalar.

To get unique vectors, we now define $\hat{u}^{(x,y),(y,x)} = \alpha^{(x,y),(y,x)}u^{(x,y),(y,x)}$ for all pairs $x, y \in A$, where $\alpha^{(x,y),(y,x)} > 0$ is a scalar such that $|\hat{u}^{(x,y),(y,x)}|_2 = 1$. Now, first note that the vectors $-\hat{u}^{(y,x),(x,y)}$ separate $\bar{R}_{(x,y)}$ from $\bar{R}_{(y,x)}$ and satisfy that $|-\hat{u}^{(y,x),(x,y)}|_2 = 1$. By the uniqueness of our separating vectors we hence get that $\hat{u}^{(x,y),(y,x)} = -\hat{u}^{(y,x),(x,y)}$. Next, we observe that for every permutation $\tau : A \to A$, the vector $\tau(u^{(x,y),(y,x)})$ separates $\bar{R}_{\tau((x,y))}$ from $\bar{R}_{\tau((y,x))}$. This follows from the fact that for every $v \in \bar{R}_{\tau((x,y))}$, there is another vector $v' \in \bar{R}_{(x,y)}$ such that $\tau(v') = v$. Moreover, it is easy to verify that $0 \leq v'u^{(x,y),(y,x)} = \tau(v')\tau(u^{(x,y),(y,x)}) = v\tau(u^{(x,y),(y,x)})$. Since an analogous argument holds for $\bar{R}_{\tau((y,x))}$, $\tau(u^{(x,y),(y,x)})$ separates these two sets from each other. By the uniqueness of the separating vectors, we infer thus that $\hat{u}^{(\tau(x),\tau(y)),(\tau(y),\tau(x))} = \tau(\hat{u}^{(x,y),(y,x)})$

Based on these two insights, we can now define the bivariate scoring function s. To this end, we fix a pair of alternatives $x^*, y^* \in A$ and set $s(i,j) = \hat{u}_k^{(x^*,y^*),(y^*,x^*)}$ for all $i, j \in \{1, \dots, m\}$, where $b(k) \in \mathcal{R}(A)$ is an arbitrary preference relation such that $i = r(b(k), x^*)$ and $j = r(b(k), y^*)$. We first show that $\hat{u}_k^{(x,y),(y,x)} = s(r(b(k), x), r(b(k), y))$ for all $x, y \in A$ and $b(k) \in \mathcal{R}(A)$. To this end, consider arbitrary alternatives $x, y \in A$ and a preference relation $b(k)$ and let $b(k')$ denote the preference relation used to define $s(i,j)$ for $i = r(b(k), x)$ and $j = r(b(k), y)$, i.e., it holds that $r(b(k), x) = r(b(k'), x^*)$ and $r(b(k), y) = r(b(k'), y^*)$. Clearly, this implies that there is a permutation $\tau : A \to A$ such that $\tau(x^*) = x$, $\tau(y^*) = y$, and $\tau(b(k')) = b(k)$. Now, by our previous insight, we have that $\hat{u}^{(x,y),(y,x)} = \tau(\hat{u}^{(x^*,y^*),(y^*,x^*)})$. In particular, it holds that $\hat{u}_k^{(x,y),(y,x)} = \tau(\hat{u}^{(x^*,y^*),(y^*,x^*)})_k = \hat{u}_{k'}^{(x^*,y^*),(y^*,x^*)}$ because $\tau(b(k')) = b(k)$. Finally, this means that $\hat{u}_k^{(x,y),(y,x)} = s(r(b(k), x), r(b(k), y))$ for all $x, y \in A$ and $b(k) \in \mathcal{R}(A)$.

It remains to show that s is indeed a pairwise scoring rule. To this end, we first note that $s(i,j) = -s(j,i)$ because $s(r(b(k), x), r(b(k), y)) = \hat{u}_k^{(x,y),(y,x)} = -\hat{u}_k^{(y,x),(x,y)} = -s(r(b(k), y), r(b(k), x))$ for all alternatives $x, y \in A$ and preference relations $b(k) \in \mathcal{R}(A)$. Furthermore, the faithfulness of f entails that $s(i,j) \geq 0$ if $i < j$. To make this more precise, fix two alternatives x, y and consider a preference relation $\succ \in \mathcal{R}(A)$ with $x \succ y$. By faithfulness, it holds that $(x,y) \in f(\succ, \{(x,y)\}) = \hat{g}(v(\succ, \{(x,y)\})$, so $v(\succ) \in \bar{R}_{(x,y)}$. This means that $\hat{u}_k^{(x,y),(y,x)} \geq 0$ for the index k with $b(k) = \succ$, so also $s(i,j) \geq 0$ if $i < j$. Finally, since $\hat{u}^{(x,y),(y,x)}$ is non-zero, there are indices $i, j \in \{1, \dots, m\}$ with $i < j$ and $s(i,j) > 0$. □

We note that the vectors $\hat{u}^{(x,y),(y,x)}$ constructed in Lemma 6 are highly symmetric, and we thus will always work with these vectors from now. For simplicity, we will denote these vectors also by $u^{(x,y),(y,x)}$ (i.e., without the hat).

We next will investigate the linear independence of the vectors $u^{(x,y),(y,x)}$ constructed in Lemma 6 as this will be crucial for the analysis of larger feasible sets.

Lemma 7. *Consider a feasible set $X = \{a_1, \dots, a_\ell\} \in \mathcal{F}(A)$ with $\ell \geq 3$ and the sets $U_1^X = \{u^{(a_1,a_i),(a_i,a_1)} \colon i \in \{2, \dots, \ell\}\}$ and $U_2^X = \{u^{(a_i,a_j),(a_j,a_i)} \colon i, j \in \{1, \dots, \ell\} \colon i < j\}$. The following claims are true:*

(1) The set U_1^X is linearly independent.

(2) If U_2^X is linearly dependent, then $u^{(a_i,a_j),(a_j,a_i)} = u^{(a_i,a_1),(a_1,a_i)} + u^{(a_1,a_j),(a_1,a_j)}$ for all $i, j \in \{2, \dots, \ell\}$ with $i < j$.

Proof. Let $X \in \mathcal{F}(A)$ denote an arbitrary feasible set with at least three alternatives and let U_1^X and U_2^X be defined as in the lemma. We will show both claims separately and start with the linear independence of U_1^X.

Claim (1): The set U_1^X is linearly independent.

For proving this claim, we consider the profiles R^{a_i} constructed in Lemma 1 and recall that $f(R^{a_i}, \{a_1, a_i\}) = \{(a_i, a_1)\}$. By definition of $\bar{R}_{(a_i,a_1)}$, this means that $v(R^{a_i})$ is contained by this set. Even more, continuity and reinforcement entail that there is $\lambda \in \mathbb{N}$ such that $f(\lambda R^{a_i} + \succ, \{a_1, a_i\}) = \{(a_i, a_1)\}$ for all $\succ \in \mathcal{R}(A)$, so $v(R^{a_i}) \in \text{int } \bar{R}_{(a_i,a_1)}$. In turn, Lemma 5 then shows that $v(R^{a_i})u^{(a_i,a_1),(a_1,a_i)} > 0$. On the other hand, it holds by local agenda consistency that $f(R^{a_i}, \{a_1, a_j\}) = \{(a_1, a_j), (a_j, a_1)\}$ for all $a_j \in X \setminus \{a_1, a_i\}$, so $v(R^{a_i}) \in \bar{R}_{(a_1,a_j)} \cap \bar{R}_{(a_j,a_1)}$. This means that $v(R^{a_i})u^{(a_1,a_j),(a_j,a_1)} = 0$, so $u^{(a_1,a_i),(a_i,a_1)}$ is linearly independent of the other vectors in U_1^X. Since a_i is chosen arbtrarily, we infer that U_1^X is linearly independent.

Claim (2): If U_2^X is linearly dependent, then $u^{(a_i,a_j),(a_j,a_i)} = u^{(a_i,a_1),(a_1,a_i)} + u^{(a_1,a_j),(a_1,a_j)}$ for all $i, j \in \{2, \dots, \ell\}$ with $i < j$.

We prove this claim in multiple steps and therefore start with an auxiliary observations on $v(R^x)u^{(y,z),(z,y)}$. As the second step, we prove the claim for the case that $|X| = 3$, and in the last step, we derive the presentation also for $|X| \geq 4$.

Step 1: The goal of this step is to better understand the relation between the profile R^x constructed in Lemma 1 and the vectors $u^{(y,z),(z,y)}$. In particular, we will show that that $v(R^x)u^{(y,z),(z,y)} = 0$ and $v(R^x)u^{(x,y),(y,x)} = v(R^x)u^{(x,z),(z,x)} > 0$ for all distinct alternatives $x, y, z \in A$. For the first claim, we note that $f(R^x, \{y, z\}) = \{(y, z), (z, y)\}$ by local agenda consistency, so $v(R^x) \in \bar{R}_{(y,z),(z,y)} \cap \bar{R}_{(z,y),(y,z)}$. This means that $v(R^x)u^{(y,z),(z,y)} = 0$. For the second claim, we observe that $f(R^x, \{x, y\}) = \{(x, y)\}$ for all $y \in A \setminus \{y\}$, so $v(R^x) \in \bar{R}_{(x,y)}$. Moreover, due to continuity and reinforcement, there is $\lambda \in \mathbb{N}$ such that $f(\lambda R^x + \succ, \{x, y\}) = \{(x, y)\})$ for all $\succ \in \mathcal{R}$. This means that $v(R^x) \in \text{int } \bar{R}_{(x,y)}$ and Lemma 5 hence entails that $v(R^x)u^{(x,y),(y,x)} > 0$. Finally, we consider the permutation τ with $\tau(y) = z$, $\tau(z) = y$, and $\tau(w) = w$ for all other alternatives $w \in A \setminus \{y, z\}$. It follows by Lemma 6 that $u_k^{(x,z),(z,x)} = \tau(u_k^{(x,y),(y,x)} = u_{k'}^{(x,y),(y,x)}$ for all rankings $b(k), b(k') \in \mathcal{R}$ with $b(k) = \tau(b(k'))$ since $r(b(k), x) = r(b(k'), x)$ and $r(b(k), y) = r(b(k'), y)$, so $u^{(x,z),(z,x)} = \tau(u^{(x,y),(y,x)})$. Since also $\tau(v(R^x)) = v(R^x)$ due to the symmetry of this profile, it follows that $v(R^x)u^{(x,y),(y,x)} = \tau(v(R^x))\tau(u^{(x,y),(y,x)}) = v(R^x)u^{(x,z),(z,x)}$.

Step 2: Next, we will show Claim (2) for the case that $|X| = 3$. To this end, we fix a feasible set $X = \{a_1, a_2, a_3\}$ and suppose that U_2^X is linearly dependent. By Claim (1), we have that U_1^X is linearly independent, so there are values λ_2, λ_3 such that $u^{(a_2,a_3),(a_3,a_2)} = \lambda_2 u^{(a_1,a_2),(a_2,a_1)} + \lambda_3 u^{(a_1,a_3),(a_3,a_1)}$. Now, consider the profile R^{a_1}, for which Step 1 shows that $v(R^{a_1})u^{(a_2,a_3),(a_3,a_2)} = 0$ and $v(R^{a_1})u^{(a_1,a_2),(a_2,a_1)} = v(R^{a_1})u^{(a_1,a_3),(a_3,a_1)} > 0$. These observations entail that $\lambda_2 = -\lambda_3$. Finally, we consider the profile R^{a_2}: it holds that $v(R^{a_2})u^{(a_2,a_3),(a_3,a_2)} = v(R^{a_2})u^{(a_2,a_1),(a_1,a_2)} > 0$ and $v(R^{a_2})u^{(a_1,a_3),(a_3,a_1)} = 0$ by Step 1. Since $u^{(a_2,a_1),(a_1,a_2)} = -u^{(a_1,a_2),(a_2,a_1)}$, we can now easily infer that $\lambda_2 = -1$ and therefore $\lambda_3 = 1$. Thus, $u^{(a_2,a_3),(a_3,a_2)} = u^{(a_2,a_1),(a_1,a_2)} + u^{(a_1,a_3),(a_3,a_1)}$.

Step 3: As the last case, we suppose that $|X| \geq 4$ and that $u^{(x,y),(y,x)} \in U_2^X$ is a vector that is linearly depending on the other vectors in U_2^X. Furthermore, let $U_2^- = U_2 \setminus \{u^{(x,y),(y,x)}\}$. By the linear dependence, there are values $\lambda_{(x',y'),(y',x')}$, not all of which are 0, such that $u^{(x,y),(y,x)} = \sum_{u^{(x',y'),(y',x')} \in U_2^-} \lambda_{(x',y'),(y',x')} u^{(x',y'),(y',x')}$. Now, let $\mathcal{T}_1 = \{\tau \in A^A \colon \tau(x) = x \wedge \tau(y) = y\}$ denote the set of permutations that map x to x and y to y. By Lemma 6, we infer for all permutations $\tau \in \mathcal{T}_1$ and preference relations $b(k)$ and $b(k') = \tau(b(k))$ that $\tau(u^{(x,y),(y,x)})_{k'} = u_k^{(x,y),(y,x)} = u_{k'}^{(x,y),(y,x)}$ because $r(b(k), x) = r(b(k'), x)$ and $r(b(k), y) = r(b(k'), y)$. Hence, it holds that $u^{(x,y),(y,x)} = \tau(u^{(x,y),(y,x)}) = \sum_{u^{(x',y'),(y',x')} \in U_2^X} \lambda_{(x',y'),(y',x')} \tau(u^{(x',y'),(y',x')})$ for every $\tau \in \mathcal{T}_1$ and therefore also that $u^{(x,y),(y,x)} = \frac{1}{(m-2)!} \sum_{\tau \in \mathcal{T}_1} \sum_{u^{(x',y'),(y',x')} \in U_2^-} \lambda_{(x',y'),(y',x')} \tau(u^{(x',y'),(y',x')})$.

Next, we aim to simplify this representation by summing up the $\lambda_{(x',y'),(y',x')}$ associated with every vector $u^{(x',y'),(y',x')}$. More formally, for every pair of alternatives $v, w \in A$ we define $\lambda'_{(v,w),(w,v)} = \frac{1}{(m-2)!} \sum_{x',y' \in X} \sum_{\tau \in \mathcal{T}_1 \colon \tau(x') = v \wedge \tau(y') = w} \lambda_{(x',y'),(y',x')}$. It is straightforward that $u^{(x,y),(y,x)} = \sum_{x',y' \in A} \lambda'_{(x',y'),(y',x')} u^{(x',y'),(y',x')}$. We first observe that $\lambda'_{(x,y),(y,x)} = \lambda'_{(y,x),(x,y)} = 0$ because $\tau(x) = x$ and $\tau(y) = y$ for all $\tau \in \mathcal{T}_1$. Next, we consider two pairs of alternatives $v, w \in A \setminus \{x, y\}$ and $x', y' \in X \setminus \{x, y\}$. There are precisely $(m-4)!$ permutations in $\mathcal{T}_1$ with $\tau(x') = v$ and $\tau(y') = w$ and another $(m-4)!$ permutations with $\tau(x') = w$ and $\tau(y') = v$. Hence, $\lambda'_{(v,w),(w,v)} = \lambda'_{(w,v),(v,w)}$. Since Lemma 6 entails that $u_k^{(v,w),(w,v)} = -u_k^{(w,v),(v,w)}$ for all $k \in \{1, \dots, m!\}$, we can now cancel the summands that concern only alternatives $v, w \in A \setminus \{x, y\}$. More formally, this means that $u^{(x,y),(y,x)} = \sum_{z \in A \setminus \{x,y\}} \lambda'_{(x,z),(z,x)} u^{(x,z),(z,x)} + \lambda'_{(z,y),(y,z)} u^{(z,y),(y,z)}$. (Note here that we can always replace $u^{(z,x),(x,z)}$ with $-u^{(x,z),(z,x)}$ and push the minus into the λ' if required).

Furthermore, for all $z, z' \in A \setminus \{x, y\}$, there are precisely $(m-3)!$ permutations with $\tau(x) = x$, $\tau(y) = y$, and $\tau(z') = z$. Hence, each vector $u^{(x,z'),(z',x)}$ (resp. $u^{(z',x),(x,z')}$) is mapped to $u^{(x,z),(z,x)}$ equally often. This implies that $\lambda'_{(x,z),(z,x)} = \lambda'_{(x,z'),(z',x)}$ for all $z, z' \in A \setminus \{x, y\}$. An analogous argument also shows that $\lambda'_{(z,y),(y,z)} = \lambda'_{(z',y),(y,z')}$ for all $z, z' \in A \setminus \{x, y\}$. Finally, by considering again the profiles R^z of Lemma 1, we derive that $\lambda_{(x,z),(z,x)} = \lambda_{(z,y),(y,z)}$ since $v(R^z)u^{(x,y),(y,x)} = 0$ and $v(R^z)u^{(x,z),(z,x))} = -v(R^z)u^{(z,y),(y,z)} < 0$. Hence, there is $\lambda \neq 0$ such that $u^{(x,y),(y,x)} = \lambda \sum_{z \in A \setminus \{x,y\}} u^{(x,z),(z,x)} + u^{(z,y),(y,z)}$.

As our next step, we will derive a representation of $u^{(x,y),(y,x)}$ based on less "intermediate" alternatives. For this, let B denote the set of alternatives z such that $u^{(x,z),(z,x)}$ and $u^{(y,z),(z,y)}$ still appear in the presentation of $u^{(x,y),(y,x)}$. In particular, $B = A \setminus \{x, y\}$ in the beginning. We will now give a construction that allows to remove one alternative from B if $|B| > 1$. Thus, choose an arbitrary alternative $z \in B \setminus \{a_1\}$ and consider the permutations τ_1 and τ_2 that only swap x and z, and y and z, respectively. Lemma 6 entails for all alternatives $v, w, v', w' \in A$ that $u^{(v',w'),(w',v')} = \tau(u^{(v,w),(w,v)})$ if $\tau(v) = v'$ and $\tau(w) = w'$, so it holds that

$$\begin{aligned} u^{(z,y),(y,z)} &= \tau_1(u^{(x,y),(y,x)}) = \lambda \sum_{w\in B} \tau_1(u^{(x,w),(w,x)}) + \tau_1(u^{(w,y),(y,w)}) \\ &= \lambda u^{(z,x),(x,z)} + \lambda u^{(x,y),(y,x)} + \lambda \sum_{w\in B\setminus\{z\}} u^{(z,w),(w,z)} + u^{(w,y),(y,w)} \end{aligned}$$

$$\begin{aligned} u^{(x,z),(z,x)} &= \tau_2(u^{(x,y),(y,x)}) = \lambda \sum_{w\in B} \tau_2(u^{(x,w),(w,x)}) + \tau_2(u^{(w,y),(y,w)}) \\ &= \lambda u^{(x,y),(y,x)} + \lambda u^{(y,z),(z,y)} + \lambda \sum_{w\in B\setminus\{z\}} u^{(x,w),(w,x)} + u^{(w,z),(z,w)}. \end{aligned}$$

By summing up these two inequalities and dividing by 2λ, we infer that

$$\begin{aligned} \frac{1}{2\lambda}u^{(x,z),(z,x)} + \frac{1}{2\lambda}u^{(z,y),(y,z)} =& \frac{1}{2}u^{(z,x),(x,z)} + \frac{1}{2}u^{(y,z),(z,y)} + u^{(x,y),(y,x)} \\ &+ \frac{1}{2} \sum_{w\in B\setminus\{z\}} u^{(x,w),(w,x)} + u^{(w,y),(y,w)} \\ =& (\frac{1}{2} - \lambda)(u^{(z,x),(x,z)} + u^{(y,z),(z,y)}) \\ &+ (\frac{1}{2} + \lambda) \sum_{w\in B\setminus\{z\}} u^{(x,w),(w,x)} + u^{(w,y),(y,w)}. \end{aligned}$$

This simplifies to $(\frac{1}{2\lambda} - \lambda + \frac{1}{2})(u^{(x,z),(z,x)} + u^{(z,y),(y,z)}) = (\frac{1}{2} + \lambda) \sum_{w\in B\setminus\{z\}} u^{(x,w),(w,x)} + u^{(w,y),(y,w)}$. Now, first note that Claim (1) shows that $u^{(x,z),(z,x)} + u^{(z,y),(y,z)} \neq 0$ as these two vectors are linear independent. Furthermore, if $\sum_{w\in B\setminus\{z\}} u^{(x,w),(w,x)} + u^{(w,y),(y,w)} = 0$, then $u^{(x,y),(y,x)} = \lambda(u^{(x,z),(z,x)} + u^{(z,y),(y,z)})$ since $u^{(x,y),(y,x)} = \lambda \sum_{w\in B} u^{(x,w),(w,x)} + u^{(w,y),(w,x)}$. By applying the permutation τ that only swaps a_1 and z, we equivalently get that $u^{(x,y),(y,x)} = \lambda(u^{(x,a_1),(a_1,x)} + u^{(a_1,y),(a_1,z)})$ and we can then proceed with the steps in the subsequent paragraph. Hence, we next assume that $\sum_{w\in B\setminus\{z\}} u^{(x,w),w,x} + u^{w,y\to y,w} \neq 0$. For this case, we note that $(\frac{1}{2\lambda} - \lambda + \frac{1}{2}) = 0$ is only true if $\lambda = -\frac{1}{2}$ or $\lambda = 1$. Now, if $\lambda = 1$, then $0 = (\frac{1}{2} + 1) \sum_{w\in B\setminus\{z\}} u^{(x,w),(w,x)} + u^{(w,y),(y,w)}$, but this contradicts that $\sum_{w\in B\setminus\{z\}} u^{(x,w),(w,x)} + u^{(w,y),(y,w)} \neq 0$. On the other hand, $\lambda \neq -\frac{1}{2}$ as otherwise $v(R^x)u^{(x,y),(y,x)} > 0$ and $\lambda v(R^x) \sum_{w\in B} u^{(x,w),(w,x)} + u^{(w,y),(y,w)} < 0$. This follows from Step 1 as $v(R^x)u^{(x,w),(w,x)} > 0$ for every $w \in A \setminus \{x\}$ and $v(R^x)u^{(w,y),(y,w)} = 0$ for all $w \in A \setminus \{w, y\}$. Hence, we can now conclude that

$$u^{(x,z),(z,x)} + u^{(z,y),(y,z)} = \frac{\frac{1}{2} + \lambda}{\frac{1}{2\lambda} - \lambda + \frac{1}{2}} \sum_{w\in B\setminus\{z\}} u^{(x,w),(w,x)} + u^{(w,y),(y,w)}.$$

By setting $\lambda' = \lambda + \lambda\frac{\frac{1}{2}+\lambda}{\frac{1}{2\lambda}-\lambda+\frac{1}{2}}$, we then get that $u^{(x,y),(y,x)} = \lambda' \sum_{w\in B\setminus\{z\}} u^{(x,w),(w,x)} + u^{(w,y),(y,w)}$ and have thus removed z from our set B. We can clearly repeat this until $B = \{a_1\}$, so $u^{(x,y),(y,x)} = \lambda(u^{(x,a_1),(a_1,x)} + u^{(a_1,y),(y,a_i)}$ for some $\lambda \neq 0$.

As last step, we need to show that $\lambda = 1$. To this end, we consider again the vector $v(R^x)$: it holds that $v(R^x)u^{(x,y),(y,x)} = v(R^x)u^{(x,a_1),(a_1,x)} > 0$ and $v(R^x)u^{(a_1,y),(y,a_1)} = 0$ due to Step 1. So, it follows that $\lambda = 1$. Finally, we consider two arbitrary alternatives a_i, a_j with $i, j \in \{2, \dots, m\}$ and $i < j$ and let τ denote a permutation with $\tau(x) = a_i$, $\tau(y) = a_j$, and $\tau(a_1) = a_1$. It follows from Lemma 6 that $u^{(a_i,a_j),(a_j,a_i)} = \tau(u^{(x,y),(y,x)}) = \tau(u^{(x,a_1),(a_1,x)}) + \tau(u^{(a_1,y),(y,a_1)}) = u^{(a_i,a_1),(a_1,a_i)} + u^{(a_1,a_j),(a_1,a_j)}$, so the lemma is proven. $\square$

A.4. Feasible Sets of Size Greater 2

We now turn to feasible sets X with $|X| > 2$ and, as a first step, relate the hyperplanes between rankings for large feasible sets with smaller feasible sets. To this end, recall that $C(X,Y) = \{\succ \in \mathcal{R}(Y)\colon \forall x, y \in X, z \in Y \setminus X\colon z \succ x \text{ if and only if } z \succ y\}$ denotes the set of rankings on Y in which the alternatives in X appear consecutively.

Lemma 8. *Consider two feasible set $X, Y \in \mathcal{F}(A)$ and two rankings $\rhd, \rhd' \in \mathcal{R}(Y)$ such that $X \subseteq Y$ and $\rhd, \rhd' \in C_(X,Y)$. The vector $u^{\rhd|_X, \rhd'|_X}$ separates $\bar{R}_\rhd$ from $\bar{R}_{\rhd'}$.*

Proof. Consider two feasible $X, Y \in \mathcal{F}(A)$ and two rankings $\rhd, \rhd' \in \mathcal{R}(X)$ that satisfy the conditions of the lemma. The key insight for the this lemma is that $\hat{g}$ is locally agenda consistent (see Lemma 3). Thus, if $\rhd \in \hat{g}(v,Y) \cap C(X,Y)$ for some $v \in \mathbb{Q}^{m!}$, then $\rhd|_X \in \hat{g}(v,X)$. As a consequence, $v \in R_\rhd$ implies that $v \in \bar{R}_{\rhd|_X}$, which equivalently means that $R_\rhd \subseteq R_{\rhd|_X}$. From this, We can infer that $\bar{R}_\rhd \subseteq \bar{R}_{\rhd|_X}$ and a symmetric reasoning shows that $\bar{R}_{\rhd'} \subseteq \bar{R}_{\rhd'|_X}$. Now, consider a non-zero vector $u^{\rhd|_X, \rhd'|_X}$ which satisfies that $vu^{\rhd|_X, \rhd'|_X} \geq 0$ if $v \in \bar{R}_{\rhd|_X}$ and $vu^{\rhd|_X, \rhd'|_X} \leq 0$ if $v \in \bar{R}_{\rhd'|_X}$. Since $\bar{R}_\rhd \subseteq \bar{R}_{\rhd|_X}$ and $\bar{R}_{\rhd'} \subseteq \bar{R}_{\rhd'|_X}$, it follows immediately that this vector also separates $\bar{R}_\rhd$ from $\bar{R}_{\rhd'}$, which thus shows the lemma. $\square$

We note that Lemma 8 is rather general, but we will only use it for a particular case: given a feasible set X with $|X| \geq 3$ and two rankings $\rhd, \rhd' \in \mathcal{R}(X)$ such that $\rhd \setminus \rhd' = \{(x,y)\}$ for some pair of alternatives $x, y \in X$, Lemma 8 shows that $u^{(x,y),(y,x)}$ also separates $\bar{R}_\rhd$ from $\bar{R}_{\rhd'}$. Or, less formally, if we want to investigate the hyperplane between two rankings that differ only in a swap, we can restrict our attention exactly to the swapped alternatives. We thus assume from now that $u^{\rhd,\rhd'} = u^{(x,y),(y,x)}$ for all rankings $\rhd, \rhd'$ with $\rhd \setminus \rhd' = \{(x,y)\}$ for some pair of alternatives $x, y \in X$. This allows us to transfer the insights of Appendix A.3 to larger feasible sets.

We next aim to find a representation of the hyperplanes for rankings $\rhd, \rhd'$ with $|\rhd \setminus \rhd'| > 1$. To this end, we want to show that $u^{\rhd,\rhd'} = \sum_{(x,y)\in\rhd\setminus\rhd'} u^{(x,y),(y,x)}$ for all $\rhd, \rhd'$. For doing so, we will heavily rely on Lemma 7; we recall thus that $U_1^X = \{u^{(a_1,a_i),(a_i,a_1)}\colon i \in \{2, \dots, \ell\}\}$ and $U_2^X = \{u^{(a_i,a_j),(a_j,a_i)}\colon i, j \in \{1, \dots, \ell\}\colon i < j\}$ for an arbitrary feasible set $X = \{a_1, \dots, a_\ell\}$. Now, a consequence of Lemma 7 is that either the set U_2^X is linear independent, or every strict superset of U_1^X is linear dependent. We consider these cases separately and focus first on the simpler case that U_1^X is a maximal linear independent set.

Lemma 9. *Consider a feasible set $X \in \mathcal{F}(A)$ with $|X| \geq 3$ and two rankings $\rhd, \rhd' \in \mathcal{F}(X)$ and suppose that the set U_2^X is linearly dependent. The vector $\sum_{(x,y)\in \rhd \setminus \rhd'}$ separates $\bar{R}_{\rhd}$ from $\bar{R}_{\rhd'}$.*

Proof. Consider a feasible set $X = \{a_1, \dots, a_\ell\} \in \mathcal{F}(A)$ such that U_2^X is linearly dependent and two rankings $\rhd, \rhd' \in \mathcal{F}(X)$. Without loss of generality, we suppose that $\rhd = (a_1, \dots, a_\ell)$ and consider a point $v \in \bar{R}_{\rhd}$. By the local agenda consistency of $\hat{g}$, it follows that $v \in \bar{R}_{(a_i, a_{i+1})}$ for all $i \in \{1, \dots, \ell - 1\}$. This means that $vu^{(a_i,a_{i+1}),(a_{i+1},a_i)} \geq 0$. We will next show that $vu^{(a_1,a_{i+1}),(a_{i+1},a_1)} \geq vu^{(a_1,a_i),(a_i,a_1)} \geq 0$. To this end, we recall that $0 \leq vu^{(a_i,a_{i+1}),(a_{i+1},a_i)} = vu^{(a_i,a_1),(a_1,a_i)} + vu^{(a_1,a_{i+1}),(a_{i+1},a_1)}$ by Lemma 7. Since $u^{(a_i,a_1),(a_1,a_i)} = -u^{(a_1,a_i),(a_i,a_1)}$, our claim now follows. Now, consider two arbitrary alternatives $a_i, a_j \in X$ with $i < j$. Using Lemma 7 and our previous observation, we infer that $vu^{(a_i,a_j),(a_j,a_i)} = vu^{(a_i,a_1),(a_1,a_i)} + vu^{(a_1,a_j),(a_j,a_1)} \geq 0$ because $vu^{(a_1,a_j),(a_j,a_1)} \geq vu^{(a_1,a_i),(a_i,a_1)}$. This means that $vu^{(x,y),(y,x)} \geq 0$ for all x, y with $x \rhd y$, so $v \sum_{(x,y)\in \rhd \setminus \rhd'} u^{(x,y),(y,x)} \geq 0$.

Furthermore, an analogous argument for $\rhd'$ shows that $vu^{(x,y),(y,x)} \geq 0$ for all $v \in \bar{R}_{\rhd'}$ and $x, y \in X$ with $x \rhd' y$. Hence, it also holds that $v \sum_{(x,y)\in \rhd' \setminus \rhd} u^{(x,y),(y,x)} \geq 0$. Finally, the lemma follows by observing that $\sum_{(x,y)\in \rhd' \setminus \rhd} u^{(x,y),(y,x)} = -\sum_{(x,y)\in \rhd \setminus \rhd'} u^{(x,y),(y,x)}$ because $(x, y) \in \rhd \setminus \rhd'$ if and only if $(y, x) \in \rhd' \setminus \rhd$ and $u^{(x,y),(y,x)} = -u^{(y,x),(x,y)}$. $\square$

From the proof of Lemma 9, it can be easily inferred that the hyperplanes separating $\bar{R}_{\rhd}$ from $\bar{R}_{\rhd'}$ are not unique if $|\rhd \setminus \rhd'| > 1$. Moreover, it can be shown that, if U_2^X is linear dependent, then f is a positional scoring rule, so this insight corresponds to the fact that there is also no unique representation of positional scoring rule by a bivariate scoring function.

We next turn to the case that U_2^X is linear independent, which is significantly more involved. To this end, we will reuse some of the ideas presented by Young and Levenglick (1978). In particular, just as Kemeny's rule, every bivariate scoring rule can be seen as a linear optimization problem over a convex set. For formalizing this idea, we associate each ranking $\rhd \in \mathcal{R}(X)$ on some feasible set $X = \{a_1, \dots, a_{|X|}\}$ with the matrix $M^{\rhd} \in \mathbb{R}^{|X|\times|X|}$, which is defined by $M^{\rhd}_{i,j} = 1$ if $a_i \rhd a_j$, $M^{\rhd}_{i,j} = -1$ if $a_j \rhd a_i$, and $M^{\rhd}_{i,j} = 0$ if $i = j$. Moreover, let $\mathcal{M}$ denote the convex hull over the matrices $M^{\rhd}$ for all $\rhd \in \mathcal{F}^X$, i.e., $p \in \mathcal{M}$ if and only if there are non-negative scalars $\lambda_{\rhd}$ for all $\rhd \in \mathcal{R}(X)$ such that $\sum_{\rhd\in\mathcal{R}(X)} \lambda_{\rhd} = 1$ and $p = \sum_{\rhd\in\mathcal{R}(X)} \lambda_{\rhd} M^{\rhd}$. Given a profile R, it can now easily be checked that a bivariate scoring rule simply solves the optimization problem $\sum_{a_j,a_k\in X} M_{j,k}\hat{s}(R, (a_j, a_k))$ subject to $M \in \mathcal{M}$. In more detail, every optimal extreme point of the underlying polytope correponds to a chosen ranking of the bivariate scoring rule.

The reason why we are interested in this representation is the following basic fact of linear optimization: if a ranking $\rhd \in \mathcal{R}(X)$ is not chosen for some profile R, then there is a neighboring extreme point $M^{\rhd'}$ in $\mathcal{M}$ that achieves a higher objective value. To make this more formal, we call two rankings $\rhd, \rhd'$ neighbors of each other if the points $M^{\rhd}$ and $M^{\rhd'}$ are contained on a facet of dimension 1 of $\mathcal{M}$, and we write $\mathit{Neighbor}(\rhd)$ for the set of neighbors of $\rhd$. Based on this notion, we can compute a bivariate scoring rules only by considering the neighbors of every ranking, i.e., $f(R, X) = \{\rhd \in \mathcal{R}(X) \colon \forall \rhd' \in \mathit{Neighbor} \colon \hat{s}(\rhd, R) \geq \hat{s}(\rhd', R)\}$.

Based on this insight, it will turn out sufficient to investigate the vectors $u^{\rhd,\rhd'}$ only for neighboring rankings $\rhd \in \mathcal{R}(X)$, $\rhd' \in \mathit{Neighbor}(\rhd)$. This leads to the question of when two rankings are neighbors in the polytope $\mathcal{M}$. This question has been answered by Young and Levenglick (1978) who note that for all feasible sets $X = \{a_1, \dots, a_{|X|}\}$ and rankings $\rhd \in \mathcal{R}(X)$, the matrices $\bar{M}^{\rhd}$ defined by $\bar{M}^{\rhd}_{i,j} = 1$ if $a_i \rhd a_j$ and $\bar{M}^{\rhd}_{i,j} = 0$ otherwise is a simple linear transformation of $M^{\rhd}$. This means that two matrices $\bar{M}^{\rhd}, \bar{M}^{\rhd'}$ are neighbors in the respective polytope $\bar{\mathcal{M}}$ if and only if $M^{\rhd}, M^{\rhd'}$ are neighbors in $\mathcal{M}$. (In fact, it can be shown that the matrices $\bar{M}^{\rhd}$ can also be used to describe bivariate scoring rules). Moreover, the polytope defined by the matrices $\bar{M}^{\rhd}, \bar{M}^{\rhd'}$ is known as the permutation polytope, for which the neighborhood relation has been characterized by Gilmore and Hoffmann (1964) and Young (1978). To explain this characterization, we define the graph $G(\rhd, \rhd')$ for two rankings $\rhd, \rhd'$ on some feasible set $X \in \mathcal{F}(A)$ by the vertex set $V = \{\{a, b\} \in X^2 \colon a \rhd b \wedge b \rhd' a\}$ and the edge set $E = \{\{\{a, b\}, \{b, c\}\} \in V^2 \colon a \neq c \wedge \{a, c\} \notin V\}$. The characterization of Hoffmann then states that two rankings $\rhd, \rhd'$ are neighbors with respect to $\bar{\mathcal{M}}$ and therefore also with respect to $\mathcal{M}$ if and only if $G(\rhd, \rhd')$ is connected.

Based on our new notation and insights, we will now also analyze the vectors $u^{\rhd,\rhd'}$ for larger feasible sets.

Lemma 10. *Consider a set of alternatives X with $|X| \geq 3$, two rankings $\rhd, \rhd \in \mathcal{R}(X)$ with $\rhd' \in \mathit{Neighbor}(\rhd)$. If the set U_2^X is linearly independent, then the vector $\sum_{(x,y) \in \rhd \setminus \rhd'} u^{(x,y),(y,x)}$ separates $\bar{R}_{\rhd}$ from $\bar{R}_{\rhd'}$.*

Proof. Consider an arbitrary feasible set $X \in \mathcal{F}(A)$ such that $|X| \geq 3$ and U_2^X is linear independent and two neighboring rankings $\rhd, \rhd' \in \mathcal{R}(X)$. Now, if $\rhd \setminus \rhd' = 1$, the claim follows from Lemma 8 and the discussion after the lemma, so we suppose that $|\rhd \setminus \rhd'| \geq 2$. In this case, we proceed in multiple steps to prove the lemma. In more detail, we show first that the vector $u^{\rhd,\rhd'}$ is linearly dependent on U_2^X, which means that there are values λ_u such that $u^{\rhd,\rhd'} = \sum_{u \in U_2^X} \lambda_u u$. In our second step, we then show that $\lambda_u = 0$ for all $u = u^{(x,y),(y,x)} \in U_2^X$ with $(x, y) \in \rhd$ if and only if $(x, y) \in \rhd'$. Hence, $u^{\rhd,\rhd'} = \sum_{(x,y) \in \rhd \setminus \rhd'} \lambda_{(x,y),(y,x)} u^{(x,y),(y,x)}$ for some values $\lambda_{(x,y),(y,x)}$. Based on this insight, we proof the lemma in the third step for the special case that $\rhd$ differs from $\rhd'$ only by pushing an alternative two positions down (or up). In Step 4, we then generalize the insights to arbitrary movements of a single alternative, and in Step 5, we finally prove the lemma.

Step 1: As first step, we will show that the vector $u^{\rhd,\rhd'}$ is linearly dependent on U_2^X. For proving this claim, we assume for contradiction that this is not true. Now, consider the matrix M that contains all vectors in U_X^2 and the vector $u^{\rhd,\rhd'}$ as rows.

Since all rows in M are linearly independent, this matrix has full rank, so its image has full dimension. This means that there is a point $v \in \mathbb{R}^{m!}$ such that $v^{(x,y),(y,x)} = 1$ for all $x, y \in X$ with $x \rhd y$ and $u^{\rhd,\rhd'} = -1$. Now, by definition of the latter vector, we immediately get that $v \notin \bar{R}_{\rhd}$. On the other hand, for every other ranking $\rhd''$, there is a pair of alternatives $x, y \in X$ such that $x \rhd y$, $y \rhd'' x$, and $\rhd \in C(\{x, y\}, X)$. Now, by the choice of v, we also know that $v \notin \bar{R}_{(y,x)}$ and, since these alternatives are consecutive in $\rhd''$, Lemma 8 shows that $v \notin \bar{R}_{\rhd''}$, too. However, this means that $v \notin \bar{R}_{\rhd''}$ for any $\rhd'' \in \mathcal{R}(X)$, which contradicts that $\bigcup_{\rhd'' \in \mathcal{R}(X)} \bar{R}_{\rhd''} = \mathbb{R}^{m!}$. This shows that our initial assumption was wrong, so $u^{\rhd,\rhd'}$ is linearly dependent on U_2^X.

Step 2: As a consequence of Step 1, there are values λ_u for all $u \in U_2^X$, not all of which are 0, such that $u^{\rhd,\rhd'} = \sum_{u \in U_2^X} \lambda_u u$. As next step, we will show that $\lambda_u = 0$ for every vector $u = u^{(x,y),(y,x)} \in U_2^X$ such that $(x, y) \in \rhd$ if and only if $(x, y) \in \rhd'$. Once again, we assume for contradiction that this is not the case, which means that there are alternatives $a, b \in X$ such that either $(a, b) \in \rhd \cap \rhd'$ or $(a, b) \notin \rhd \cup \rhd$ but $\lambda_u \neq 0$ for $u = u^{(a,b),(b,a)}$. We subsequently focus on the case that $(a, b) \in \rhd \cap \rhd'$; if $(a, b) \notin \rhd \cup \rhd$, we can simply replace $u^{(a,b),(b,a)}$ with $u^{(b,a),(a,b)}$ and λ_u with $-\lambda_u$ in the presentation of $u^{\rhd,\rhd'}$ to arrive at the case that $(b, a) \in \rhd \cap \rhd'$.

Now, consider a vector $v \in \mathbb{R}^{m!}$ such that $vu = 0$ for all $u \in U_2^X \setminus \{u^{(a,b),(b,a)}\}$ and $vu^{(a,b),(b,a)} = 1$; such a vector exists as the set U_2^X is linearly independent. By contrast, we know that $u^{\rhd,\rhd'}$ linearly depends on U_2^X and that $\lambda_u \neq 0$. This means that $vu^{\rhd,\rhd'} \neq 0$ and we suppose first that $vu^{\rhd,\rhd'} < 0$. In this case, let $v' \in \mathbb{R}^{m!}$ denote a vector such that $u^{(a,b),(b,a)} = 0$ and $v'u^{(x,y),(y,x)} > 0$ for all other $x, y \in X$ with $x \rhd y$. Moreover, let $\epsilon > 0$ be a sufficiently small value that $v'' = v + \epsilon v'$ satisfies $v''u^{\rhd,\rhd'} < 0$. Now, since $v''u^{\rhd,\rhd'} < 0$, we infer that $v'' \notin \bar{R}_{\rhd}$. Furthermore, for every ranking $\rhd'' \in \mathcal{R}(X) \setminus \{\rhd\}$, there is a consecutive pair of alternatives x, y with $x \rhd'' y$ and $y \rhd x$. It is easy to verify that $v''u^{(y,x),(x,y)} > 0$, so $v'' \notin \bar{R}_{(x,y)}$ and Lemma 8 entails that $v'' \notin \bar{R}_{\rhd''}$. However, this means that $v'' \notin \bar{R}_{\rhd''}$ for any $\rhd'' \in \mathcal{R}(X)$, which contradicts that $\bigcup_{\rhd'' \in \mathcal{R}(X)} \bar{R}_{\rhd''} = \mathbb{R}^{m!}$. This is the desired contradiction, so $vu^{\rhd,\rhd'} < 0$ is not possible. Now, for the case that $vu^{\rhd,\rhd'} > 0$, we can simply replace $\rhd$ with $\rhd'$ in our construction. Then, we will construct a vector v'' such that $v'' \notin \bar{R}_{\rhd'}$ since $v''u^{\rhd,\rhd'} > 0$ and $v'' \notin \bar{R}_{\rhd''}$ for any other ranking $\rhd'' \in \mathcal{R}^X$ because there is a pair of consecutive alternatives (x, y) in $\rhd''$ such that $u^{(y,x),(x,y)} > 0$. Hence, it is also not possible that $vu^{\rhd,\rhd'} > 0$, so our initial assumption that $\lambda_u \neq 0$ must have been wrong.

Step 3: As third step, we will prove the lemma for rankings $\rhd, \rhd'$ such that $\rhd$ differs from $\rhd'$ by only moving an alternative for two positions. We thus suppose that $\rhd = \dots, a, b, c, \dots$, and $\rhd' = \dots, b, c, a, \dots$, i.e., $\rhd's$ is derived from $\rhd$ by moving a two positions down (the case of moving a up is symmetric). Now, by our last two steps, we know that $u^{\rhd,\rhd'} = \lambda_{(a,b),(b,a)} u^{(a,b),(b,a)} + \lambda_{(a,c),(c,a)} u^{(a,c),(c,a)}$ for some $\lambda_{(a,b),(b,a)}, \lambda_{(a,c),(c,a)} \in \mathbb{R}$. We will next show that $\lambda_{(a,b),(b,a)} = \lambda_{(a,c),(c,a)} > 0$. To this end, consider the vector $v \in \mathbb{R}^{m!}$ such that $vu^{(a,b),(b,a)} = 1$, $vu^{(b,c),(c,b)} = 1$, $vu^{(c,a),(a,c)} = 1$, and $vu^{(x,y),(y,x)} = 1$ for all other pairs of alternatives $x, y \in X$ with $x \rhd y$. Moreover, let τ denote the permutation defined by $\tau(a) = b$, $\tau(b) = c$, $\tau(c) = a$, and $\tau(x) = x$ for all other alternatives. It is easy to see that $\tau(v)\tau(u^{(x,y),(y,x)}) = vu^{(x,y),(y,x)}$ for all $x, y \in X$ since the alternatives a, b, c appear consecutively in $\rhd$ and a, b, c "form a cycle" with respect to v. Next, we define the vector $v^* = \frac{1}{3}(v + \tau(v) + \tau(\tau(v)))$ and observe that $v^*u^{(x,y),(y,x)} = vu^{(x,y),(y,x)}$ for all $x, y \in X$ due to the last insight. Moreover, it holds that $v^* = \tau(v^*)$, so a, b, and c are symmetric to each other in v^*.

Our goal is now to show that $v^* \in \bar{R}_{\rhd} \cap \bar{R}_{\rhd'}$ because we then get that $v^*u^{\rhd,\rhd'} = 0$. Since $v^*u^{(a,b),(b,a)} = 1$ and $v^*u^{(a,c),(c,a)} = -v^*u^{(c,a),(a,c)} = -1$, this proves that $\lambda_{(a,b),(b,a)} = \lambda_{(a,c),(c,a)}$. For doing so, consider an arbitrary ranking $\rhd''$ such that $v^* \in \bar{R}_{\rhd''}$; such a ranking exists as $\bigcup_{\rhd'' \in \mathcal{R}(X)} \bar{R}_{\rhd''} = \mathbb{R}^{m!}$. Now, suppose there is a pair of alternatives $x, y \in X$ with $\{x, y\} \not\subseteq \{a, b, c\}$ such that $x \rhd'' y$ and $y \rhd x$. By going now through the ranking $\rhd''$ starting at x, we can also find a consecutive pair of alternatives x', y' with $x' \rhd'' y'$ and $y' \rhd x'$. However, it holds that $v^*u^{(y',x'),(x',y')} = 1$, so $v^* \notin \bar{R}_{(x',y')}$ and Lemma 8 shows that $v^* \notin \bar{R}_{\rhd''}$ either. This shows that if $v^* \in \bar{R}_{\rhd''}$, then $x \rhd'' y$ if and only if $x \rhd y$ for all $x, y \in X$ with $\{x, y\} \not\subseteq \{a, b, c\}$. Consequently, $\rhd''$ can only contradict $\rhd$ on the order on a, b, and c. Moreover, using again Lemma 8 and the vectors $u^{(a,b),(b,a)}$, $u^{(b,c),(c,b)}$, and $u^{(c,a),(a,c)}$, we get that $v^* \notin \bar{R}_{\rhd''}$ for $\rhd'' \in \{(\dots, b, a, c, \dots), (\dots, a, c, b, \dots), (\dots, c, b, a, \dots)\}$. This means that $v^* \in \bar{R}_{\rhd''}$ for $\rhd'' = \rhd$, $\rhd'' = \rhd'$, or $\rhd'' = (\dots, c, a, b, \dots)$. Finally, by the symmetry of v^* and of the sets $\bar{R}_{\rhd}$, we derive that $v^* \in \bar{R}_{\rhd}$ if and only if $\tau(v^*) = v^* \in \bar{R}_{\rhd'}$ if and only if $\tau(v^*) = v^* \in \bar{R}_{(\dots,c,a,b,\dots)}$. Hence, v^* is in all three sets, so $v^* \in \bar{R}_{\rhd} \cap \bar{R}_{\rhd'}$ and $\lambda_{(a,b),(b,a)} = \lambda_{(a,c),(c,a)}$.

Since $u^{\rhd,\rhd'}$ is by definition a non-zero vector, $\lambda_{(a,b),(b,a)} = \lambda_{(a,c),(c,a)} \neq 0$. We will next show that $\lambda_{(a,b),(b,a)} = \lambda_{(a,c),(c,a)} > 0$. To see this, consider a point $v \in \mathbb{R}^{m!}$ such that $vu^{(x,y),(y,x)} = 1$ for all $x, y \in X$ with $x \rhd y$; in particular $vu^{(a,b),(b,a)} = vu^{(a,c),(c,a)} = 1$. Following a similar reasoning as in Steps 1 and 2, we infer that $v \in \bar{R}_{\rhd}$ and $v \notin \bar{R}_{\rhd''}$ for any other ranking $\rhd'' \in \mathcal{R}(X)$. This entails that $vu^{\rhd,\rhd'} = \lambda_{(a,b),(b,a)} + \lambda_{(a,c),(c,a)} \geq 0$, which is only the case if $\lambda_{(a,b),(b,a)} = \lambda_{(a,c),(c,a)} > 0$. Finally, since our separating vectors are invariant under scaling, the vector $u^{(a,b),(b,a)} + u^{(a,c),(c,a)}$ separates $\bar{R}_{\rhd}$ from $\bar{R}_{\rhd'}$.

Step 4: For our fourth step, we will generalize the insights of the last step to arbitrary shifts. For this, we assume that $\rhd = \dots, a, b_1, \dots, b_\ell, \dots$ and $\rhd' = \dots, b_1, \dots, b_\ell, a, \dots$, i.e., we derive $\rhd'$ from $\rhd$ by pushing down a for several positions. We note again that the case of moving up a is symmetric. If $\ell = 2$, we know from Step 3 that the vector $u^{(a,b_1),(b_1,a)} + u^{(a,b_2),(b_2,a)}$ separates $\bar{R}_{\rhd}$ from $\bar{R}_{\rhd'}$. We thus assume that $\ell \geq 3$. By Steps 1 and 2, we already have that $u^{\rhd,\rhd'} = \sum_{i \in \{1,\dots,\ell\}} \lambda_{(a,b_i),(b_i,a)} u^{(a,b_i),(b_i,a)}$ for some $\lambda_{(a,b_i),(b_i,a)} \in \mathbb{R}$. We next suppose for contradiction that there is $i \in \{1, \dots, \ell - 1\}$ such that $\lambda_{(a,b_i),(b_i,a)} < \lambda_{(a,b_{i+1}),(b_{i+1},a)}$. This means that there is an integer $k \in \mathbb{N}$ such that $k\lambda_{(a,b_{i+1}),(b_{i+1},a)} > k\lambda_{(a,b_i),(b_i,a)} + \sum_{x \in \{1,\dots,\ell\} \setminus \{i+1\}} \lambda_{(a,b_x),(b_x,a)}$. Now, consider a vector $v \in \mathbb{R}^{m!}$ such that $vu^{(a,b_i),(b_i,a)} = k+1$, $vu^{(a,b_{i+1}),(b_{i+1},a)} = -k$, $vu^{(b_j,b_{j'}),(b_{j'},b_j)} = k+1$ for all $j, j' \in \{1, \dots, \ell\}$ with $j < j'$, and $vu^{(x,y),(y,x)} = 1$ for all other alternatives $x, y \in X$ with $x \rhd y$. Such a vector exists as U_2^X is linearly independent. Our goal is to show that $v \notin \bar{R}_{\rhd''}$ for all $\rhd'' \in \mathcal{R}(X)$ as this contradicts that $\bigcup_{\rhd'' \in \mathcal{R}(X)} \bar{R}_{\rhd''} = \mathbb{R}^{m!}$. Hence, the initial assumption that $\lambda_{(a,b_i),(b_i,a)} < \lambda_{(a,b_{i+1}),(b_{i+1},a)}$ must have been wrong.

To this end, we first note that $vu^{\rhd,\rhd'} = v\sum_{j\in\{1,\dots,\ell\}}\lambda_{(a,b_j),(b_j,a)}u^{(a,b_j),(b_j,a)} = -k\lambda_{(a,b_{i+1}),(b_{i+1},a)} + k\lambda_{(a,b_i),(b_i,a)} + \sum_{x\in\{1,\dots,\ell\}\setminus\{i+1\}}\lambda_{(a,b_x),(b_x,a)} < 0$, so $v\notin\bar{R}_{\rhd}$. Now, consider an arbitrary ranking $\rhd''\neq\rhd$ and note that there is a consecutive pair of alternatives x,y in $\rhd''$ such that $x\rhd'' y$ and $y\rhd x$; otherwise $\rhd=\rhd''$. Now, if $x\neq b_{i+1}$ or $y\neq a$, then it holds that $vu^{(y,x),(x,y)}>0$. So, $v\notin\bar{R}_{(x,y)}$ and, by Lemma 8, $v\notin\bar{R}_{\rhd''}$ either. Hence, the only consecutive pair in $\rhd''$ that is ordered differently than in $\rhd$ is (a,b_{i+1}), i.e., we have $b_{i+1}\rhd'' a$. Next, let X^+ and X^- denote the alternatives that are ranked above and below a in $\rhd''$, respectively, and note that all alternatives in X^+ and X^- must be arranged as in $\rhd$ due to our previous insight. Moreover, there cannot be an alternative x with $x\rhd a$ in X^- because then $\rhd''=\dots,b_{i+1},a,x'$ for some x' with $x'\rhd a$. However, then $v\notin\bar{R}_{\rhd''}$ because $vu^{(x,a),(a,x)}=1$, so $x\rhd a$ implies $x\in X^+$.

Next, we proceed with another case distinction and suppose that there is an alternative $b_j\in X^-$ with $j\leq i$. By our previous insights, this means that such an alternative b_j is directly below a in $\rhd''$, i.e., $\rhd''=\dots,b_{i+1},a,b_j,\dots$. Now, consider the ranking $\hat{\rhd}=\dots,b_j,b_{i+1},a,\dots$ that is derived from $\rhd''$ by pushing up b_j two positions. It holds that $vu^{\hat{\rhd},\rhd''}=vu^{(b_j,b_{i+1}),(b_{i+1},b_j)}+vu^{(b_j,a),(a,b_j)}=k+1-1>0$ due to Step 3 and the construction of v, so $v\notin\bar{R}_{\rhd''}$. As second case, suppose that there is no alternative $b_j\in X^-$ with $j\leq i$. This implies that $b_i\in X^+$ and, since these alternatives are ordered just as in $\rhd$, we have that $\rhd''=\dots,b_i,b_{i+1},a,\dots$. In this case, we consider the ranking $\hat{\rhd}=\dots,a,b_i,b_{i+1},\dots$ and compute that $vu^{\hat{\rhd},\rhd''}=vu^{(a,b_i),(b_i,a)}+vu^{(a,b_{i+1}),(b_{i+1},a)}=k+1-k>0$. It thus follows again that $v\notin\bar{R}_{\rhd''}$. However, this shows $v\notin\bar{R}_{\rhd''}$ for any ranking $\rhd''\in\mathcal{R}(X)\setminus\{\rhd\}$ either. This gives the desired contradiction and hence, the assumption that $\lambda_{(a,b_i),(b_i,a)}<\lambda_{(a,b_{i+1}),(b_{i+1},a)}$.

As second case, we suppose that there is an index $i\in\{1,\dots,\ell-1\}$ such that $\lambda_{(a,b_i),(b_i,a)}>\lambda_{(a,b_{i+1}),(b_{i+1},a)}$. In this case, we note that the vector $-\sum_{j\in\{1,\dots,\ell\}}\lambda_{(a,b_j),(b_j,a)}u^{(a,b_j),(b_j,a)}$ separates $\bar{R}_{\rhd'}$ from $\bar{R}_{\rhd}$. Moreover, it holds that $-\lambda_{(a,b_i),(b_i,a)}<-\lambda_{(a,b_{i+1}),(b_{i+1},a)}$, so we can now apply the same argument as before by simply replacing $\rhd$ with $\rhd'$. Hence, we can conclude that $\lambda_{(a,b_i),(b_i,a)}=\lambda_{(a,b_j),(b_j,a)}$ for all $i,j\in\{1,\dots,\ell\}$. Moreover, since $u^{\rhd,\rhd'}\neq 0$, not all of these λs are 0. Finally, by considering the vector v with $vu^{x,y}=1$ if $x\rhd y$, we can infer just as in Step 3 that $\lambda_{(a,b_i),(b_i,a)}>0$ for all $i\in\{1,\dots,\ell\}$. Finally, by rescaling, we thus get that the vector $\sum_{j\in\{1,\dots,\ell\}}u^{(a,b_j),(b_j,a)}$ separates $\bar{R}_{\rhd}$ from $\bar{R}_{\rhd'}$.

Step 5: As last point, we consider arbitrary rankings $\rhd,\rhd'\in\mathcal{R}(X)$ such that $\rhd'\in \mathit{Neighbor}(\rhd)$ and we will show that $\sum_{(x,y)\in\rhd\setminus\rhd'}u^{(x,y),(y,x)}$ separates $\bar{R}_{\rhd}$ from $\bar{R}_{\rhd'}$. To this end, we recall that $u^{\rhd,\rhd'}=\sum_{(x,y)\in\rhd\setminus\rhd'}\lambda_{(x,y),(y,x)}u^{(x,y),(y,x)}$ due to Steps 1 and 2. Hence, this step follows again by showing that all $\lambda_{(x,y),(y,x)}$ are equal and positive. For this, we will use the characterization of Gilmore and Hoffmann (1964), which entails that the graph $G(\rhd,\rhd')$ (defined before this lemma) is connected as $\rhd$ and $\rhd'$ are neighbors. Now, consider two connected vertices $\{a,b\}$ and $\{a,c\}$ in $G(\rhd,\rhd')$. Since these sets are connected vertices of $G(\rhd,\rhd')$, we know that $a\rhd b$ if and only if $b\rhd' a$, $a\rhd c$ if and only if $c\rhd' a$, and $b\rhd c$ if and only if $b\rhd' c$. This entails that $(\rhd|_{\{a,b,c\}},\rhd'|_{\{a,b,c\}})\in\{((a,b,c),(b,c,a)),((a,c,b),(c,b,a)),((b,c,a),(a,b,c)),((c,b,a),(a,c,b))\}$. Since all these cases are symmetric, we subsequently suppose that $a \rhd b \rhd c$ and $b \rhd' c \rhd' a$ (we note that this is not the same case as in Step 3 as there can be alternatives between a, b, and c). We next aim to show that $\lambda_{(a,b),(b,a)} = \lambda_{(a,c),(c,a)}$. Since $G(\rhd, \rhd')$ is connected and the vertices of this graph correspond to $\rhd \setminus \rhd'$, we can repeatedly apply this argument to infer that $\lambda_{(x,y),(y,x)} = \lambda_{(x',y'),(y',x')}$ for all $(x, y), (x', y)' \in \rhd \setminus \rhd'$.

To prove this claim, we consider the vector v such that $vu^{(a,b),(b,a)} = vu^{(b,c),(c,b)} = vu^{(c,a),(a,c)} = 1$ and $vu^{(x,y),(y,x)} = 0$ for all other pairs of alternatives $x, y \in X$. Just as in Step 3, we can also find a vector v^* that is perfectly symmetric with respect to a, b, and c by defining $v^* = \frac{1}{3}(v + \tau(v) + \tau(\tau(v)))$ for the permutation τ with $\tau(a) = b$, $\tau(b) = c$, $\tau(c) = a$, and $\tau(x) = x$ for all other alternatives $x \in A \setminus \{a, b, c\}$. In particular, it is easy to check that $v^*u^{(a,b),(b,a)} = v^*u^{(b,c),(c,b)} = v^*u^{(c,a),(a,c)} = 1$ and $v^*u^{(x,y),(y,x)} = 0$ still holds. Our goal is to show that $v^* \in \bar{R}_{\rhd} \cap \bar{R}_{\rhd'}$ as this implies that $v^*u^{\rhd,\rhd'} = v^* \sum_{(x,y)\in\rhd\setminus\rhd'} \lambda_{(x,y),(y,x)} u^{(x,y),(y,x)} = \lambda_{(a,b),(b,a)} - \lambda_{(a,c),(c,a)} = 0$.

To this end, we observe that $v^* \notin \bar{R}_{\rhd''}$ for any $\rhd''$ with $b \rhd'' a \rhd'' c$ because of Step 4. In more detail, it holds for the ranking $\hat{\rhd}$, which is derived from $\rhd''$ by moving a over b, that $v^*u^{\hat{\rhd},\rhd''} = v^*u^{(a,b),(b,a)} = 1$. Hence, $v^* \notin \bar{R}_{\rhd''}$. A similar argument also rules out that $v^* \in \bar{R}_{\rhd''}$ for any $\rhd'' \in \mathcal{R}(X)$ with $c \rhd'' b \rhd'' a$ and $a \rhd'' c \rhd'' b$. Now, let $\rhd''$ denote a ranking with $v^* \in \bar{R}_{\rhd''}$; such a ranking has to exist since $\bigcup_{\rhd''\in\mathcal{R}(X)} = \bar{R}_{\rhd''}$. Moreover, we assume that $a \rhd'' b \rhd'' c$ as all other cases are symmetric. Finally, let x, y denote two consecutive alternatives in $\rhd''$ such that $x \rhd'' y$ and $\{x, y\} \not\subseteq \{a, b, c\}$. Our goal is to show that $v^* \in \bar{R}_{\hat{\rhd}}$ for the ranking $\hat{\rhd}$ derived from $\rhd''$ by swapping x and y. By repeatedly applying this argument, we derive that $v^* \in \bar{R}_{\bar{\rhd}}$ for every ranking $\bar{\rhd}$ with $a \bar{\rhd} b \bar{\rhd} c$. Moreover, since a, b, c are completely symmetric in v^* and the sets $\bar{R}_{\bar{\rhd}}$ are also symmetric, this entails that $v^* \in \bar{R}_{\bar{\rhd}}$ for all $\bar{\rhd}$ with $b \bar{\rhd} c \bar{\rhd} a$, which proves that $v^* \in \bar{R}_{\rhd} \cap \bar{R}_{\rhd'}$.

To prove this claim, we assume for contradiction that $v^* \notin \bar{R}_{\hat{\rhd}}$. This means that there is another ranking $\rhd^*$ such that $v^*\hat{u}^{\hat{\rhd},\rhd^*} < 0$ due to Lemma 5. To derive a contradiction, we introduce two auxiliary vectors: v_1 is a vector such that $v_1u^{(b,c),(c,b)} = -1$, $v_1vu^{(c,a),(a,c)} = -2$, and $v_1u^{(x,y),(y,x)} = 0$ for all other pairs of alternatives $x, y \in X$, and v_2 is a vector such that $v_2u^{(a,b),(b,a)} = v_2u^{(b,c),(c,b)} = v_2u^{(c,a),(a,c)} = 0$, and $v_2u^{(x,y),(y,x)} = 1$ for all other pairs of alternatives $x, y \in X$ with $x \hat{\rhd} y$. Again, such vectors exist as U_2^X is linearly independent. Next, we define v_1^* as $v_1^* = v^* + \epsilon v_1$, where $\epsilon > 0$ is so small that $v^*u^{\rhd_1,\rhd_2} < 0$ implies $v_1^*u^{\rhd_1,\rhd_2} < 0$ for all rankings $\rhd_1, \rhd_2 \in \mathcal{R}(X)$. By Lemma 5, this means that if $v^* \notin \bar{R}_{\rhd_1}$ for some ranking $\rhd_1$, then $v_1^* \notin \bar{R}_{\rhd_1}$, too. Moreover, we claim that $v_1^* \notin \bar{R}_{\rhd_1}$ for any ranking $\rhd_1$ with $b \rhd_1 c \rhd_1 a$ or $c \rhd_1 a \rhd_1 b$. For proving this, we first assume that $b \rhd_1 c \rhd_1 a$ and consider the ranking $\rhd_2$ which is derived by moving a over b. By Step 4 and the definition of v_1^*, we have that $v_1^*u^{\rhd_2,\rhd_1} = v_1^*u^{(a,b),(b,a)} + v_1^*u^{(a,c),(c,a)} = 1 - (1 - 2\epsilon) = 2\epsilon > 0$, so $v^* \notin \bar{R}_{\rhd_1}$. Similarly, if $c \rhd_1 a \rhd_1 b$, we consider the ranking $\rhd_2$ derived by moving c below b and infer that $v_1^*u^{\rhd_2,\rhd_1} = v_1^*u^{(a,c),(c,a)} + v_1^*u^{(b,c),(c,b)} = -(1 - 2\epsilon) + (1 - \epsilon) = \epsilon > 0$. Putting everything together, this shows that v_1^* can only be in sets $\bar{R}_{\rhd_1}$ for rankings $\rhd_1$ with $a \rhd_1 b \rhd_1 c$.

As the next step, we consider the vector v_2^* defined by $v_2^* = v_1^* + \epsilon' v_2$, where $\epsilon > 0$ is again so small that $v_1^*u^{\rhd_1,\rhd_2} < 0$ implies $v_2^*u^{\rhd_1,\rhd_2} < 0$ for all $\rhd_1, \rhd_2 \in \mathcal{R}(X)$. Now, consider an arbitrary ranking $\rhd_1$ such that $v_2^* \in \bar{R}_{\rhd_1}$. By the choice of ϵ' and our previous insights, we get that $a \rhd_1 b \rhd_1 c$ and $\rhd_1 \neq \hat{\rhd}$ (because $v^* \notin \bar{R}_{\hat{\rhd}}$). Furthermore, since $\rhd_1 \neq \hat{\rhd}$, there is at least one consecutive pair of alternatives $x, y \in X$ such that $x \rhd_1 y$, $y \mathrel{\hat{\rhd}} x$. Because $a \rhd_1 b \rhd_1 c$ and $a \mathrel{\hat{\rhd}} b \mathrel{\hat{\rhd}} c$, we also infer that $x, y \notin \{a, b, c\}$. Finally, it follows now that $v_2^* u^{(x,y),(y,x)} = \epsilon' v_2 u^{(x,y),(y,x)} < 0$ by definition, so Lemma 8 shows that $v_2^* \notin \bar{R}_{\rhd_1}$. However, this means that $v_2^* \notin \bar{R}_{\bar{\rhd}}$ for any $\bar{\rhd} \in \mathcal{R}(X)$, which is a contradiction. Hence, $v^* \in \bar{R}_{\rhd''}$ implies $v^* \in \bar{R}_{\hat{\rhd}}$.

In turn, we can now infer that $v^* \in \bar{R}_{\rhd_1}$ for all $\rhd_1 \in \mathcal{R}(X)$ with $a \rhd_1 b \rhd_1 c$. By the symmetry of v^* and our sets $\bar{R}_{\rhd_1}$, this means that $v^* \in \bar{R}_{\rhd_1}$ for all $\rhd_1 \in \mathcal{R}(X)$ with $b \rhd_1 c \rhd_1 a$, too. In particular, we thus have that $v^* \in \bar{R}_{\rhd} \cap \bar{R}_{\rhd'}$, so $v^* u^{\rhd_1, \rhd_2} = \lambda_{(a,b),(b,a)} - \lambda_{(a,c),(c,a)} = 0$. Hence, we derive now that $\lambda_{(a,b),(b,a)} = \lambda_{(a,c),(c,a)}$. By applying this argument to all edges of the graph $G(\rhd, \rhd')$, we then infer that $\lambda_{(x,y),(y,x)} = \lambda_{(x',y'),(y',x')}$ for all pairs of alternatives $(x, y), (x', y') \in \rhd \setminus \rhd'$. Next, not all $\lambda_{(x,y),(y,x)}$ can be 0 as $u^{\rhd, \rhd'}$ is a non-zero vector. Finally, by considering again the vector v with $vu^{(x,y),(y,x)} = 1$ for all $x, y \in X$ with $x \rhd y$, we infer that all $\lambda_{(x,y),(y,x)} > 0$ and rescaling proves the lemma. □

We are finally ready to prove our main theorem.

Theorem 1. *An SPF is a bivariate scoring rule if and only if it satisfies anonymity, neutrality, continuity, faithfulness, reinforcement, and local agenda consistency.*

Proof. Since the direction from right to left was proven in the main body, we focus on the converse. Thus, let f denote an SPF that satisfies all given axioms. First, by Lemma 3, there is a function $\hat{g}$ defined on $\mathbb{Q}^{m!} \times \mathcal{F}(A)$ that inherits all desirable properties of f and that satisfies $f(R, X) = \hat{g}(v(R), X)$ for all profiles $R \in \mathcal{R}^*$. Next, we define the sets $R_{\rhd} = \{v \in \mathbb{Q}^{m!} \colon \rhd \in \hat{g}(v, X)\}$ for every feasible set $X \in \mathcal{F}(A)$ and every ranking $\rhd \in \mathcal{R}(X)$. Moreover, we let $\bar{R}_{\rhd}$ denote the closure of every $R_{\rhd}$ with respect to $\mathbb{R}^{m!}$. It follows that $\hat{g}(v, X) = \{\rhd \in \mathcal{R}(X) \colon v \in R_{\rhd}\} \subseteq \{\rhd \in \mathcal{R}(X) \colon v \in \bar{R}_{\rhd}\}$. Now, in Lemmas 3 to 5, we show that we can describe these sets by non-zero vectors $u^{\rhd, \rhd'}$ that satisfy $vu^{\rhd, \rhd'} \geq 0$ if $v \in \bar{R}_{\rhd}$ and $vu^{\rhd, \rhd'} \leq 0$ if $v \in \bar{R}_{\rhd'}$: it holds that $\bar{R}_{\rhd} = \{v \in \mathbb{R}^{m!} \colon \forall \rhd' \in \mathcal{R}(X) \setminus \{\rhd\} \colon vu^{\rhd, \rhd'}\}$.

If $|X| = 2$, this means that the set $\bar{R}_{(x,y)}$ is the halfspace $\{v \in \mathbb{R}^{m!} \colon vu^{(x,y),(y,x)} \geq 0\}$. Moreover, Lemma 6 shows that these vectors can be described by a bivariate scoring function s: there is a bivariate scoring function s such that $u_k^{(x,y),(y,x)} = s(r(b(k), x), r(b(k), y)$ for all $x, y \in A$ and $k \in \{1, \dots, m!\}$. Consequently, it holds that $\bar{R}_{(x,y)} = \{v \in \mathbb{R}^{m!} \colon vu^{(x,y),(y,x)} \geq 0\} = \{v \in \mathbb{R}^{m!} \colon \hat{s}(v, (x, y)) \geq \hat{s}(v, (y, x)\}$, where $\hat{s}(v, (x, y)) = \sum_{k=1}^{m!} v_k s(r(b(k), x), r(b(k), y))$. This means that $\hat{g}(R, \{x, y\}) \subseteq \{\rhd \in \mathcal{R}(\{x, y\}) \colon \hat{s}(v, (x, y)) \geq \hat{s}(v, (y, x)\}$.

Now, for agendas of larger size, we restrict the set $\bar{R}_{\rhd}$ further: we define $\bar{R}_{\rhd}^N$ as $\bar{R}_{\rhd}^N = \{v \in \mathbb{R}^{m!} \colon \forall \rhd' \in \mathit{Neighbor}(\rhd) \colon vu^{\rhd, \rhd'} \geq 0\}$ and note that $\bar{R}_{\rhd} \subseteq \bar{R}_{\rhd}^N$. By Lemmas 9 and 10, we know that the vector $\sum_{(x,y) \in \rhd \setminus \rhd'} u^{(x,y),(y,x)}$ separates $\bar{R}_{\rhd}$ from $\bar{R}_{\rhd'}$ for all neighboring rankings $\rhd, \rhd'$ on some feasible set X. Since we know that the vectors $u^{(x,y),(y,x)}$ can be described by a bivariate scoring function s, this means that $\bar{R}_{\rhd}^N = \{v \in \mathbb{R}^{m!} \colon \forall \rhd' \in \mathit{Neighbor}(\rhd) \colon \hat{s}(\rhd, v) \geq \hat{s}(\rhd', v)\}$.

Next, we note that $\bar{R}^N_\rhd = \{v \in \mathbb{R}^{m!} \colon \forall \rhd' \in \mathit{Neighbor}(\rhd) \colon \hat{s}(\rhd, v) \geq \hat{s}(\rhd', v)\} = \{v \in \mathbb{R}^{m!} \colon \forall \rhd' \in \mathcal{R}(X)(\rhd) \colon \hat{s}(\rhd, v) \geq \hat{s}(\rhd', v)\}$ because we we can rewrite the right hand side as a linear optimization problem. In more detail, as explained before Lemma 10, we can associate every ranking $\rhd$ on a feasible set $X = \{a_1, \dots, a_\ell\}$ with the matrix $M^\rhd$ defined by $M^\rhd_{i,j} = 1$ if $a_i \rhd a_j$, $M^\rhd_{i,j} = -1$ if $a_j \rhd a_i$, and $M^\rhd_{i,j} = 0$ otherwise. Moreover, we define the set $\mathcal{M}$ as the convex hull over all $M^\rhd$ for $\rhd \in \mathcal{R}(X)$ and rewrite $\{v \in \mathbb{R}^{m!} \colon \forall \rhd' \in \mathcal{R}(X) \colon \hat{s}(\rhd, v) \geq \hat{s}(\rhd', v)\} = \{v \in \mathbb{R}^{m!} \colon M^\rhd \in \operatorname{argmax}_{M \in \mathcal{M}} \sum_{i,j \in \{1,\dots,\ell\}} M_{i,j} \hat{s}(v, (a_i, a_j))\}$ for every ranking $\rhd \in \mathcal{R}(X)$. It is a well-known fact in linear optimization that each non-optimal extreme point (the $M^\rhd$ in our case) has a neighbor with higher objective value. Since the neighbor relation of the matrices in $\mathcal{M}$ is equivalent to the neighborhood relationship of our rankings, we infer that $\{v \in \mathbb{R}^{m!} \colon \forall \rhd' \in \mathit{Neighbor}(\rhd) \colon \hat{s}(\rhd, v) \geq \hat{s}(\rhd', v)\} = \{v \in \mathbb{R}^{m!} \colon \forall \rhd' \in \mathcal{R}(X)(\rhd) \colon \hat{s}(\rhd, v) \geq \hat{s}(\rhd', v)\}$. Putting everything together, we derive that $f(R, X) = \hat{g}(v(R), X) \subseteq \{\rhd \in \mathcal{R}(X) \colon v(R) \in \bar{R}^N_\rhd\} = \{\rhd \in \mathcal{R}(X) \colon \forall \rhd' \in \mathit{Neighbor}(\rhd) \colon \hat{s}(v(R), \rhd) \geq \hat{s}(v(R), \rhd')\} = \{\rhd \in \mathcal{R}(X) \colon \forall \rhd' \in \mathcal{R}(X) \colon \hat{s}(v(R), \rhd) \geq \hat{s}(v(R), \rhd')\}$ (where $\hat{s}(v, \rhd) = \sum_{(x,y) \in \rhd} \hat{s}(v, (x, y)))$ for every profile $R \in \mathcal{R}^*$ and feasible set $X \in \mathcal{F}(A)$. Or, put differently, this proves that f returns always a subset of the bivariate scoring rule f' induced by s.

Finally, suppose for contradiction that $f(R, X) \subsetneq f'(R, X)$ for some profile $R \in \mathcal{R}^*$ and feasible set $X \in \mathcal{F}(A)$, and let $\rhd \in f'(R, X) \setminus f(R, X)$. Since f' satisfies all axioms requires of Lemma 2, there is a profile R' such that $f'(R', X) = \{\rhd\}$ and by our subset inclusion, we also have that $f(R', X) = \{\rhd\}$. In turn, by the reinforcement of f', it follows for every $\lambda \in \mathbb{N}$ that $f(\lambda R + R', X) = f'(\lambda R + R', X) = \{\rhd\}$. This, however, contradicts the continuity of f, which requires that there is a $\lambda \in \mathbb{N}$ such that $f(\lambda R + R', X) \subseteq f(R, X)$. Hence, it must hold that $f(R, X) = f'(R, X)$ for all profiles $R \in \mathcal{R}^*$ and feasible sets $X \in \mathcal{F}(A)$ and f is the bivariate scoring rule induced by s. $\square$