Strategyproof Social Decision Schemes on Super Condorcet Domains

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ABSTRACT

One of the central economic paradigms in multi-agent systems is that agents should not be better off by acting dishonestly. In the context of collective decision-making, this axiom is known as strategyproofness and turned out to be rather prohibitive, even when allowing for randomization. In particular, Gibbard's random dictatorship theorem shows that only rather unattractive social decision schemes (SDSs) satisfy strategyproofness on the full domain of preferences. In this paper, we obtain more positive results by investigating strategyproof SDSs on the Condorcet domain, which consists of all preference profiles that admit a Condorcet winner. In more detail, we show that, if the number of voters n is odd, every strategyproof and non-imposing SDS on the Condorcet domain can be represented as a mixture of dictatorial SDSs and the Condorcet rule (which chooses the Condorcet winner with probability 1). Moreover, we prove that, if n is odd, only mixtures of dictatorial SDSs satisfy strategyproofness and non-imposition on every sufficiently connected superset of the Condorcet domain. Finally, we extend these results to an even number of voters by characterizing the set of group-strategyproof and non-imposing SDSs on the Condorcet domain and its supersets. These characterizations strengthen the random dictatorship theorem and establish that the Condorcet domain is essentially a maximal domain that allows for attractive strategyproof SDSs.

KEYWORDS

Randomized Social Choice; Strategyproofness; Domain restriction; Condorcet domain

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1 INTRODUCTION

Strategyproofness—no agent should be better of by acting dishonestly—is one of the central economic paradigms in multi-agent systems [7, 35, 51]. A significant problem for such systems is collective decision-making, which aims at identifying socially desirable outcomes by letting voters express their preferences over the possible alternatives. A multitude of theorems has shown that even rather basic properties of voting rules cannot be satisfied simultaneously. In this context, strategyproofness is known to be a particularly restrictive axiom. This is exemplified by the Gibbard-Satterthwaite theorem which states that dictatorships are the only deterministic voting rules that satisfy strategyproofness and non-imposition (i.e., every alternative is elected for some preference profile). Since dictatorships are not acceptable for most applications, this result is commonly considered an impossibility theorem.

One of the most successful escape routes from the Gibbard-Satterthwaite impossibility is to restrict the domain of feasible preference profiles. For instance, Moulin [33] prominently showed that there are attractive strategyproof voting rules on the domain of single-peaked preference profiles, and various other restricted domains of preferences have been considered since then [e.g., 4, 22, 36, 47]. The idea behind domain restrictions is that the voters' preferences often obey structural constraints and thus, not all preference profiles are likely or plausible. A particularly significant constraint is the existence of a Condorcet winner which is an alternative that is favored to every other alternative by a majority of the voters. Apart from its natural appeal, this concept is important because there is strong empirical evidence that real-world elections usually admit Condorcet winners [25, 30, 44]. This motivates the study of the Condorcet domain which consists precisely of the preference profiles that have a Condorcet winner. Note that the Condorcet domain is a superset of several important domains such as those of single-peaked and single-dipped preferences when the number of voters is odd. There are several results showing the existence of attractive strategyproof voting rules on the Condorcet domain. In particular, Campbell and Kelly [10] characterize the Condorcet rule, which always picks the Condorcet winner, as the only strategyproof, non-imposing, and non-dictatorial voting rule on the Condorcet domain if the number of voters is odd.

In this paper, we focus on *randomized* voting rules, so-called social decision schemes (SDSs). Gibbard [27] has shown that randomization unfortunately does not allow for much more leeway beyond the negative consequences of the Gibbard-Satterthwaite theorem: random dictatorships, which select each voter with a fixed probability and elect the favorite alternative of the chosen voter, are the only SDSs on the full domain that satisfy strategyproofness and non-imposition (which in the randomized setting requires that every alternative is chosen with probability 1 for some preference profile). Thus, these SDSs are merely "mixtures of dictatorships".

In order to circumvent this negative result, we are interested in large domains that allow for strategyproof and non-imposing SDSs apart from random dictatorships. A natural candidate for this is the

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Condorcet domain and, indeed, we show that the Condorcet domain is essentially a maximal domain that allows for strategyproof, non-imposing, and "non-randomly dictatorial" social choice. In more detail, we prove that, if the number of voters n is odd, every strategyproof and non-imposing SDS on the Condorcet domain can be represented as a mixture of dictatorial SDSs and the Condorcet rule (which chooses the Condorcet winner with probability 1). This result entails that the Condorcet rule is the only strategyproof, non-imposing, and completely "non-randomly dictatorial" SDS on the Condorcet domain for odd n. Moreover, we show that—under a mild connectedness condition on the domain—random dictatorships are the only strategyproof and non-imposing SDSs on every superset of the Condorcet domain. Unfortunately, our theorems do not extend to an even number of voters because a single voter cannot change the Condorcet winner in this case. We address this problem by characterizing the set of group-strategyproof and non-imposing SDSs on the Condorcet domain and most of its supersets.

In summary, our results demonstrate two important insights: *(i)* the Condorcet domain is essentially a a maximal domain that allows for strategyproof, non-randomly dictatorial, and non-imposing SDSs, and *(ii)* the (deterministic) Condorcet rule is the most appealing strategyproof voting rule on this domain, even if we allow for randomization. Our characterizations can also be seen as attractive complements to classic negative results for the full domain, whereas our results for supersets of the Condorcet domain significantly strengthen these negative results. In particular, our theorems imply statements by Barberà [2] and Campbell and Kelly [10] as well as the Gibbard-Satterthwaite theorem [26, 50] and the random dictatorship theorem [27] when the number of voters is odd.

2 RELATED WORK

Restricting the domain of preference profiles in order to circumvent classic impossibility theorems has a long tradition and remains an active research area to date. In particular, the existence of attractive deterministic voting rules that satisfy strategyproofness has been shown for a number of domains. Classic examples include the domains of single-peaked [33], single-dipped [3], and single-crossing [47] preference profiles. More recent possibility results focus on broader but more technical domains such as the domains of multi-dimensionally single-peaked or semi single-peaked preference profiles [e.g., 4, 14, 34, 43]. On the other hand, domain restrictions are also used to strengthen impossibility results by proving them for smaller domains [e.g, 1, 28, 48]. In more recent research, the possibility and impossibility results converge by giving precise conditions under which a domain allows for strategyproof and non-dictatorial deterministic voting rules [14, 15, 18, 46].

While similar results have also been put forward for SDSs, this setting is not as well understood. For instance, Ehlers, Peters, and Storcken [20] have shown the existence of attractive strategyproof SDSs on the domain of single-peaked preference profiles [see also 41, 42]. The existence of strategyproof and non-imposing SDSs other than random dictatorships has also been investigated for a variety of other domains [39, 40, 45]. Chatterji, Sen, and Zeng [16] and Chatterji and Zeng [17] identify criteria for deciding whether a domain admits such SDSs.

The strong interest in restricted domains also led to the study of many computational problems for restricted domains [e.g., 6, 9, 19, 21, 23, 37, 38]. For instance, Bredereck, Chen, and Woeginger [9] give an algorithm for recognizing whether a preference profile is single-crossing, which can be used to decide whether the positive results apply for a given domain. For the Condorcet domain, this recognition problem can be solved efficiently because it is easy to verify the existence of a Condorcet winner.

Finally, observe that all aforementioned results are restricted to *Cartesian* domains, i.e., domains of the form $\mathcal{D} = \mathcal{X}^n$, where $\mathcal{X}$ is a set of preference relations. However, the Condorcet domain is not Cartesian. In this sense, the only results directly related to ours are the ones by Campbell and Kelly [10] and their follow-up work [11, 12, 32]. These papers can be seen as predecessors of our work since they investigate strategyproof deterministic voting rules on the Condorcet domain.

3 PRELIMINARIES

Let $N = \{1, \dots, n\}$ denote a finite set of voters and $A = \{a, b, \dots\}$ be a finite set of m alternatives. Every voter $i \in N$ is equipped with a *preference relation* R_i which is a complete, transitive, and anti-symmetric binary relation on A. We define $\mathcal{R}$ as the set of all preference relations on A. A *preference profile* $R \in \mathcal{R}^n$ consists of the preference relations of all voters $i \in N$. A domain of preference profiles $\mathcal{D}$ is a subset of the full domain $\mathcal{R}^n$. When writing preference profiles, we represent preference relations as comma-separated lists and indicate the set of voters who share a preference relation directly before the preference relation. For instance, $\{1, 2\}\colon a, b, c$ indicates that voters 1 and 2 prefer a to b to c. We omit the brackets for singleton sets.

The main object of study in this paper are *social decision schemes (SDSs)* which are voting rules that may use randomization to determine the winner of an election. More formally, an SDS maps every preference profile R of a domain $\mathcal{D}$ to a lottery over the alternatives, which determines the winning chance of every alternative. A *lottery* p is a probability distribution over the alternatives, i.e., $p(x) \geq 0$ for all $x \in A$ and $\sum_{x\in A} p(x) = 1$. We define $\Delta(A)$ as the set of all lotteries over A. An SDS on a domain $\mathcal{D}$ is then a function of the form $f : \mathcal{D} \to \Delta(A)$. Hence, SDSs are a generalization of deterministic voting rules which choose an alternative with probability 1 in every preference profile. The term $f(R, x)$ denotes the probability assigned to x by the lottery $f(R)$. For every set $X \subseteq A$ and lottery p, we define $p(X) = \sum_{x\in A} p(x)$; in particular $f(R, X) = \sum_{x\in X} f(R, x)$. Finally, an SDS $f : \mathcal{D} \to \Delta(A)$ is a *mixture of SDSs* $g_1, \dots, g_k$ if there are values $\lambda_i \geq 0$ for $i \in \{1, \dots, k\}$ such that $\sum_{i=1}^{k} \lambda_i = 1$ and $f(R) = \sum_{i=1}^{k} \lambda_i g_i(R)$ for all profiles $R \in \mathcal{D}$.

A natural desideratum for an SDS $f : \mathcal{D} \to \Delta(A)$ is *non-imposition* which merely requires that every alternative can be chosen with probability 1, i.e., for every alternative $x \in A$, there is a preference profile $R \in \mathcal{D}$ such that $f(R, x) = 1$. A prominent strengthening of non-imposition is *ex post* efficiency. For defining this axiom, we say an alternative $x \in A$ *Pareto-dominates* another alternative $y \in A \setminus \{x\}$ in a preference profile R if xR_iy for all voters $i \in N$. Then, an SDS $f : \mathcal{D} \to \Delta(A)$ is *ex post efficient* if $f(R, x) = 0$ for all alternatives $x \in A$ and preference profiles $R \in \mathcal{D}$ such that x is Pareto-dominated in R.

3.1 Stochastic Dominance and Strategyproofness

Strategic manipulation is one of the central issues in social choice theory: voters might be better off by voting dishonestly. Since satisfactory collective decisions require the voters' true preferences, SDSs should incentivize honest voting. In order to formalize this, we need to specify how voters compare lotteries over alternatives. The most prominent approach for this is based on (first order) stochastic dominance [e.g., 20, 27, 39]. Let the *upper contour set* $U(R_i, x) = \{y \in A \colon yR_ix\}$ be the set of alternatives that voter i weakly prefers to x in R. Then, (first order) stochastic dominance states that a voter i prefers a lottery p to another lottery q, denoted by $p \succsim_i q$, if $p(U(R_i, x)) \geq q(U(R_i, x))$ for all $x \in A$. Note that the stochastic dominance relation is transitive but not complete. Using stochastic dominance to compare lotteries is appealing because $p \succsim_i q$ holds iff p guarantees voter i at least as much expected utility than q for every utility function that is ordinally consistent with his preference relation R_i.

Based on stochastic dominance, we define multiple variants of strategyproofness. The standard notion requires that a single voter cannot benefit by lying about his true preferences. Formally, we say an SDS $f : \mathcal{D} \to \Delta(A)$ is *strategyproof* if $f(R) \succsim_i f(R')$ for all preference profiles $R, R' \in \mathcal{D}$ and voters $i \in N$ such that $R_j = R'_j$ for all $j \in N \setminus \{i\}$. Conversely, an SDS is called *manipulable* if it is not strategyproof. A convenient property of strategyproofness is that mixtures of strategyproof SDSs are again strategyproof.

A natural weakening of strategyproofness is local strategyproofness, which only disincentivizes manipulations by swapping two alternatives. In order to formalize this concept, we define $R^{i:yx}$ as the preference profile derived from another profile R by reinforcing y against x in the preference relation of voter i. Note that this definition requires that $x \neq y$ and that voter i ranks x directly above y in R. Then, an SDS $f : \mathcal{D} \to \Delta(A)$ is *locally strategyproof* if $f(R) \succsim_i f(R')$ for all voters $i \in N$, alternatives $x, y \in A$, and preference profiles $R, R' \in \mathcal{D}$ such that $R' = R^{i:yx}$. Local strategyproofness is an attractive weakening of strategyproofness since empirical results suggest that voters often manipulate by reporting such small and inconspicuous lies [24, 31].

In order to disincentivize *groups* of voters from manipulating, we need a stronger strategyproofness notion: an SDS $f : \mathcal{D} \to \Delta(A)$ is *group-strategyproof* if for all preference profiles $R, R' \in \mathcal{D}$ and all non-empty sets of voters $I \subseteq N$ with $R_j = R'_j$ for all $j \in N \setminus I$, there is a voter $i \in I$ such that $f(R) \succsim_i f(R')$. Conversely, an SDS is *group-manipulable* if it is not group-strategyproof. Note that group-strategyproofness implies strategyproofness and strategyproofness implies local strategyproofness.

3.2 γ-Randomly Dictatorial SDSs

The random dictatorship theorem shows that random dictatorships are the only non-imposing and strategyproof SDSs on the full domain [27]. For defining these functions, we call an SDS d_i *dictatorial* if it always assigns probability 1 to the most preferred alternative of voter i. Then, a *random dictatorship* f is a mixture of *dictatorial SDSs*. While they are more attractive than dictatorships, random dictatorships are often undesirable because they cannot compromise. We therefore interpret the random dictatorship theorem as a negative result.

An escape route from the random dictatorship theorem is to consider restricted domains of preference profiles. However, random dictatorships are still strategyproof and non-imposing on any reasonable subdomain and thus, the strategyproof and non-imposing SDSs that are not random dictatorships are of particular interest. Unfortunately, it is not straightforward how to define these "non-randomly dictatorial" SDSs because mixtures of strategyproof SDSs are again strategyproof. For instance, the SDS that returns with probability 0.99 the outcome of a random dictatorship and otherwise the outcome of another strategyproof SDS is strategyproof and no random dictatorship but it clearly feels "randomly dictatorial". To address this problem, Brandt, Lederer, and Romen [8] introduced the notion of γ-randomly dictatorial SDSs. A strategyproof SDS $f : \mathcal{D} \to \Delta(A)$ is *γ-randomly dictatorial* if $\gamma \in [0, 1]$ is the maximal value such that f can be represented as $f = \gamma d + (1-\gamma)g$, where d is a random dictatorship and g is a strategyproof SDS on $\mathcal{D}$. Note that if $\gamma = 1$, f is a random dictatorship independently of the choice of g, and if $\gamma < 1$, the maximality of γ implies that g is 0-randomly dictatorial. Intuitively, the notion of γ-random dictatorships quantifies how close an SDS is to a random dictatorship. Hence, 0-randomly dictatorial SDSs can be seen as a rigorous randomized equivalent to deterministic non-dictatorial voting rules.

Note that γ-random dictatorships are defined with respect to a specific strategyproofness notion, but the general idea of this axiom is independent of such details. Thus, we can use other strategyproofness notions to derive new variants of γ-random dictatorships. In particular, we call a locally strategyproof SDS $f : \mathcal{D} \to \Delta(A)$ *locally γ-randomly dictatorial* if $\gamma \in [0, 1]$ is the maximal value such that f can be represented as $f = \gamma d + (1-\gamma)g$, where d is a random dictatorship and g is a locally strategyproof SDS on $\mathcal{D}$. The intuition behind locally γ-randomly dictatorial SDSs is the same as for γ-random dictatorships, but local γ-random dictatorships are easier to handle because local strategyproofness is a much more restricted concept than strategyproofness. Also, since strategyproofness implies local strategyproofness, every γ-randomly dictatorial SDS is locally γ'-randomly dictatorial for $\gamma' \geq \gamma$.

3.3 Super Condorcet Domains

Since there are no attractive strategyproof rules on the full domain, we investigate the Condorcet domain and its super domains with respect to the existence of such SDSs. For defining these domains, we first have to introduce additional terminology. The *majority margin* $g_R(x, y) = |\{i \in N \colon xR_iy\}| - |\{i \in N \colon yR_ix\}|$ indicates how many more voters prefer x to y in the profile R than vice versa. Based on the majority margins, we define the *Condorcet winner* of a profile R as the alternative x such that $g_R(x, y) > 0$ for all $y \in A \setminus \{x\}$. Since the existence of a Condorcet winner is not guaranteed, we focus on the *Condorcet domain* $\mathcal{D}_C = \{R \in \mathcal{R}^n \colon \text{there is a Condorcet winner in } R\}$ which contains all profiles with a Condorcet winner. As explained in the introduction, this domain is of interest because of the observation that real-world elections frequently omit a Condorcet winner. A particularly natural

SDS on the Condorcet domain is the *Condorcet rule (COND)* which assigns always probability 1 to the Condorcet winner.

In this paper, we also investigate *super Condorcet domains*, which are domains $\mathcal{D}$ such that $\mathcal{D}_C \subseteq \mathcal{D}$. For our analysis, we need to define additional properties of super Condorcet domains. First, a super Condorcet domain $\mathcal{D}$ is *strict* if there is a profile $R \in \mathcal{D}$ such that for every alternative $x \in A$, there is another alternative $y \in A \setminus \{x\}$ with $g_R(y, x) > 0$. In other words, every strict super Condorcet domain contains at least one profile that does not even admit a weak Condorcet winner, i.e., an alternative that beats or ties every other alternative. If n is odd, every super Condorcet domain $\mathcal{D}$ with $\mathcal{D} \neq \mathcal{D}_C$ is strict because no majority ties are possible.

Super Condorcet domains $\mathcal{D}$ can be disconnected with respect to strategyproofness: there can be a profile $R \in \mathcal{D}$ that differs from every other profile $R' \in \mathcal{D} \setminus \{R\}$ in at least two preference relations. This is problematic for our analysis because strategyproofness has no implications for such isolated profiles. We address this issue by introducing connected super Condorcet domains. To this end, we first introduce the idea of ad-paths: an *ad-path* from a profile R to another profile R' in a domain $\mathcal{D}$ is a sequence of preference profiles $(R^1, \dots, R^l)$ such that $R^1 = R$, $R^l = R'$, $R^k \in \mathcal{D}$ for all $k \in \{1, \dots, l\}$, and the profile R^{k+1} evolves out of R^k by swapping two alternatives $x, y \in A$ in the preference relation of a voter $i \in N$, i.e, $R^{k+1} = (R^k)^{i:yx}$ for all $k \in \{1, \dots, l-1\}$. Then, a super Condorcet domain $\mathcal{D}$ is *connected* if for all alternatives $x \in A$ and profiles $R \in \mathcal{D} \setminus \mathcal{D}_C$, there is an ad-path from R to a profile $R' \in \mathcal{D}_C$ on which x is never swapped with another alternative. The intuition behind this conditions is the following: if n is odd, every alternative $x \in A$ loses the majority comparison against another alternative y in every profile $R \in \mathcal{D}_C \setminus \mathcal{D}$. The idea of connectedness is now that we can turn y into the Condorcet winner by applying a sequence of swaps that do not involve x. Note that connectedness is weaker than Sato's non-restoration property [49], which requires that there is an add-path between any two profiles along which no voter has to swap a pair of alternatives twice.

Finally, we introduce the concept of Condorcet-consistency: an SDS f on a super Condorcet domain $\mathcal{D}$ is *Condorcet-consistent* if $f(R) = COND(R)$ for all profiles $R \in \mathcal{D}_C \subseteq \mathcal{D}$. Hence, Condorcet-consistent SDSs extend the Condorcet rule to larger domains.

4 CHARACTERIZATIONS

In this section, we present our characterizations of strategyproof and group-strategyproof SDSs on super Condorcet domains. First, we discuss characterizations of strategyproof and non-imposing SDSs on connected super Condorcet domains for odd n and show that the Condorcet domain is essentially the only super Condorcet domain that allows for a strategyproof, 0-randomly dictatorial, and non-imposing SDS. We then extend these result to an even number of voters by strengthening strategyproofness to group-strategyproofness. Due to space restrictions, we defer the involved proofs of Lemmas 1, 3 and 4 to the supplementary material.

In order to prove our results, we first investigate the relationship between non-imposition and *ex post* efficiency on super Condorcet domains. It is well-known that these axioms are equivalent for strategyproof SDSs on the full domain, and we show next a similar result for (connected) super Condorcet domains.

Lemma 1. *Assume $m \geq 3$. The following claims are true.*

1. *Every non-imposing and locally strategyproof SDS on a connected super Condorcet domain is* ex post *efficient if n is odd.*
2. *Every non-imposing and group-strategyproof SDS on a super Condorcet domain is* ex post *efficient.*

Proof sketch. We first prove the claim relying on group-strategyproofness. Thus, let $\mathcal{D}$ denote an arbitrary super Condorcet domain, and let f denote a group-strategyproof and non-imposing SDS f on $\mathcal{D}$. Assume for contradiction that f violates *ex post* efficiency, which means that there are a profile $R \in \mathcal{D}$ and two alternatives $x, y \in A$ such that x Pareto-dominates y in R but $f(R, y) > 0$. Moreover, there is a profile $R' \in \mathcal{D}$ such that $f(R', x) = 1$ because f is non-imposing. Finally, consider a profile $R'' \in \mathcal{D}$ in which all voters top-rank x. Group-strategyproofness between R' and R'' requires that $f(R'', x) = 1$ as otherwise, the set of all voters N can group-manipulate by deviating from R'' to R'. On the other hand, group-strategyproofness between R and R'' requires that $f(R'', x) < 1$; otherwise, the set of all voters N can group-manipulate by deviating from R to R'' because all voters $i \in N$ prefer x to y and thus $f(R'', U(R_i, x)) > f(R, U(R_i, x))$. These observations contradict each other, which proves that f is *ex post* efficient. A similar argument shows that every locally strategyproof and non-imposing SDS on a connected super Condorcet domain is *ex post* efficient as we can break up the group-manipulations into multiple swaps. □

4.1 Results based on Strategyproofness

Our first result shows that every strategyproof and non-imposing SDS on the Condorcet domain is a mixture of a random dictatorship and the Condorcet rule if n is odd. As a byproduct, we obtain a characterization of the Condorcet rule as the only strategyproof, non-imposing, and 0-randomly dictatorial SDS on the Condorcet domain for odd n. By contrast, we show for every strict and connected super Condorcet domain $\mathcal{D}$ that random dictatorships are the only strategyproof and non-imposing SDSs on $\mathcal{D}$ if n is odd.

In order to derive these characterizations, we first investigate the notion of local γ-random dictatorships. In particular, we are interested in the locally 0-randomly dictatorial, locally strategyproof, and non-imposing SDSs on a domain $\mathcal{D}$ because these are the new SDSs gained by considering a restricted domain. Indeed, we prove next that these SDSs characterize the set of all locally strategyproof and non-imposing SDSs for connected super Condorcet domains.

Lemma 2. *Assume $n \geq 3$ is odd and $m \geq 3$. An SDS f on a connected super Condorcet domain $\mathcal{D}$ is locally strategyproof and non-imposing iff it is a random dictatorship or there are $\gamma \in [0, 1)$, a random dictatorship d, and a locally 0-randomly dictatorial, locally strategyproof, and non-imposing SDS g on $\mathcal{D}$ such that $f = \gamma d + (1 - \gamma) g$.*

Proof. Assume $n \geq 3$ is odd and consider a connected super Condorcet domain $\mathcal{D}$. We first show the direction from left to right and let f thus denote a locally strategyproof and non-imposing SDS on $\mathcal{D}$. Using the definition of local γ-random dictatorships, there is a maximal $\gamma \in [0, 1]$ such that f can be represented as $f = \gamma d + (1 - \gamma) g$, where d is a random dictatorship and g is another locally strategyproof SDS on $\mathcal{D}$. If $\gamma = 1$, this means that f is a random dictatorship and thus, the lemma holds. On the other hand, if $\gamma < 1$, g is locally 0-randomly dictatorial because of the maximality of γ; otherwise, we could represent g as $g = \gamma' d' + (1 - \gamma')h$, where $\gamma' > 0$, d' is a random dictatorship, and h is another locally strategyproof SDS on $\mathcal{D}$. Consequently, $f = \gamma d + (1-\gamma)(\gamma' d' + (1-\gamma')h)$ which means that f is $\gamma + (1 - \gamma)\gamma'$-randomly dictatorial. This contradicts the maximality of γ and thus, g must be locally 0-randomly dictatorial. Finally, observe that g inherits non-imposition from f. In more detail, there is for every alternative $x \in A$ a profile $R \in \mathcal{D}$ such that $f(R, x) = 1$ and it holds for these profiles also that $g(R, x) = 1$ because $f(R, x) = \gamma d(R, x) + (1-\gamma)g(R, x) < 1$ if $g(R, x) < 1$. Hence, for $\gamma < 1$, every γ-randomly dictatorial SDS f can be represented as a mixture of a random dictatorship and a locally strategyproof, locally 0-randomly dictatorial, and non-imposing SDS.

For the other direction, note first that random dictatorships on $\mathcal{D}$ are locally strategyproof since they satisfy this axiom also on the full domain and non-imposing because $\mathcal{D}$ contains all profiles in which the voters agree on the best alternative. Therefore, we focus on an SDS f that is a mixture of a random dictatorship d and another locally strategyproof, locally 0-randomly dictatorial, and non-imposing SDS g, i.e., there is $\gamma \in [0, 1)$ such that $f = \gamma d + (1-\gamma)g$. First note that f is locally strategyproof on $\mathcal{D}$ because mixtures of locally strategyproof SDSs are also locally strategyproof. Moreover, f is *ex post* efficient because Lemma 1 shows that both d and g satisfy this axiom. Consequently, f is non-imposing as it chooses an alternative with probability 1 if it top-ranked by every voter. This shows that every mixture of a random dictatorship and a locally strategyproof, locally 0-randomly dictatorial, and non-imposing SDS on a connected super Condorcet domain is again locally strategyproof and non-imposing. □

As a consequence of Lemma 2, the set of locally 0-randomly dictatorial, locally strategyproof, and non-imposing SDSs $\mathcal{G}$ of a connected super Condorcet domain $\mathcal{D}$ characterizes the set of all locally strategyproof and non-imposing SDSs on $\mathcal{D}$ because each such SDS is a random dictatorship or a mixture of a random dictatorship and an SDS $g \in \mathcal{G}$. Hence, a characterization of the set $\mathcal{G}$ for a connected super Condorcet domain $\mathcal{D}$ immediately implies a characterization of the set of all locally strategyproof and non-imposing SDSs on $\mathcal{D}$. In particular, if there is no locally strategyproof, non-imposing, and locally 0-randomly dictatorial SDS on $\mathcal{D}$, random dictatorships are the only locally strategyproof and non-imposing SDSs on this domain.

Motivated by Lemma 2, we investigate the set of locally 0-randomly dictatorial, locally strategyproof, and non-imposing SDSs for connected super Condorcet domains.

Lemma 3. *Assume $n \geq 3$ is odd and $m \geq 3$. A non-imposing and locally strategyproof SDS on a connected super Condorcet domain is locally 0-randomly dictatorial iff it is Condorcet-consistent.*

Proof sketch. The direction from right to left is trivial: for a Condorcet-consistent SDS, the best alternative of every voter can have probability 0 because another alternative might be the Condorcet winner. Consequently, every Condorcet-consistent and locally strategyproof SDS on a connected super Condorcet domain is locally 0-randomly dictatorial. The inverse direction is much more difficult to prove. The key insight is that every locally strategyproof, locally 0-randomly dictatorial, and non-imposing SDS on a connected super Condorcet domain $\mathcal{D}$ has to choose an alternative with probability 1 whenever it is top-ranked by $n-1$ voters. Departing from this insight, we derive inductively that an alternative is also chosen with probability 1 whenever it is top-ranked by more than half of the voters. Finally, we infer from this observation that every locally strategyproof, locally 0-randomly dictatorial, and non-imposing SDS on $\mathcal{D}$ is Condorcet-consistent. □

Lemma 3 has multiple important consequences. First of all, it identifies Condorcet-consistent SDSs as counterparts to random dictatorships on the large class of connected super Condorcet domains because every locally strategyproof, non-imposing, and locally 0-randomly dictatorial SDS is Condorcet-consistent, whereas only random dictatorships are locally 1-randomly dictatorial. Even more, Theorems 1 and 2 show that this insight is also true if we use strategyproofness and 0-random dictatorships instead of the local variants. This observation has an intuitive explanation: while random dictatorships never compromise, Condorcet-consistent SDSs can be seen as maximally compromising.

Next, we use Lemma 3 to characterize the set of strategyproof and non-imposing SDSs on the Condorcet domain for odd n.

Theorem 1. *Assume n is odd and $m \geq 3$. An SDS on the Condorcet domain is strategyproof and non-imposing iff it is a mixture of a random dictatorship and the Condorcet rule.*

Proof. Assume that the number of voters is odd and $m \geq 3$. First, note that the theorem is trivial if $n = 1$ because the Condorcet domain coincides with the full domain in this case. Hence, it is straightforward to see that an SDS is strategyproof and non-imposing iff it always chooses the best alternative of the single voter with probability 1. This SDS is equivalent to both the Condorcet rule and the dictatorial SDS of this voter.

Next, suppose that $n \geq 3$. We first show that every strategyproof and non-imposing SDS on the Condorcet domain is a mixture of a random dictatorship and the Condorcet rule. For this, we characterize the Condorcet rule as the only locally strategyproof, locally 0-randomly dictatorial, and non-imposing SDS on the Condorcet domain. Note here that the Condorcet rule is the only Condorcet-consistent SDS on $\mathcal{D}_C$ and hence, Lemma 3 shows that it is the only SDS that can satisfy the given axioms. Since it is easy to verify that the Condorcet rule is non-imposing, 0-locally randomly dictatorial, and locally strategyproof, these axioms indeed characterize it. Next, we derive from Lemma 2 that mixtures of random dictatorships and the Condorcet rule are the only locally strategyproof and non-imposing SDSs on the Condorcet domain. Finally, this means that every strategyproof and non-imposing SDS on $\mathcal{D}_C$ can be represented as a mixture of a random dictatorship and the Condorcet rule because strategyproofness implies local strategyproofness.

For the inverse direction, observe that mixtures of strategyproof SDSs are strategyproof. Since the Condorcet rule and all random dictatorships are strategyproof on the Condorcet domain, mixtures of these SDSs are thus also strategyproof. Moreover, the Condorcet rule and all random dictatorships choose an alternative with probability 1 if it is unanimously top-ranked. This implies that mixtures of these SDSs satisfy non-imposition on the Condorcet domain. □

Note that Theorem 1 immediately implies that the Condorcet rule is the only 0-randomly dictatorial SDS on the Condorcet domain that satisfies strategyproofness and non-imposition if $n \geq 3$ is odd and $m \geq 3$. This corollary generalizes Theorem 1 of Campbell and Kelly [10] who have characterized the Condorcet rule with equivalent axioms in the deterministic setting. Furthermore, this insight highlights the appeal of the Condorcet rule on the Condorcet domain because every other strategyproof and non-imposing SDS is merely a mixture of the Condorcet rule and a random dictatorship. Hence, there is a unique, attractive, and strategyproof voting rule if we restrict our attention to the Condorcet domain.

A natural follow-up question to Theorem 1 is whether we can find larger domains for which there are strategyproof and non-imposing SDSs other than random dictatorships. We find a negative answer to this problem if n is odd: on every strict and connected super Condorcet domain $\mathcal{D}$, only random dictatorships are strategyproof and non-imposing. Note that here, strictness implies only that $\mathcal{D} \neq \mathcal{D}_C$ because n is odd. Hence, the subsequent theorem shows that the Condorcet domain is essentially a maximal domain that allows for attractive strategyproof SDSs.

Theorem 2. *Assume n is odd and $m \geq 3$. An SDS on a strict and connected super Condorcet domain is strategyproof and non-imposing iff it is a random dictatorship.*

Proof. Assume n is odd and $m \geq 3$ and consider an arbitrary connected and strict super Condorcet domain $\mathcal{D}$. Since the Condorcet domain coincides with the full domain if $n = 1$, there is no strict super Condorcet domain in this case and we thus suppose that $n \geq 3$. First, note that, on $\mathcal{D}$, random dictatorships satisfy strategyproofness since they satisfy this axiom even on the full domain, and non-imposition because $\mathcal{D}$ contains all profiles in which all voters agree on the best alternative. We hence focus on the inverse direction: every strategyproof and non-imposing SDS on $\mathcal{D}$ is a random dictatorship. We prove this by showing that there is no locally strategyproof and Condorcet-consistent SDS because the conjunction of Lemmas 2 and 3 implies then that random dictatorships are the only locally strategyproof and non-imposing SDSs on $\mathcal{D}$. Since strategyproofness implies local strategyproofness, this proves the theorem.

Hence, assume for contradiction that there is a locally strategyproof and Condorcet-consistent SDS $f : \mathcal{D} \to \Delta(A)$. Since $\mathcal{D}$ is a strict super Condorcet domain, there is a profile $R \in \mathcal{D}$ such that every alternative strictly loses a majority comparison in R. In particular, there is no Condorcet winner in R. We prove next that there is no feasible outcome for R. Note for this that $\mathcal{D}$ is connected and thus, there is for every alternative $x \in A$ an ad-path π from R to a profile $R^x \in \mathcal{D}_C$ such that x is not swapped with any other alternative on π. Since $R^x \in \mathcal{D}_C$ and f is Condorcet-consistent, it follows for the Condorcet winner c in R^x that $f(R^x, c) = 1$. Moreover, $c \neq x$ because there is no Condorcet winner in R and x is not swapped with any alternative in the construction of R^x. Hence, a repeated application of local strategyproofness shows that $f(R, x) = f(R^x, x) = 0$. Finally, since x can be chosen arbitrarily, we derive that $f(R, y) = 0$ for all alternatives $y \in A$. This contradicts the definition of an SDS, which means that there is no locally strategyproof and Condorcet-consistent SDS on $\mathcal{D}$. □

First, note that Theorem 2 generalizes the random dictatorship theorem from the full domain to every strict and connected super Condorcet domain if n is odd. In particular, this result shows that adding even a single profile to the Condorcet domain can turn the positive result of Theorem 1 into a negative one because the resulting domain is a strict and connected super Condorcet domain. This follows, for instance, by considering the subsequent domain $\mathcal{D}_1 = \mathcal{D}_C \cup \{R^*\}$. The preference profile R^* is shown below, where $I = \{4, 6, \dots, n-1\}$, $J = \{5, 7, \dots, n\}$.

R^*:	1: $a, b, c, \dots$	2: $b, c, a, \dots$	3: $c, a, b, \dots$
	I: $a, b, c, \dots$	J: $c, b, a, \dots$	

The domain $\mathcal{D}_1$ is a strict and connected super Condorcet domain because we can go from R^* to a profile in the Condorcet domain if voter 1 swaps a and b, if voter 2 swaps b and c, and if voter 3 swaps a and c. Hence, Theorem 2 shows that random dictatorships are the only strategyproof and non-imposing SDSs on $\mathcal{D}_1$.

Remark 1. An important consequence of Theorems 1 and 2 is that, if n is odd, every strategyproof and non-imposing SDS on a connected super Condorcet domain can be represented as a mixture of deterministic voting rules, each of which is strategyproof and non-imposing. This is sometimes called deterministic extreme point property and it is remarkable that many important domains satisfy this condition [45]. On one side, this shows that randomization does not lead to completely new strategyproof SDSs. On the other hand, the deterministic extreme point property allows for a natural interpretation of strategyproof and non-imposing SDSs: we decide by chance which deterministic voting rule is executed.

Remark 2. The connectedness condition is required for Theorem 2 because there are domains $\mathcal{D}$ with $\mathcal{D}_C \subsetneq \mathcal{D}$ that allow for non-imposing and strategyproof SDSs that are no random dictatorships. For example, consider the domain $\mathcal{D}_2$ which is derived by adding a single preference profile R^1 to the Condorcet domain. If R^1 differs from every profile in $\mathcal{D}_C$ in the preference relations of at least two voters, an arbitrary outcome can be returned for R^1 without violating strategyproofness.

Remark 3. The proofs of Theorems 1 and 2 show that these characterizations also hold if we use local strategyproofness instead of strategyproofness. Consequently, local strategyproofness is equivalent to strategyproofness for non-imposing SDSs on connected super Condorcet domain if n is odd. Similar results have been shown for a number of other domains, e.g., the domain of single-peaked profiles, the domain of single-dipped profiles, or the full domain [see, e.g., 13, 27, 29, 49].

Remark 4. If n is even, there are larger domains than $\mathcal{D}_C$ that allow for strategyproof, non-imposing, and 0-randomly dictatorial SDSs. An example of such a domain is the shifted Condorcet domain $\mathcal{D}_C^{R_i}$ which contains a profile R iff there is a Condorcet winner in (R, R_i), i.e., in the profile derived by adding a fixed preference relation R_i to R. An SDS on $\mathcal{D}_C^{R_i}$ that satisfies all our requirements is $f(R) = COND(R, R_i)$ because the domain $\{(R, R_i) \colon R \in \mathcal{D}_C^{R_i}\}$ is a subset of the Condorcet domain for $n + 1$ voters.

4.2 Results based on Group-strategyproofness

Maybe the strongest restrictions of Theorems 1 and 2 is that they only apply if the number of voters n is odd. Indeed, Theorem 1 does not hold for even n because the counterexamples by

	Full domain	Strict super Condorcet domains	Condorcet domain
Deterministic, strategyproof, and non-imposing voting rules	Dictatorships [26, 50]	*Dictatorships*$^\diamond$ *(Consequence of Theorem 2)*	Dictatorships and the Condorcet rule* [10]
Strategyproof and non-imposing SDSs	Random dictatorships [27]	*Random dictatorships*$^\diamond$ *(Theorem 2)*	*Mixtures of random dictatorships and the Condorcet rule** *(Theorem 1)*
Group-strategyproof and non-imposing SDSs	Dictatorial SDSs [2]	*Dictatorial SDSs (Theorem 3)*	*Dictatorial SDSs and the Condorcet rule (Theorem 3)*

Table 1: Comparison of results for the full domain, strict super Condorcet domains, and the Condorcet domain. Each row characterizes a set of SDSs for the full domain, every strict super Condorcet domain, and the Condorcet domain, respectively. All results require that there are $m \geq 3$ alternatives and results marked with an asterisk ($*$) only hold if there is an odd number of voters. The results marked with a diamond ($\diamond$) require that n is odd and that the strict super Condorcet domain is additionally connected. New results are italicized.

Merrill [32] show that there are strategyproof, non-imposing, and 0-randomly dictatorial SDSs $f : \mathcal{D}_C \rightarrow \Delta(A)$ other than the Condorcet rule if n is even. The reason for this is that a single voter cannot change the Condorcet winner if n is even and consequently, the Condorcet domain breaks down into several disconnected subdomains. A natural approach to restore the connectedness is to use group-strategyproofness because this axiom allows to change the Condorcet winner with a group-manipulation. We therefore characterize now the set of group-strategyproof and non-imposing SDSs on super Condorcet domains independently of the parity of n. In more detail, we show that only dictatorial SDSs and the Condorcet rule satisfy these axioms on the Condorcet domain, whereas only dictatorships are group-strategyproof and non-imposing on strict super Condorcet domains.

The proofs of these results are very similar to the proofs of Theorem 1 and Theorem 2. In particular, we first generalize Lemma 3 by investigating when a group-strategyproof and non-imposing SDS is non-dictatorial.

Lemma 4. *Assume $n \geq 3$ and $m \geq 3$. A group-strategyproof and non-imposing SDS on a super Condorcet domain is non-dictatorial iff it is Condorcet-consistent.*

Proof sketch. It is obvious that a Condorcet-consistent SDS is non-dictatorial as the best alternative of a voter gets probability 0 if another alternative is the Condorcet winner. For the other direction, we prove that every group-strategyproof, non-imposing, and non-dictatorial SDS on a super Condorcet domain chooses an alternative x with probability 1 whenever $n-1$ voters report x as their favorite option. Based on this insight, we derive analogously to the proof of Lemma 3 that such SDSs are Condorcet-consistent. ◻

Similar to Lemma 3, Lemma 4 identifies Condorcet-consistent SDSs as counterparts to dictatorships. However, Lemma 4 is more general as we characterize non-dictatorial SDSs instead of local 0-random dictatorships and no connectedness condition on the domain is required. This is possible since we use group-strategyproofness instead of local strategyproofness.

Next, we employ this lemma to characterize the set of group-strategyproof and non-imposing SDSs on the Condorcet domain and every strict super Condorcet domain.

Theorem 3. *Assume $m \geq 3$. An SDS on the Condorcet domain is group-strategyproof and non-imposing iff it is a dictatorial SDS or the Condorcet rule. An SDS on a strict super Condorcet domain is group-strategyproof and non-imposing iff it is a dictatorial SDS.*

Proof. The theorem consists of two independent characterizations of group-strategyproof and non-imposing SDSs for the Condorcet domain and for strict super Condorcet domains, respectively. We prove these results separately.

Claim 1: An SDS on the Condorcet domain is group-strategyproof and non-imposing iff it is a dictatorial SDS or the Condorcet rule.

We first show the direction from left to right and thus consider a group-strategyproof and non-imposing SDS f on the Condorcet domain. We need to show that f is the Condorcet rule or a dictatorial SDS. First, note that if $n \geq 3$, this follows immediately from Lemma 4 because the Condorcet rule is the only Condorcet-consistent SDS on the Condorcet domain. Hence, this lemma shows that f is either dictatorial if it is not Condorcet-consistent, or the Condorcet rule. Moreover, if $n \leq 2$, our claim follows from the fact that f is *ex post* efficient (see point (2) in Lemma 1). In more detail, if $n = 1$, the best alternative x of the single voter always Pareto-dominates every other alternative and thus, $f(R, x) = 1$. If $n = 2$, a profile R is only in the Condorcet domain if the two voters agree on the best alternative x. Hence, alternative x Pareto-dominates every other alternative, which means again that $f(R, x) = 1$. In both cases, the single top-ranked alternative is also the Condorcet winner, which means that f is the Condorcet rule (which is equivalent to the dictatorial SDSs in this case).

For the other direction, we need to show that all dictatorial SDSs and the Condorcet rule are group-strategyproof and non-imposing. First, note that all these SDSs are by definition non-imposing on the Condorcet domain because they are even deterministic. Next, every dictatorial SDS d_i is group-strategyproof because only a single voter i can change the outcome. However, this voter has no incentive to manipulate as his best alternative is already chosen with probability 1, which means that any other outcome makes this voter worse.

Finally, we show that the Condorcet rule is group-strategyproof. Assume for contradiction that this is not the case, i.e., there are preference profiles $R, R' \in \mathcal{D}_C$ and a set of voters $I \subseteq N$ such that $R_j = R'_j$ for all $j \in N \setminus I$ and $COND(R) \not\succsim_i COND(R')$ for all voters $i \in I$. If the Condorcet winner in R and R' is the same, this is clearly no manipulation since $COND(R) = COND(R')$. Hence, assume that the Condorcet winner c of R is not the same alternative as the Condorcet winner c' of R'. This means that $g_R(c, c') > 0$ and $g_{R'}(c', c) > 0$, which is only possible if there is a manipulator $i \in I$ with cR_ic'. However, this voter prefers $COND(R)$ to $COND(R')$, contradicting that $COND(R) \not\succsim_j COND(R')$ for all $j \in I$. Thus, the Condorcet rule is group-strategyproof on the Condorcet domain.

Claim 2: An SDS on a strict super Condorcet domain is group-strategyproof and non-imposing iff it is dictatorial. First, note that there is not strict super Condorcet domain if $n \leq 2$. In more detail, if $n = 1$, every profile has a Condorcet winner, which means that there is no super Condorcet domain with $\mathcal{D} \neq \mathcal{D}_C$. Furthermore, if $n = 2$, the best alternative x of the first voter satisfies in every preference profile $R \in \mathcal{R}^2$ that $g_R(x, y) \geq 0$ for all $y \in A \setminus \{x\}$ and thus, there cannot be a profile in which every alternative strictly loses the majority comparison to another alternative. Hence, we suppose in the sequel that $n \geq 3$ and consider an arbitrary strict super Condorcet domain $\mathcal{D}$. Observe first that the same arguments as for Claim 1 show that every dictatorship is group-strategyproof and non-imposing on $\mathcal{D}$. Thus, we focus on proving that every group-strategyproof and non-imposing SDS on $\mathcal{D}$ is dictatorial. For this, we show that there is no group-strategyproof and Condorcet-consistent SDS on $\mathcal{D}$ because Lemma 4 shows then that only dictatorial SDSs can be group-strategyproof and non-imposing on $\mathcal{D}$.

Therefore, suppose for contradiction that f is a group-strategyproof and Condorcet-consistent SDS on $\mathcal{D}$. Subsequently, we use the fact that $\mathcal{D}$ is a strict super Condorcet domain: there is profile $R \in \mathcal{D}$ such that every alternative $x \in A$ strictly loses the pairwise comparison against another alternative $y \in A \setminus \{x\}$, i.e., such that $g_R(y, x) > 0$. Next, consider an alternative $x \in A$ such that $f(R, x) > 0$; such an alternative must exist because $\sum_{x \in A} f(R, x) = 1$. We show that the voters $I = \{i \in N : yR_ix\}$ can group-manipulate by reporting y as their favorite alternative. In particular, this manipulation leads to a profile $R' \in \mathcal{D}_C$ in which y is the Condorcet winner because $|I| > \frac{n}{2}$. Thus, Condorcet-consistency requires that $f(R, y) = 1$. However, it holds that $f(R, U(R_i, y)) < 1 = f(R', U(R_i, y))$ for all $i \in I$ because $x \notin U(R_i, y)$ and $f(R, x) > 0$. Hence, this step is indeed a group-manipulation for the voters $i \in I$, contradicting that f is group-strategyproof. Therefore, there is no group-strategyproof and Condorcet-consistent SDS on $\mathcal{D}$ and Lemma 4 entails that only dictatorial SDSs can satisfy group-strategyproofness and non-imposition on strict super Condorcet domains. □

Theorem 3 generalizes Theorems 1 and 2 to super Condorcet domains for an even number of voters by using group-strategyproofness. In particular, it entails that the Condorcet rule is the only group-strategyproof, non-imposing, and non-dictatorial SDS on the Condorcet domain. Moreover, the second part of the theorem shows that the Condorcet domain is essentially a maximal domain that allows for a group-strategyproof and non-imposing SDS apart of dictatorships. Note here that, if n is odd, every super Condorcet domain $\mathcal{D}$ with $\mathcal{D} \neq \mathcal{D}_C$ is strict and Theorem 3 hence shows for odd n that *no* superset of the Condorcet domain admits group-strategyproof and non-imposing SDSs other than dictatorships.

Remark 5. The results of Barberà [2] imply that every group-strategyproof and non-imposing SDS on the full domain is a dictatorship. Hence, Theorem 3 and Barberà's results share a common idea: group-strategyproof and non-imposing SDSs cannot use randomization to determine the winner. However, whereas only undesirable SDSs are group-strategyproof and non-imposing on $\mathcal{R}^n$, the attractive Condorcet rule satisfies these axioms on $\mathcal{D}_C$.

5 CONCLUSION

We study strategyproof and non-imposing social decision schemes (SDSs) on the Condorcet domain (which consists of all preference profiles with a Condorcet winner) and its supersets. These domains are highly relevant because empirical results suggest that real-world election commonly admit a Condorcet winner. In contrast to the full domain, there are attractive strategyproof SDSs on the Condorcet domain: we show that, if the number of voters n is odd, every strategyproof and non-imposing SDS on the Condorcet domain can be represented as a mixture of a random dictatorship and the Condorcet rule. An immediate consequence of this insight is that the Condorcet rule is the only strategyproof, non-imposing, and 0-randomly dictatorial SDS on the Condorcet domain if n is odd. Furthermore, we show that, if n is odd, only random dictatorships are strategyproof and non-imposing on every sufficiently connected superset of the Condorcet domain. This means that the Condorcet domain is essentially a maximal domain which allows for a 0-randomly dictatorial, non-imposing, and strategyproof SDS. Finally, we extend our results to even n by using group-strategyproofness: we prove that the Condorcet rule is the only non-dictatorial, group-strategyproof, and non-imposing SDS on the Condorcet domain, and that no SDS satisfies these axioms on larger domains.

Our results for the Condorcet domain show an astonishing similarity to classic results for the full domain, but have a more positive flavor. For instance, while the random dictatorship theorem shows that only mixtures of dictatorial SDSs are strategyproof and non-imposing on the full domain, we prove in Theorem 1 that mixtures of dictatorial SDSs and the Condorcet rule are the only strategyproof and non-imposing SDSs on the Condorcet domain (if the number of voters is odd). A more exhaustive comparison between results for the full domain and for the Condorcet domain is given in Table 1. In particular, our results highlights the important role of the Condorcet rule on the Condorcet domain: even if we allow for randomization, it is still the most appealing strategyproof voting rule. Thus, our theorems make a strong case for choosing the Condorcet winner whenever it exists.

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APPENDIX A: OMITTED PROOFS

In this appendix, we discuss the omitted proofs of Lemmas 1, 3 and 4. In many of the subsequent proofs, we use additional notation. In particular, we use the $*$ symbol in preference relations to indicate that all missing alternatives can be ordered arbitrarily. For instance, $1 : a, b, *$ means that voter 1 prefers a to b to every other alternative, and that the alternatives in $A \setminus \{a, b\}$ can be ordered arbitrarily. Furthermore, the *rank* $r(R_i, x) = |\{y \in A \colon xR_iy\}|$ of an alternative x with respect to a preference relation R_i denotes the number of alternatives that voter i weakly prefers to x. Less formally, if an alternative is the k-th best alternative of voter i, its rank with respect to R_i is k. In particular, $r(R_i, x) = r(R_i, y) + 1$ means that voter i places y directly over x in his preference relation.

Proof of Lemma 1

In this section, we prove Lemma 1. Note that the proof sketch in the main body contains actually a complete proof for the statement on group-strategyproof SDSs, and thus, we focus on the analogous statement for locally strategyproof SDSs. For this, we first investigate the consequences of local strategyproofness in more detail and discuss a characterization of local strategyproofness based on non-perversity and localizedness. Here, we say an SDS $f : \mathcal{D} \to \Delta(A)$ is *non-perverse* if $f(R^{i:yx}, y) \geq f(R, y)$ for all preference profiles $R \in \mathcal{D}$, alternatives $x, y \in A$, and voters $i \in N$ such that $R^{i:yx} \in \mathcal{D}$. Furthermore, an SDS $f : \mathcal{D} \to \Delta(A)$ is *localized* if $f(R^{i:yx}, z) = f(R, z)$ for all preference profiles $R \in \mathcal{D}$, distinct alternatives $x, y, z \in A$, and voters $i \in N$ such that $R^{i:yx} \in \mathcal{D}$. Intuitively, non-perversity requires that the probability of an alternative increases if it is reinforced, and localizedness demands that the probability of an alternative does not change if it is not involved in a swap. Gibbard [27], who first used these axioms, has proven that the conjunction of non-perversity and localizedness is equivalent to strategyproofness on the full domain. We show a variant of this claim for restricted domains: non-perversity and localizedness are equivalent to local strategyproofness for every domain.

Lemma 5. *Let $\mathcal{D} \subseteq R^n$ denote an arbitrary domain. An SDS $f : \mathcal{D} \to \Delta(A)$ is locally strategyproof iff it is localized and non-perverse.*

Proof. Consider an arbitrary domain $\mathcal{D}$ and let f denote an SDS on this domain. We need to show that f is locally strategyproof if it is localized and non-perverse, and that f is non-perverse and localized if it is locally strategyproof.

Claim 1: If f is locally strategyproof, it is localized and non-perverse.

First, assume that $f : \mathcal{D} \to \Delta(A)$ is a locally strategyproof SDS. Moreover, consider two arbitrary preference profiles $R, R' \in \mathcal{D}$, a voter $i \in N$, and two alternatives $x, y \in A$ such that $R' = R^{i:yx}$. Note that local strategyproofness implies that $f(R) \succsim_i f(R')$ and $f(R') \succsim_i' f(R)$.

We first show that f satisfies localizedness, i.e., that $f(R, z) = f(R', z)$ for all alternatives $z \in A \setminus \{x, y\}$. Observe for this that the assumption $R' = R^{i:yx}$ entails that $U(R_i, z) = U(R_i', z)$ for all $z \in A \setminus \{x, y\}$ and that $U(R_i, y) = U(R_i', x)$. Thus, local strategyproofness from R to R' and from R' to R implies that $f(R, U(R_i, z)) = f(R', U(R_i', z))$ for all $z \in A \setminus \{x, y\}$ and that $f(R, U(R_i, y)) = f(R', U(R_i', x))$. For instance, the latter is true because local strategyproofness implies that $f(R, U(R_i, y)) \geq f(R', U(R_i, y))$ and $f(R', U(R_i', x)) \geq f(R, U(R_i', x))$. Furthermore, note that for every alternative $z \in A$ and every preference relation $\bar{R}_i$, the set $U(\bar{R}_i, z) \setminus \{z\}$ is either empty (if z is the best alternative of voter i) or equal to the upper contour set $U(\bar{R}_i, z')$ of the alternative z' with $r(\bar{R}_i, z') = r(\bar{R}_i, z) - 1$. Hence, it holds for all alternatives $z \in A \setminus \{x, y\}$ with $r(R_i, z) \neq r(R_i, y) + 1$ that $f(R, U(R_i, z) \setminus \{z\}) = f(R', U(R_i', z) \setminus \{z\})$ because either $U(R_i, z) \setminus \{z\} = U(R_i', z) \setminus \{z\} = \emptyset$ or there is another alternative $z' \in A \setminus \{x, y\}$ such that $U(R_i, z) \setminus \{z\} = U(R_i, z') = U(R_i', z') = U(R_i', z) \setminus \{z\}$. On the other hand, it also holds for the alternative $z \in A \setminus \{x, y\}$ with $r(R_i, z) = r(R_i, y) + 1$ that $f(R, U(R_i, z) \setminus \{z\}) = f(R', U(R_i', z) \setminus \{z\})$ because $U(R_i, z) \setminus \{z\} = U(R_i, y) = U(R_i', x) = U(R_i', z) \setminus \{z\}$. Hence, f is localized because we can calculate for all alternatives $z \in A \setminus \{x, y\}$ that

$$\begin{aligned} f(R, z) &= f(R, U(R_i, z)) - f(R, U(R_i, z) \setminus \{z\}) \\ &= f(R', U(R_i', z)) - f(R', U(R_i', z) \setminus \{z\}) = f(R', z). \end{aligned}$$

Finally, we show that f is non-perverse, i.e., that $f(R', y) \geq f(R, y)$. Using local strategyproofness from R' to R, we derive that $f(R', U(R_i', y)) \geq f(R, U(R_i', y))$, and localizedness implies that $f(R', z) = f(R, z)$ for all $z \in A \setminus (\{x, y\})$. Since $U(R_i', y) \setminus \{y\} \subseteq A \setminus \{x, y\}$, it follows that f is non-perverse because

$$\begin{aligned} f(R', y) &= f(R', U(R_i', y)) - f(R', U(R_i', y) \setminus \{y\}) \\ &\geq f(R, U(R_i', y)) - f(R, U(R_i', y) \setminus \{y\}) = f(R, y). \end{aligned}$$

Claim 2: If f is localized and non-perverse, it is locally strategyproof.

Suppose that $f : \mathcal{D} \to \Delta(A)$ is a localized and non-perverse SDS. We need to show that this SDS is locally strategyproof and consider thus two preference profiles $R, R' \in \mathcal{D}$, a voter $i \in N$, and two alternatives $x, y \in A$ such that $R' = R^{i:yx}$. Our goal is to prove that $f(R) \succsim_i f(R')$ which requires that $f(R, U(R_i, z)) \geq f(R', U(R_i, z))$ for all $z \in A$. First, note that $f(R, z) = f(R', z)$ for all $z \in A \setminus \{x, y\}$ because of localizedness. This implies that $f(R, U(R_i, z)) = f(R', U(R_i, z))$ for all $z \in A \setminus \{x\}$ with zR_ix because $x, y \notin U(R_i, z)$ and thus $U(R_i, z) \subseteq A \setminus \{x, y\}$. Furthermore, for all $z \in A \setminus \{x\}$ with xR_iz, we can compute that $f(R, U(R_i, z)) = 1 - f(R, A \setminus U(R_i, z)) = 1 - f(R', A \setminus U(R_i, z)) = f(R', U(R_i, z))$ because $\{x, y\} \subseteq U(R_i, z)$ and thus $A \setminus U(R_i, z) \subseteq A \setminus \{x, y\}$. Finally, non-perversity shows for x that $f(R, x) \geq f(R', x)$. Since localizedness implies that $f(R, w) = f(R', w)$ for all $w \in U(R_i, x) \setminus \{x\}$, it follows that $f(R, U(R_i, x)) \geq f(R', U(R_i, x))$. Hence, it indeed holds that $f(R, U(R_i, z)) \geq f(R', U(R_i, z))$ for all $z \in A$ and f is therefore locally strategyproof. ◻

Lemma 5 clarifies the consequences of local strategyproofness on restricted domains and shows that we can use equivalently non-perversity and localizedness, which are easier to handle. We use this insight next to prove that every locally strategyproof and non-imposing SDS on a connected super Condorcet domain is *ex post* efficient if n is odd. In the proof of this lemma, we use the notion of k-unanimity: an SDS $f : \mathcal{D} \to \Delta(A)$ is *k-unanimous* if $f(R, x) = 1$ for all profiles $R \in \mathcal{D}$ and alternatives $x \in A$ such that x is top-ranked by at least $n - k$ voters in R. In particular, we show next that every locally strategyproof SDS on a connected super

Condorcet domain is 0-unanimous if it is non-imposing and that it violates 0-unanimity if it violates *ex post* efficiency.

Lemma 1. *Assume $m \geq 3$. The following claims are true.*

1. *Every non-imposing and locally strategyproof SDS on a connected super Condorcet domain is* ex post *efficient if n is odd.*
2. *Every non-imposing and group-strategyproof SDS on a super Condorcet domain is* ex post *efficient.*

Proof. Note that the proof of the second statement can be found in the main body and hence, we only discuss the result about locally strategyproof SDS. Hence, assume that n is odd and let f denote a locally strategyproof and non-imposing SDS on an arbitrary connected super Condorcet domain $\mathcal{D}$. We assume for contradiction that f fails *ex post* efficiency and derive a contradiction in two steps: first, we show f needs to be 0-unanimous, and secondly, we show that this cannot be the case if a Pareto-dominated alternative is assigned positive probability in a profile. Since these claims contradict each other, it follows that every locally strategyproof and non-imposing SDS on $\mathcal{D}$ satisfies *ex post* efficiency. In the subsequent proof, we always ensure that we stay in the Condorcet domain, which also means that we stay in $\mathcal{D}$ since $\mathcal{D}_C \subseteq \mathcal{D}$.

Claim 1: Every locally strategyproof and non-imposing SDS $f : \mathcal{D} \to \Delta(A)$ satisfies 0-unanimity.

Choose an arbitrary alternative $x \in A$ and let $R \in \mathcal{D}$ denote a profile in which all voters report x as their favorite choice. We need to show that $f(R, x) = 1$ and consider for this a profile $R^1 \in \mathcal{D}$ with $f(R^1, x) = 1$. Such a profile exists since f is non-imposing. If $R^1 \in \mathcal{D} \setminus \mathcal{D}_C$, we first use the connectedness of $\mathcal{D}$ to go into the Condorcet domain. In more detail, this axiom states that there is an ad-path from R^1 to a profile $R^2 \in \mathcal{D}_C$ such that x is not swapped with any alternative along this ad-path. Consequently, a repeated application of localizedness along this ad-path shows that $f(R^2, x) = f(R^1, x) = 1$. On the other hand, if R^1 is already in the Condorcet domain, we just define $R^2 = R^1$.

Let c denote the Condorcet winner in R^2. We proceed with a case distinction with respect to whether $c = x$ or not. First, assume that x is the Condorcet winner in R^2. In this case, we let all voters repeatedly swap up x until it is the best alternative of every voter. Since we only reinforce x, it stays the Condorcet winner during these steps. Moreover, non-perversity entails that the probability of x is non-decreasing. Thus, this process results in a profile R^3 in which x is unanimously top-ranked and $f(R^3, x) = 1$. As last step, we let all voters reorder all alternatives in $A \setminus \{x\}$ one after another to derive the profile R. It is easy to see that we can use swaps that do not involve x for this, and thus, localizedness implies that $f(R, x) = 1$. Hence, the case $x = c$ is proven.

For the second case, we assume that $c \neq x$ and define $I = \{i \in N : cR_i^2 x\}$ as the set of voters who prefer c to x in R^2. We let all voters in I repeatedly swap up c until it is their best alternative. Because c is only reinforced during these steps, it stays the Condorcet winner. Furthermore, we never swap c and x because of the definition of I. Hence, this process terminates in a profile R^3 with $f(R^3, x) = 1$ because of localizedness. As second step, we let the voters $i \in I$ one after another swap up x until it is their second best alternative and the voters in $N \setminus I$ swap up x until it is their best alternative. During these steps, c stays the Condorcet winner as all voters in I report it as their best alternative, which means that we do not leave $\mathcal{D}$. Moreover, non-perversity entails that the probability of x cannot decrease during these steps, and therefore, $f(R^4, x) = 1$ for the resulting profile R^4. As next step, we let the voters in I swap x and c one after another, which results in a profile R^5 in which x is the Condorcet winner because all voters report it as their favorite choice. This process does not leave the Condorcet domain as every voter top-ranks either c or x and thus, one of these alternatives is top-ranked by more than half of the voters during each step because n is odd. Finally, we deduce that $f(R^5, x) = 1$ because non-perversity requires that $f(R^5, x) \geq f(R^4, x)$. As last step, we transform R^5 into R by reordering the alternatives in $A \setminus \{x\}$ in the preference relations of all voters. Since x is not involved in these swaps, it follows that $f(R, x) = 1$ because of localizedness. Because x and R are arbitrarily chosen, it follows that f is 0-unanimous.

Claim 2: Every locally strategyproof SDS $f : \mathcal{D} \to \Delta(A)$ that violates *ex post* efficiency fails 0-unanimity.

Next, assume that f violates *ex post* efficiency, which means that there is a profile $R \in \mathcal{D}$ and alternatives $x, y \in A$ such that $xR_i y$ for all voters $i \in N$ but $f(R, y) > 0$. We show that x is not chosen with probability 1 if it is top-ranked by every voter. If $R \in \mathcal{D} \setminus \mathcal{D}_C$, we use again the connectedness of $\mathcal{D}$ to go to a profile $R^1 \in \mathcal{D}_C$. In particular, there is an ad-path from R to a profile $R^1 \in \mathcal{D}_C$ such that y is not swapped with any other alternative along this path. Hence, localizedness implies that $f(R^1, y) = f(R, y) > 0$, and x still Pareto-dominates y in R^1 as we never swap these alternatives. If $R \in \mathcal{D}_C$, we define $R^1 = R$.

Next, we use a case distinction with respect to the Condorcet winner c in R^1. First, assume that x is the Condorcet winner in R^1. In this case, we can sequentially swap up x in the preference relation of every voter until it is unanimously top-ranked. Because y is not affected by these steps, localizedness implies that $f(R^2, y) = f(R^1, y) > 0$ for the resulting profile R^2. However, this means that f violates 0-unanimity as $f(R^2, x) \neq 1$ even though x is unanimously top-ranked in R^2.

As second case, assume that $c \neq x$ and let $I = \{i \in N : cR_i^1 x\}$ denote the voters who prefer c to x in R^1. Note also that $c \neq y$ because all voters prefer x to y. As first step, we let the voters $i \in I$ swap up c until it is their best alternative. We do not leave the Condorcet domain during these steps because we only reinforce the Condorcet winner. Furthermore, y is not involved in any swap since $cR_i^1 x$ and $xR_i^1 y$ for all voters $i \in I$, and thus, this step leads to a profile R^2 with $f(R^2, y) = f(R^1, y) > 0$ because of localizedness. As next step, we let all voters $i \in I$ reinforce x until it is their second best alternative and all voters $i \in N \setminus I$ reinforce x until it is their best alternative. Alternative c stays the Condorcet winner during these steps since it is always top-ranked by all voters in I, and localizedness implies for the resulting profile R^3 that $f(R^3, y) = f(R^2, y) > 0$ because all voters prefer x to y in R^2. Finally, we derive the profile R^4 by sequentially swapping c and x in the preferences of the voters $i \in I$. This process does not leave the Condorcet domain because in every intermediate profile, only c and x are top-ranked. This means that one of these alternatives is the Condorcet winner because n is odd and thus, either c or x is top-ranked by more than half of the voters. Moreover, y is not involved in these swaps and localizedness implies therefore that $f(R^4, y) = f(R^3, y) > 0$.

However, x is now top-ranked by all voters but $f(R^4, x) \neq 1$, which means that f fails 0-unanimity. ◻

Proof of Lemma 3

Next, we discuss the proof of Lemma 3. Since the proof of this result is rather involved, we need to introduce several auxiliary lemmas and we also use the insights of the preceding subsection about local strategyproofness.

As first step, we show that every locally strategyproof and non-imposing SDS on a connected super Condorcet domain satisfies strong symmetry conditions between profiles in which only two alternatives are top-ranked. For formally stating this lemma, we denote with $N_x^R = \{i \in N : r(R_i, x) = 1\}$ the set of voters who report x as their favorite alternative in R. Moreover, we define $S^{xIy} = \{R \in \mathcal{R}^n : N_x^R = I \wedge N_y^R = N \setminus I\}$ for all subsets of voters $I \subseteq N$ and alternatives $x, y \in A$ as the domain that contains precisely the preference profiles in which all voters in I report x as their best alternative and all voters in $N \setminus I$ report y as their best alternative. Note that $S^{xIy} \subseteq \mathcal{D}_C$ for all subsets of voters I and alternatives $x, y \in A$ if n is odd because either x or y is top-ranked by more than half of the voters and thus, one of these alternatives is the Condorcet winner for every profile $R \in S^{xIy}$.

Lemma 6. *Assume $n \geq 3$ is odd and $m \geq 3$. Furthermore, consider a locally strategyproof and non-imposing SDS f on a connected super Condorcet domain. For all sets of voters $I \subseteq N$ with $\emptyset \subsetneq I \subsetneq N$, alternatives $w, x, y, z \in A$ with $w \neq x$ and $y \neq z$, and preference profiles $R \in S^{wIx}$, $R' \in S^{yIz}$, it holds that $f(R, w) = f(R', y)$, $f(R, x) = f(R', z)$, and $f(R, w) + f(R, x) = 1$.*

Proof. Consider a connected super Condorcet domain $\mathcal{D}$ and let $f : \mathcal{D} \to \Delta(A)$ denote a locally strategyproof and non-imposing SDS. Furthermore, consider an arbitrary set of voters I with $\emptyset \subsetneq I \subsetneq N$. We prove the lemma in three steps: first, we show that $f(R) = f(R')$ and $f(R, x) + f(R, y) = 1$ for all alternatives $x, y \in A$ with $x \neq y$ and all preference profiles $R, R' \in S^{xIy}$. This is helpful as we now only need to show that there are preference profiles $R \in S^{wIx}$, $R' \in S^{yIz}$ such that $f(R, w) = f(R', y)$, $f(R, x) = f(R', z)$, and $f(R, x) + f(R, y) = 1$. As second step, we consider three distinct alternatives $x, y, z \in A$ and show that $f(R, x) = f(R', z)$ for all $R \in S^{xIy}$, $R' \in S^{zIy}$. Based on these two insights, we prove the lemma as last step.

Step 1: $f(R) = f(R')$ **and** $f(R, x) + (R, y) = 1$ **for all distinct alternatives** $x, y \in A$ **and preference profiles** $R, R' \in S^{xIy}$

Let x and y denote two distinct alternatives and consider a profile $R^* \in \mathcal{D}_C \subseteq \mathcal{D}$ such that $r(R_i^*, x) = 1$ and $r(R_i^*, y) = 2$ for all voters $i \in I$, and $r(R_i^*, y) = 1$ and $r(R_i^*, x) = 2$ for all voters $i \in N \setminus I$. Note that $f(R^*, x) + f(R^*, y) = 1$ because f is *ex post* efficient (see Lemma 1) and all alternatives $z \in A \setminus \{x, y\}$ are Pareto-dominated in R^*. Moreover, consider an arbitrary preference profile $R \in S^{xIy}$. We will show that $f(R^*) = f(R)$ which proves this step since R is chosen arbitrarily. Note that we do not have to worry about leaving $\mathcal{D}$ during the subsequent transformations because we do not change the favorite alternatives of the voters. Since either I or $N \setminus I$ contains more than half of the voters, it follows therefore that there is always a Condorcet winner. For transforming R^* into R, we consider two auxiliary profiles R^1 and R^2: in R^1, the voters $i \in I$ report R_i^* and the voters $i \in N \setminus I$ report R_i. The profile R^2 is constructed inversely: the voters $i \in I$ report R_i and the voters $i \in N \setminus I$ report R_i^*. Next, we show that $f(R, x) = f(R^1, x) = f(R^*, x)$. Note for this that we can use pairwise swaps to transform R^* to R^1 and that no swap involves y because we only need to reorder the alternatives $A \setminus \{y\}$ in the preference relations of the voters $i \in N \setminus I$. Thus, it follows from localizedness that $f(R^1, y) = f(R^*, y)$. Moreover, y Pareto-dominates all other alternatives but x in R^1 as it is second-ranked by all voters in I and top-ranked by the voters in $N \setminus I$. Hence, Lemma 1 implies that $f(R^1, x) + f(R^1, y) = 1$ which means that $f(R^1, x) = 1 - f(R^1, y) = 1 - f(R^*, y) = f(R^*, x)$. As next step, we transfrom R^1 into R by reordering the alternatives in $A \setminus \{x\}$ in the preference relations of voters i. We can use for this again pairwise swaps that do not involve x and thus, localizedness implies that $f(R, x) = f(R^1, x) = f(R^*, x)$. Moreover, we can use a symmetric argument to derive that $f(R, y) = f(R^2, y) = f(R^*, y)$. Therefore, we conclude that $f(R) = f(R^*)$ and $f(R, x) + f(R, y) = 1$ for all preference profiles $R \in S^{xIy}$.

Step 2: $f(R, x) = f(R', z)$ **for all distinct alternatives** $x, y, z \in A$ **and preference profiles** $R \in S^{xIy}$, $R' \in S^{zIy}$

For proving this claim, let x, y, z denote three distinct alternatives and consider the profiles R^3 and R^4.

R^3:	I: $x, z, y, *$	$N \setminus I$: $y, x, z, *$
R^4:	I: $z, x, y, *$	$N \setminus I$: $y, x, z, *$

We start the analysis at the profile $R^3 \in S^{xIy}$. In particular, note that Step 1 implies for R^3 that $f(R^3, x) + f(R^3, y) = 1$. Next, we derive the profile R^4 by letting every voter $i \in I$ swap x and z one after another. If y is the Condorcet winner in R^3, it is clear that this process does not leave the Condorcet domain as y is not involved in these steps. If x is the Condorcet winner, then x or z are the Condorcet winner during each step. The reason for this is that $g_{R^3}(z, w) = g_{R^3}(x, w) > 0$ for every alternative $w \in A \setminus \{x, z\}$. This means that x is the Condorcet winner as long as a majority of the voters prefers x to z and otherwise, z is the Condorcet winner. Moreover, localizedness implies that $f(R^4, y) = f(R^3, y)$ since y was not involved in any swap during these steps. Finally, Step 1 implies that $f(R^4, w) = 0$ for all $w \in A \setminus \{y, z\}$ because $R^4 \in S^{zIy}$. Hence, we infer that $f(R^4, z) = 1 - f(R^4, y) = 1 - f(R^3, y) = f(R^3, x)$. Since $R^3 \in S^{xIy}$ and $R^4 \in S^{zIy}$, we can now use the insights of the first step to deduce that $f(R, x) = f(R', z)$ for all distinct alternatives $x, y, z \in A$ and preference profiles $R \in S^{xIy}$ and $R' \in S^{zIy}$.

Step 3: $f(R, w) = f(R', y)$, $f(R, x) = f(R', z)$ **and** $f(R, w) + f(R, x) = 1$ **for all alternatives** $w, x, y, z \in A$ **with** $w \neq x$ **and** $y \neq z$ **and all preference profiles** $R \in S^{wIx}$, $R' \in S^{yIz}$

Let w, x, y, z denote arbitrary alternatives with $w \neq x$ and $y \neq z$. First note that Step 1 proves the claim if $w = y$ and $x = z$. Furthermore, the case that $x = z$ and $w \neq y$ follows from Step 1 and Step 2: Step 2 entails that $f(R, w) = f(R', y)$ for all $R \in S^{wIx}$ and $R' \in S^{yIx}$ and then, we infer from Step 1 that $f(R, x) = f(R', x)$ because all other alternatives must have probability 0 in both R and R'. Note that we can use the same argument also for the case that $w = y$ and $x \neq z$. The reason for this is that $S^{wIx} = S^{xN \setminus Iw}$ and $S^{yIz} = S^{zN \setminus Iy}$, i.e., we can just revert the roles of I and $N \setminus I$ to apply the same argument as in the previous case. If $\{w, x\} \cap \{y, z\} = \emptyset$, the lemma follows by applying the previous ideas twice: we know that $f(R,w) = f(\bar{R},y) = f(R',y)$ and $f(R,x) = f(\bar{R},x) = f(R',z)$ for all $R \in S^{wIx}$, $\bar{R} \in S^{yIx}$, and $R' \in S^{yIz}$. Finally, the last case is that $w = z$ or $x = y$ or both. We focus on the case that $w = z$, i.e., we assume that $R \in S^{wIx}$ and $R' \in S^{yIw}$; the case $x = y$ follows by a symmetric argument. In this case, we use an auxiliary alternative v and two auxiliary profiles $R^1 \in S^{vIx}$, $R^2 \in S^{vIw}$: we know that $f(R,w) = f(R^1,v) = f(R^2,v) = f(R',y)$ and $f(R,x) = f(R^1,x) = f(R^2,w) = f(R',w)$ for all preference profiles $R \in S^{wIx}$, $R' \in S^{yIw}$ by applying the previous argument three times. Note that this argument also covers the case that $w = z$ and $x = y$ because y can be chosen arbitrarily in $A \setminus \{w\}$. This concludes the proof as all cases are covered. □

Lemma 6 is helpful for our analysis because determining the probabilities for a single preference profile $R \in S^{xIy}$ specifies the outcome for a whole class of profiles. Next, we show a technical auxiliary lemma that discusses how the probabilities change if a voter swaps to alternatives $w, x \in A$ in two related preference profiles R and $\bar{R}$.

Lemma 7. *Assume $n \geq 2$ and $m \geq 3$, and let f denote a locally strategyproof SDS on an arbitrary domain $\mathcal{D} \subseteq \mathcal{R}^n$. Moreover, consider preference profiles $R, \bar{R}, R', \bar{R}' \in \mathcal{D}$, two distinct voters $i, j \in N$, and alternatives $w, x, y, z \in A$ such that $w \neq x$, $y \neq z$, $\{w,x\} \neq \{y,z\}$, $\bar{R} = R^{i:xw}$, $R' = R^{j:zy}$, and $\bar{R}' = \bar{R}^{j:zy}$. It holds that $f(\bar{R},w) - f(R,w) = f(\bar{R}',w) - f(R',w)$ and $f(\bar{R},x) - f(R,x) = f(\bar{R}',x) - f(R',x)$.*

Proof. Assume that $n \geq 2$ and $m \geq 3$, and let f denote a locally strategyproof SDS on an arbitrary domain $\mathcal{D}$. It is for this lemma crucial that local strategyproofness implies localizedness for every domain (see Lemma 5). Next, suppose that there are preference profiles $R, \bar{R}, R', \bar{R}' \in \mathcal{D}$, voters $i, j \in N$, and alternatives $w, x, y, z \in A$ such that $w \neq x$, $y \neq z$, $\{w,x\} \neq \{y,z\}$, $\bar{R} = R^{i:xw}$, $R' = R^{j:zy}$, and $\bar{R}' = \bar{R}^{j:zy}$. First, note that the assumption $\{w,x\} \neq \{y,z\}$ means that $w \notin \{y,z\}$ or $x \notin \{y,z\}$ or both. We assume without loss of generality that $w \notin \{y,z\}$ as both cases are symmetric. Hence, localizedness between R and R' implies that $f(R',w) = f(R,w)$ and localizedness between $\bar{R}$ and $\bar{R}'$ that $f(\bar{R}',w) = f(\bar{R},w)$ because $R' = R^{j:zy}$ and $\bar{R}' = \bar{R}^{j:zy}$. Therefore, we infer immediately that $f(\bar{R},w) - f(R,w) = f(\bar{R}',w) - f(R',w)$, which proves the first claim of this lemma.

Next, observe that our assumptions entail that $\bar{R}' = (R')^{i:xw}$ and we suppose that $\bar{R} = R^{i:xw}$. Thus, localizedness implies that $f(\bar{R}',v) = f(R',v)$ and $f(\bar{R},v) = f(R,v)$ for all $v \in A \setminus \{w,x\}$. Hence, we can compute that

$$\begin{aligned}
&f(\bar{R},x) - f(R,x)\\
&= \Big(1 - f(\bar{R}, A \setminus \{x\})\Big) - \Big(1 - f(R, A \setminus \{x\})\Big)\\
&= \Big(f(R,w) - f(\bar{R},w)\Big)\\
&= \Big(f(R',w) - f(\bar{R}',w)\Big)\\
&= \Big(1 - f(\bar{R}', A \setminus \{x\})\Big) - \Big(1 - f(R', A \setminus \{x\})\Big)\\
&= f(\bar{R}',x) - f(R',x).
\end{aligned}$$

This proves the second equality of the lemma. □

We will use Lemma 7 later on as it relates the probabilities between two profiles that only differ in a single swap, even after applying several swaps. Next, we discuss what it means to be locally 0-randomly dictatorial on connected super Condorcet domains.

Lemma 8. *Assume $n \geq 3$ is odd and $m \geq 3$ and let f denote a locally strategyproof and locally 0-randomly dictatorial SDS on a connected super Condorcet domain $\mathcal{D}$. For every voter $i \in N$, there are preference profiles $R, R' \in \mathcal{D}$ and alternatives $x, y \in A$ such that $R' = R^{i:yx}$, $r(R_i,x) = 1$, $r(R_i,y) = 2$, and $f(R',y) = f(R,y)$.*

Proof. Consider a connected super Condorcet domain $\mathcal{D}$ and let f denote a locally strategyproof and locally 0-randomly dictatorial SDS on $\mathcal{D}$. Furthermore, suppose for contradiction that there is a voter $i \in N$ such that $f(R',y) - f(R,y) \neq 0$ for all preference profiles $R, R' \in \mathcal{D}$ and alternatives $x, y \in A$ such that $R' = R^{i:yx}$, $r(R_i,x) = 1$, and $r(R_i,y) = 2$. First note that non-perversity entails for all these profiles that $f(R',y) - f(R,y) \geq 0$. Consequently, there is an $\epsilon > 0$ such that $f(R',y) - f(R,y) \geq \epsilon$ for all preference profiles $R, R' \in \mathcal{D}$ and alternatives $x, y \in A$ such that $R' = R^{i:yx}$, $r(R_i,x) = 1$, and $r(R_i,y) = 2$. Our goal is to show that f is locally γ-randomly dictatorial for $\gamma \geq \epsilon$. This contradicts the assumption that f is a local 0-random dictatorship and hence, there must be profiles for voter i that satisfy all conditions of the lemma.

For deriving the contradiction, we proceed in multiple steps. In Step 1, we show that $f(R,x) \geq \epsilon$ for all preference profiles $R \in \mathcal{D}$ and alternatives $x \in A$ such that voter i prefers x the most in R. Based on this observation, it follows that $\epsilon < 1$ because otherwise, f chooses always the best alternative of voter i with probability 1. However, this means that f is the dictatorial SDS of voter i and therefore, it is 1-randomly dictatorial if $\epsilon = 1$. Since this contradicts the assumption that f is locally 0-randomly dictatorial, we can assume that $0 < \epsilon < 1$. For this case, we show in Step 2 that $g = \frac{1}{1-\epsilon} f - \frac{\epsilon}{1-\epsilon} d_i$ (where d_i is the dictatorial SDS of voter i) is a well-defined and locally strategyproof SDS. Hence, f can be represented as $f = \epsilon d_i + (1-\epsilon) g$, which shows that f is locally γ-randomly dictatorial for some $\gamma \geq \epsilon$. This is in conflict with our assumptions and thus, there must be profiles $R, R' \in \mathcal{D}$ and alternatives $x, y \in A$ such that $f(R,y) = f(R',y)$, $R' = R^{i:yx}$, $r(R_i,x) = 1$, and $r(R_i,y) = 2$.

Step 1: $f(R,x) \geq \epsilon$ for every preference profile $R \in \mathcal{D}$ and every alternative $x \in A$ such that $r(R_i,x) = 1$.

As first step, we show that $f(R,x) \geq \epsilon$ for all preference profiles $R \in \mathcal{D}$ and alternatives $x \in A$ such that voter i top-ranks x in R. Thus, consider an arbitrary profile $R \in \mathcal{D}$ and let x denote voter i's best alternative. As first step, we construct a profile $R^1 \in \mathcal{D}_C$ such that $f(R,x) = f(R^1,x)$ and $r(R_i,x) = 1$. If R is already in the Condorcet domain, we just define $R^1 = R$. Otherwise, we can find an ad-path from $R \in \mathcal{D} \setminus \mathcal{D}_C$ to a profile $R^1 \in \mathcal{D}_C$ along which x is not swapped with any other alternative because $\mathcal{D}$ is connected. Consequently, a repeated application of localizedness along the path from R to R^1 implies that $f(R^1,x) = f(R,x)$ and x is still voter i's best alternative in R^1.

In the sequel, we show that $f(R^1,x) \geq \epsilon$, which proves this step since $f(R,x) = f(R^1,x)$. For this, we use a case distinction with respect to the Condorcet winner c in R^1 and assume first that $c \neq x$. In this case, consider the profile R^2 derived from R^1 by swapping voter i's best alternative x with his second best alternative y. Since x is not the Condorcet winner in R^1, it follows that $R^2 \in \mathcal{D}_C \subseteq \mathcal{D}$: either the Condorcet winner is not affected by swapping x and y or it is reinforced if $c = y$. Our assumptions on f imply that $f(R^1, x) - f(R^2, x) \geq \epsilon$ and it holds that $f(R^2, x) \geq 0$ because of the definition of SDSs. Hence, $f(R^1, x) \geq \epsilon$ which proves this case.

As second case, suppose that $c = x$. In this case, we choose an arbitrary alternative $y \in A \setminus \{x\}$ and define $J = \{j \in N : xR^1_j y\}$ as the set of voters who prefer x to y in R^1. As first step, consider the profile R^2 derived from R^1 by letting all voters $j \in J$ reinforce y until it is directly below x. Since no swap in the construction of R^2 involves x, we do not leave the Condorcet domain. Also, this observation implies that $f(R^2, x) = f(R^1, x)$ because of localizedness. Moreover, $g_{R^2}(y, z) \geq g_{R^2}(x, z) > 0$ for all alternatives $z \in A \setminus \{x, y\}$ because $xR^2_j z$ implies $yR^2 z$ and x is the Condorcet winner in R^2. We use this fact to construct the next profile R^3 by letting the voters $j \in J \setminus \{i\}$ swap x and y one after another. This construction does not leave the Condorcet domain: since n is odd and $g_{R^2}(x, z) = g_{R^2}(y, z) > 0$ for all $z \in A \setminus \{x, y\}$, x or y is the Condorcet winner in every intermediate profile. Moreover, in the final profile R^3, all voters in $N \setminus \{i\}$ prefer y to x and thus, y is now the Condorcet winner because $n \geq 3$. Hence, it follows from the last case that $f(R^3, x) \geq \epsilon$ because voter i top-ranks x in R^3 and x is not the Condorcet winner in this profile. Furthermore, we never reinforce x in the construction of R^3, and thus, non-perversity implies that $f(R^3, x) \leq f(R^2, x) = f(R^1, x)$. Combining the last two observations shows that $f(R^1, x) \geq \epsilon$ and thus, f always chooses the best alternative of voter i with a probability of at least ϵ.

Step 2: If $\epsilon < 1$, then $g = \frac{1}{1-\epsilon} f - \frac{\epsilon}{1-\epsilon} d_i$ is a well-defined and locally strategyproof SDS.

Assume that $\epsilon < 1$ and consider the function $g = \frac{1}{1-\epsilon} f - \frac{\epsilon}{1-\epsilon} d_i$, where d_i is the dictatorial SDS of voter i. We need to show that g is a well-defined and locally strategyproof SDS because f is then not locally 0-randomly dictatorial. First, we show that g is a well-defined SDS, i.e., that $g(R, x) \geq 0$ for all alternatives $x \in A$ and preference profiles $R \in \mathcal{D}$, and that $\sum_{x \in A} g(R, x) = 1$ for all preference profiles $R \in \mathcal{D}$. For proving the first claim, consider a preference profile $R \in \mathcal{D}$ and an alternative $x \in A$. We use a case distinction with respect to whether x is voter i's best alternative in R. If this is the case, Step 1 implies that $f(R, x) \geq \epsilon$ and $d_i(R, x) = 1$ because d_i chooses the best alternative of voter i. Hence, we can compute that $g(R, x) = \frac{1}{1-\epsilon} f(R, x) - \frac{\epsilon}{1-\epsilon} d_i(R, x) \geq \frac{\epsilon}{1-\epsilon} - \frac{\epsilon}{1-\epsilon} = 0$. On the other hand, if x is not voter i's best alternative, it follows that $d_i(R, x) = 0$. This means that $g(R, x) = \frac{1}{1-\epsilon} f(R, x) - \frac{\epsilon}{1-\epsilon} d_i(R, x) = \frac{1}{1-\epsilon} f(R, x) \geq 0$, which shows that g only assigns non-negative values to the alternatives. Next, we show that the probabilities assigned by g sum up to 1 for every preference profile $R \in \mathcal{D}$. This follows immediately by the definition of this SDS since $\sum_{x \in A} g(R, x) = \frac{1}{1-\epsilon} \sum_{x \in A} f(R, x) - \frac{\epsilon}{1-\epsilon} \sum_{x \in A} d_i(R, x) = \frac{1}{1-\epsilon} - \frac{\epsilon}{1-\epsilon} = 1$ for every profile $R \in \mathcal{D}$. Hence, g is a well-defined SDS.

Next, we show that g is locally strategyproof. Because Lemma 5 shows that local strategyproofness is equivalent to non-perversity and localizedness, it suffices to show that g satisfies the latter two axioms. Hence, consider two preference profiles $R, R' \in \mathcal{D}$, two alternatives $x, y \in A$, and a voter $j \in N$ such that $R' = R^{j:yx}$. First, note that g is localized because

$$
\begin{aligned}
g(R, z) &= \frac{1}{1-\epsilon} f(R, z) - \frac{\epsilon}{1-\epsilon} d_i(R, z) \\
&= \frac{1}{1-\epsilon} f(R', z) - \frac{\epsilon}{1-\epsilon} d_i(R', z) = g(R', z)
\end{aligned}
$$

for all $z \in A \setminus \{x, y\}$. This is true since f and d_i are locally strategyproof and therefore localized. As second point, we need to show that g is non-perverse, i.e., that $g(R', y) \geq g(R, y)$ for all preference profiles $R, R' \in \mathcal{D}$, alternatives $x, y \in A$, and voters $j \in N$ such that $R' = R^{j:yx}$. Note for this that $f(R', y) \geq f(R, y)$ for all these profiles because f is locally strategyproof and thus also non-perverse. Moreover, if $r(R_i, x) > 1$ or $j \neq i$, it follows that $d_i(R, y) = d_i(R', y)$. Combining these observations shows that

$$
\begin{aligned}
g(R', y) &= \frac{1}{1-\epsilon} f(R', y) - \frac{\epsilon}{1-\epsilon} d_i(R', y) \\
&\geq \frac{1}{1-\epsilon} f(R, y) - \frac{\epsilon}{1-\epsilon} d_i(R, y) = g(R, y).
\end{aligned}
$$

Hence, g is non-perverse in this case. On the other hand, if $r(R_i, x) = 1$ and $j = i$, it holds that $d_i(R, y) = 0 \neq 1 = d_i(R', y)$. Moreover, our assumptions on f imply for this case that $f(R', y) - f(R, y) \geq \epsilon$. Thus, we can compute that

$$
\begin{aligned}
g(R', y) &= \frac{1}{1-\epsilon} f(R', y) - \frac{\epsilon}{1-\epsilon} d_i(R', y) \\
&\geq \frac{1}{1-\epsilon} (\epsilon + f(R, y)) - \frac{\epsilon}{1-\epsilon} \\
&= \frac{1}{1-\epsilon} f(R, y) \\
&= g(R, y).
\end{aligned}
$$

This inequality shows also for the second case that g is non-perverse and consequently, it is locally strategyproof. Hence, f is not locally 0-randomly dictatorial because $f = \epsilon d_i + (1 - \epsilon) g$. In particular, this representation entails that f is locally γ-randomly dictatorial for $\gamma \geq \epsilon$. This means that the initial assumption is false and there are for every voter $i \in N$ two preference profiles $R, R' \in \mathcal{D}$ and alternatives $x, y \in A$ such that $f(R, y) = f(R', y)$, $R' = R^{i:yx}$, $r(R_i, x) = 1$, and $r(R_i, y) = 2$. □

As a consequence of Lemma 8, there are for every locally 0-randomly dictatorial and locally strategyproof SDS f on the Condorcet domain and every voter $i \in N$ two preference profiles $R, R' \in \mathcal{D}_C$ and two alternatives $x, y \in A$ such that $R' = R^{i:yx}$, $f(R', y) - f(R, y) = 0$, and voter i prefers x the most and y the second most in R. We show next that this is also true for every connected super Condorcet domain $\mathcal{D}$, i.e., even if $\mathcal{D}_C \subsetneq \mathcal{D}$, we can find for every voter i suitable profiles R and R' in $\mathcal{D}_C$.

Lemma 9. *Assume $n \geq 3$ is odd and $m \geq 3$ and let f denote a locally strategyproof and locally 0-randomly dictatorial SDS on a connected super Condorcet domain $\mathcal{D}$. For every voter $i \in N$, there are preference profiles $R, R' \in \mathcal{D}_C \subseteq \mathcal{D}$ and alternatives $x, y \in A$ such that $R' = R^{i:yx}$, $r(R_i, x) = 1$, $r(R_i, y) = 2$, and $f(R', y) = f(R, y)$.*

Proof. Assume that $n \geq 3$ is odd and $m \geq 3$, and let $\mathcal{D}$ denote a connected super Condorcet domain. Furthermore, consider a locally strategyproof and locally 0-randomly dictatorial SDS $f : \mathcal{D} \to \Delta(A)$. We need to show that for every voter $j \in N$ there are preference profiles $R, R' \in \mathcal{D}_C$ and alternatives $x, y \in A$ such that $R' = R^{j:yx}$, $r(R_j, x) = 1$, $r(R_j, y) = 2$, and $f(R, y) = f(R', y)$. First, note that Lemma 8 proves that there are such profiles $R, R' \in \mathcal{D}$ for every voter. If these profiles are for all $j \in N$ also in the Condorcet domain, the lemma is already proven. Hence, suppose that there is a voter $i \in N$ such that $\{R, R'\} \not\subseteq \mathcal{D}_C$ for all profiles $R, R' \in \mathcal{D}$ and alternatives $x, y \in A$ with $R' = R^{i:yx}$, $r(R_i, x) = 1$, $r(R_i, y) = 2$, and $f(R, y) = f(R', y)$. Next, we consider two such profiles R and R' and we suppose that $R \notin \mathcal{D}_C$ because the case that $R' \notin \mathcal{D}_C$ is symmetric. Our goal is to transform the profiles R, R' into two new profiles $\bar{R}$ and $\bar{R}'$, respectively, that satisfy all conditions of the lemma, i.e., $\{\bar{R}, \bar{R}'\} \subseteq \mathcal{D}_C$, $\bar{R}' = \bar{R}^{i:yx}$, $r(\bar{R}_i, x) = 1$, $r(\bar{R}_i, y) = 2$, and $f(\bar{R}, y) = f(\bar{R}', y)$. Note for this that the assumption $f(R, y) = f(R', y)$ implies also that $f(R, x) = f(R', x)$ because of localizedness.

As first step in the construction of $\bar{R}$ and $\bar{R}'$, we use the connectedness of $\mathcal{D}$: there is an ad-path from R to a profile $R^1 \in \mathcal{D}_C$ that does not swap x with another alternative. Similarly, if $R' \notin \mathcal{D}_C$, there is also a profile $R^2 \in \mathcal{D}_C$ derived from R' without swapping x; if $R' \in \mathcal{D}_C$, we set $R^2 = R'$. Since R^1 is derived from R without swapping x with another alternative, it follows that $U(R^1_j, x) = U(R_j, x)$ for all voters $j \in N$. In particular, this means that x is not the Condorcet winner in R^1 because it is not the Condorcet winner in R. Analogously, it follows that $U(R^2_j, x) = U(R'_j, x)$ for all voters $j \in N$. This entails that $U(R^2_j, x) = U(R^1_j, x)$ for all $j \in N \setminus \{i\}$ because $R_j = R'_j$ for these voters, and $U(R^1_i, x) \subseteq U(R^2_i, x)$ because $R' = R^{i:yx}$. In particular, these observations imply that x is also not the Condorcet winner in R^2, and that $r(R^1_i, x) = 1$, $r(R^2_i, y) = 1$, and $r(R^2_i, x) = 2$. Finally, localizedness shows that $f(R^1, x) = f(R, x) = f(R', x) = f(R^2, x)$ because x is not swapped with another alternative in the construction of R^1 or R^2.

Next, let c denote the Condorcet winner in R^1 and define the set $I = \{j \in N \colon cR^1_j x\}$ of voters who prefer c to x in R^1. Analogously, we define c' as the Condorcet winner in R^2 and $J = \{j \in N \colon c'R^2_j x\}$ as the set of voters who prefer c' to x in R^2. Note that $i \notin I$ because voter i top-ranks x in R^1. We derive the profile $\bar{R}$ from R^1 as follows: we let all voters $j \in I$ swap up c until it is their best alternative. This step leads to an intermediate profile R^3, and we do not leave $\mathcal{D}_C \subseteq \mathcal{D}$ during the construction of this profile because we only reinforce the Condorcet winner c. Furthermore, it follows from the definition of I that x is not involved in any swap, which means that and $f(R^3, x) = f(R^1, x)$ because of localizedness. Finally, we construct the profile $\bar{R}$ based on R^3 by reordering the alternatives $z \in A \setminus \{x\}$ in voter i's preference relation as in R^2, i.e., voter i's preference relation only differs from R^2_i in the fact that he prefers x to y. Since all voters in I prefer c the most in R^3, we do not leave $\mathcal{D}_C$ during this step. Furthermore, localizedness implies that $f(\bar{R}, x) = f(R^3, x) = f(R, x)$.

As second point, we construct the profile $\bar{R}'$ based on R^2. In particular, we need to ensure that $\bar{R}'_j = \bar{R}_j$ for all voters $j \in N \setminus \{i\}$ and introduce therefore multiple auxiliary profiles. Hence, consider the profile R^4 derived from R^2 by letting all voters $j \in J$ push up c' until it is their best alternative, all voters $j \in I \cap J$ push up c until it is directly below c', and all voters $j \in I \setminus J$ push up c until it is their best alternative. Note that voter i does not change his preference relation during these steps: it holds that $i \notin I$ because $r(R^1_i, x) = 1$, and if $i \in J$, then $c' = y$ because alternative y is the only alternative in $A \setminus \{x\}$ with $yR^2_i x$. However, voter i already top-ranks y in R^2 and thus, it follows that $R^4_i = R^2_i$. Furthermore, we do not leave the Condorcet domain during these steps because we never weaken the Condorcet winner c'. Finally, observe that x is not involved in any swap: we have already shown that $R^4_i = R^2_i$, and the definition of the sets I and J combined with the fact that $U(R^1_j, x) = U(R^2_j, x)$ for all $j \in N \setminus \{i\}$ implies that these voters do not swap x with another alternative. Hence, localizedness entails that $f(R^4, x) = f(R^2, x)$.

As next step, we construct the profile R^5 based on R^4 by letting all voters in $I \cap J$ swap c and c' one after another. Note that this intersection is non-empty because $|I| > \frac{n}{2}$ and $|J| > \frac{n}{2}$. Furthermore, this process does not leave the Condorcet domain because n is odd and $g_{R^4}(c, z) > 0$, $g_{R^4}(c', z) > 0$ for all $z \in A \setminus \{c, c'\}$, which means that either c or c' is the Condorcet winner in every intermediate profile. In particular, c is the Condorcet winner in the final profile as it is top-ranked by all voters in I. As final points on R^5, observe that localizedness implies that $f(R^5, x) = f(R^4, x)$, and that $R^5_i = R^4_i = R^2_i$ because $i \notin I$.

As last step, we use the observation that all voters in I report c as their best alternative in both R^5 and $\bar{R}$ to construct the final profile $\bar{R}'$. In more detail, we derive this profile from R^5 by letting all voters $j \in N \setminus \{i\}$ reorder their preference relations as in $\bar{R}$. We can transform R^5 into $\bar{R}'$ without leaving $\mathcal{D}_C$ by only using pairwise swaps because all voters in I always report c as their best alternative, which ensures that c is the Condorcet winner. Furthermore, observe that $U(R^5_j, x) = U(\bar{R}_j, x)$ for all voters $j \in N \setminus \{i\}$ because we did never swap x with another alternative during the construction of these profiles. Hence, we can construct the profile $\bar{R}'$ without swapping x with another alternative in the preference relations of these voters and localizedness implies consequently that $f(\bar{R}', x) = f(R^5, x)$. Finally, observe that, by construction, $\bar{R}'_j = \bar{R}_j$ for all $j \in N \setminus \{i\}$, and $\bar{R}'_i = R^2_i = \bar{R}^{i:yx}_i$. Hence, it indeed holds that $\bar{R}' = \bar{R}^{i:yx}$ and $\bar{R}, \bar{R}' \in \mathcal{D}_C$. Furthermore, $f(\bar{R}, x) = f(R, x) = f(R', x) = f(\bar{R}', x)$ because we never swapped x with another alternative in our construction and it follows therefore from localizedness that $f(\bar{R}', y) = 1 - \sum_{z \in A \setminus \{y\}} f(\bar{R}', z) = 1 - \sum_{z \in A \setminus \{y\}} f(\bar{R}, z) = f(\bar{R}, y)$. This proves the lemma. ◻

Lemma 9 is important because it shows that the behavior of 0-randomly dictatorial SDSs does not depend on the exact connected super Condorcet domain. In particular, regardless of the considered super Condorcet domain, there are for every voter preference profiles in the Condorcet domain that satisfy all conditions of Lemma 8 and thus, our subsequent analysis can focus on profiles in $\mathcal{D}_C$.

In the next lemma, we show that every 0-randomly dictatorial, locally strategyproof, and non-imposing SDS on a connected super Condorcet domain is 1-unanimous if $n \geq 3$ is odd. Recall here that an SDS $f : \mathcal{D} \to \Delta(A)$ is k-unanimous if $f(R, x) = 1$ for all alternatives $x \in A$ and preference profiles $R \in \mathcal{D}$ such that $|N^R_x| \geq n - k$.

Lemma 10. *Assume $n \geq 3$ is odd and $m \geq 3$. Every locally 0-randomly dictatorial, locally strategyproof, and non-imposing SDS on a connected super Condorcet domain is 1-unanimous.*

Proof. Assume $n \geq 3$ is odd and $m \geq 3$, and let $\mathcal{D}$ denote a connected super Condorcet domain. Furthermore, let f denote a locally 0-randomly dictatorial SDS on $\mathcal{D}$ that satisfies local strategyproofness and non-imposition. We need to show that $f(R,x) = 1$ for every preference profile $R \in \mathcal{D}$ and every alternative $x \in A$ such that $|N^R_x| \geq n-1$. First, note that f meets all requirements of Lemma 1, which implies that an alternative is chosen with probability 1 if it is top-ranked by all voters because every other alternative is then Pareto-dominated. Therefore, it suffices to focus on profiles $R \in \mathcal{D}$ and alternatives $x \in A$ such that $|N^R_x| = n-1$.

Consider for this an arbitrary voter $i \in N$. We show in the sequel that $f(R,x) = 1$ for all preference profiles $R \in \mathcal{D}$ and alternatives $x \in A$ such that $N^R_x = N \setminus \{i\}$. Since i is chosen arbitrarily, this claim proves the lemma. As next step, we use the insights of Lemma 9: for the considered voter i, there are preference profiles $R, \bar{R} \in \mathcal{D}_C$ and two alternatives $x, y \in A$ such that $\bar{R} = R^{i:yx}$, $r(R_i, x) = 1$, $r(R_i, y) = 2$, and $f(R, y) = f(\bar{R}, y)$. We proceed with a case distinction with respect to the profile R and its Condorcet winner c.

Case 1: There is $z \in A$ such that $N \setminus \{i\} \subseteq N^z_R$.

As first case, suppose that there is an alternative $z \in A$ such that $N \setminus \{i\} \subseteq N^z_R$, i.e., all voters in $N \setminus \{i\}$ report z as their best alternative in R. If $z = x$, then x Pareto-dominates every other alternative because every voter top-ranks x. Hence, $f(R,x) = 1$ because of Lemma 1. Then, our assumptions imply that $f(\bar{R}, y) = f(R, y) = 0$, even though $\bar{R} \in S^{xN\setminus\{i\}y}$. Thus, Lemma 6 applies and shows that $f(R', w) = 1$ for all profiles $R' \in \mathcal{D}_C$ and alternatives $w \in A$ such that $N^{R'}_w = N \setminus \{i\}$. The reason for this is that for all these profiles R', there is an alternative $v \in A \setminus \{w\}$ such that $R' \in S^{wN\setminus\{i\}v}$. Moreover, the case that $z = y$ follows by exchanging the roles of $\bar{R}$ and R in the above argument.

Hence, suppose that $z \in A \setminus \{x, y\}$. In this case, we use Lemma 6 to deduce that $f(R, y) = 0$ since $R \in S^{zN\setminus\{i\}x}$. Our assumptions entail therefore that $f(\bar{R}, y) = f(R, y) = 0$. Since $\bar{R} \in S^{zN\setminus\{i\}y}$, we can again use Lemma 6 to derive that $f(R', w) = 1$ for all preference profiles $R' \in \mathcal{D}_C$ and alternatives $w \in A$ such that $N \setminus \{i\}$ is precisely the set of voters who report w as their favorite choice in R. Hence, the claim is in this case proven.

Case 2: There is no $z \in A$ such that $N \setminus \{i\} \subseteq N^R_x$ and $c \in A \setminus \{x, y\}$.

As second case, suppose that the voters in $N \setminus \{i\}$ do not agree on a best alternative and that the Condorcet winner c in the profile R is not in $\{x, y\}$. In this case, we derive two new the profiles R' and $\bar{R}'$ from R and $\bar{R}$, respectively, such that $R' \in S^{cN\setminus\{i\}x}$, $\bar{R}' = (R')^{i:yx}$, $r(R'_i, x) = 1$, $r(R'_i, y) = 2$, and $f(R', y) = f(\bar{R}', y)$. Then, the insights of Case 1 show that f is 1-unanimous. For constructing these new preference profiles, we repeatedly apply Lemma 7 to push up c in the preference of every voter $j \in N \setminus \{i\}$ until it is their best alternative. In more detail, we construct two sequences of preference profiles $(R^1, \dots, R^k)$ and $(\bar{R}^1, \dots, \bar{R}^k)$ such that $R^1 = R$, $R^k = R'$, $\bar{R}^1 = \bar{R}$, $\bar{R}^k = \bar{R}'$, and for every index $l \in \{1, \dots, k-1\}$ there is a voter $j \in N \setminus \{i\}$ and an alternative $z \in A \setminus \{c\}$ such that $R^{l+1} = (R^l)^{j:cz}$ and $\bar{R}^{l+1} = (\bar{R}^l)^{j:cz}$. Less formally, in each step, the new profile is derived by swapping up c in the preference of some voter j in both R^l and $\bar{R}^l$. It is straightforward to construct such a sequence of preference profiles: we repeatedly identify a voter $j \in N \setminus \{i\}$ who does not top-rank c and reinforce this alternative in his preference relation.

Since $\bar{R} = R^{i:yx}$, it also holds that $\bar{R}^l = (R^l)^{i:yx}$ because we use the same sequence of swaps to derive R^l from R and $\bar{R}^l$ from $\bar{R}$. Hence, Lemma 7 implies for every step that $f(R^{l+1}, y) - f(\bar{R}^{l+1}, y) = f(R^l, y) - f(\bar{R}^l, y)$. In particular, it is important for this that $c \notin \{x, y\}$ to ensure that $\{c, z\} \neq \{x, y\}$ (where z denote the alternatives with which c is swapped in the current step). By repeatedly using the last insight, it follows that $f(R', y) - f(\bar{R}', y) = f(R, y) - f(\bar{R}, y) = 0$. Furthermore, since we only reinforce the Condorcet winner, we do not leave the Condorcet domain. Hence, it is easy to verify that the profiles R' and $\bar{R}'$ indeed satisfy all requirements of Case 1, which means that f is 1-unanimous.

Case 3: There is no $z \in A$ such that $N \setminus \{i\} \subseteq N^R_x$ and $c \in \{x, y\}$.

As last case, we assume again that the voters $j \in N \setminus \{i\}$ do not agree on a best alternative. However, this time we suppose additionally that $c \in \{x, y\}$, i.e., the Condorcet winner is affected in the derivation of $\bar{R}$. For this case, we define the sets $I = \{j \in N \setminus \{i\} \colon xR_iy\}$ and $J = \{j \in N \setminus \{i\} \colon yR_ix\}$, which partition the voters in $N \setminus \{i\}$ according to their preference on x and y in R. We use these sets to construct auxiliary profiles just as in the last case. In more detail, we derive two new profiles R^1 and $\bar{R}^1$ from R and $\bar{R}$, respectively, by pushing up x and y in the preference relations of the voters $j \in N \setminus \{i\}$ without changing their relative order until they are the best two alternatives. These profiles are shown below and, just as in Case 2, a repeated application of Lemma 7 proves that $f(\bar{R}^1, y) - f(R^1, y) = f(\bar{R}, y) - f(R, y) = 0$. Furthermore, Lemma 1 implies that $f(R^1, x) + f(R^1, y) = 1$ and $f(\bar{R}^1, x) + f(\bar{R}^1, y) = 1$ because all other alternatives are Pareto-dominated. Hence, it follows that $f(\bar{R}^1) = f(R^1)$. As last point, it should be mentioned that this process does not leave the Condorcet domain as the Condorcet winner $c \in \{x, y\}$ is never weakened.

R^1:	I: $x, y, *$	J: $y, x, *$	i: $x, y, *$
$\bar{R}^1$:	I: $x, y, *$	J: $y, x, *$	i: $y, x, *$

As next step, we choose an arbitrary alternative $z \in A \setminus \{x, y\}$, and let the voter i reinforce this alternative until it is his second best one. Applying this step to R^1 and $\bar{R}^1$ results in the profiles R^2 and $\bar{R}^2$, respectively. Since $R^1, R^2 \in S^{yJx}$ and $\bar{R}^1, \bar{R}^2 \in S^{xIy}$, it follows from Lemma 6 that $f(R^2) = f(R^1) = (\bar{R}^1) = f(\bar{R}^2)$. Finally, note that we do not leave the Condorcet domain in the construction of R^2 and $\bar{R}^2$ since x or y is always top-ranked by more than half of the voters.

R^2:	I: $x, y, *$	J: $y, x, *$	i: $x, z, y, *$
$\bar{R}^2$:	I: $x, y, *$	J: $y, x, *$	i: $y, z, x, *$

Thereafter, we consider the profiles R^3 and $\bar{R}^3$ derived from R^2 and $\bar{R}^2$, respectively, by letting voter i swap his best two alternatives. First note that $R^3, \bar{R}^3 \in \mathcal{D}_C$ because $g_{R^3}(x, z) = g_{\bar{R}^3}(x, z) = g_{R^3}(y, z) = g_{\bar{R}^3}(y, z) = n-2 > 0$ and thus, this swap does not affect whether x or y is the Condorcet winner. Furthermore, localizedness between R^2 and R^3 implies that $f(R^3, y) = f(R^2, y)$ and localizedness between $\bar{R}^2 = \bar{R}^3$ implies that $f(\bar{R}^3, x) = f(\bar{R}^2, x)$. Moreover, we already know that $f(R^2) = f(\bar{R}^2)$ and non-perversity between R^3 and $\bar{R}^3$ shows that $f(R^3, x) \geq f(\bar{R}^3, x)$. Hence, we conclude that $f(R^3, x) \geq f(\bar{R}^3, x) = f(\bar{R}^2, x) = f(R^2, x)$. This means that $f(R^3) = f(R^2)$ because $f(R^2, y) + f(R^2, x) = 1$, $f(R^3, y) = f(R^2, y)$, and $f(R^3, x) \geq f(R^2, x)$. In particular, this shows that $f(R^3, z) = 0$.

R^3: I: $x, y, *$ J: $y, x, *$ i: $z, x, y, *$

$\bar{R}^3$: I: $x, y, *$ J: $y, x, *$ i: $z, y, x, *$

As last point, we let all voters $j \in J$ swap x and y to derive the profile R^4 from R^3. If x is the Condorcet winner in R^3, it is obvious that this process stays in the Condorcet domain. On the other hand, if y is the Condorcet winner in R^3, we also do not leave the Condorcet domain because $g_{R^3}(x,w) = g_{R^3}(y,w) > 0$ for all alternatives $w \in A \setminus \{x, y\}$. Hence, y is the Condorcet winner during these steps as long as a majority of voters prefers y to x, and x is the Condorcet winner otherwise. Next, note that $f(R^4, z) = f(R^3, z) = 0$ because of localizedness.

R^4: I: $x, y, *$ J: $x, y, *$ i: $z, x, y, *$

Since $R^4 \in S^{xN\setminus\{i\}z}$, $f(R^4, z) = 0$ implies that $f(R', w) = 1$ for all preference profiles $R \in \mathcal{D}_C$ and alternatives $w \in A$ such that $N_w^{R'} = N \setminus \{i\}$ because of Lemma 6. Hence, f is also in this case 1-unanimous, which proves the lemma. □

As next step, we strengthen the consequences of Lemma 10 by showing that every 1-unanimous and locally strategyproof SDS on a connected super Condorcet domain also satisfies $\frac{n-1}{2}$-unanimity.

Lemma 11. *Assume $n \geq 3$ is odd and $m \geq 3$. Every 1-unanimous and locally strategyproof SDS on a connected super Condorcet domain is $\frac{n-1}{2}$-unanimous.*

Proof. Let f denote a 1-unanimous and locally strategyproof SDS on a connected super Condorcet domain $\mathcal{D}$. We prove the lemma by an induction on $k \in \{1, \dots, \frac{n-3}{2}\}$ and show that if f is k-unanimous, it is also $k+1$-unanimous. As first step, observe that the induction basis $k = 1$ is true because we assume that f is 1-unanimous.

Next, suppose that f is k-unanimous for a fixed $k \in \{1, \dots, \frac{n-3}{2}\}$. We need to show that f is $k+1$-unanimous, i.e., that $f(R,x) = 1$ for every profile $R \in \mathcal{D}$ and every alternative $x \in A$ such that $|N_x^R| = n-k-1$. Note that $k+1$-unanimity also applies for profiles $R \in \mathcal{D}$ and alternatives $x \in A$ with $|N_x^R| > n-k-1$, but k-unanimity immediately guarantees for these profiles that $f(R,x) = 1$. Hence, it suffices to focus on profiles $R \in \mathcal{D}$ and alternatives $x \in A$ such that $|N_x^R| = n-k-1$.

As first step, we show that $f(R,x) = 1$ for all alternatives $x, y \in A$ and profiles $R \in S^{xIy}$ with $|I| = n-k-1$. For proving this, consider an arbitrary set of voters $I \subsetneq N$ with $|I| = n-k-1$, let h denote an arbitrary voter in $N \setminus I$, and define the set $J = N \setminus (I \cup \{h\})$. Note that $J \neq \emptyset$ because $k \geq 1$ implies that $|I| \leq n-2$. Finally, consider the profiles R^1 to R^4 shown below and note that $\{R^1, R^2, R^3, R^4\} \subseteq \mathcal{D}_C$ because $|I| > \frac{n}{2}$ and the top-ranked alternatives of these voters is thus the Condorcet winner. Our goal is to show that $f(R^4, x) = 1$ because Lemma 6 implies then our intermediate claim.

R^1: I: $x, z, y, *$ J: $z, y, x, *$ h: $y, x, z, *$

R^2: I: $x, z, y, *$ J: $z, y, x, *$ h: $x, y, z, *$

R^3: I: $z, x, y, *$ J: $z, y, x, *$ h: $y, x, z, *$

R^4: I: $x, z, y, *$ J: $y, z, x, *$ h: $y, x, z, *$

First, observe that f satisfies *ex post* efficiency because of Lemma 1 and thus, $f(R^1, w) = 0$ for all alternatives $w \in A \setminus \{x, y, z\}$. Next, $f(R^1, z) = 0$ because voter h could manipulate by deviating to R^2 otherwise. In more detail, $f(R^2, x) = 1$ because of k-unanimity and thus, localizedness from R^2 to R^1 requires that $f(R^1, z) = 0$. We can use a similar argument between R^1 and R^3 to derive that $f(R^1, y) = 0$. Note for this that $f(R^3, z) = 1$ because of k-unanimity and that we can derive R^1 from R^3 by letting the voters $i \in I$ sequentially swap x and z. This process does not leave the Condorcet domain as either x or z are the Condorcet winner in every intermediate profile. This is true because n is odd and $|I| > \frac{n}{2}$ implies that $g_{R^3}(x,w) > 0$ and $g_{R^3}(z,w) > 0$ for all alternatives $w \in A \setminus \{x, z\}$. Moreover, no swap involves y and thus, we derive that $f(R^1, y) = f(R^3, y) = 0$ because of localizedness. Hence, it follows that $f(R^1, x) = 1$ because $f(R^1, w) = 0$ for all other alternatives $w \in A \setminus \{x\}$. Finally, we derive the profile R^4 from R^1 by swapping y and z in the preference relations of the voters $i \in I$. Because these swaps do not affect x, it stays the Condorcet winner and localizedness implies that $f(R^4, x) = f(R^1, x) = 1$. Finally, observe that $R^4 \in S^{xIy}$, which means that Lemma 6 applies and shows that $f(R,x) = 1$ for all alternatives $x, y \in A$ and preference profiles $R \in S^{xIy}$.

As next step, consider an arbitrary alternative $x \in A$ and a profile $R \in \mathcal{D}$ such that $|N_x^R| = n-k-1$. It remains to show that $f(R,x) = 1$ and we consider for this an auxiliary profile R' which satisfies that $R'_i = R_i$ for all voters $i \in N_x^R$, and all voters $i \in N \setminus N_x^R$ report an alternative $y \in A \setminus \{x\}$ as their favorite option and x as their least preferred option. It follows from the previous observations that $f(R', x) = 1$ because $R' \in S^{xN_x^R y}$. Starting at R', we let each voter $i \in N \setminus N_x^R$ use swaps to transform his preference relation into R_i. In particular, we never need to weaken x during these swaps as all voters in $N \setminus N_x^R$ report it as their least preferred alternative in R'. Hence, it follows from non-perversity and localizedness that $f(R,x) = f(R',x) = 1$. Also, it should be mentioned that all intermediate profiles are in the Condorcet domain because x is always top-ranked by all voters in N_x^R. Finally, since the alternative x and the profile R are chosen arbitrarily, it follows that f satisfies $k+1$-unanimity. Hence, the induction step is proven and thus, we derive that f is $\frac{n-1}{2}$-unanimous. □

Note that the inductive argument in this proof breaks down for $k = \frac{n-1}{2}$ because the profile R^1 is not in the Condorcet domain anymore. However, for larger domains, such as the full domain, one can repeat this inductive argument until $|I| = 1$ to derive that there is no 1-unanimous and locally strategyproof SDS on these domains.

Finally, we use the insights of this section to prove Lemma 3.

Lemma 3. *Assume $n \geq 3$ is odd and $m \geq 3$. A non-imposing and locally strategyproof SDS on a connected super Condorcet domain is locally 0-randomly dictatorial iff it is Condorcet-consistent.*

Proof. Assume $n \geq 3$ is odd and consider an arbitrary connected super Condorcet domain $\mathcal{D}$. Moreover, let f denote a locally strategyproof and non-imposing SDS on $\mathcal{D}$. If f is Condorcet-consistent, it is easy to see that it locally 0-randomly dictatorial because the best alternative of every voter can have probability 0. In more detail, it holds for all voters $i \in N$ and alternatives $x, y \in A$ that $f(R, y) = 0$ for all preference profiles $R \in \mathcal{S}^{xN\setminus\{i\}y}$. Consequently, f cannot be represented as a mixture of a random dictatorship and another locally strategyproof SDS because this representation implies that there is a voter whose best alternative always gets positive probability. This shows that every locally strategyproof and Condorcet-consistent SDS is locally 0-randomly dictatorial.

For the inverse direction, suppose that f is locally strategyproof, locally 0-randomly dictatorial, and non-imposing. Moreover, assume for contradiction that f violates Condorcet-consistency, i.e., that there is a profile $R \in \mathcal{D}_C$ with Condorcet winner c such that $f(R, c) \neq 1$. This means that there is another alternative $d \in A \setminus \{c\}$ with $f(R, d) > 0$. As next step, we let all voters $i \in N$ with cR_id swap up c until it is their best alternative to derive the profile R'. Since we only reinforce the Condorcet winner, we do not leave the Condorcet domain and $f(R', d) > 0$ follows from localizedness because no swap involves d. Consequently, $f(R', c) < 1$ even though c is top-ranked by more than half of the voters in R'. This is true as more than half of the voters prefer c to d in R because it is the Condorcet winner. However, this contradicts our previous insights: f satisfies all requirements of Lemma 10 and Lemma 11, and thus it has to choose an alternative with probability 1 if it is top-ranked by more than half of the voters. Hence, we have derived a contradiction, which implies that the assumption that f fails Condorcet-consistency is wrong. ◻

Proof of Lemma 4

The last result that we need to prove is Lemma 4, and we proceed for this similar to to the proof of Lemma 3. As first step, we discuss a generalization of Lemma 6 to our new setting. Note that the proof of the new lemma can be easily derived from the proof of Lemma 6 and therefore, we only explain the differences. In particular, note for the subsequent lemma that $S^{xIy} \subseteq \mathcal{D}_C$ if $|I| \neq \frac{n}{2}$.

Lemma 12. *Assume $n \geq 3$ and $m \geq 3$ and let f denote a group-strategyproof and non-imposing SDS on a super Condorcet domain $\mathcal{D}$. For all sets of voters $I \subseteq N$ with $|I| \notin \{0, \frac{n}{2}, n\}$, alternatives $w, x, y, z \in A$ with $w \neq x$ and $y \neq z$, and preference profiles $R \in S^{wIx}$, $R' \in S^{yIz}$, it holds that $f(R, w) = f(R', y)$, $f(R, x) = f(R', z)$, and $f(R, w) + f(R, x) = 1$.*

Proof. We prove this result by pointing out how to change the proof of Lemma 6. Hence, let $\mathcal{D}$ denote a super Condorcet domain and let f denote a group-strategyproof and non-imposing SDS on $\mathcal{D}$. Furthermore, consider a set of voters $I \subseteq N$ with $|I| \notin \{0, \frac{n}{2}, n\}$. Just as for Lemma 6, we prove this result in three steps. First, we show that for all distinct alternatives $x, y \in A$ that $f(R, x) + f(R, y) = 1$ and $f(R) = f(R')$ for all profiles $R, R' \in S^{xIy}$. We can use exactly the same argument as in Lemma 6 to prove this claim because the applied construction does not change the Condorcet winner.

As second step, we show that $f(R, x) = f(R', z)$ for all distinct alternatives $x, y, z \in A$ and preference profiles $R \in S^{xIy}$ and $R' \in S^{zIy}$. In particular, we prove this step by considering the same profiles R^3 and R^4 as in the proof of Lemma 6. First, note that $R^3 \in S^{xIy}$ and thus, the insights of the first step imply that $f(R^3, w) = 0$ for all $w \in A \setminus \{x, y\}$. Next, we let all voters in I simultaneously swap x and z to derive R^4. The insights of the first step apply again and show that $f(R^4, w) = 0$ for all $w \in A \setminus \{y, z\}$. Also, since we let all voters in I change their preference relation simultaneously, we do not leave the Condorcet domain since $I \neq \frac{n}{2}$. Finally, note that group-strategyproofness implies that $f(R^3, y) = f(R^4, y)$: if $f(R^3, y) > f(R^4, y)$, then the voters $i \in I$ can group-manipulate by deviating from R^3 to R^4 and if $f(R^3, y) < f(R^4, y)$, these voters can group-manipulate by deviating from R^4 to R^3. Combining all observations shows that $f(R^4, z) = 1 - f(R^4, y) = 1 - f(R^3, y) = f(R^3, x)$. Finally, using the insights of the first step, it follows that $f(R, x) = f(R', z)$ for all $R \in S^{xIy}$, $R' \in S^{zIy}$.

R^3:	I: $x, z, y, *$	$N \setminus I$: $y, x, z, *$
R^4:	I: $z, x, y, *$	$N \setminus I$: $y, x, z, *$

As last step, we repeatedly use the insights of the last two paragraphs to prove the lemma. Here, we can use again the same argumentation as in Step 3 of the proof of Lemma 6 because all considered profiles are in the Condorcet domain. ◻

Next, we begin to investiate the behavior of non-dictatorial, non-imposing, and group-strategyproof SDSs on super Condorcet domains. First, we show that all such SDSs are 1-unanimous.

Lemma 13. *Assume $n \geq 3$ and $m \geq 3$. Every group-strategyproof, non-dictatorial, and non-imposing SDS on a super Condorcet domain is 1-unanimous.*

Proof. Assume that f is a group-strategyproof, non-dictatorial, and non-imposing SDS on a super Condorcet domain $\mathcal{D}$. Furthermore, suppose for contradiction that f is not 1-unanimous. Since Lemma 1 shows that f satisfies *ex post* efficiency, this means that there is a profile $R^* \in \mathcal{D}$, a voter $i \in N$, and two alternatives $a, b \in A$ such that voter i reports a as his favorite alternative in R^*, all other voters report b as their favorite alternative, but $f(R^*, b) < 1$. Since, $R^* \in S^{bN\setminus\{i\}a}$, this means that $f(R^*, a) > 0$ because $f(R^*, x) = 0$ for all other alternatives $x \in A \setminus \{a, b\}$. In the sequel, we show that this means that f is the dictatorial SDS for voter i. This contradicts our assumptions and therefore f has to satisfy 1-unanimity.

As first step, we show that $f(R, x) = 1$ for all alternatives $x, y \in A$ and preference profiles $R \in S^{yN\setminus\{i\}x}$. Consider for this the profiles R^1 and R^2 depicted in the sequel and note that $R^1, R^2 \in \mathcal{D}_C \subseteq \mathcal{D}$ since all voters in $N \setminus \{i\}$ agree on a best alternative in these profiles.

R^1:	i: $x, y, z, *$	$N\setminus\{i\}$: $z, y, x, *$
R^2:	i: $y, x, z, *$	$N\setminus\{i\}$: $y, z, x, *$

We show in the sequel that $f(R^1, x) = 1$ by contradiction, i.e., we suppose that $f(R^1, x) < 1$. First, observe that $f(R^1, x) > 0$ because Lemma 12 shows that $f(R^1, x) = f(R^*, a) > 0$ since $R^1 \in S^{zN\setminus\{i\}x}$ and $R^* \in S^{bN\setminus\{i\}a}$. Moreover, this lemma also entails that $f(R^1, w) = 0$ for all alternatives $w \in A \setminus \{x, z\}$. Thus, $f(R^1, x) < 1$ implies that $f(R^1, z) > 0$. In particular, this means that $f(R, U(R_j, y)) < 1$ for all voters $j \in N$. Hence, the set of all voters N can group-manipulate by deviating to R^2 because claim (2) of Lemma 1 implies that $f(R^2, y) = 1$. Consequently, it indeed holds that $f(R^1, x) = 1$ because otherwise, f is group-manipulable. Finally, Lemma 12 implies now that $f(R, x) = 1$ for all alternatives $x, y \in A$ and preference profiles $R \in S^{yN\setminus\{i\}x}$.

As second step, we show that f always chooses voter i's best alternative with probability 1. Hence, consider an arbitrary preference profile $R \in \mathcal{D}$ and let x denote voter i's best alternative in R. Furthermore, we choose an arbitrary alternative $y \in A \setminus \{x\}$ and consider a profile $R' \in \mathcal{D}_C \subseteq \mathcal{D}$ such that $R'_i = R_i$, and $r(R'_j, x) = m$ and $r(R'_j, y) = 1$ for all voters $j \in N \setminus \{i\}$. Since $R' \in S^{yN\setminus\{i\}x}$, it follows from our previous observations that $f(R', x) = 1$ even though all voters in $N \setminus \{i\}$ report x as their worst alternative. This implies that $f(R, x) = 1$ because otherwise, the voters in $N \setminus \{i\}$ can group-manipulate by deviating from R' to R. Because the profile

$R \in \mathcal{D}$ is chosen arbitrarily, f always chooses the best alternative of voter i with probability 1 and thus, it is the dictatorial SDS of this voter. This contradicts the assumption that f is non-dictatorial, which proves that f is 1-unanimous. □

Next, we use Lemma 13 to prove Lemma 4. Note that we use Lemma 11 in the proof because it is straightforward to adapt this lemma to the case for even n if we use group-strategyproofness instead of strategyproofness.

Lemma 4. *Assume $n \geq 3$ and $m \geq 3$. A group-strategyproof and non-imposing SDS on a super Condorcet domain is non-dictatorial iff it is Condorcet-consistent.*

Proof. Consider an arbitrary super Condorcet domain $\mathcal{D}$ for $n \geq 3$ and suppose that f is a group-strategyproof and non-imposing SDS on $\mathcal{D}$. First, note that if f is also Condorcet-consistent, it is obviously non-dictatorial. The reason for this is that $f(R, x) = 1$ holds for every preference profile $R \in \mathcal{D}_C$ in which only $n - 1$ voters prefer x the most. Consequently, no voter $i \in N$ is a dictator for f as there is a profile in which voter i's best alternative gets probability 0.

For the inverse direction, suppose that f is group-strategyproof, non-imposing, and non-dictatorial. First, note that Lemma 13 shows that f is 1-unanimous. Next, it is straightforward to adapt the proof of Lemma 11 to derive that f is even $\lfloor \frac{n-1}{2} \rfloor$. Based on this insight, it is now easy to show by contradiction that f is Condorcet-consistent: suppose that there is a profile $R \in \mathcal{D}_C$ with Condorcet winner c but $f(R, c) \neq 1$. Hence, there is an alternative $d \in A \setminus \{c\}$ with $f(R, d) > 0$. This means in particular that $f(R, U(R_i, c)) < 1$ for all voters $i \in N$ with cR_id. As a consequence, these voters can group-manipulate by making c into their best alternative. This step results in a profile R' in which more than half of the voters top-rank c and hence, our previous observation implies that $f(R', c) = 1$. This is a group-manipulation for the voters i with cR_id, which means that the initial assumption is wrong. Thus, every group-strategyproof, non-imposing, and non-dictatorial SDS on a super Condorcet domain is Condorcet-consistent. □