Consistent Probabilistic Social Choice

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DSS
Decision Sciences & Systems

Preliminaries

- Finite set of alternatives A — $\{a,b,c\}$
  - A is not fixed
- Linear preference relations $\succcurlyeq\, \in \mathscr{L}(A)$ — $a \succcurlyeq b \succcurlyeq c$
- Fractional preference profiles $R \in \Delta(\mathscr{L}(A)) = \mathscr{R}|_A$
  - $\Delta(\mathscr{L}(A))$ denotes the $(|A|!\text{-}1)$-dimensional unit simplex
  - implicitly assumes anonymity and homogeneity
  - will only consider rational fractions

½	⅓	⅙
a	a	b
b	c	c
c	b	a

- Fractional collective preference $R(x,y) = \sum_{\succcurlyeq \in \mathscr{L}(A):\, x \succcurlyeq y} R(\succcurlyeq)$ — $R(a,b)=⅚$
- Majority margin $g_R(x,y) = R(x,y) - R(y,x)$ — $g_R(a,b)=⅔$
- (Weak) Condorcet winner x with $g_R(x,y) \geq 0$ for all y — a

Probabilistic Social Choice Functions

- A *probabilistic social choice function (PSCF)* f maps a preference profile $R \in \mathscr{R}|_A$ to a non-empty subset of $\Delta(A)$.
  - f is (upper hemi-)continuous (*continuity*)
  - $f(R)$ is a convex set (*convexity*)
  - $R \in \mathscr{R}|_{\{x,y\}}$ and $R(x,y)=1$ imply $f(R) = \{x\}$ (*unanimity*)
  - $\{R \in \mathscr{R}|_A : |f(R)|=1\}$ is dense in $\mathscr{R}|_A$ (*decisiveness*)
- Non-probabilistic SCFs are PSCFs where, for all $R \in \mathscr{R}|_A$, $f(R)=\Delta(X)$ for some $X \subseteq A$.
- The axioms we propose for PSCFs coincide with classic axioms for the special case of non-probabilistic SCFs.

Random Dictatorship

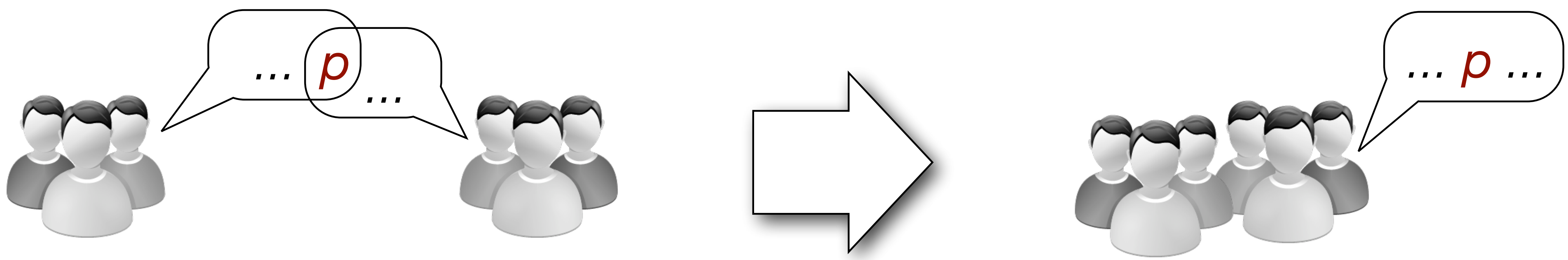
- One agent is picked uniformly at random and his most preferred alternative is implemented as the social choice.
- $RD(R) = \{ \sum_{\succcurlyeq \in \mathscr{L}(A)} R(\succcurlyeq) \cdot \max_{\succcurlyeq}(A) \}$

½	⅓	⅙
a	a	b
b	c	c
c	b	a

$RD(R) = \{⅚\, a + ⅙\, b\}$

- *RD* is a (single-valued) PSCF.
  - It satisfies continuity, convexity, unanimity, and decisiveness.

Population-Consistency



Whenever two disjoint electorates agree on a lottery,
this lottery should also be chosen by the union of both electorates.

Population-Consistency

½	½
a	*b*
b	*c*
c	*a*

R

½ *a* + ½ *b*

½	½
a	*b*
c	*c*
b	*a*

R'

½ *a* + ½ *b*

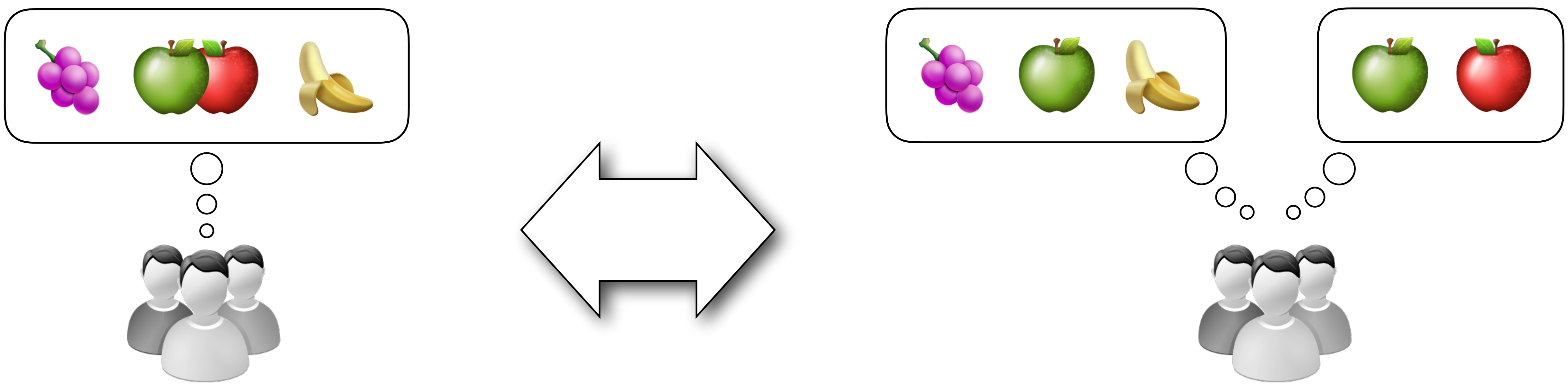
¼	¼	½
a	*a*	*b*
b	*c*	*c*
c	*b*	*a*

$\frac{1}{2} R + \frac{1}{2} R'$

½ *a* + ½ *b*

- $f(R) \cap f(R') \subseteq f(\lambda R + (1-\lambda)R')$
  - strong population-consistency requires equality (not only inclusion) whenever left-hand-side is non-empty
  - first proposed by Smith (1973), Young (1974), Fine & Fine (1974)
  - also known as “reinforcement” (Moulin, 1988)
  - variants used by Fishburn, Merlin, Myerson, Saari, etc.

Composition-Consistency



Decomposable preference profiles are treated component-wise.

Composition-Consistency

$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$
a	a	b
b'	b	b'
b	b'	a

R

$\frac{1}{2}\,a + \frac{1}{3}\,b + \frac{1}{6}\,b'$

$\frac{1}{2}$	$\frac{1}{2}$
a	b
b	a

$R|_{A'}$

$\frac{1}{2}\,a + \frac{1}{2}\,b$

$\frac{1}{3}$	$\frac{2}{3}$
b'	b
b	b'

$R|_B$

$\frac{2}{3}\,b + \frac{1}{3}\,b'$

- $f(R|_{A'}) \times_b f(R|_B) = f(R)$
  - Laffond, Laslier, and Le Breton (1996)
  - Cloning-consistency precursors: Arrow and Hurwicz (1972), Maskin (1979), Moulin (1986), Tideman (1987)

Chevalier de Borda

Non-Probabilistic Social Choice

Marquis de Condorcet

- All scoring rules satisfy population-consistency. (Smith 1973; Young, 1974)
- No Condorcet extension satisfies population-consistency. (Young and Levenglick, 1978)
- Many Condorcet extensions satisfy composition-consistency. (Laffond et al., 1996)
- No Pareto-optimal scoring rule satisfies composition-consistency. (Laslier, 1996)
- Theorem: There is no SCF that satisfies population-consistency and composition-consistency.
- But: These two axioms uniquely characterize a PSCF.


Germain Kreweras

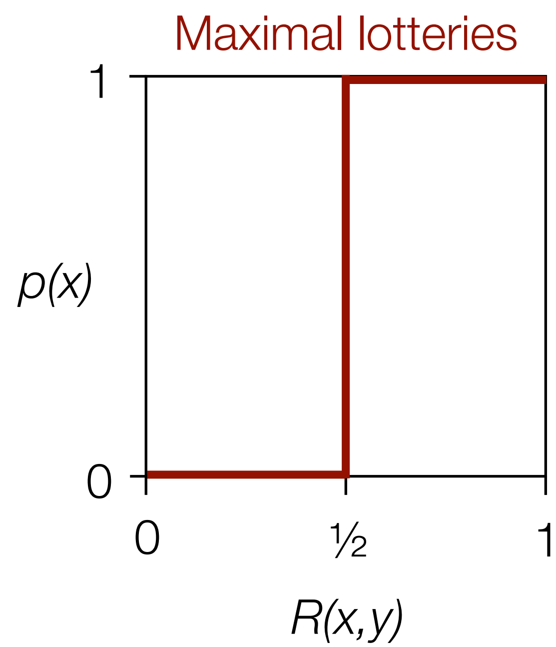
Maximal Lotteries

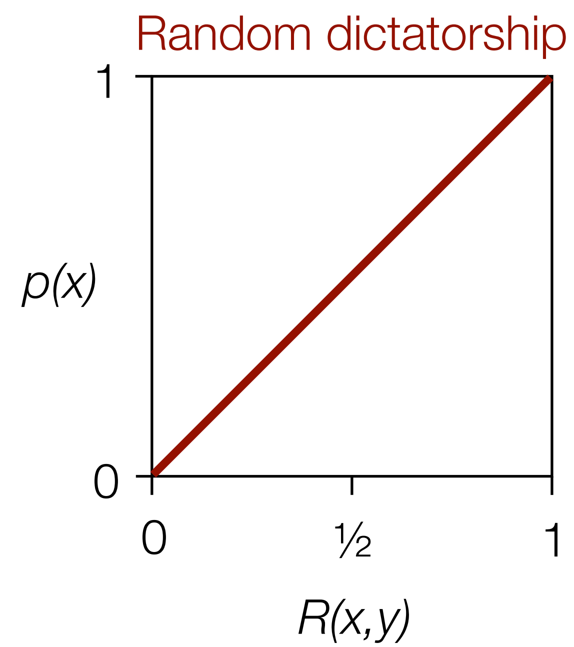

Peter C. Fishburn

- Kreweras (1965) and Fishburn (1984)
  - rediscovered by Laffond et al. (1993), Felsenthal and Machover (1992), Fisher and Ryan (1995), Rivest and Shen (2010)
- Extend g_R to lotteries: $g_R(p,q) = \sum_{x,y} p(x) \cdot q(y) \cdot g_R(x,y)$
  - fractional collective preferences over lotteries
- p is a maximal lottery, $p \in ML(R)$, if $g_R(p,q) \geq 0$ for all $q \in \Delta(A)$.
  - probabilistic Condorcet winner
  - always exists due to Minimax Theorem (v. Neumann, 1928)
- Set of profiles with unique maximal lotteries is open and dense.
  - set of profiles with multiple maximal lotteries is negligible
  - always unique for odd number of voters (Laffond et al., 1997)
  - generalized uniqueness conditions by Le Breton (2005)

Examples

- Two alternatives





- g_R can be interpreted as a symmetric zero-sum game.
  - Maximal lotteries are mixed minimax strategies.

2	**2**	**1**
a	*b*	*c*
b	*c*	*a*
c	*a*	*b*

	a	*b*	*c*
a	0	1	-1
b	-1	0	3
c	1	-3	0

  - The unique maximal lottery is *3/5 a + 1/5 b + 1/5 c*.

Main Result

- Theorem: A PSCF f satisfies population-consistency and composition-consistency iff f=ML.
- Proof structure:
  - Composition-consistency implies neutrality.
  - Two-alternative characterization (via three-alternative profiles)
  - Condorcet-consistency around uniform profile
  - $f \subseteq ML$.
    - Assume for contradiction that f yields a lottery that is not maximal.
    - Construct a Condorcet profile in which a uniform lottery is returned.
    - Derive a density violation.
  - $ML \subseteq f$.
    - For any vertex of the set of maximal lotteries in a profile, construct a sequence of profiles that converges to the original profile and whose unique maximal lotteries converge to the original maximal lottery.
    - Apply continuity and convexity.

Two-Alternative Proof

- Lemma: Let f be a composition-consistent PSCF and $A=\{x,y\}$. $p \in f(R)$ with $p \neq x,y$ implies $f(R)= \Delta(A)$.
  - Proof: Let $r=R(x,y)$.

r	**1-r**
x	y
y	x

$\lambda x + (1-\lambda)y$

r	**1-r**
x	y
x'	x'
y	x

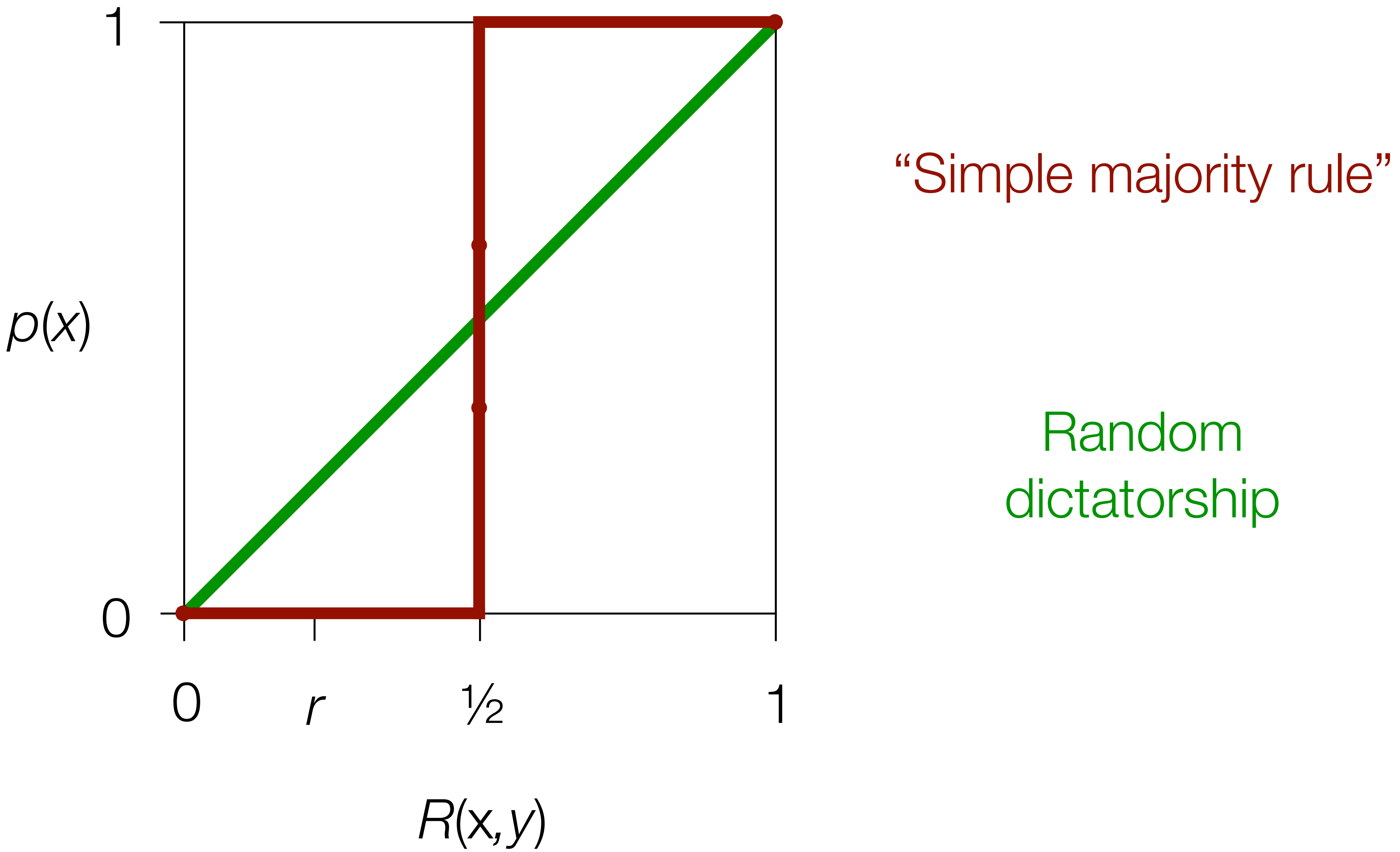
$\lambda^2 x + \lambda(1-\lambda)x' + (1-\lambda)y$

1	**1-r**
x	y
y	x

$\lambda^2 x + (1-\lambda^2)y$

  - Hence, $\lambda^2 x + (1-\lambda^2)y \in f(R)$.
  - Repeated application, continuity, and convexity imply the statement.
- As a consequence, *RD* violates composition-consistency.

Two-Alternative Proof (ctd.)



Remarks

- Independence of axioms
  - population, not composition: random dictatorship *RD*
  - composition, not population: maximal lotteries variant ML^3
- *ML* almost always satisfy strong population-consistency
- Composition-consistency can be weakened to cloning-consistency when also requiring Condorcet-consistency
  - *RD* satisfies cloning-consistency
- *ML* also satisfy agenda-consistency (Sen's α and γ)
- Axioms imply Fishburn's C2 (pairwiseness) as well as Condorcet-consistency.

Remarks (ctd.)

- Possible non-probabilistic interpretation of outcomes as fractional division (e.g., budget division, time shares)
  - Axioms are equally natural.
- Pareto-dominated alternatives always get zero probability in every maximal lottery.
  - In fact, *ML* is even SD-efficient (Aziz et al., 2012).
- *ML* does not require asymmetry, completeness, or even transitivity of preferences.
  - Random dictatorship requires unique maximum.
- In assignment domain, *ML* are known as popular mixed matchings (Kavitha et al., 2011).
- *ML* can be efficiently computed via linear programming.

	Maximal Lotteries	Random Serial Dictatorship	Borda's Rule
population-consistency	✅	only for strict prefs	✅
agenda-consistency	✅	✅	—
cloning-consistency	✅ even composition-consistency	✅	—
Condorcet-consistency	✅	—	—
(weak SD-) **strategyproofness**	—	✅	—
weak group-strategyproofness	✅	✅	—
(weak SD-) **participation**	✅ even for groups	✅ even very strongly	✅
(SD-) **efficiency**	✅	only for strict prefs otherwise only *ex post*	✅
efficient computability	✅	#P-complete in P for strict prefs	✅

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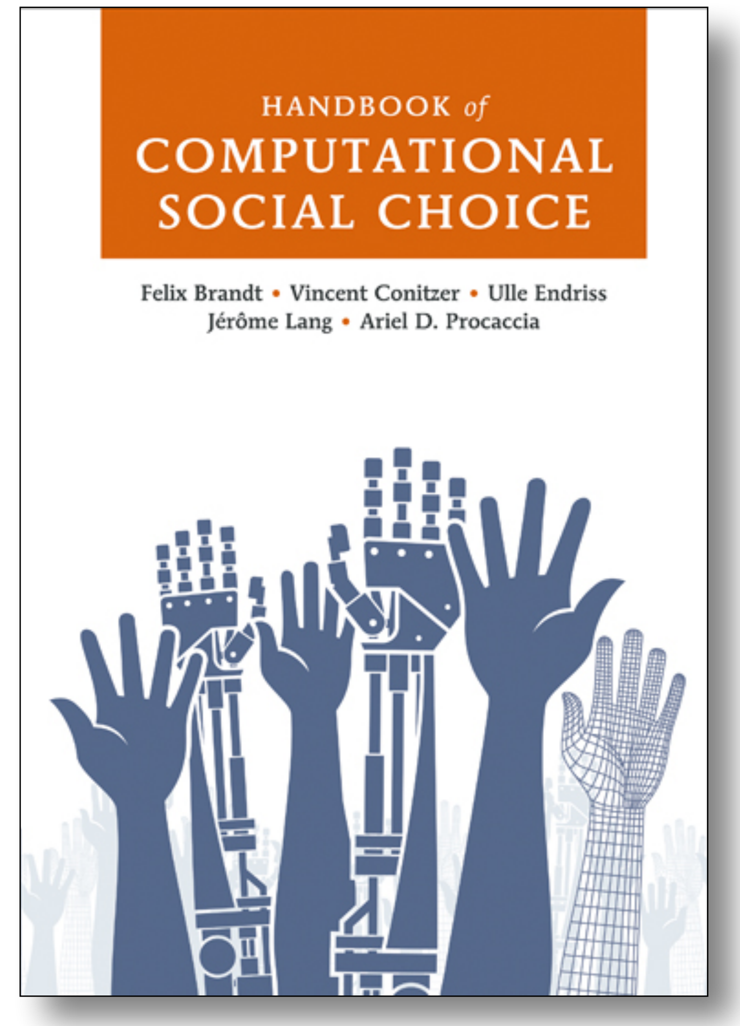

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