Strategyproof Social Decision Schemes on Super Condorcet Domains

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Abstract

One of the central economic paradigms in multi-agent systems is that agents should not be better off by acting dishonestly. In the context of collective decision-making, this axiom is known as strategyproofness and turned out to be rather prohibitive, even when allowing for randomization. In particular, Gibbard’s random dictatorship theorem shows that only rather unattractive social decision schemes (SDSs) satisfy strategyproofness on the full domain of preferences. In this paper, we obtain more positive results by investigating strategyproof SDSs on the Condorcet domain, which consists of all preference profiles that admit a Condorcet winner. In more detail, we show that, if the number of voters $n$ is odd, every strategyproof and non-imposing SDS on the Condorcet domain can be represented as a mixture of dictatorial SDSs and the Condorcet rule (which chooses the Condorcet winner with probability 1). Moreover, we prove that, if $n$ is odd, only mixtures of dictatorial SDSs satisfy strategyproofness and non-imposition on every sufficiently connected superset of the Condorcet domain. This strengthens the random dictatorship theorem and establishes that the Condorcet domain is essentially a maximal domain that allows for attractive strategyproof SDSs. We finally extend our results to an even number of voters by characterizing the set of group-strategyproof and non-imposing SDSs on the Condorcet domain and its supersets.

Introduction

Strategyproofness—no agent should be better of by acting dishonestly—is one of the central economic paradigms in multi-agent systems (see, e.g., Nisan et al. 2007; Shoham and Leyton-Brown 2009; Brandt et al. 2016). A significant problem for such systems is collective decision-making, which aims at identifying socially desirable outcomes by letting voters express their preferences over the possible alternatives. A multitude of theorems has shown that even rather basic properties of voting rules cannot be satisfied simultaneously. In this context, strategyproofness is known to be a particularly restrictive axiom. This is exemplified by the Gibbard-Satterthwaite theorem which states that dictatorships are the only deterministic voting rules that satisfy strategyproofness and non-imposition (i.e., every alternative is elected for some preference profile). Since dictatorships are not acceptable for most applications, this result is usually considered an impossibility theorem.

One of the most successful escape routes from the Gibbard-Satterthwaite impossibility is to restrict the domain of feasible preference profiles. For instance, Moulin (1980) prominently showed that there are attractive strategyproof voting rules on the domain of single-peaked preference profiles, and various other restricted domains of preferences have been considered since then (see, e.g., Barberà, Gul, and Stacchetti 1993; Peremans and Storcken 1999; Saporiti 2009; Elkind, Lackner, and Peters 2017). The idea behind domain restrictions is that the voters’ preferences often obey structural constraints and thus, not all preference profiles are likely or plausible. A particularly significant constraint is the existence of a Condorcet winner, which is an alternative that is favored to every other alternative by a majority of the voters. Apart from its natural appeal, this concept is important because there is strong empirical evidence that real-world elections usually admit Condorcet winners (see, e.g., Regenwetter et al. 2006; Laslier 2010; Gehrlein and Lepelley 2011). This motivates the study of the Condorcet domain, which consists precisely of the preference profiles that have a Condorcet winner. Note that the Condorcet domain is a superset of several important domains such as those of single-peaked and single-dipped preferences when the number of voters is odd. There are several results showing the existence of attractive strategyproof voting rules on the Condorcet domain. In particular, Campbell and Kelly (2003) characterize the Condorcet rule, which always picks the Condorcet winner, as the only strategyproof, non-imposing, and non-dictatorial voting rule on the Condorcet domain if the number of voters is odd.

In this paper, we focus on randomized voting rules, so-called social decision schemes (SDSs). Gibbard (1977) has shown that randomization unfortunately does not allow for much more leeway beyond the negative consequences of the Gibbard-Satterthwaite theorem: random dictatorships, which select each voter with a fixed probability and elect the favorite alternative of the chosen voter, are the only SDSs on the full domain that satisfy strategyproofness and non-imposition (which in the randomized setting requires that every alternative is chosen with probability 1 for some preference profile). Thus, these SDSs are merely “mixtures of dictatorships”. In order to circumvent this negative result, we are interested in large domains that allow for strategyproof and non-imposing SDSs apart from random dictatorships. A natural candidate for this is the Condorcet domain and,
indeed, we show that the Condorcet domain is essentially a maximal domain that allows for strategyproof, non-imposing, and “non-randomly dictatorial” social choice. In more detail, we prove that, if the number of voters \( n \) is odd, every strategyproof and non-imposing SDS on the Condorcet domain can be represented as a mixture of dictatorial SDSs and the Condorcet rule (which chooses the Condorcet winner with probability 1). This result entails that the Condorcet rule is the only strategyproof, non-imposing, and completely “non-randomly dictatorial” SDS on the Condorcet domain for odd \( n \). Moreover, we show that—under a mild connectedness condition on the domain—random dictatorships are the only strategyproof and non-imposing SDS on every superset of the Condorcet domain. Unfortunately, our theorems do not extend to an even number of voters because a single voter cannot change the Condorcet winner in this case. We address this problem by characterizing the set of group-strategyproof and non-imposing SDSs on the Condorcet domain and most of its supersets. Our characterizations on the Condorcet domain can be seen as attractive complements to classic negative results for the full domain, whereas our results for supersets of the Condorcet domain significantly strengthen these negative results. In particular, our theorems imply statements by Barberà (1979) and Campbell and Kelly (2003) as well as the Gibbard-Satterthwaite theorem (Gibbard 1973; Satterthwaite 1975) and the random dictatorship theorem (Gibbard 1977) when the number of voters is odd. The relationships to results for deterministic voting rules hold because every such rule can be seen as an SDS that always returns degenerate lotteries.

**Related Work**

Restricting the domain of preference profiles in order to circumvent classic impossibility theorems has a long tradition and remains an active research area to date. In particular, the existence of attractive deterministic voting rules that satisfy strategyproofness has been shown for a number of domains. Classic examples include the domains of single-peaked (Moulin 1980), single-dipped (Barberà, Berga, and Moreno 2012), and single-crossing (Saporiti 2009) preference profiles. More recent possibility results focus on broader but more technical domains such as the domains of multi-dimensionally single-peaked or semi single-peaked preference profiles (e.g., Barberà, Gul, and Stacchetti 1993; Nehring and Puppe 2007; Chatterji, Sanver, and Sen 2013; Reffgen 2015). On the other hand, domain restrictions are also used to strengthen impossibility results by proving them for smaller domains (e.g., Aswal, Chatterji, and Sen 2003; Sato 2010; Gopakumar and Roy 2018). In more recent research, the possibility and impossibility results converge by giving precise conditions under which a domain allows for strategyproof and non-dictatorial deterministic voting rules (e.g., Chatterji and Sen 2011; Chatterji, Sanver, and Sen 2013; Roy and Storcken 2019; Chatterji and Zeng 2021).

While similar results have also been put forward for SDSs, this setting is not as well understood. For instance, Ehlers, Peters, and Storcken (2002) have shown the existence of attractive strategyproof SDSs on the domain of single-peaked preference profiles (see also Peters et al. 2014; Pycia and Unver 2015). The existence of strategyproof and non-imposing SDSs other than random dictatorships has also been investigated for a variety of other domains (e.g., Peters et al. 2017; Roy and Sadhukhan 2020; Peters, Roy, and Sadhukhan 2021). Chatterji, Sen, and Zeng (2014) and Chatterji and Zeng (2018) pursue a more general approach and identify criteria for deciding whether a domain admits such SDSs.

The strong interest in restricted domains also led to the study of many computational problems for restricted domains (e.g., Conitzer 2009; Faliszewski et al. 2011; Bredereck, Chen, and Woeginger 2013; Brandt et al. 2015; Elkind, Lackner, and Peters 2016; Peters 2017; Peters and Lackner 2020). For instance, Bredereck, Chen, and Woeginger (2013) give an algorithm for recognizing whether a preference profile is single-crossing, which can be used to decide whether the positive results apply for a given domain.

Finally, observe that all aforementioned results are restricted to Cartesian domains, i.e., domains of the form \( D = X^n \), where \( X \) is a set of preference relations. However, the Condorcet domain is not Cartesian. In this sense, the only results directly related to ours are the ones by Campbell and Kelly (2003) and their follow-up work (Merrill 2011; Campbell and Kelly 2015, 2016). These papers can be seen as predecessors of our work since they investigate strategyproof deterministic voting rules on the Condorcet domain.

**Preliminaries**

Let \( N = \{1, \ldots, n\} \) denote a finite set of voters and \( A = \{a, b, \ldots\} \) a finite set of \( m \) alternatives. Every voter \( i \in N \) is equipped with a preference relation \( R_i \), which is a complete, transitive, and anti-symmetric binary relation on \( A \). We define \( R \) as the set of all preference relations on \( A \). A preference profile \( R \in R^n \) consists of one preference relation for each voter \( i \in N \). A domain of preference profiles \( D \) is a subset of the full domain \( R^n \). When writing preference profiles, we represent preference relations as comma-separated lists and indicate the set of voters who share a preference relation directly before the preference relation. For instance, \( \{1, 2\} : a, b, c \) indicates that voters 1 and 2 prefer \( a \) to \( b \) to \( c \). We omit the brackets for singleton sets.

The main object of study in this paper are social decision schemes (SDSs) which are voting rules that may use randomization to determine the winner of an election. More formally, an SDS maps every preference profile \( R \) of a domain \( D \) to a lottery over the alternatives, which determines the winning chance of every alternative. A lottery \( p \) is a probability distribution over the alternatives, i.e., \( p(x) \geq 0 \) for all \( x \in A \) and \( \sum_{x \in A} p(x) = 1 \), and the set of all lotteries over \( A \) is denoted by \( \Delta(A) \). An SDS on a domain \( D \) is then a function of the form \( f : D \rightarrow \Delta(A) \). The term \( f(R, x) \) denotes the probability assigned to \( x \) by the lottery \( f(R) \) and for every set \( X \subseteq A \), we define \( f(R, X) = \sum_{x \in X} f(R, x) \). Note that SDSs are a generalization of deterministic voting rules which choose an alternative with probability 1 in every preference profile. A natural requirement for an SDS \( f : D \rightarrow \Delta(A) \) is non-imposition which states that for every alternative \( x \in A \), there is a preference profile \( R \in D \) such that \( f(R, x) = 1 \). Less formally, this axiom merely requires that every alterna-
Stochastic Dominance and Strategyproofness

Strategic manipulation is one of the central issues in social choice theory: voters might be better off by voting dishonestly. Since satisfactory collective decisions require the voters’ true preferences, SDSs should incentivize honest voting. In order to formalize this, we need to specify how voters compare lotteries over alternatives. The most prominent approach for this is based on stochastic dominance (see, e.g., Gibbard 1977; Ehlers, Peters, and Storcken 2002; Peters, Roy, and Sadhukhan 2021). Let the upper contour set \( U(R_i,x) = \{ y \in A : yR_i x \} \) be the set of alternatives that voter \( i \) weakly prefers to \( x \) in \( R \). Then, stochastic dominance states that a voter \( i \) prefers a lottery \( p \) to another lottery \( q \), denoted by \( p \succeq_i q \), if \( p(U(R_i,x)) \geq q(U(R_i,x)) \) for all \( x \in A \). Note that the stochastic dominance relation is transitive but not complete. Using stochastic dominance to compare lotteries is appealing because it allows voters to compare lotteries in a way that is ordinal consistent with their preferences.

Based on stochastic dominance, we define multiple variants of strategyproofness. The standard notion requires that a single voter cannot benefit by lying about his true preferences. Formally, we say an SDS \( f : D \rightarrow \Delta(A) \) is strategyproof if \( f(R) \succeq_i f(R') \) for all preference profiles \( R, R' \in D \) and voters \( i \in N \). Conversely, an SDS is called manipulable if it is not strategyproof. A convenient property of strategyproofness is that mixtures of strategyproof SDSs are strategyproof, i.e., the set of strategyproof SDSs is convex for every domain.

A natural weakening of strategyproofness is local strategyproofness, which only disincentivizes manipulations by swapping two alternatives. In order to formalize this concept, we define \( R^{x\rightarrow y} \) as the preference profile derived from another profile \( R \) by reinforcing \( y \) against \( x \) in the preference relation of voter \( i \). Note that this definition requires that \( x \neq y \) and that voter \( i \) ranks \( x \) directly above \( y \) in \( R \). Then, an SDS \( f : D \rightarrow \Delta(A) \) is locally strategyproof if \( f(R) \succeq_i f(R') \) for all voters \( i \in N \), alternatives \( x, y \in A \), and preference profiles \( R, R' \in D \) such that \( R' = R^{x\rightarrow y} \). Local strategyproofness is an attractive weakening of strategyproofness since empirical results suggest that voters often manipulate by reporting such inconspicuous lies (see, e.g., Fischbacher and Föllmi-Heusi 2013; Mennle et al. 2015).

In order to dis incentivize groups of voters from manipulating, we need a stronger strategyproofness notion: an SDS \( f : D \rightarrow \Delta(A) \) is group-strategyproof if for all preference profiles \( R, R' \in D \) and all non-empty sets of voters \( I \subseteq N \) with \( R_I = R'_I \) for all \( j \in N \setminus I \), there is a voter \( i \in I \) such that \( f(R) \succeq_i f(R') \). Conversely, an SDS is group-manipulable if it is not group-strategyproof. Note that group-strategyproofness implies strategyproofness and strategyproofness implies local strategyproofness.

\( \gamma \)-Randomly Dictatorial SDSs

The random dictatorship theorem shows that random dictatorships are the only non-imposing and strategyproof SDSs on the full domain (see, e.g., Gibbard 1977). Informally, a random dictatorship \( f \) is a mixture of dictatorial SDSs. A dictatorial SDS \( d_i \) always chooses the most preferred alternative of a single voter \( i \in N \) with probability 1. Formally, an SDS \( f \) is a random dictatorship if there are weights \( \alpha_i \geq 0 \) for all voters \( i \in N \) such that \( \sum_{i \in N} \alpha_i = 1 \) and \( f = \sum_{i \in N} \alpha_i d_i \). While they are more attractive than dictatorships, random dictatorships are often undesirable because they do not allow for any compromise. For instance, if the voters agree on a second best alternative but strongly disagree on the best one, it seems reasonable to compromise on the second best alternative. However, this is not possible with random dictatorships and thus, we here interpret the random dictatorship theorem as a negative result.

An escape route from the random dictatorship theorem is to consider restricted domains of preference profiles. However, random dictatorships are still strategyproof and non-imposing on any reasonable subdomain and thus, the strategyproof and non-imposing SDSs that are no random dictatorships are of particular interest. Unfortunately, it is not straightforward how to define these “non-randomly dictatorial” SDSs because mixtures of strategyproof SDSs are again strategyproof. For instance, the SDS that returns with probability 0.99 the outcome of a random dictatorship and otherwise the outcome of another strategyproof SDS is strategyproof and no random dictatorship but it clearly feels “randomly dictatorial”. To address this problem, Brandt, Lederer, and Romen (2021) introduced the notion of \( \gamma \)-randomly dictatorial SDSs. A strategyproof SDS \( f : D \rightarrow \Delta(A) \) is \( \gamma \)-randomly dictatorial if \( \gamma \in [0,1] \) is the maximal value such that \( f \) can be represented as \( f = \gamma d + (1-\gamma)g \), where \( d \) is a random dictatorship and \( g \) is a strategyproof SDS on \( D \). Note that if \( \gamma < 1 \), the maximality of \( \gamma \) implies that \( g \) is 0-randomly dictatorial. On the other hand, if \( \gamma = 1 \), \( f \) is a random dictatorship and the choice of \( g \) does not matter. Intuitively, the notion of \( \gamma \)-random dictatorships quantifies how close an SDS is to a random dictatorship. Hence, 0-randomly dictatorial SDSs can be seen as a rigorous randomized equivalent to deterministic non-dictatorial voting rules.

Note that \( \gamma \)-random dictatorships are defined with respect to a specific strategyproofness notion, but the general idea of this axiom is independent of such details. Thus, we can use other strategyproofness notions to derive new variants of \( \gamma \)-dictatorships. In particular, we call a locally strategyproof SDS \( f : D \rightarrow \Delta(A) \) locally \( \gamma \)-randomly dictatorial if \( \gamma \in [0,1] \) is the maximal value such that \( f \) can be represented as \( f = \gamma d + (1-\gamma)g \), where \( d \) is a random dictatorship and \( g \) is a locally strategyproof SDS on \( D \). The intuition behind locally \( \gamma \)-randomly dictatorial SDSs is the same as for \( \gamma \)-random dictatorships, but local \( \gamma \)-random dictatorships are easier to handle because local strategyproofness is a much more restricted concept than strategyproofness. Also, since strategyproofness implies local strategyproofness, every \( \gamma \)-randomly dictatorial SDS is locally \( \gamma' \)-randomly dictatorial for \( \gamma' \geq \gamma \).
Super Condorcet Domains

Since there are no attractive strategyproof rules on the full domain, we investigate the Condorcet domain and its super domains with respect to the existence of such SDSs. For defining these domains, we first have to introduce additional terminology. The majority margin $g_R(x, y) = |\{i \in N: x_i R y_i\} - |\{i \in N: y_i R x_i\}|$ indicates how many more voters prefer $x$ to $y$ in the profile $R$ than vice versa. Based on the majority margins, we define the Condorcet winner of a profile $R$ as the alternative $x$ such that $g_R(x, y) > 0$ for all $y \in A \setminus \{x\}$. Since the existence of a Condorcet winner is not guaranteed, we are interested in the Condorcet domain $\mathcal{D}_C = \{R \in \mathbb{R}^n: \text{there is a Condorcet winner in } R\}$ which contains all profiles with a Condorcet winner. A particularly natural SDS on the Condorcet domain is the Condorcet rule (COND) which assigns always probability 1 to the Condorcet winner.

In this paper, we also investigate super Condorcet domains, which are domains $D$ such that $\mathcal{D}_C \subseteq D$. For our analysis, we need to define additional properties of super Condorcet domains. First, a super Condorcet domain $D$ is strict if there is a profile $R \in D$ such that for every alternative $x \in A$, there is another alternative $y \in A \setminus \{x\}$ with $g_R(y, x) > 0$. In other words, every strict super Condorcet domain contains at least one profile that does not even admit a weak Condorcet winner, i.e., an alternative that beats or ties every other alternative. If $n$ is odd, every super Condorcet domain $D$ with $D \neq \mathcal{D}_C$ is strict, but this is not true when there are majority ties.

Super Condorcet domains $D$ can be disconnected with respect to strategyproofness: there can be a profile $R \in D$ that differs from every other profile $R' \in D \setminus \{R\}$ in at least two preference relations. This is problematic for our analysis because strategyproofness has no implications for such an isolated profile $R$. We address this issue by introducing connected super Condorcet domains. To this end, we first introduce the idea of ad-paths: an ad-path from a profile $R$ to another profile $R'$ in a domain $D$ is a sequence of preference profiles $\{R_0, \ldots, R_{l-1}\}$ such that $R_0 = R$, $R_{l-1} = R'$, $R_k \in D$ for all $k \in \{1, \ldots, l-1\}$, and the profile $R_{k+1}$ evolves out of $R_k$ by swapping two alternatives $x, y \in A$ in the preference relation of a voter $i \in N$, i.e., $R_{k+1} = (R_k)\{i\}^{xy}$ for all $k \in \{1, \ldots, l-1\}$. Then, a super Condorcet domain $D$ is connected if for all alternatives $x \in A$ and profiles $R \in D \setminus \mathcal{D}_C$, there is an ad-path from $R$ to a profile $R' \in \mathcal{D}_C$ on which $x$ is never swapped with another alternative. Less formally, this means that for every profile $R$ outside of the Condorcet domain and every alternative $x \in A$, we can apply a sequence of swaps, none of which involves $x$, to transform $R$ into a profile in the Condorcet domain. Furthermore, observe that the notion of connected super Condorcet domains is only reasonable for an odd number of voters $n$; for even $n$, the Condorcet domain is the only connected super Condorcet domain because of majority ties.

Finally, we introduce the concept of Condorcet-consistency because the Condorcet rule is not well-defined on super Condorcet domains: an SDS $f$ on a super Condorcet domain $D$ is Condorcet-consistent if $f(R) = \text{COND}(R)$ for all profiles $R \in \mathcal{D}_C \subseteq D$. Hence, Condorcet-consistent SDSs extend the Condorcet rule to larger domains.

Characterizations

In this section, we present our characterizations of strategyproof and group-strategyproof SDSs on super Condorcet domains. First, we discuss characterizations of strategyproof and non-imposing SDSs on connected super Condorcet domains for odd $n$ and show that the Condorcet domain is essentially the only super Condorcet domain that allows for a strategyproof, 0-randomly dictatorial, and non-imposing SDS. We then extend these results to an even number of voters by strengthening strategyproofness to group-strategyproofness. We defer the proofs of all results to the supplementary material because of space restrictions and only discuss proof sketches here.

Results based on Strategyproofness

Our first result shows that every strategyproof and non-imposing SDS on the Condorcet is a mixture of a random dictatorship and the Condorcet rule if $n$ is odd. As a byproduct, we obtain a characterization of the Condorcet rule as the only strategyproof, non-imposing, and 0-randomly dictatorial SDS on the Condorcet domain for odd $n$. By contrast, we show for every strict and connected super Condorcet domain $D$ that random dictatorships are the only strategyproof and non-imposing SDSs on $D$ if $n$ is odd.

In order to derive these characterizations, we rely on the notion of locally $\gamma$-randomly dictatorial SDSs. In particular, we are interested in the locally 0-randomly dictatorial, locally strategyproof, and non-imposing SDSs on a domain $D$ because these are the new SDSs gained by considering a restricted domain. Moreover, these SDSs characterize the set of all locally strategyproof and non-imposing SDSs for all sufficiently connected domains. We prove this claim for connected super Condorcet domains, but it holds for all commonly studied domains.

Lemma 1. Assume $n \geq 3$ is odd and $m \geq 3$. An SDS $f$ on a connected super Condorcet domain is locally strategyproof and non-imposing iff it is a random dictatorship or there are $\gamma \in [0, 1)$, a random dictatorship $d$, and a locally $\gamma$-randomly dictatorial, locally strategyproof, and non-imposing SDS $g$ on $D$ such that $f = \gamma d + (1 - \gamma)g$.

The direction from left to right follows from the definition of local $\gamma$-random dictatorships: for every locally strategyproof SDS $f$, there is a maximal $\gamma$ such that $f$ can be represented as $f = \gamma d + (1 - \gamma)g$, where $d$ is a random dictatorship and $g$ is a locally strategyproof SDS. If $\gamma = 1$, $f$ is a random dictatorship. On the other hand, if $\gamma < 1$, the maximality of $\gamma$ entails that $g$ is locally 0-randomly dictatorial. Since $g$ also inherits non-imposition from $f$, there is a suitable representation for every non-imposing and locally strategyproof SDS on a connected super Condorcet domain. For the other direction, note that every random dictatorship is locally strategyproof and non-imposing on every connected super Condorcet domain. On the other hand, if $f = \gamma d + (1 - \gamma)g$ for $\gamma \in [0, 1)$, a random dictatorship $d$, and a locally strategyproof and non-imposing SDS $g$, it follows that $f$ is also locally strategyproof because mixtures of locally strategyproof SDSs are locally strategyproof on
every domain. Moreover, it is easy to see that $f$ is also non-imposing since both $g$ and $d$ satisfy this axiom.

As a consequence of Lemma 1, the set of locally 0-randomly dictatorial, locally strategyproof, and non-imposing SDSs $G$ of a connected super Condorcet domain $D$ characterizes the set of all locally strategyproof and non-imposing SDSs on $D$ because each such SDS is a random dictatorship or a mixture of a random dictatorship and an SDS $g \in G$. Hence, a characterization of the set $G$ for a connected super Condorcet domain $D$ immediately implies a characterization of the set of all locally strategyproof and non-imposing SDSs on $D$. In particular, if there is no locally strategyproof, non-imposing, and locally 0-randomly dictatorial SDS on $D$, random dictatorships are the only locally strategyproof and non-imposing SDSs on this domain.

Motivated by Lemma 1, we investigate the set of locally 0-randomly dictatorial, locally strategyproof, and non-imposing SDSs for connected super Condorcet domains. As a first step, we present a characterization of locally 0-randomly dictatorial SDSs on these domains.

**Lemma 2.** Assume $n \geq 3$ is odd and $m \geq 3$. A non-imposing and locally strategyproof SDS $f$ on a connected super Condorcet domain is locally 0-randomly dictatorial iff it is Condorcet-consistent.

The direction from right to left is trivial: if $f$ is a Condorcet-consistent SDS, the best alternative of every voter can have probability 0 because it might not be the Condorcet winner. Consequently, every Condorcet-consistent and locally strategyproof SDS on a connected super Condorcet domain is locally 0-randomly dictatorial. The inverse direction is much more difficult to prove. The key insight is that every locally strategyproof, locally 0-randomly dictatorial, and non-imposing SDS on a connected super Condorcet domain $D$ has to choose an alternative with probability 1 whenever it is top-ranked by $n - 1$ voters. Departing from this insight, we derive inductively that an alternative is also chosen with probability 1 whenever it is top-ranked by more than half of the voters. Finally, we infer from this observation that every locally strategyproof, locally 0-randomly dictatorial, and non-imposing SDS on $D$ is Condorcet-consistent.

Lemma 2 has multiple important consequences. First of all, it identifies Condorcet-consistent SDSs as counterparts to random dictatorships on the large class of connected super Condorcet domains because every locally strategyproof, non-imposing, and locally 0-randomly dictatorial SDS is Condorcet-consistent, whereas only random dictatorships are locally 1-randomly dictatorial. Even more, Theorem 1 and Theorem 2 show that this insight is also true if we use strategyproofness and 0-random dictatorships instead of the local variants. This observation has an intuitive explanation: while random dictatorships never compromise, Condorcet-consistent SDSs can be seen as maximally compromising. Furthermore, we can use Lemma 2 to characterize the set of strategyproof and non-imposing SDSs on the Condorcet domain for odd $n$.

**Theorem 1.** Assume $n$ is odd and $m \geq 3$. An SDS on the Condorcet domain is strategyproof and non-imposing iff it is a mixture of a random dictatorship and the Condorcet rule.

The proof of this result follows almost directly from Lemma 1 and Lemma 2. In particular, the definition of Condorcet-consistency implies that the Condorcet rule is the only Condorcet-consistent SDS on the Condorcet domain and thus, Lemma 2 shows that it is also the only SDS that can simultaneously satisfy local strategyproofness, non-imposition, and local 0-random dictatorship. Since the Condorcet rule indeed satisfies all these axioms, it is the only locally strategyproof, locally 0-randomly dictatorial, and non-imposing SDS on the Condorcet domain for odd $n$. Hence, we can use Lemma 1 to derive that every locally strategyproof and non-imposing SDS is a mixture of dictatorial SDSs and the Condorcet rule. Finally, since all these SDSs are also strategyproof and strategyproofness implies local strategyproofness, the theorem follows.

Note that Theorem 1 immediately implies that the Condorcet rule is the only 0-randomly dictatorial SDS on the Condorcet domain that satisfies strategyproofness and non-imposition if $n \geq 3$ is odd and $m \geq 3$. This corollary generalizes Theorem 1 of Campbell and Kelly (2003) who have characterized the Condorcet rule with equivalent axioms in the deterministic setting. Furthermore, this insight highlights the appeal of theCondorcet rule on the Condorcet domain because every other strategyproof and non-imposing SDS is merely a mixture of the Condorcet rule and a random dictatorship. Hence, there is a unique, attractive, and strategyproof method for choosing the winner of an election if we restrict our attention to the Condorcet domain.

A natural follow-up question to Theorem 1 is whether we can find larger domains for which there are strategyproof and non-imposing SDSs other than random dictatorships. We find a negative answer to this problem if $n$ is odd: on every strict and connected super Condorcet domain $D$, only random dictatorships are strategyproof and non-imposing. Note that here, strictness implies only that $D \neq D_C$ because $n$ is odd. Hence, the subsequent theorem shows that the Condorcet domain is essentially a maximal domain that allows for attractive strategyproof SDSs.

**Theorem 2.** Assume $n$ is odd and $m \geq 3$. An SDS on a strict and connected super Condorcet domain is strategyproof and non-imposing iff it is a random dictatorship.

The proof of this result follows almost directly from Lemma 1 and Lemma 2. In particular, we only need to show that there is no locally strategyproof and Condorcet-consistent SDS on any strict and connected super Condorcet domain $D$ because Lemma 2 then shows that there is no locally strategyproof, non-imposing, and locally 0-randomly dictatorial SDS on this domain. Thus, we can use Lemma 1 to derive that only random dictatorships satisfy local strategyproofness and non-imposition, and consequently, the theorem follows because all these SDSs are also strategyproof.

First, note that Theorem 2 generalizes the random dictatorship theorem from the full domain to every strict and connected super Condorcet domain if $n$ is odd. Moreover, since deterministic voting rules can be seen as SDSs which choose in every preference profile an alternative with probability 1, our result also generalizes the Gibbard-Satterthwaite theorem to these smaller domains if $n$ is odd.
lar, Theorem 2 shows that adding even a single profile to the Condorcet domain can turn the partial positive result of Theorem 1 into a negative one because the resulting domain is a strict and connected super Condorcet domain. This follows, for instance, by considering the subsequent domain $D_1 = D_C \cup \{R^*\}$. The preference profile $R^*$ is shown below, where $I = \{4, 6, \ldots, n - 1\}$, $J = \{5, 7, \ldots, n\}$, and the $*$ indicates that all missing alternatives can be ordered arbitrarily below $a$, $b$, and $c$.

It is easy to verify that $D_1$ is a strict and connected super Condorcet domain because we can go from $R^*$ to a profile in the Condorcet domain if voter 1 swaps $a$ and $b$, if voter 2 swaps $b$ and $c$, and if voter 3 swaps $a$ and $c$. Hence, Theorem 2 entails that random dictatorships are the only strategyproof and non-imposing SDSs on $D_1$.

**Remark 1.** An important consequence of Theorems 1 and 2 is that, if $n$ is odd, every strategyproof and non-imposing SDS on a connected super Condorcet domain can be represented as a mixture of deterministic voting rules, each of which is strategyproof and non-imposing. This is sometimes called deterministic extreme point property and it is remarkable that many important domains satisfy this condition (see Roy and Sadhukhan 2020). On one side, this shows that randomization does not lead to completely new strategyproof SDSs. On the other hand, the deterministic extreme point property allows for a natural interpretation of strategyproof and non-imposing SDSs: we decide by chance which deterministic voting rule is executed. Also, note that there are domains that violate this condition (see Chatterji, Sen, and Zeng 2014).

**Remark 2.** There are domains $D$ with $D_C \subseteq D$ that allow for non-imposing and strategyproof SDSs that are no random dictatorships. For example, consider the domain $D_2$ which is derived by adding a single preference profile $R^1$ to the Condorcet domain. If $R^1$ differs from every profile in $D_C$ in the preference relations of at least two voters, an arbitrary outcome can be returned for $R^1$ without violating strategyproofness. This shows that the connectedness condition is required for Theorem 2.

**Remark 3.** The proofs of Theorem 1 and Theorem 2 show that these characterizations also hold if we use local strategyproofness instead of strategyproofness. Consequently, local strategyproofness is equivalent to strategyproofness for every connected super Condorcet domain if $n$ is odd. Equivalent results have been shown for a number of other domains, e.g., the domain of single-peaked profiles, the domain of single-dipped profiles, or the full domain (see, e.g., Gibbard 1977; Carroll 2012; Sato 2013; Kumar et al. 2021). Furthermore, since local strategyproofness is equivalent to strategyproofness on every connected super Condorcet domain, the notions of local $\gamma$-random dictatorships and $\gamma$-random dictatorships are equivalent, too. Hence, all local and global notions can be used interchangeably for our results.

**Remark 4.** If $n$ is even, there are larger domains than the Condorcet domain that allow for strategyproof, non-imposing, and 0-randomly dictatorial SDSs. An example of such a domain is the shifted Condorcet domain $D_C^R$, which contains a profile $R$ if there is a Condorcet winner in $(R, R_i)$, i.e., in the profile derived by adding a fixed preference relation $R_i$ to $R$. An SDS on $D_C^R$ that satisfies all our requirements is $f(R) = COND(R, R_i)$ because the domain $\{(R, R_i) : R \in D_C^R\}$ is a subset of the Condorcet domain for $n + 1$ voters.

**Results based on Group-strategyproofness**

Maybe the strongest restrictions of Theorem 1 and Theorem 2 is that both results only apply if the number of voters $n$ is odd. Indeed, Theorem 1 does not hold for even $n$ because the counterexamples by Merrill (2011) show that there are strategyproof, non-imposing, and 0-randomly dictatorial SDSs $f : D_C \rightarrow \Delta(A)$ other than the Condorcet rule if $n$ is even. The reason for this is that a single voter cannot change the Condorcet winner if $n$ is even and consequently, the Condorcet domain breaks down into several disconnected subdomains. A natural approach to restore the connectedness is to use group-strategyproofness instead of strategyproofness because this axiom allows to change the Condorcet winner with a group-manipulation. We therefore characterize now the set of group-strategyproof and non-imposing SDSs on super Condorcet domains independently of the parity of $n$. In more detail, we show that only dictatorial SDSs and the Condorcet rule satisfy these axioms on the Condorcet domain, whereas only dictatorships are group-strategyproof and non-imposing on strict super Condorcet domains.

The proofs of these results are very similar to the proofs of Theorem 1 and Theorem 2. In particular, we first generalize Lemma 2 by characterizing when a group-strategyproof and non-imposing SDS is non-dictatorial.

**Lemma 3.** Assume $n \geq 3$ and $m \geq 3$. A group-strategyproof and non-imposing SDS on a super Condorcet domain is non-dictatorial iff it is Condorcet-consistent.

It is obvious that a Condorcet-consistent SDS is non-dictatorial as the best alternative of a voter gets probability 0 if it is not the Condorcet winner. For the other direction, we prove that every group-strategyproof, non-imposing, and non-dictatorial SDS on a super Condorcet domain chooses an alternative $x$ with probability 1 whenever $n - 1$ voters report $x$ as their favorite option. Based on this insight, we derive—analogously to the proof of Lemma 2—that such SDSs are Condorcet-consistent.

Similar to Lemma 2, Lemma 3 identifies Condorcet-consistent SDSs as counterparts to dictatorships since all group-strategyproof, non-imposing, and non-dictatorial SDSs on super Condorcet domains are Condorcet-consistent. However, Lemma 3 is more general as we characterize non-dictatorial SDSs instead of local 0-random dictatorships and no connectedness condition on the domain is required. This is possible since we use group-strategyproofness instead of local strategyproofness. Next, we employ this lemma to characterize the set of group-strategyproof and non-imposing SDSs on the Condorcet domain and every strict super Condorcet domain.
**Theorem 3.** Assume $m \geq 3$. An SDS on the Condorcet domain is group-strategyproof and non-imposing if and only if it is a dictator SD or the Condorcet rule. An SDS on a strict super Condorcet domain is group-strategyproof and non-imposing if and only if it is a dictator SDS.

The proof of the claim for the Condorcet domain follows immediately from Lemma 3 since the Condorcet rule is the only Condorcet-consistent SDS on the Condorcet domain. Hence, only dictatorial SDSs and the Condorcet rule can be group-strategyproof and non-imposing and it is easy to verify that all these SDSs indeed satisfy both axioms. Furthermore, we prove the claim for strict super Condorcet domains by showing that there is no group-strategyproof and Condorcet-consistent SDS on these domains because Lemma 3 entails then that every group-strategyproof and non-imposing SDS is dictatorial.

Theorem 3 generalizes many consequences of Theorem 1 and Theorem 2 to super Condorcet domains for an even number of voters by using group-strategyproofness. For instance, it entails that the Condorcet rule is the only group-strategyproof, non-imposing, and non-dictatorial SDS on the Condorcet domain and thus generalizes an analogous implication of Theorem 1. Moreover, the second part of the theorem shows that the Condorcet rule is essentially a maximal domain that allows for a group-strategyproof and non-imposing SDS apart of dictatorships. In particular, if $n$ is odd, every super Condorcet domain $D$ with $D \neq D_C$ is strict because no majority ties are possible. Hence, Theorem 3 shows for odd $n$ that no superset of the Condorcet domain admits group-strategyproof and non-imposing SDSs other than dictatorships.

**Remark 5.** The results of Barberà (1979) imply that every group-strategyproof and non-imposing SDS on the full domain is a dictatorship. Hence, Theorem 3 and Barberà’s results share a common idea: group-strategyproof and non-imposing SDSs cannot use randomization to determine the winner. However, whereas only undesirable SDSs are group-strategyproof and non-imposing on the full domain, the attractive Condorcet rule satisfies these axioms on $D_C$.

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**Conclusion**

We studied strategyproof and non-imposing social decision schemes (SDSs) on the Condorcet domain (which consists of all preference profiles with a Condorcet winner) and its supersets. In contrast to the full domain, there are attractive strategyproof SDSs on the Condorcet domain. In particular, we show that, if the number of voters $n$ is odd, every strategyproof and non-imposing SDS on the Condorcet domain can be represented as a mixture of a random dictatorship and the Condorcet rule. An immediate consequence of this insight is that the Condorcet rule is the only strategyproof, non-imposing, and 0-randomly dictatorial SDS on the Condorcet domain if $n$ is odd. Furthermore, we show that, if $n$ is odd, only random dictatorships are strategyproof and non-imposing on every sufficiently connected superset of the Condorcet domain. This shows that the Condorcet domain is essentially a maximal domain which allows for a 0-randomly dictatorial, non-imposing, and strategyproof SDS. Finally, we extend our results to an even number of voters by using group-strategyproofness: we show that only the Condorcet rule and dictatorial SDSs are group-strategyproof and non-imposing on the Condorcet domain, whereas only dictatorial SDSs satisfy these axioms on larger domains.

Our results for the Condorcet domain show an astonishing similarity to classic results for the full domain, but have a more positive flavor. For instance, while the random dictatorship theorem shows that only mixtures of dictatorial SDSs are strategyproof and non-imposing on the full domain, we prove in Theorem 1 that mixtures of dictatorial SDSs and the Condorcet rule are the only strategyproof and non-imposing SDSs on the Condorcet domain (if the number of voters is odd). A more exhaustive comparison between results for the full domain and for the Condorcet domain is given in Table 1. In particular, our results highlights the important role of the Condorcet rule in the Condorcet domain and thus, they make a strong case for choosing the Condorcet winner whenever it exists. Furthermore, our results for strict super Condorcet domains strengthen the classic results by showing that they even hold in small domains.

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**Table 1:** Comparison of results for the full domain, strict super Condorcet domains, and the Condorcet domain. Each row characterizes a set of SDSs for the full domain, every strict super Condorcet domain, and the Condorcet domain, respectively. All results require that there are $m \geq 3$ alternatives and results marked with an asterisk (*) only hold if there is an odd number of voters. The results marked with a diamond (⋄) require that $n$ is odd and that the strict super Condorcet domain is additionally connected. New results are italicized.
References


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Appendix A: Proofs of Theorems 1 and 2

In this appendix, our goal is to prove Theorems 1 and 2. As the proofs of these results are rather involved, we introduce several auxiliary lemmas which are organized in subsections to indicate related ideas. In particular, we first investigate the implications of local strategyproofness and non-imposition for connected super Condorcet domains. Based on these insights, we next analyze the behavior of local 0-randomly dictatorial SDSs and prove Lemma 2. As a last step, we derive Lemma 1, Theorem 1, and Theorem 2.

In many of the subsequent proofs, we use additional notation. In particular, we use the ∗ symbol in preference relations to indicate that all missing alternatives can be ordered arbitrarily. For instance, \( 1: a, b, ∗ \) means that voter 1 prefers \( a \) to \( b \) to every other alternative, and that the alternatives in \( A \setminus \{ a, b \} \) can be ordered arbitrarily. Furthermore, the rank \( r(R_i, x) = \{ y \in A : xR_i y \} \) of an alternative \( x \) with respect to a preference relation \( R_i \) denotes the number of alternatives that voter \( i \) weakly prefers to \( x \). Less formally, if an alternative \( x \) is localized and non-perverse, it is locally strategyproof.

**Local Strategyproofness on Connected Super Condorcet Domains**

As first step, we investigate the consequences of local strategyproofness for connected super Condorcet domains. Hence, we present a characterization of local strategyproofness based on non-perversity and localizability. An SDS \( f : D \to \Delta(A) \) is non-pervasive if \( f(R^{wy}, y) \geq f(R, y) \) for all preference profiles \( R \in D \), alternatives \( x, y \in A \), and voters \( i \in N \) such that \( R^{wy} \in D \). Furthermore, an SDS \( f : D \to \Delta(A) \) is localized if \( f(R^{wy}, z) = f(R, z) \) for all preference profiles \( R \in D \), distinct alternatives \( x, y, z \in A \), and voters \( i \in N \) such that \( R^{wy} \in D \). Intuitively, non-perversity requires that the probability of an alternative increases if it is reinforced, and localizability requires that the probability of an alternative does not change if it is not involved in a swap. Gibbard (1977), who first used these axioms, has proven that the conjunction of non-perversity and localizability is equivalent to strategyproofness on the full domain. We show a variant of this claim for restricted domains: non-perversity and localizability are equivalent to local strategyproofness for every domain.

**Lemma 4.** Let \( D \subseteq R^n \) denote an arbitrary domain. An SDS \( f : D \to \Delta(A) \) is locally strategyproof if and only if it is localized and non-pervasive.

**Proof.** Consider an arbitrary domain \( D \) and let \( f \) denote an SDS on this domain. We need to show that \( f \) is locally strategyproof if it is localized and non-pervasive, and that \( f \) is non-pervasive and localized if it is locally strategyproof.

**Claim 1: If \( f \) is locally strategyproof, it is localized and non-pervasive.**

First, assume that \( f : D \to \Delta(A) \) is a locally strategyproof SDS. Moreover, consider two arbitrary preference profiles \( R, R' \in D \), a voter \( i \in N \), and two alternatives \( x, y \in A \) such that \( R' = R^{wy} \). Note that local strategyproofness implies that \( f(R) \succ_i f(R') \) and \( f(R') \succ_i f(R) \).

We first show that \( f \) satisfies localizedness, i.e., that \( f(R, z) = f(R', z) \) for all alternatives \( z \in A \setminus \{ x, y \} \). Observe for this that the assumption \( R' = R^{wy} \) entails that \( U(R_i, z) = U(R_i', z) \) for all \( z \in A \setminus \{ x, y \} \) and that \( U(R_i, y) = U(R_i', x) \). Thus, local strategyproofness from \( R \) to \( R' \) and from \( R' \) to \( R \) implies that \( f(R, U(R_i, y)) = f(R', U(R_i', x)) \) for all \( z \in A \setminus \{ x, y \} \) and that \( f(R, U(R_i, y)) = f(R', U(R_i', x)) \). For instance, the latter is true because local strategyproofness implies that \( f(R, U(R_i, y)) \geq f(R', U(R_i', y)) \) and \( f(R', U(R_i', x)) \geq f(R, U(R_i, x)) \). Furthermore, note that for every alternative \( z \in A \) and every preference relation \( R_i \), the set \( U(R_i', z) \) is either empty (if \( z \) is the best alternative of voter \( i \)) or equal to the upper contour set \( U(R, z) \) of the alternative \( z' \) with \( r(R_i, z') = r(R_i, z) - 1 \). Hence, it holds for all alternatives \( z \in A \setminus \{ x, y \} \) with \( r(R_i, z) \neq r(R_i, y) + 1 \) that \( f(R, U(R_i, z) \setminus \{ z \}) = f(R', U(R_i', z) \setminus \{ z \}) \) because either \( U(R_i, z) \setminus \{ z \} = U(R_i', z) \setminus \{ z \} = \emptyset \) or there is another alternative \( z' \in A \setminus \{ x, y \} \) such that \( U(R_i, z') \setminus \{ z \} = U(R_i', z') \setminus \{ z \} \). On the other hand, it also holds for the alternative \( z \in A \setminus \{ x, y \} \) with \( r(R_i, z) = r(R_i, y) + 1 \) that \( f(R, U(R_i, z) \setminus \{ z \}) = f(R', U(R_i', z) \setminus \{ z \}) \) because \( U(R_i, z) \setminus \{ z \} = U(R_i, y) = U(R_i', x) = U(R_i', z) \setminus \{ z \} \). Hence, \( f \) is localized because we can calculate for all alternatives \( z \in A \setminus \{ x, y \} \) that

\[
\begin{align*}
&f(R, z) = f(R, U(R_i, z) \setminus \{ z \}) = f(R', U(R_i, z) \setminus \{ z \}) \quad \text{for all } \{ x, y \} \subseteq A \setminus \{ x, y \}.
&\end{align*}
\]

Finally, we show that \( f \) is non-pervasive, i.e., that \( f(R', y) \geq f(R, y) \). Using local strategyproofness from \( R' \) to \( R \), we derive that \( f(R', U(R_i', y)) \geq f(R, U(R_i, y)) \), and localizability implies that \( f(R', z) = f(R, z) \) for all \( z \in A \setminus \{ x, y \} \). Since \( U(R_i', y) \setminus \{ y \} \subseteq A \setminus \{ x, y \} \), it follows that \( f \) is non-pervasive because

\[
\begin{align*}
&f(R', y) = f(R', U(R_i', y) \setminus \{ y \}) = f(R, U(R_i, y) \setminus \{ y \}) \quad \text{for all } \{ x, y \} \subseteq A \setminus \{ x, y \}.
&\end{align*}
\]

Claim 2: If \( f \) is localized and non-pervasive, it is locally strategyproof.

Suppose that \( f : D \to \Delta(A) \) is a localized and non-pervasive SDS. We need to show that this SDS is locally strategyproof and consider thus two preference profiles \( R, R' \subseteq D \), a voter \( i \in N \), and two alternatives \( x, y \in A \) such that \( R' = R^{wy} \). Our goal is to prove that \( f(R) \succ_i f(R') \) which requires that \( f(R, U(R_i, z)) \geq f(R', U(R_i, z)) \) for all \( z \in A \). First, note that \( f(R, z) = f(R', z) \) for all \( z \in A \setminus \{ x, y \} \) because of localizedness. This implies that \( f(R, U(R_i, z)) = f(R', U(R_i, z)) \) for all \( z \in A \setminus \{ x, x \} \) with \( xR_i x \) because \( x, y \notin U(R_i, z) \) and thus \( U(R_i, z) \subseteq A \setminus \{ x, y \} \). Furthermore, for all \( z \in A \setminus \{ x \} \) with \( xR_i z \), we can compute that \( f(R, U(R_i, z)) = 1 - f(R, A \setminus U(R_i, z)) = 1 - f(R', A \setminus U(R_i, z)) \) because \( x, y \subseteq U(R_i, z) \) and thus \( A \setminus U(R_i, z) \subseteq A \setminus \{ x, y \} \). Finally, non-perversity shows for \( x \) that \( f(R, x) \geq f(R', x) \). Since localizedness implies that \( f(R, w) = f(R', w) \) for all \( w \in U(R_i, x) \setminus \{ x \} \), it follows that \( f(R, U(R_i, x)) \geq f(R', U(R_i, x)) \). Hence, it
indeed holds that \( f(R, U(R_z, z)) \geq f(R', U(R_z, z)) \) for all \( z \in A \), which shows that \( f \) is locally strategyproof. \( \square \)

Lemma 4 clarifies the consequences of local strategyproofness on restricted domains and shows that we can use equivalently non-perversity and localizedness, which are easier to handle. Next, we investigate locally strategyproof and non-imposing SDSs on connected super Condorcet domains. For phrasing and proving the next lemma, we need to introduce some additional terminology. We say an alternative \( x \in A \) Pareto-dominates any other alternative \( y \in A \setminus \{ x \} \) in a preference profile \( R \) if \( x R y \) for all voters \( i \in N \). Conversely, an alternative is Pareto-optimal if it is not Pareto-dominated by another alternative. Finally, an SDS \( f : D \rightarrow \Delta(A) \) is Pareto-optimal if it is not Pareto-dominated by another alternative. Moreover, \( f \) is deterministic if it is deterministic for all alternatives and profiles. Furthermore, we call an SDS \( f : D \rightarrow \Delta(A) \) k-unanimous if \( f(R, x) = 1 \) for all alternatives \( x \in A \) and preference profiles \( R \in D \) such that at least \( n - k \) voters rank \( x \) in \( R \). k-unanimity is a natural strengthening of the common idea of unanimity, which requires that an alternative is chosen with probability \( 1 \) if it is unanimously top-ranked. We show next that every non-imposing and locally strategyproof SDS on a connected super Condorcet domain is Pareto-optimal and therefore also 0-unanimous if \( n \) is odd.

**Lemma 5.** Assume \( n \) is odd and \( m \geq 3 \). Every non-imposing and locally strategyproof SDS on a connected super Condorcet domain satisfies Pareto-optimality.

**Proof.** Assume \( n \) is odd and let \( f \) denote a locally strategyproof and non-imposing SDS on an arbitrary connected super Condorcet domain \( D \). We assume for contradiction that \( f \) fails Pareto-optimality and derive a contradiction in two steps: first, we show \( f \) needs to be 0-unanimous, and secondly, we show that this cannot be the case if a Pareto-dominated alternative is assigned positive probability in a profile. Since these claims contradict each other, it follows that every locally strategyproof and non-imposing SDS on \( D \) satisfies Pareto-optimality. In the subsequent proof, we always ensure that we stay in the Condorcet domain, which also means that we stay in \( D \) since \( D_C \subseteq D \).

**Claim 1:** Every locally strategyproof and non-imposing SDS \( f : D \rightarrow \Delta(A) \) satisfies 0-unanimity.

Choose an arbitrary alternative \( x \in A \) and let \( R \in D \) denote a profile in which all voters report \( x \) as their favorite. We need to show that \( f(R, x) = 1 \) and consider for this a profile \( R^1 \in D \) with \( f(R^1, x) = 1 \). Such a profile exists since \( f \) is non-imposing. If \( R^1 \in D \setminus D_C \), we first use the connectedness of \( D \) to go into the Condorcet domain. In more detail, this axiom states that there is an ad-path from \( R^1 \) to a profile \( R^2 \in D_C \) such that \( x \) is not swapped with any alternative along this ad-path. Consequently, a repeated application of localizedness along this ad-path shows that \( f(R^2, x) = f(R^1, x) = 1 \). On the other hand, if \( R^1 \) is already in the Condorcet domain, we just define \( R^2 = R^1 \).

Let \( c \) denote the Condorcet winner in \( R^2 \). We proceed with a case distinction with respect to whether \( c = x \) or not. First, assume that \( x \) is the Condorcet winner in \( R^2 \). In this case, we let all voters repeatedly swap up \( x \) until it is the best alternative of every voter. Since we only reinforce \( x \), it stays the Condorcet winner during these steps. Moreover, non-perversity entails that the probability of \( x \) is non-decreasing. Thus, this process results in a profile \( R^3 \) in which \( x \) is unanimously top-ranked and \( f(R^3, x) = 1 \). As last step, we let all voters reorder all alternatives in \( A \setminus \{ x \} \) one after another to derive the profile \( R \). It is easy to see that we can use pairwise swaps for this, and thus, localizedness implies that \( f(R, x) = 1 \). Also, these steps stay in the Condorcet domain because \( x \) is always unanimously top-ranked. Hence, the case \( x = c \) is proven.

For the second case, we assume that \( c \neq x \) and define \( I = \{ i \in N : cR^\circ_i x \} \) as the set of voters who prefer \( c \) to \( x \) in \( R^2 \). Next, we let all voters in \( I \) repeatedly swap up \( c \) until it is their best alternative. Because \( c \) is only reinforced during these steps, it stays the Condorcet winner. Furthermore, we never swap \( c \) and \( x \) because of the definition of \( I \). Hence, this process terminates in a profile \( R^4 \) with \( f(R^4, x) = 1 \) because of localizedness. As second step, we let the voters \( i \in I \) one after another swap up \( x \) until it is their second best alternative and the voters in \( N \setminus I \) swap up \( x \) until it is their best alternative. During these steps, \( c \) stays the Condorcet winner as all voters in \( I \) report it as their best alternative and thus, we do not leave the Condorcet domain. Moreover, non-perversity entails that the probability of \( x \) cannot decrease during these steps, and therefore, \( f(R^4, x) = 1 \) for the resulting profile \( R^4 \). As next step, we let the voters in \( I \) swap \( x \) and \( c \) one after another, which results in a profile \( R^5 \) in which \( x \) is the Condorcet winner because all voters report it as their favorite choice. This process does not leave the Condorcet domain as every voter top-ranks either \( c \) or \( x \) and thus, one of these alternatives is top-ranked by more than half of the voters during each step because \( n \) is odd. Finally, we deduce that \( f(R^5, x) = 1 \) because non-perversity requires that \( f(R^5, x) \geq f(R^4, x) \). As last step, we transform \( R^5 \) into \( R \) by reordering the alternatives in \( A \setminus \{ x \} \) in the preference relations of all voters. Since \( x \) is not involved in any of these swaps, it follows that \( f(R, x) = 1 \) because of localizedness, and we do not leave the Condorcet domain as \( x \) is always unanimously top-ranked. Because \( x \) and \( R \) are arbitrarily chosen, it follows that \( f \) is 0-unanimous.

**Claim 2:** Every locally strategyproof SDS \( f : D \rightarrow \Delta(A) \) that violates Pareto-optimality fails 0-unanimity.

Next, assume that \( f \) violates Pareto-optimality, which means that there is a profile \( R \in D \) and alternatives \( x, y \in A \) such that \( x R y \) for all voters \( i \in N \) but \( f(R, y) > 0 \). We show that \( x \) is not chosen with probability \( 1 \) if it is top-ranked by every voter. If \( R \in D \setminus D_C \), we use again the connectedness of \( D \) to go to a profile \( R^3 \in D_C \). In particular, there is an ad-path from \( R \) to a profile \( R^1 \in D_C \) such that \( y \) is not swapped with any other alternative along this path. Hence, localizedness implies that \( f(R, y) = f(R, y) > 0 \), and \( x \) still Pareto-dominates \( y \) in \( R^2 \) as we never swap these alternatives. If \( R \in D_C \), we define \( R^1 = R \).

Next, we use a case distinction with respect to the Condorcet winner \( c \) in \( R^1 \). First, assume that \( x \) is the Condorcet
As next step, we let all voters prefer the Condorcet winner in $R^3$. In this case, we can sequentially swap up $x$ and down $y$ in the preference relation of every voter until it is unanimously top-ranked. Because $y$ is not affected by these steps, localizedness implies that $f(R^2, y) = f(R^3, y) > 0$ for the resulting profile $R^2$. However, this means that $f$ violates 0-unanimity as $f(R^2, x) \neq 1$ even though $x$ is unanimously top-ranked in $R^2$.

As second case, assume that $c \neq x$ and let $I = \{i \in N : cR^1_i x\}$ denote the voters who prefer $c$ to $x$ in $R^1$. Note also that $c \neq y$ because all voters prefer $x$ to $y$. As first step, we let the voters $i \in I$ swap $x$ with $y$ until it is their best alternative. We do not leave the Condorcet domain during these steps because we only reinforce the Condorcet winner.

Furthermore, $y$ is not involved in any swap since $cR^1_i x$ and $xR^1_i y$ for all voters $i \in I$, and thus, this step leads to a profile $R^2$ with $f(R^2, y) = f(R^1, y) > 0$ because of localizedness.

As next step, we let all voters $i \in I$ reinforce $x$ until it is their second best alternative and all voters $i \in N \setminus I$ reinforce $x$ until it is their best alternative. Alternative $c$ stays the Condorcet winner during these steps since it is always top-ranked by all voters in $I$, and localizedness implies for the resulting profile $R^2$ that $f(R^2, y) = f(R^2, y) > 0$ because all voters prefer $x$ to $y$ in $R^2$. Finally, we derive the profile $R^3$ by sequentially swapping $c$ and $x$ in the preferences of the voters $i \in I$. This process does not leave the Condorcet domain because in every intermediate profile, only $c$ and $x$ are top-ranked. This means that one of these alternatives is the Condorcet winner because $n$ is odd and thus, either $c$ or $x$ is top-ranked by more than half of the voters. Moreover, $y$ is not involved in these swaps and localizedness implies therefore that $f(R^3, y) = f(R^3, y) > 0$. However, $x$ is now top-ranked by all voters but $f(R^3, x) \neq 1$, which means that $f$ fails 0-unanimity.

Lemma 5 is one of our basic tools that will be used in almost all subsequent proofs. For instance, we use this result now to show that every locally strategy-proof and non-imposing SDS on a connected super Condorcet domain satisfies strong symmetry conditions between profiles in which only two alternatives are top-ranked. For formally stating this lemma, we denote with $N^3_{1,2} = \{i \in N : cR^1_i x\}$ the set of voters who report $x$ as their favorite alternative in $R^3$. Moreover, we define $S^{{\leq}1,2} = \{R \in R^3 : N^3_{1,2} = I \wedge N^3_y = N \setminus I\}$ for all subsets of voters $I \subseteq N$ and alternatives $x, y \in A$ as the domain that contains precisely the preference profiles in which all voters in $I$ report $x$ as their best alternative and all voters in $N \setminus I$ report $y$ as their best alternative. Note that $S^{{\leq}1,2} \subseteq D_C$ for all subsets of voters $I$ and alternatives $x, y \in A$ if $n$ is odd because either $x$ or $y$ is top-ranked by more than half of the voters and thus, one of these alternatives is the Condorcet winner for every profile $R \in S^{{\leq}1,2}$.

**Lemma 6.** Assume $n \geq 3$ is odd and $n \geq 3$. Furthermore, consider a locally strategy-proof and non-imposing SDS $f$ on a connected super Condorcet domain. For all sets of voters $I \subseteq N$ with $0 \subseteq I \subseteq N$, alternatives $w, x, y, z \in A$ with $w \neq x$ and $y \neq z$, and preference profiles $R \in S^{{\leq}1,2}$, it holds that $f(R, w) = f(R', y)$, $f(R, x) = f(R', z)$, and $f(R, w) + f(R, x) = 1$.

**Proof.** Consider a connected super Condorcet domain $D$ and let $f : D \rightarrow \Delta(A)$ denote a locally strategy-proof and non-imposing SDS. Furthermore, consider an arbitrary set of voters $I$ with $0 \subseteq I \subseteq N$. We prove the lemma in three steps: first, we show that $f(R) = f(R')$ and $f(R, x) + f(R, y) = 1$ for all alternatives $x, y \in A$ with $x \neq y$ and all preference profiles $R, R' \in S^{{\leq}1,2}$. This is helpful as we now only need to show that there are preference profiles $R \in S^{{\leq}1,2}$, $R' \in S^{{\geq}1,2}$ such that $f(R, w) = f(R', y)$, $f(R, x) = f(R', z)$, and $f(R, x) + f(R, y) = 1$. As second step, we consider three distinct alternatives $x, y, z \in A$ and show that $f(R, x) = f(R', z)$ for all $R \in S^{{\leq}1,2}$, $R' \in S^{{\leq}1,2}$. Using these two insights, we prove the lemma as last step.

**Step 1:** $f(R) = f(R')$ and $f(R, x) + f(R, y) = 1$ for all distinct alternatives $x, y \in A$ and preference profiles $R, R' \in S^{{\leq}1,2}$.

Let $x$ and $y$ denote two distinct alternatives and consider a profile $R^* \in \mathcal{D}_C \subseteq D$ such that $r(R^*_i, x) = 1$ and $r(R^*_i, y) = 2$ for all voters $i \in I$, and $r(R^*_i, y) = 1$ and $r(R^*_i, x) = 2$ for all voters $i \in N \setminus I$. Note that $f(R^*, x) + f(R^*, y) = 1$ because $f$ is Pareto-optimal (see Lemma 5) and all alternatives $z \in A \setminus \{x, y\}$ are Pareto-dominated in $R^*$. Moreover, consider an arbitrary preference profile $R \in S^{{\leq}1,2}$. We will show that $f(R^*) = f(R)$ which proves this step since $R$ is chosen arbitrarily. Note that we do not have to worry about leaving $D$ during the subsequent transformations because we do not change the favorite alternatives of the voters. Since either $I$ or $N \setminus I$ contains more than half of the voters, it follows therefore that there is always a Condorcet winner. For transforming $R^*$ into $R$, we consider two auxiliary profiles $R^1$ and $R^2$: in $R^1$, the voters $i \in I$ report $R^*_i$ and the voters $i \in N \setminus I$ report $R^*_i$. The profile $R^2$ is constructed inversely: the voters in $i \in I$ report $R^*_i$ and the voters $i \in N \setminus I$ report $R^*_i$. Next, we show that $f(R, x) = f(R^1, x) = f(R^*, x)$. Note for this that we can use pairwise swaps to transform $R^*$ to $R^2$ and that no swap involves $y$ because we only need to reorder the alternatives $A \setminus \{y\}$ in the preference relations of the voters $i \in N \setminus I$. Thus, it follows from localizedness that $f(R^1, x) = f(R^*, x)$. Moreover, $y$ Pareto-dominates all other alternatives but $x$ in $R^2$ as it is second-ranked by all voters in $I$ and top-ranked by the voters in $N \setminus I$. Hence, Lemma 5 implies that $f(R^1, x) + f(R^1, y) = 1$ which means that $f(R^1, x) = 1 - f(R^1, y) = 1 - f(R^*, y) = f(R^*, x)$. As next step, we transform $R^1$ into $R$ by reordering the alternatives in $A \setminus \{x\}$ in the preference relations of voters $i$. We can use for this again pairwise swaps that do not involve $x$ and thus, localizedness implies that $f(R, x) = f(R^1, x) = f(R^*, x)$. Moreover, we can use a symmetric argument to derive that $f(R, y) = f(R^2, y) = f(R^*, y)$. Therefore, we conclude that $f(R) = f(R^*)$ and $f(R, x) + f(R, y) = 1$ for all preference profiles $R \in S^{{\leq}1,2}$.

**Step 2:** $f(R, x) = f(R', z)$ for all distinct alternatives $x, y \in A$ and preference profiles $R \in S^{{\leq}1,2}$, $R' \in S^{{\leq}1,2}$.

For proving this claim, let $x, y, z \in A$ denote three distinct alternatives and consider the profiles $R^1$ and $R^2$.

$R^1$: $I: x, y, z$, $N \setminus I: y, x, z$, $R^2$: $I: x, x, y$, $N \setminus I: y, x, z$. 

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We start the analysis at the profile $R^3 \in S^{x,y,z}$. In particular, note that Step 1 implies for $R^3$ that $f(R^3, x) + f(R^3, y) = 1$. Next, we derive the profile $R^4$ by letting every voter $i \in I$ swap $x$ and $z$ one after another. If $y$ is the Condorcet winner in $R^3$, it is clear that this process does not leave the Condorcet domain as $y$ is not involved in these steps. If $x$ is the Condorcet winner, then $x$ or $z$ are the Condorcet winner during each step. The reason for this is that $g_R(z, w) = g_R(x, w) > 0$ for every alternative $w \in A \setminus \{x, z\}$. This means that $x$ is the Condorcet winner as long as a majority of the voters prefers $x$ to $z$ and otherwise, $z$ is the Condorcet winner. Moreover, localization implies that $f(R^3, y) = f(R^3, y)$ since $y$ was not involved in any swap during these steps. Finally, Step 1 implies that $f(R^4, w) = 0$ for all $w \in A \setminus \{y, z\}$ because $R^4 \in S^{x,y,z}$. Hence, we infer that $f(R^4, z) = 1 - f(R^3, y) = 1 - f(R^3, x)$. Since $R^3 \in S^{x,y,z}$ and $R^4 \in S^{x,y,z}$, we can now use the insights of the first step to deduce that $f(R, x) = f(R', z)$ for all distinct alternatives $x, y, z \in A$ and preference profiles $R \in S^{x,y,z}$ and $R' \in S^{x,y,z}$.

Step 3: $f(R, w) = f(R', y), f(R, x) = f(R', z)$ and $f(R, w) + f(R, x) = 1$ for all alternatives $w, x, y, z \in A$ with $w \neq x$ and $y \neq z$ and all preference profiles $R \in S^{w,x,z} \setminus R \in S^{y,z}$. Let $w, x, y, z$ denote arbitrary alternatives with $w \neq x$ and $y \neq z$. First note that Step 1 proves the claim if $w = y$ and $x = z$. Furthermore, the case that $x = z$ and $w \neq y$ follows from Step 1 and Step 2: Step 2 entails that $f(R, w) = f(R', y)$ for all $R \in S^{w,x,z}$ and $R' \in S^{y,z}$ and then, we infer from Step 1 that $f(R, x) = f(R', y)$ because all other alternatives must have probability 0 in both $R$ and $R'$. Note that we can use the same argument also for the case that $w = y$ an $x \neq z$. The reason for this is that $S^{w,x,z} = S^{x,N\setminus I}$ and $S^{y,z} = S^{y,N\setminus I}$, i.e., we can just revert the roles of $I$ and $N \setminus I$ to apply the same argument as in the previous case. If $\{w, x\} \cap \{y, z\} = \emptyset$, the lemma follows by applying the previous ideas twice: we know that $f(R, w) = f(R', y)$, $f(R', y) = f(R', y)$, and $f(R, x) = f(R', z)$. For all $R \in S^{w,x,z}$, $R' \in S^{y,z}$, and $R' \in S^{y,z}$. Finally, the last case is that $w = z$ or $x = y$ or both. We focus on the case that $w = z$, i.e., we assume that $R \in S^{w,x,z}$ and $R' \in S^{y,z}$; the case $x = y$ follows by a symmetric argument. In this case, we use an auxiliary variable $v$ and auxiliary profiles $R^3 \in S^{w,x,z}$, $R^2 \in S^{y,z}$, we know that $f(R, w) = f(R', v) = f(R', y)$, and $f(R, x) = f(R', x) = f(R', w) = f(R', v)$ for all preference profiles $R \in S^{w,x,z}$, $R' \in S^{y,z}$. This means that all cases are covered.

Lemma 6 is helpful for our analysis because determining the probabilities for a single preference profile $R \in S^{x,y,z}$ specifies the outcome for a whole class of profiles. As last result of this section, we show a technical auxiliary lemma that discusses how the probabilities change if a voter swaps to alternatives $w, x \in A$ in two related preference profiles $R$ and $R'$.
Proof. Consider a connected super Condorcet domain \( D \) and let \( f \) denote a locally strategyproof and locally 0-randomly dictatorial SDS on \( D \). Furthermore, suppose for contradiction that there is a voter \( i \in N \) such that \( f(R',y) - f(R,y) \neq 0 \) for all preference profiles \( R,R' \in D \) and alternatives \( x,y \in A \) such that \( R' = R^{u,y}, r(R_i,x) = 1 \), and \( r(R_i,y) = 2 \). First note that non-perversity entails for all these profiles that \( f(R',y) - f(R,y) \geq 0 \). Consequently, there is an \( \epsilon > 0 \) such that \( f(R',y) - f(R,y) \geq \epsilon \) for all preference profiles \( R,R' \in D \) and alternatives \( x,y \in A \) such that \( R' = R^{u,y}, r(R_i,x) = 1 \), and \( r(R_i,y) = 2 \). Our goal is to show that \( f \) is locally \( \gamma \)-randomly dictatorial for \( \gamma = \epsilon \). This contradicts the assumption that \( f \) is a local 0-random dictatorship and hence, there must be profiles for voter \( i \) that satisfy all conditions of the lemma.

For deriving the contradiction, we proceed in multiple steps. In Step 1, we show that \( f(R,x) \geq \epsilon \) for all preference profiles \( R \in D \) and every alternative \( x \in A \) such that \( r(R_i,x) = 1 \).

As first step, we show that \( f(R,x) \geq \epsilon \) for all preference profiles \( R \in D \) and alternatives \( x \in A \) such that voter \( i \) prefers \( x \) the most in \( R \). Based on this observation, it follows that \( \epsilon < 1 \) because otherwise, \( f \) chooses always the best alternative of voter \( i \) with probability 1. However, this means that \( f \) is the dictatorial SDS of voter \( i \) and therefore, it is 1-randomly dictatorial if \( \epsilon = 1 \). Since this contradicts the assumption that \( f \) is locally 0-randomly dictatorial, we can assume that \( 0 < \epsilon < 1 \). For this case, we show in Step 2 that \( g = \frac{1}{1-\epsilon} f - \frac{\epsilon}{1-\epsilon} d_i \) is a well-defined and locally strategyproof SDS. Hence, \( f \) can be represented as \( f = d_i + (1-\epsilon) g \), which shows that \( f \) is locally \( \gamma \)-randomly dictatorial for some \( \gamma \geq \epsilon \). This is in conflict with our assumptions and thus, there must be profiles \( R,R' \in D \) and alternatives \( x,y \in A \) such that \( f(R',y) = f(R,y), R' = R^{u,y}, r(R_i,x) = 1 \), and \( r(R_i,y) = 2 \).

Step 1: \( f(R,x) \geq \epsilon \) for every preference profile \( R \in D \) and every alternative \( x \in A \) such that \( r(R_i,x) = 1 \).

As first step, we show that \( f(R,x) \geq \epsilon \) for all preference profiles \( R \in D \) and alternatives \( x \in A \) such that voter \( i \) top-ranks \( x \) in \( R \). Thus, consider an arbitrary profile \( R \in D \) and let \( x \) denote voter \( i \)'s best alternative. As first step, we construct a profile \( R^1 \in D_C \) such that \( f(R,x) = f(R^1,x) \) and \( r(R_i,x) = 1 \). If \( R \) is already in the Condorcet domain, we just define \( R^1 = R \). Otherwise, we can find an ad-path from \( R \in D \setminus D_C \) to a profile \( R^1 \in D_C \) along which \( x \) is not swapped with any other alternative because \( D \) is connected. Consequently, a repeated application of localizedness along the path from \( R \) to \( R^1 \) implies that \( f(R^1,x) = f(R,x) \) and \( x \) is still voter \( i \)'s best alternative in \( R^1 \).

In the sequel, we show that \( f(R^1,x) \geq \epsilon \), which proves this step since \( f(R,x) = f(R^1,x) \). For this, we use a case distinction with respect to the Condorcet winner \( c \) in \( R^1 \) and assume first that \( c \neq x \). In this case, consider the profile \( R^2 \) derived from \( R^1 \) by swapping voter \( i \)'s best alternative \( x \) with his second best alternative \( y \). Since \( x \) is not the Condorcet winner in \( R^1 \), it follows that \( R^2 \in D_C \subseteq D \): either the Condorcet winner is not affected by swapping \( x \) and \( y \) or it is reinforced if \( c = y \). Our assumptions on \( i \) imply that \( f(R^2,x) - f(R^2,y) \geq \epsilon \) and it holds that \( f(R^2,x) \geq 0 \) because of the definition of SDSs. Hence, \( f(R^1,x) \geq \epsilon \) which proves this case.

As second case, suppose that \( c = x \). In this case, we choose an arbitrary alternative \( y \in A \setminus \{x\} \) and define \( J = \{j \in N : xR_j^1 y\} \) as the set of voters who prefer \( x \) to \( y \) in \( R^1 \). As first step, consider the profile \( R^2 \) derived from \( R^1 \) by letting all voters \( j \in J \) reinforce \( y \) until it is directly below \( x \). Since no swap in the construction of \( R^2 \) involves \( x \), we do not leave the Condorcet domain. Also, this observation implies that \( f(R^2,x) = f(R^1,x) \) because of localizability. Moreover, \( g_{R^2}(x,z) = g_{R^1}(x,z) = 0 \) for all alternatives \( z \in A \setminus \{x,y\} \) because \( xR^2_{xy} \) implies \( yR^2_{x} \) and \( x \) is the Condorcet winner in \( R^2 \). We use this fact to construct the next profile \( R^3 \) by letting the voters \( j \in J \setminus \{i\} \) swap \( x \) and \( y \) one after another. This construction does not leave the Condorcet domain: since \( n \) is odd and \( g_{R^2}(x,z) = g_{R^1}(x,z) = 0 \) for all \( z \in A \setminus \{x,y\} \), \( x \) or \( y \) is the Condorcet winner in every intermediate profile. Moreover, in the final profile \( R^3 \), all voters in \( N \setminus \{i\} \) prefer \( y \) to \( x \) and \( y \) is now the Condorcet winner because \( n \geq 3 \). Hence, it follows from the last case that \( f(R^3,x) \geq \epsilon \) because voter \( i \) top-ranks \( x \) in \( R^2 \) and \( x \) is not the Condorcet winner in this profile. Furthermore, we only weaken \( x \) in the construction of \( R^3 \), and thus, non-perversity implies that \( f(R^2,x) \leq f(R^3,x) = f(R^1,x) \). Combining the last two observations shows that \( f(R^1,x) \geq \epsilon \) and thus, \( f \) always chooses the best alternative of voter \( i \) with a probability of at least \( \epsilon \).

Step 2: If \( \epsilon < 1 \), then \( g = \frac{1}{1-\epsilon} f - \frac{\epsilon}{1-\epsilon} d_i \) is a well-defined and locally strategyproof SDS.

Assume that \( \epsilon < 1 \) and consider the function \( g = \frac{1}{1-\epsilon} f - \frac{\epsilon}{1-\epsilon} d_i \), where \( d_i \) is the dictatorial SDS of voter \( i \). We need to show that \( g \) is a well-defined and locally strategyproof SDS because \( f \) is then not locally 0-randomly dictatorial. First, we show that \( g \) is a well-defined SDS, i.e., that \( g(R,x) \geq 0 \) for all alternatives \( x \in A \) and preference profiles \( R \in D \), and that \( \sum_{x \in A} g(R,x) = 1 \) for all preference profiles \( R \in D \). For proving the first claim, consider a preference profile \( R \in D \) and an alternative \( x \in A \). We use a case distinction with respect to whether \( x \) is voter \( i \)'s best alternative in \( R \). If this is the case, step 1 implies that \( f(R,x) \geq \epsilon \) and \( d_i(R,x) = 1 \) because \( d_i \) chooses the best alternative of voter \( i \). Hence, we can compute that \( g(R,x) = \frac{1}{1-\epsilon} f(R,x) - \frac{\epsilon}{1-\epsilon} d_i(R,x) \geq \frac{\epsilon}{1-\epsilon} - \frac{\epsilon}{1-\epsilon} = 0 \). On the other hand, if \( x \) is not voter \( i \)'s best alternative, it follows that \( d_i(R,x) = 0 \). This means that \( g(R,x) = \frac{1}{1-\epsilon} f(R,x) - \frac{\epsilon}{1-\epsilon} d_i(R,x) = \frac{1}{1-\epsilon} f(R,x) \geq 0 \), which shows that \( g \) only assigns non-negative values to the alternatives. Next, we show that the probabilities assigned by \( g \) sum up to 1 for every preference profile \( R \in D \). This follows immediately by the definition of this SDS since \( \sum_{x \in A} g(R,x) = \frac{1}{1-\epsilon} \sum_{x \in A} f(R,x) - \frac{\epsilon}{1-\epsilon} \sum_{x \in A} d_i(R,x) = \frac{1}{1-\epsilon} - \frac{\epsilon}{1-\epsilon} = 1 \) for every profile \( R \in D \). Hence, \( g \) is a well-defined SDS.

Next, we show that \( g \) is locally strategyproof. Because Lemma 4 shows that local strategyproofness is equivalent to non-perversity and localizability, it suffices to show that \( g \) satisfies the latter two axioms. Hence, consider two preference profiles \( R,R' \in D \), two alternatives \( x,y \in A \), and a voter \( j \in N \) such that \( R^1 = R^{u,y} \). First, note that \( g \) is localized because
Assume Lemma 9. i.e., even if $D_y$ the most and $R_R,R_R$ and such that $f$ is locally strategyproof and thus also non-perverse. Moreover, if $r(R_i, x) > 1$ or $j \neq i$, it follows that $d_i(R, y) = d_i(R', y)$. Combining these observations shows that

$$g(R', y) = \frac{1}{1 - \epsilon} f(R', y) - \frac{\epsilon}{1 - \epsilon} d_i(R', y)$$

$$\geq \frac{1}{1 - \epsilon} f(R, y) - \frac{\epsilon}{1 - \epsilon} d_i(R, y) = g(R, y).$$

Hence, $g$ is non-perverse in this case. On the other hand, if $r(R_i, x) = 1$ and $j = i$, it holds that $d_i(R, y) = 0 \neq 1 = d_i(R', y)$. Moreover, our assumptions on $f$ imply for this case that $f(R', y) - f(R, y) \geq \epsilon$. Thus, we can compute that

$$g(R', y) = \frac{1}{1 - \epsilon} f(R', y) - \frac{\epsilon}{1 - \epsilon} d_i(R', y)$$

$$\geq \frac{1}{1 - \epsilon} (\epsilon + f(R, y)) - \frac{\epsilon}{1 - \epsilon}$$

$$= \frac{1}{1 - \epsilon} f(R, y)$$

$$= g(R, y).$$

This inequality shows also for the second case that $g$ is non-perverse and consequently, it is locally strategyproof. Hence, $f$ is not locally $0$-randomly dictatorial because $f = \epsilon d_i + (1 - \epsilon) g$. In particular, this representation entails that $f$ is locally $\gamma$-randomly dictatorial for $\gamma \geq \epsilon$. This means that the initial assumption is false and there are for every voter $i \in N$ two preference profiles $R, R' \in D$ and alternatives $x, y \in A$ such that $f(R, y) = f(R', y), R' = R'^{\gamma y}, r(R_i, x) = 1, r(R_i, y) = 2$. And $r(R, y) = f(R', y)$. Note that $f(R, y) = f(R', y)$ implies also that $f(R, x) = f(R', x)$ because of localizedness.

As first step in the construction of $R$ and $R'$, we use the connectedness of $D$: there is an ad-path from $R$ to a profile $R^1 \in D_C$ that does not swap $x$ with another alternative. Similarly, if $R^2 \notin D_C$, there is also a profile $R^2 \in D_C$ derived from $R'$ without swapping $x$; if $R^2 \in D_C$, we set $R^2 = R'$. Since $R^1$ is derived from $R$ without swapping $x$ with another alternative, it follows that $U(R^1, x) = U(R, x)$ for all voters $j \in N$. In particular, this means that $x$ is not the Condorcet winner in $R^1$ because it is not the Condorcet winner in $R$. Analogously, it follows that $U(R^2, x) = U(R', x)$ for all voters $j \in N$. This entails that $U(R^2, x) = U(R^1, x)$ for all $j \in N \setminus \{i\}$ because $r_j = r_j^i$ for these voters, and $U(R^1, x) \subseteq U(R^2, x)$ because $R^2 = R'^{\gamma y}$. In particular, these observations imply that $x$ is also not the Condorcet winner in $R^2$, and that $r(R^2, x) = 1, r(R^2, y) = 1$, and $r(R^2, x) = 2$. Finally, localizedness shows that $f(R^1, x) = f(R, x) = f(R', x) = f(R^2, x)$ because $x$ is not swapped with another alternative in the construction of $R^1$ or $R^2$.

Next, let $c$ denote the Condorcet winner in $R^1$ and define the set $I = \{j \in N : c \not\in R^1\}$ of voters who prefer $c$ to $x$ in $R^1$. Analogously, we define $c'$ as the Condorcet winner in $R^2$ and $J = \{j \in N : c' \not\in R^2\}$ as the set of voters who prefer $c'$ to $x$ in $R^2$. Note that $i \notin I$ because voter $i$ top-ranks $x$ in $R^1$. We derive the profile $R$ from $R^1$ as follows: we let all voters $j \in I$ swap up $x$ until it is their best alternative. This step leads to an intermediate profile $R^0$, and we do not leave $D_C \subseteq D$ during the construction of this profile because we only reinforce the Condorcet winner $c$. Furthermore, it follows from the definition of $I$ that $x$ is not involved in any swap, which means that $f(R^1, x) = f(R^1, x)$ because of localizedness. Finally, we construct the profile $R$ based on $R^0$ by reordering the alternatives $z \in A \setminus \{x\}$ in voter $i$’s preference relation as in $R^0$, i.e., voter $i$’s preference relation only differs from $R^0$ in the fact that he prefers $x$ to $y$. Since all voters in $I$ prefer $c$ the most in $R^0$, we do not leave $D_C$ during this step. Furthermore, localizedness implies that $f(R, x) = f(R, x) = f(R, x) = f(R, x)$. 

**Proof.** Assume that $n \geq 3$ is odd and $m \geq 3$ and let $D$ denote a connected super Condorcet domain. Furthermore, consider a locally strategyproof and locally $0$-randomly dictatorial SDS $f : D \rightarrow \Delta(A)$. We need to show that for every voter $j \in N$ there are preference profiles $R, R' \in D_C$ and alternatives $x, y \in A$ such that $f(R', y) = f(R, y), r(R_i, y) = 1, r(R_i, x) = 2$, and $f(R, y) = f(R', y)$. First, note that Lemma 8 proves that there are such profiles $R, R' \in D$ for every voter. If these profiles are for all $j \in N$ also in the Condorcet domain, the lemma is already proven. Hence, suppose that there is a voter $i \in N$ such that $\{R, R'\} \not\subseteq D_C$ for all profiles $R, R' \in D$ and alternatives $x, y \in A$ with $R' = R'^{\gamma y}, r(R_i, x) = 1, r(R_i, y) = 2$, and $f(R, y) = f(R', y)$. Next, we consider two such profiles $R$ and $R'$ and we suppose that $R \not\in D_C$ because the case that $R' \not\in D_C$ is symmetric. Our goal is to transform the profiles $R, R'$ into two new profiles $R$ and $R'$, respectively, that satisfy all conditions of the lemma, i.e., $\{R, R'\} \subseteq D_C, R' = R'^{\gamma y}, \bar{R}(R_i, x) = 1, \bar{R}(R_i, y) = 2$, and $f(R, y) = f(R', y)$. Note for this that the assumption $f(R, y) = f(R', y)$ implies also that $f(R, x) = f(R', x)$ because of localizedness.
As second point, we construct the profile \( \tilde{R}' \) based on \( R^2 \). In particular, we need to ensure that \( \tilde{R}' = \tilde{R}_j \) for all voters \( j \in N \setminus \{i\} \) and introduce therefore multiple auxiliary profiles. Hence, consider the profile \( R^d \) derived from \( R^2 \) by letting all voters \( j \in J \) push up \( e' \) until it is their best alternative, all voters \( j \in I \cap J \) push up \( e \) until it is directly below \( e' \), and all voters \( j \in I \setminus J \) push up \( e \) until it is their best alternative. Note that voter \( i \) does not change his preference relation during these steps: it holds that \( i \notin I \) because \( r(R'_i, x) = 1 \), and if \( i \in J \), then \( e' = y \) because alternative \( y \) is the only alternative in \( A \setminus \{x\} \) with \( yR'_2 x \). However, voter \( i \) already top-ranks \( y \) in \( R^2 \) and thus, it follows that \( R^d_i = R^2_i \). Furthermore, we do not leave the Condorcet domain during these steps because we never weaken the Condorcet winner \( e' \).

Finally, observe that \( x \) is not involved in any swap: we have already shown that \( R^d_i = R^2_i \), and the definition of the sets \( I \) and \( J \) combined with the fact that \( U(R'_i, x) = U(R^2_i, x) \) for all \( j \in N \setminus \{i\} \) implies that these voters do not swap \( x \) with another alternative. Hence, localizedness entails that \( f(R^d, x) = f(R^2, x) \).

As next step, we construct the profile \( R^c \) based on \( R^4 \) by letting all voters in \( I \cap J \) swap \( e \) and \( e' \) one after another. Note that this intersection is non-empty because \( |I| > \frac{n}{2} \) and \( |J| > \frac{n}{2} \). Further, this process does not leave the Condorcet domain because \( n \) is odd and \( g_{R_4}(c, z) > 0 \) and \( g_{R_4}(e', z) > 0 \) for all \( z \in A \setminus \{c, e'\} \), which means that either \( c \) or \( e' \) is the Condorcet winner in every intermediate profile. In particular, \( c \) is the Condorcet winner in the final profile as it is top-ranked by all voters in \( I \). As final points on \( R^c \), observe that localizedness implies that \( f(R^c, x) = f(R^4, x) \), and that \( R^c_i = R^2_i \) because \( i \notin I \).

As last step, we use the observation that all voters in \( I \) report \( c \) as their best alternative in both \( R^2 \) and \( R \) to construct the final profile \( \tilde{R}' \). In more detail, we derive this profile from \( R^2 \) by letting all voters \( j \in N \setminus \{i\} \) reorder their preference relations as in \( R \). We can transform \( R^2 \) into \( R' \) without leaving \( D_C \) by only using pairwise swaps because all voters in \( I \) always report \( c \) as their best alternative, which ensures that \( c \) is the Condorcet winner. Furthermore, observe that \( U(R'_i, x) = U(R_i, x) \) for all voters \( j \in N \setminus \{i\} \) because \( x \) was never swapped with another alternative during the construction of these profiles. Hence, we can construct the profile \( \tilde{R}' \) without swapping \( x \) with another alternative in the preference relations of these voters and localizedness implies consequently that \( f(\tilde{R}', x) = f(R', x) \). Finally, observe that, by construction, \( \tilde{R}'_j = R_j \) for all \( j \in N \setminus \{i\} \), and \( \tilde{R}'_i = R^2_i = R_i^{y|x} \). Hence, it indeed holds that \( \tilde{R}' = R_i^{y|x} \) and \( \tilde{R}' \in D_C \). Furthermore, \( f(\tilde{R}, x) = f(R, x) = f(R', x) = f(R', x) \) because we never swapped \( x \) with another alternative in our construction and it follows therefore from localizedness that \( f(\tilde{R}', y) = 1 - \sum_{z \in A \setminus \{y\}} f(R', z) = 1 - \sum_{z \in A \setminus \{y\}} f(R, z) = f(R, y) \). This proves the lemma.

Lemma 9 is important because it shows that the behavior of 0-randomly dictatorial SDSs does not depend on the exact connected super Condorcet domain. In particular, regardless of the considered super Condorcet domain, there are for every voter preference profiles in the Condorcet domain that satisfy all conditions of Lemma 8 and thus, our subsequent analysis can focus on profiles in \( D_C \).

In the next lemma, we show that every 0-randomly dictatorial, locally strategyproof, and non-imposing SDS on a connected super Condorcet domain is 1-unanimous if \( n \geq 3 \) is odd. Recall that an SDS \( f : D \rightarrow \Delta(A) \) is \( k \)-unanimous if \( f(R, x) = 1 \) for all alternatives \( x \in A \) and preference profiles \( R \in D \) such that \( |N^R_x| \geq n - k \).

**Lemma 10.** Assume \( n \geq 3 \) is odd and \( m \geq 3 \). Every locally 0-randomly dictatorial, locally strategyproof, and non-imposing SDS on a connected super Condorcet domain is 1-unanimous.

**Proof.** Assume \( n \geq 3 \) is odd and \( m \geq 3 \) and let \( D \) denote a connected super Condorcet domain. Furthermore, let \( f \) denote a locally 0-randomly dictatorial SDS on \( D \) that satisfies local strategyproofness and non-imposition. We need to show that \( f(R, x) = 1 \) for every preference profile \( R \in D \) and every alternative \( x \in A \) such that \( |N^R_x| \geq n - 1 \).

Consider for this an arbitrary voter \( i \in N \). We show in the sequel that \( f(R, x) = 1 \) for all preference profiles \( R \in D \) and alternatives \( x \in A \) such that \( |N^R_x| = n - 1 \).

Since \( i \) is chosen arbitrarily, this claim proves the lemma. As next step, we use the insights of Lemma 9: for the considered voter \( i \), there are preference profiles \( R, \tilde{R} \in D_C \) and two alternatives \( x, y \in A \) such that \( \tilde{R} = R^{w|x}, x \in A \) satisfies \( R(R_i, x) = 1, R(R_i, y) = 2, \) and \( f(R, y) = f(\tilde{R}, y) \). We proceed with a case distinction with respect to the profile \( R \) and its Condorcet winner \( c \).

**Case 1:** There is \( z \in A \) such that \( N \setminus \{i\} \subseteq N^R_z \).

As first case, suppose that there is an alternative \( z \in A \) such that \( N \setminus \{i\} \subseteq N^R_z \), i.e., all voters in \( N \setminus \{i\} \) report \( z \) as their best alternative in \( R \). If \( z = x \), then \( z \) Pareto-dominates every other alternative because every voter top-ranks \( x \). Hence, \( f(R, x) = 1 \) because of Lemma 5. Then, our assumptions imply that \( f(R, y) = f(R, y) = 0 \), even though \( R \in S^{z \in N \setminus \{i\}} \). Thus, Lemma 6 applies and shows that \( f(R', w) = 1 \) for all profiles \( R' \in D \) and alternatives \( w \in A \) such that \( N^R_w = N \setminus \{i\} \). The reason for this is that for all these profiles \( R' \), there is an alternative \( v \in A \setminus \{w\} \) such that \( R' \in S^{w \in N \setminus \{i\}} \). Moreover, the case that \( z = y \) follows by exchanging the roles of \( R \) and \( R' \) in the above argument.

Hence, suppose that \( z \in A \setminus \{x, y\} \). In this case, we use Lemma 6 to deduce that \( f(R, y) = 0 \) since \( R \in S^{z \in N \setminus \{i\}} \). Our assumptions entail therefore that \( f(R, y) = f(R, y) = 0 \). Since \( R \in S^{z \in N \setminus \{i\}} \), we can again use Lemma 6 to derive that \( f(R', w) = 1 \) for all preference profiles \( R' \in D \) and alternatives \( w \in A \) such that \( N \setminus \{i\} \) is precisely the set of voters who report \( w \) as their favorite choice in \( R \). Hence, the claim is in this case proven.

**Case 2:** There is no \( z \in A \) such that \( N \setminus \{i\} \subseteq N^R_z \) and \( c \in A \setminus \{x, y\} \).
As second case, suppose that the voters in $N \setminus \{i\}$ do not agree on a best alternative and suppose that the Condorcet winner $c$ in the profile $R$ satisfies that $c \in A \setminus \{x, y\}$. In this case, we derive two new the profiles $R'$ and $R''$ from $R$ and $F$, respectively, such that $R' \in S^{c \setminus \{i\}}$ with $R' = (R^j)^i_{yz}$, $r(R', x) = 1$, $r(R', y) = 2$, and $f(R', y) = f(R', y)$. Then, the insights of Case 1 show that $f$ is 1-unanimous. For constructing these new preference profiles, we repeatedly identify a voter in $F$. Construct auxiliary profiles just as in the last case. In more detail, we construct two sequences of preference profiles $(R^1, \ldots, R^k)$ and $(R^1', \ldots, R^k')$ such that $R^i = R$, $R^k = R'$, $R^k = R'$, and for every index $l \in \{1, \ldots, k-1\}$ there is a voter $j \in N \setminus \{i\}$ and an alternative $z \in A \setminus \{c\}$ such that $R^{l+1}_j = (R^j)^i_{yz}$ and $R^{l+1}_j = (R^j)^i_{yz}$. Let's formalize this. In each step, the new profile is derived by swapping up $c$ in the preference of some voter $j$ in both $R'$ and $R''$. It is straightforward to construct such a sequence of preference profiles: we repeatedly identify a voter in $j \in N \setminus \{i\}$ for which $c$ is not toprank $c$ and reinforce this alternative in his preference relation.

Since $R = R^{i}_{y,z}$, our assumptions imply that $R^{i}_{y,z}$ because we use the same sequence of swaps to derive $R'$ from $R$ and $R''$ from $R$. As we can use for each step Lemma 7 to derive that $f(R^{i+1}, y) - f(R^{i+1}, y) = f(R', y) - f(R', y)$. In particular, it is important for this that $c \notin \{x, y\}$ to ensure that $\{i, z\} \cap \{x, y\} \leq 1$ (where $z$ denote the alternatives with which $c$ is swapped in the current step). By repeatedly using the last insight, it follows that $f(R', y) - f(R', y) = f(R', y) - f(R', y) = 0$. Furthermore, since we only reinforce the Condorcet winner, we do not leave the Condorcet domain. Hence, it is easy to verify that the profiles $R'$ and $R''$ indeed satisfy all requirements of Case 1, which means that $f$ is 1-unanimous.

**Case 3: There is no $z \in A$ such that $N \setminus \{i\} \subseteq N_{yz}$ and $c \in \{x, y\}$.**

As last case, we assume again that the voters $j \in N \setminus \{i\}$ do not agree on a best alternative. However, this time we suppose additionally that $c \in \{x, y\}$, i.e., the Condorcet winner is affected in the derivation of $R$. For this case, we define the sets $I = \{j \in N \setminus \{i\} : x \in \{j \in N \setminus \{i\} : y \in R\}$ and $J = \{j \in N \setminus \{i\} : y \in R\}$, which partition the voters in $N \setminus \{i\}$ according to their preference on $x$ and $y$ in $R$. We use these sets to construct auxiliary profiles just as in the last case. In more detail, we derive two new profiles $R'$ and $R''$ from $R$ and $R$, respectively, by pushing up $x$ and $y$ in the preference relations of the voters $j \in N \setminus \{i\}$ without changing their relative order until they are the best two alternatives. These profiles are shown below and, just as in Case 2, a repeated application of Lemma 7 proves that $f(R^i, y) - f(R^i, y) = f(R', y) - f(R', y) = 0$. Furthermore, Lemma 5 implies that $f(R^i, x) + f(R', y) = 1$ and $f(R^i, x) + f(R', y) = 1$ because all other alternatives are Pareto-dominated. Hence, it follows that $f(R^i) = f(R^i)$. As last point, it should be mentioned that this process does not leave the Condorcet domain as the Condorcet winner $c \in \{x, y\}$ is never weakened.

As next step, we choose an arbitrary alternative $z \in A \setminus \{x, y\}$, and let the voter $i$ reinforce this alternative until it is his second best one. Applying this step to $R$ and $R'$ results in the profiles $R''$ and $R''$, respectively. Since $R''$ and $R''$ is $S^{c \setminus \{i\}}$ and $R''$, $R''$ is $S^{c \setminus \{i\}}$ if follows from Lemma 6 that $f(R^i) = f(R^i) = f(R''')$. Finally, note that we do not leave the Condorcet domain in the construction of $R'$ and $R''$ since $x$ or $y$ is always top-ranked by more than half of the voters.

**Thereafter, consider the profiles $R$ and $R'$ derived from $R''$ and $R''$, respectively, by letting voter $i$ swap his best two alternatives. First note that $R''$, $R'' \in D_C$ because $g_{R''}(x, z) = g_{R''}(x, z) = g_{R''}(y, z) = g_{R''}(y, z) = n > 2$ and thus, this swap does not affect whether $x$ or $y$ is the Condorcet winner. Furthermore, localizedness between $R''$ and $R''$ implies that $f(R''', y) = f(R''', y)$ and localizedness between $R''$ and $R''$ implies that $f(R''', x) = f(R''', x)$. Moreover, we already know that $f(R''', y) = f(R''', y)$ and non-perversity between $R''$ and $R''$ shows that $f(R''', x) \geq f(R''', x)$. Hence, we conclude that $f(R''', x) \geq f(R''', x) = f(R''', x) = f(R''', x) = f(R''', x) = f(R''', x)$. Finally, this means that $f(R''') = f(R''')$ because $f(R''', y) + f(R''', x) = 1, f(R''', y) = f(R''', y), and f(R''', x) \geq f(R''', x)$. In particular, this shows that $f(R''', z) = 0$.

**As last point, we let all voters $j \in J$ swap $x$ and $y$ to derive the profile $R''$ from $R''$. If $x$ is the Condorcet winner in $R''$, it is obvious that this process stays in the Condorcet domain. On the other hand, if $y$ is the Condorcet winner in $R''$, we also do not leave the Condorcet domain because $g_{R''}(x, w) = g_{R''}(y, w) > 0$ for all alternatives $w \in A \setminus \{x, y\}$. Hence, $y$ is the Condorcet winner during these steps as long as a majority of voters prefers $y$ to $x$, and $x$ is the Condorcet winner otherwise. Next, note that $f(R''', z) = f(R''', z) = 0$ because of localizedness.

**Since $R'' \in S^{c \setminus \{i\}}$, $f(R''', z) = 0$ implies that $f(R''', w) = 1$ for all preference profiles $R \in D$ and alternatives $w \in A$ such that $N_{R''}^w = N \setminus \{i\}$ because of Lemma 6. Since voter $i$ is chosen arbitrarily, it follows from the last three cases that $f$ is indeed 1-unanimous.**

As next step, we strengthen the consequences of Lemma 10 by showing that every 1-unanimous and locally strategyproof SDS on a connected super Condorcet domain also satisfies $\frac{n-1}{2}$-unanimity.

**Lemma 11. Assume $n \geq 3$ is odd and $m \geq 3$. Every 1-unanimous and locally strategyproof SDS on a connected super Condorcet domain is $\frac{n-1}{2}$-unanimous.**

**Proof.** Let $f$ denote a 1-unanimous and locally strategyproof SDS on a connected super Condorcet domain $D$. We prove the lemma by an induction on $k \in \{1, \ldots, \frac{n-1}{2}\}$ and show that if $f$ is $k$-unanimous, it is also $k + 1$-unanimous. As first
step, observe that the induction basis $k = 1$ is true because we assume that $f$ is $1$-unanimous.

Next, suppose that $f$ is $k$-unanimous for a fixed $k \in \{1, \ldots, \frac{n-1}{2}\}$. We need to show that $f$ is $k+1$-unanimous, i.e., that $f(R, x) = 1$ for every profile $R \in \mathcal{D}$ and every alternative $x \in A$ such that $|N(R)| = n - k - 1$. Note that $k + 1$-unanimity also applies for profiles $R \in \mathcal{D}$ and alternatives $x \in A$ with $|N(R)| > n - k - 1$, but $k$-unanimity immediately guarantees for these profiles that $f(R, x) = 1$. Hence, it suffices to focus on profiles $R \in \mathcal{D}$ and alternatives $x \in A$ such that $|N(R)| = n - k - 1$.

As first step, we show that $f(R, x) = 1$ for all alternatives $x \in A$ and profiles $R \in S^{x \downarrow y}$ with $|I| = n - k - 1$. For proving this, consider an arbitrary set of voters $I \subseteq N$ with $|I| = n - k - 1$, let $h$ denote an arbitrary voter in $N \setminus I$, and define the set $J = N \setminus \{I \cup \{h\}\}$. Note that $J \neq \emptyset$ because $k \geq 1$ implies that $|I| \leq n - 2$. Finally, consider the profiles $R^1$ to $R^4$ shown below and note that $\{R^1, R^2, R^3, R^4\} \subseteq \mathcal{D}_C$ because $|I| > \frac{n}{2}$ and the top-ranked alternatives of these voters is thus the Condorcet winner. Our goal is to show that $f(R^i, x) = 1$ because Lemma 6 implies then our intermediate claim.

<table>
<thead>
<tr>
<th>$R^1$</th>
<th>$R^2$</th>
<th>$R^3$</th>
<th>$R^4$</th>
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<tbody>
<tr>
<td>$I: x, y, z, \ast$</td>
<td>$J: z, y, x, \ast$</td>
<td>$h: h, x, y, z, \ast$</td>
<td>$J: y, z, x, \ast$</td>
</tr>
<tr>
<td>$R^2$</td>
<td>$I: x, z, y, \ast$</td>
<td>$J: y, x, z, \ast$</td>
<td>$h: y, x, z, \ast$</td>
</tr>
<tr>
<td>$R^3$</td>
<td>$I: z, x, y, \ast$</td>
<td>$J: z, y, x, \ast$</td>
<td>$h: x, y, z, \ast$</td>
</tr>
<tr>
<td>$R^4$</td>
<td>$I: z, y, x, \ast$</td>
<td>$J: y, z, x, \ast$</td>
<td>$h: y, x, z, \ast$</td>
</tr>
</tbody>
</table>

First, observe that $f$ satisfies Pareto-optimality because of Lemma 5 and thus, $f(R^1, w) = 0$ for all alternatives $w \in A \setminus \{x, y, z\}$. Next, $f(R^1, z) = 0$ because voter $h$ could manipulate by deviating to $R^2$ otherwise. In more detail, $f(R^2, x) = 1$ because of $k$-unanimity and thus, localizedness from $R^2$ to $R^1$ requires that $f(R^2, z) = 0$. We can use a similar argument between $R^1$ and $R^3$ to derive that $f(R^3, y) = 0$. Note for this that $f(R^3, z) = 1$ because of $k$-unanimity and that we can derive $R^3$ from $R^2$ by letting the voters $i \in I$ sequentially swap $x$ and $z$. This process does not leave the Condorcet domain as either $x$ or $z$ are the Condorcet winner in every intermediate profile. This is true because $n$ is odd and $|I| > \frac{n}{2}$ implies that $g_{R^3}(x, w) > 0$ and $g_{R^3}(z, w) > 0$ for all alternatives $w \in A \setminus \{x, y, z\}$. Moreover, no swap involves $y$ and thus, we derive that $f(R^4, y) = f(R^1, y) = 0$ because of localizedness. Hence, it follows that $f(R^1, x) = 1$ because $f(R^1, w) = 0$ for all other alternatives $w \in A \setminus \{x\}$. Finally, we derive the profile $R^1$ from $R^1$ by swapping $y$ and $z$ in the preference relations of the voters $i \in I$. Because these swaps do not affect $x$, it stays the Condorcet winner and localizedness implies that $f(R^1, x) = f(R^1, x) = 1$. Finally, observe that $R^4 = S^{x \downarrow y}$, which means that Lemma 6 applies and shows that $f(R, x) = 1$ for all alternatives $x, y \in A$ and preference profiles $R \in S^{x \downarrow y}$.

As next step, consider an arbitrary alternative $x \in A$ and a profile $R \in \mathcal{D}$ such that $|N(R)| = n - k - 1$. It remains to show that $f(R, x) = 1$ and we consider for this an auxiliary profile $R'$ which satisfies that $R'_i = R_i$ for all voters $i \in N \setminus N^R$, and all voters $i \in N \setminus N^R$ report an alternative $y \in A \setminus \{x\}$ as their favorite option and $x$ as their least preferred option. It follows from the previous observations that $f(R', x) = 1$ because $R' \in S^{x \downarrow y}$. Starting at $R'$, we let each voter $i \in N \setminus N^R$ use swaps to transform his preference relation into $R_i$. In particular, we never need to weaken $x$ during these swaps as all voters in $N \setminus N^R$ report it as their least preferred alternative in $R_i$. Hence, it follows from non-perversity and localizedness that $f(R, x) = f(R', x) = 1$. Also, it should be mentioned that all intermediate profiles are in the Condorcet domain because $x$ is always top-ranked by all voters in $N^R$. Finally, since the alternative $x$ and the profile $R$ are chosen arbitrarily, it follows that $f$ satisfies $k+1$-unanimity. Hence, the induction step is proven and thus, we derive that $f$ is $\frac{n-1}{2}$-unanimous.

Note that the inductive argument in this proof breaks down for $k = \frac{n-1}{2}$ because the profile $R^i$ is not in the Condorcet domain anymore. However, for larger domains, such as the full domain, one can repeat this inductive argument until $|I| = 1$ to derive that there is no $1$-unanimous and locally strategyproof SDS on these domains.

Finally, we use the insights of this section to prove Lemma 2.

**Lemma 2.** Assume $n \geq 3$ is odd and $m \geq 3$. A non-imposing and locally strategyproof SDS $f$ on a connected super Condorcet domain is locally $0$-randomly dictatorial iff it is Condorcet-consistent.

**Proof.** Assume $n \geq 3$ is odd and consider an arbitrary connected super Condorcet domain $\mathcal{D}$. Moreover, let $f$ denote a locally strategyproof and non-imposing SDS on $\mathcal{D}$. If $f$ is Condorcet-consistent, it is easy to see that it locally $0$-randomly dictatorial because the best alternative of every voter can have probability 0. In more detail, it holds for all voters $i \in N$ and alternatives $x, y \in A$ that $f(R, y) = 0$ for all preference profiles $R \in S^{x \downarrow y}$. Consequently, $f$ cannot be represented as a mixture of a random dictatorship and another locally strategyproof SDS because this representation implies that there is a voter whose best alternative always gets positive probability. This shows that every locally strategyproof and Condorcet-consistent SDS is locally $0$-randomly dictatorial.

For the inverse direction, suppose that $f$ is locally strategyproof, locally $0$-randomly dictatorial, and non-imposing. Moreover, suppose for contradiction that $f$ violates Condorcet-consistency, i.e., there is a profile $R \in \mathcal{D}_C$ with Condorcet winner $c$ such that $f(R, c) \neq 1$. This means that there is another alternative $d \in A \setminus \{c\}$ with $f(R, d) > 0$. As next step, we let all voters $i \in N$ with $cR_id$ swap up until it is their best alternative to derive the profile $R'$. Since we only reinforce the Condorcet winner, we do not leave the Condorcet domain and $f(R', d) > 0$ follows from localizedness because no swap involves $d$. Consequently, $f(R', c) < 1$ even though $c$ is top-ranked by more than half of the voters in $R'$. This is true as more than half of the voters prefer $d$ to $R$ because it is the Condorcet winner. However, this contradicts our previous insights: $f$ satisfies all requirements of Lemma 10 and Lemma 11, and thus it has to choose an alternative with probability 1 if it is top-ranked by more than half of the voters. Hence, we have derived a contradiction, which implies that the assumption that $f$ fails Condorcet-consistency is wrong. □
Proofs of Theorems 1 and 2

We are almost ready to show Theorems 1 and 2. The last auxiliary lemma required for proving these results is Lemma 1.

**Lemma 1.** Assume $n \geq 3$ is odd and $m \geq 3$. An SDS $f$ on a connected super Condorcet domain is locally strategyproof and non-imposing if and only if it is a random dictatorship or there are $\gamma \in [0, 1)$, a random dictatorship $d$, and a locally 0-randomly dictatorial, locally strategyproof, and non-imposing SDS $g$ on $D$ such that $f = \gamma d + (1 - \gamma) g$.

**Proof.** Assume $n \geq 3$ is odd and consider a connected super Condorcet domain $D$. We first show the direction from left to right and let $f$ therefore denote a locally strategyproof and non-imposing SDS on $D$. Using the definition of local $\gamma$-random dictatorships, there is a maximal $\gamma \in [0, 1)$ such that $f$ can be represented as $f = \gamma d + (1 - \gamma) g$, where $d$ is a random dictatorship and $g$ is another locally strategyproof SDS on $D$. If $\gamma = 1$, this means that $f$ is a random dictatorship and thus, the lemma holds in this case. On the other hand, if $\gamma < 1$, $g$ is locally 0-randomly dictorial because of the maximality of $\gamma$; otherwise, we could represent $g$ as $g = \gamma' d' + (1 - \gamma') h$, where $\gamma' > 0$, $d'$ is a random dictatorship, and $h$ is another locally strategyproof SDS on $D$. Consequently, $f = \gamma d + (1 - \gamma) (\gamma' d' + (1 - \gamma') h)$ which means that $f$ is $\gamma + (1 - \gamma) \gamma'$-randomly dictorial.

This contradicts the maximality of $\gamma$ and thus, $g$ must be locally 0-randomly dictorial. Finally, observe that $g$ inherits non-imposition form $f$. In more detail, there is for every alternative $x \in A$ a profile $R \in D$ such that $f(R, x) = 1$ and it holds for these profiles also that $g(R, x) = 1$; otherwise, $f(R, x) = \gamma d(R, x) + (1 - \gamma) g(R, x) < 1$ because $g(R, x) < 1$. Hence, if $\gamma < 1$, $f$ can be indeed represent as a mixture of a random dictatorship and a locally strategyproof, locally 0-randomly dictorial, and non-imposing SDS on $D$.

For the other direction, note first that every random dictatorship is locally strategyproof on $D$ because they satisfy this axiom even on the full domain, and these SDSs are non-imposing on $D$ because they choose every alternative with probability 1 if it is unanimously top-ranked. Hence, if an SDS $f$ is random dictatorship on $D$, it satisfies all requirements of this lemma. Therefore, suppose that the SDS $f$ is a mixture of a random dictatorship $d$ and another locally strategyproof, locally 0-randomly dictorial, and non-imposing SDS $g$, i.e., there is $\gamma \in [0, 1)$ such that $f = \gamma d + (1 - \gamma) g$.

First note that $f$ is locally strategyproof on $D$ because mixtures of locally strategyproof SDSs are also locally strategyproof. In more detail, consider two arbitrary preference profiles $R, R' \in D$ for which there are two alternatives $x, y \in A$ and a voter $i \in N$ such that $R'_i = R'^{xy}$. Since $d$ and $g$ are locally strategyproof, it follows that $g(R) \succeq_i g(R')$ and $d(R) \succeq_i d(R')$. This means equivalently that $g(R, U(R_i, x)) \geq g(R', U(R_i, x))$ and $d(R, U(R_i, x)) \geq d(R', U(R_i, x))$ for all $x \in A$. Hence, it is straightforward to verify that $f(R, U(R_i, x)) = \gamma d(R, U(R_i, x)) + (1 - \gamma) g(R, U(R_i, x)) \geq \gamma d(R', U(R_i, x)) + (1 - \gamma) g(R', U(R_i, x)) = f(R', U(R_i, x))$ for all $x \in A$, which shows that $f(R) \succeq_i f(R')$. Moreover, $f$ is non-imposing because both $d$ and $g$ are non-imposing and locally strategyproof. Hence, Lemma 5 implies that $d$ and $g$ choose an alternative $x$ with probability 1 if it is unanimously top-ranked and thus, $f$ also chooses $x$ with probability 1 in such profiles. This shows that every mixture of a random dictatorship and a locally strategyproof, locally 0-randomly dictorial, and non-imposing SDS on a connected super Condorcet domain is again locally strategyproof and non-imposing.

Next, we discuss the proof of Theorem 1, which follows almost immediately from Lemma 1 and Lemma 2.

**Theorem 1.** Assume $n \geq 3$ is odd and $m \geq 3$. An SDS on the Condorcet domain is strategyproof and non-imposing if and only if it is a mixture of a random dictatorship and the Condorcet rule.

**Proof.** Assume that the number of voters is odd and $m \geq 3$. First, note that the theorem is trivial if $n = 1$ because every single voter profile has a Condorcet winner. Therefore, Gibbard’s random dictatorship theorem shows that an SDS is strategyproof and non-imposing if it always chooses the best alternative of the single voter with probability 1. Since this SDS is equivalent to both the Condorcet rule and the dictatorial SDS of this voter, the theorem holds if $n = 1$.

Next, suppose that $n \geq 3$. We first show that every strategyproof and non-imposing SDS on the Condorcet domain is a mixture of a random dictatorship and the Condorcet rule. For this, we characterize the Condorcet rule as the only locally strategyproof, locally 0-randomly dictorial, and non-imposing SDS on the Condorcet domain. Hence, note that the Condorcet rule is non-imposing because an alternative is chosen with probability 1 if it is the Condorcet winner, and it satisfies even full strategyproofness because a voter can only change the Condorcet winner to a less preferred alternative. In more detail, a voter $i \in N$ can only change the Condorcet winner $c$ of a profile $R \in D_C$ to an alternative $c' \neq c$ because $g_R(c) > 0$ and the sign of this majority margin flips if $c'$ is the Condorcet winner after the manipulation. Hence, Lemma 2 shows that an SDS on the Condorcet domain is locally strategyproof, locally 0-randomly dictorial, and non-imposing if it is the Condorcet rule because no other SDS on the Condorcet domain is Condorcet-consistent. Next, we can use Lemma 1 to derive that mixtures of random dictatorships and the Condorcet rule are the only locally strategyproof and non-imposing SDSs on the Condorcet domain. Finally, this means that every strategyproof and non-imposing SDS on $D_C$ can be represented as a mixture of a random dictatorship and the Condorcet rule because strategyproofness implies local strategyproofness.

For the inverse direction, observe that mixtures of strategyproof SDSs are strategyproof. Since the Condorcet rule and all random dictatorships are strategyproof on the Condorcet domain, it follows that mixtures of these SDSs are also strategyproof. Moreover, the Condorcet rule and all random dictatorships choose an alternative with probability 1 if it is unanimously top-ranked. Hence, mixtures of these SDSs also choose an alternative with probability 1 if every voter top-ranks it and therefore, these SDSs satisfy non-imposition. Thus, an SDS on the Condorcet domain is strategyproof and non-imposing if it is a mixture of a random dictatorship and the Condorcet rule.
Similar to Theorem 1, we can also derive Theorem 2 from Lemma 1 and Lemma 2. In particular, these lemmas imply Theorem 2 if we show that there is no locally strategyproof and Condorcet-consistent SDS on a strict and connected super Condorcet domain because then Lemma 2 shows that there is also no locally strategyproof, locally 0-randomly dictatorial, and non-imposing SDS on this domain.

**Theorem 2.** Assume $n$ is odd and $m \geq 3$. An SDS on a strict and connected super Condorcet domain is strategyproof and non-imposing iff it is a random dictatorship.

**Proof.** Assume $n$ is odd and $m \geq 3$ and consider an arbitrary connected and strict super Condorcet domain $D$. Note that strictness only requires that $D \not= D_C$ because $n$ is odd. Hence, there is no strict super Condorcet domain if $n = 1$ because every single voter profile has a Condorcet winner. Therefore, we suppose in the sequel that $n \geq 3$.

Next, it is straightforward that random dictatorships satisfy non-imposition and strategyproofness on $D$ since they satisfy these axioms even on the full domain. Hence, we focus on the inverse direction: every strategyproof and non-imposing SDS on $D$ is a random dictatorship. We prove this by showing that there is no locally strategyproof and Condorcet-consistent SDS. Then, Lemma 2 implies that there is no locally strategyproof, locally 0-randomly dictatorial, and non-imposing SDS, and Lemma 1 shows therefore that random dictatorships are the only locally strategyproof and non-imposing SDSs on $D$. Since strategyproofness implies local strategyproofness, this proves the theorem.

Hence, assume for contradiction that there is a locally strategyproof and Condorcet-consistent SDS $f : D \to \Delta(A)$. Since $D$ is a strict super Condorcet domain, there is a profile $R \in D \setminus D_C$. We show in the sequel that there is no feasible outcome for $R$. Note for this that $D$ is also connected and thus, there is for every alternative $x \in A$ an ad-path $\pi$ from $R$ to a profile $R' \in D_C$ such that $x$ is not swapped with any other alternative on $\pi$. Consider this profile $R'$ for an alternative $z \in A$. Since $R' \in D_C$ and $f$ is Condorcet-consistent, it follows for the Condorcet winner $c$ in $R'$ that $f(R', c) = 1$. Moreover, $c \neq z$ because $z$ is not the Condorcet winner in $R$ and it is not swapped with any alternative in the construction of $R'$. Hence, a repeated application of localizedness implies that $f(R, z) = f(R', z) = 0$. Finally, since $z$ was chosen arbitrarily, we derive that $f(R, x) = 0$ for all alternatives $x \in A$. This contradicts the definition of an SDS and thus, there is no locally strategyproof and Condorcet-consistent SDS on $D$. $\square$

**Appendix B: Proof of Theorem 3**

The last result that we need to prove is Theorem 3. For showing this result, we proceed similar to to the proofs of Theorem 1 and Theorem 2: first, we investigate the consequences of group-strategyproofness on super Condorcet domains and next what it means to be non-dictatorial on these domains. Finally, we use these insights to characterize the set of group-strategyproof and non-imposing SDSs for the Condorcet domain and every strict super Condorcet domain.

**Group-strategyproofness on Super Condorcet Domains**

Analogous to the proofs of Theorem 1 and Theorem 2, we discuss first the consequences of group-strategyproofness on super Condorcet domains. As first point, note that group-strategyproofness implies local strategyproofness and therefore also non-perversity and localizedness. Furthermore, we show next that every group-strategyproof and non-imposing SDS on a super Condorcet domain is Pareto-optimal.

**Lemma 12.** Assume $m \geq 3$. Every non-imposing and group-strategyproof SDS on a super Condorcet domain is Pareto-optimal.

**Proof.** Consider an arbitrary super Condorcet domain $D$ and let $f$ denote a group-strategyproof and non-imposing SDS on $D$. Moreover, suppose that $f$ violates Pareto-optimality, i.e., that there are a profile $R \in D$ and two alternatives $x, y \in A$ such that $x R_y y$ for all voters $i \in N$ but $f(R, y) > 0$. First, note that there is a profile $R' \in D$ such that $f(R, x) = 1$ because $f$ is non-imposing. As next step, consider the profile $R'' \in D_C \subseteq D$ derived from $R'$ by letting all voters $R$ make $x$ to their best alternative. Group-strategyproofness from $R''$ to $R'$ implies that $f(R'', x) = 1$ because otherwise, the voters could manipulate by reverting this step. However, this means that the set of all voters $N$ can group-manipulate by deviating from $R$ to $R'$: it holds for all voters $i \in N$ that $f(R, U(R_i, x)) < 1 = f(R'', U(R_i, x))$ because $f(R, y) > 0$ and $y \not\in U(R_i, x)$. Hence, no voter prefers $f(R)$ to $f(R''), i.e., f(R) \not\succ_i f(R'')$ for all $i \in N$, which means that this step is indeed a successful group-manipulation. This contradicts the assumption that $f$ is group-strategyproof, which implies that $f$ satisfies Pareto-optimality. $\square$

Note that Lemma 12 uses group-strategyproofness to generalize Lemma 5: firstly, the new result holds also for even $n$ and secondly, it is not restricted to connected super Condorcet domains. As next step, we discuss a generalization of Lemma 6 to our new setting. Note that the proof of the new lemma can be easily derived from the proof of Lemma 6 and therefore, we only explain the differences. In particular, note for the subsequent lemma that $S^{w_{1y}} \subseteq D_C$ if $|I| \not= 1/2$.

**Lemma 13.** Assume $n \geq 3$ and $m \geq 3$ and let $f$ denote a group-strategyproof and non-imposing SDS on a super Condorcet domain $D$. For all sets of voters $I \subseteq N$ with $|I| \not\in \{0, \frac{1}{2}, n\}$, alternatives $w, x, y, z \in A$ with $w \not= x$ and $y \not= z$, and preference profiles $R \in S^{w_{1y}, z}$, it holds that $f(R, w) = f(R', y), f(R, x) = f(R', z)$, and $f(R, w) + f(R, x) = 1$.

**Proof.** We prove this result by pointing out how to change the proof of Lemma 6. Hence, let $D$ denote a super Condorcet domain and let $f$ denote a group-strategyproof and non-imposing SDS on $D$. Furthermore, consider a set of voters $I \subseteq N$ with $|I| \not\in \{0, \frac{1}{2}, n\}$. Just as for Lemma 6, we prove this result in three steps. First, we show that for all distinct alternatives $x, y \in A$ that $f(R, x) + f(R, y) = 1$ and $f(R) = f(R')$ for all profiles $R, R' \in S^{w_{1y}}$. We can use exactly the same argument as in Lemma 6 to prove this.
claim because the applied construction does not change the Condorcet winner.

As second step, we show that \( f(R, x) = f(R', z) \) for all distinct alternatives \( x, y, z \in A \) and preference profiles \( R \in S^{21}y \) and \( R' \in S^{21}y \). In particular, we prove this step by considering the same profiles \( R^3 \) and \( R^4 \) as in the proof of Lemma 6. First, note that \( R^3 \in S^{21}y \) and thus, the insights of the first step imply that \( f(R^3, w) = 0 \) for all \( w \in A \setminus \{x, y\} \). Next, let all voters in \( I \) simultaneously swap \( x \) and \( z \) to derive \( R^4 \). The insights of the first step apply again and show that \( f(R^4, w) = 0 \) for all \( w \in A \setminus \{y, z\} \). Also, since we let all voters in \( I \) change their preference relation simultaneously, we do not leave the Condorcet domain since \( I \neq \frac{n}{2} \). Finally, note that group-strategyproofness implies that \( f(R^3, y) = f(R^4, y) \): if \( f(R^3, y) > f(R^4, y) \), then the voters \( i \in I \) can group-manipulate by deviating from \( R^3 \) to \( R^3 \) and if \( f(R^3, y) < f(R^4, y) \), these voters can group-manipulate by deviating from \( R^4 \) to \( R^3 \). Combining all observations shows that \( f(R^4, z) = 1 - f(R^4, y) = 1 - f(R^3, y) = f(R^3, x) \).

As second step, we use Lemma 13 to prove Lemma 3. Note that \( f(R^4, y) = f(R^3, y) \) for all \( x, y, z \in A \). Thus, \( f(R^4, y) \) is non-dictatorial, which means that \( f(R^4, y) = 1 \). Furthermore, we choose an arbitrary alternative \( y \in A \setminus \{x\} \) and consider a profile \( R' \in D' \) such that \( R'_i = R_i \) and \( r(R'_i, x) = m \) and \( r(R'_i, y) = 1 \) for all voters \( j \in N \setminus \{i\} \). Since \( R' \in S^{1}y^{i} \), it follows from our previous observations that \( f(R', x) = 1 \) even though all voters in \( N \setminus \{i\} \) report \( x \) as their worst alternative. This implies that \( f(R, x) = 1 \) because otherwise, the voters in \( N \setminus \{i\} \) can group-manipulate by deviating from \( R' \) to \( R \). This is true since \( x \) is the worst alternative of these voters in \( R' \). Because the profile \( R \in D \) is chosen arbitrarily, \( f \) always chooses the best alternative of voter \( i \) with probability 1 and thus, it is the dictatorial SDS of this voter. This contradicts the assumption that \( f \) is non-dictatorial, which implies that \( f \) is 1-unanimous.

Non-dictatorial SDSs on Super Condorcet Domains

Our next goal is to investigate the consequences of what it means to be non-dictatorial and group-strategyproof on a super Condorcet domain. In particular, we show Lemma 3 in this section. As first step, we prove that every group-strategyproof, non-dictatorial, and non-imposing SDS on a super Condorcet domain is 1-unanimous.

**Lemma 14.** Assume \( n \geq 3 \) and \( m \geq 3 \). Every group-strategyproof, non-dictatorial, and non-imposing SDS on a super Condorcet domain is 1-unanimous.

**Proof.** Assume that \( f \) is a group-strategyproof, non-dictatorial, and non-imposing SDS on a super Condorcet domain \( D \). Furthermore, suppose for contradiction that \( f \) is not 1-unanimous. Since Lemma 12 shows that \( f \) satisfies Pareto-optimality, this means that there is a profile \( R \in D \), a voter \( i \in N \), and two alternatives \( a, b \in A \) such that voter \( i \)'s reports \( a \) as his favorite alternative in \( R^* \), all other voters report \( b \) as their favorite alternative, but \( f(R^*, a) > 0 \). In the sequel, we show that this assumption means that \( f \) is the dictatorial SDS for voter \( i \). This contradicts our assumptions and therefore \( f \) has to satisfy 1-unanimity.

As first point, we show that \( f(R, x) = 1 \) for all alternatives \( x, y \in A \) and preference profiles \( R \in S^{1}y^{i} \). Consider for this the profiles \( R^1 \) and \( R^2 \) depicted in the sequel and note that \( R^1, R^2 \in D' \subset D \) since all voters in \( N \setminus \{i\} \) agree on a best alternative in these profiles.

\[
R^1: \quad i: x, y, z, * \quad N \setminus \{i\}: z, y, x, *
\]

\[
R^2: \quad i: y, x, z, * \quad N \setminus \{i\}: y, z, x, *
\]

We show in the sequel that \( f(R^1, x) = 1 \) by contradiction, i.e., we suppose that \( f(R^1, x) < 1 \). First, observe that \( f(R^1, x) > 0 \) because Lemma 13 shows that \( f(R^1, x) = f(R^*, a) > 0 \) since \( R^1 \in S^{1}y^{i} \) and \( R^* \in S^{1}y^{i} \). Moreover, this lemma also entails that \( f(R^1, w) = 0 \) for all alternatives \( w \in A \setminus \{x, z\} \). Thus, \( f(R^1, x) < 1 \) implies that \( f(R^1, z) > 0 \). In particular, this means that \( f(R, U(R_j, y)) < 1 \) for all voters \( j \in N \). Hence, the set of all voters \( N \) can group-manipulate by deviating to \( R^2 \) because Lemma 12 implies that \( f(R^2, y) = 1 \). Consequently, it indeed holds that \( f(R^1, x) = 1 \) because otherwise, \( f \) is group-manipulable. Finally, Lemma 13 implies now that \( f(R, x) = 1 \) for all alternatives \( x, y \in A \) and preference profiles \( R \in S^{1}y^{i} \).

As second step, we show that \( f \) always chooses voter \( i \)'s best alternative with probability 1. Hence, consider an arbitrary preference profile \( R \in D \) and let \( x \) denote voter \( i \)'s best alternative in \( R \). Furthermore, we choose an arbitrary alternative \( y \in A \setminus \{x\} \) and consider a profile \( R' \in D' \subset D \) such that \( R'_i = R_i \) and \( r(R'_i, x) = m \) and \( r(R'_i, y) = 1 \) for all voters \( j \in N \setminus \{i\} \). Since \( R' \in S^{1}y^{i} \), it follows from our previous observations that \( f(R', x) = 1 \) even though all voters in \( N \setminus \{i\} \) report \( x \) as their worst alternative. This implies that \( f(R, x) = 1 \) because otherwise, the voters in \( N \setminus \{i\} \) can group-manipulate by deviating from \( R' \) to \( R \). This is true since \( x \) is the worst alternative of these voters in \( R' \). Because the profile \( R \in D \) is chosen arbitrarily, \( f \) always chooses the best alternative of voter \( i \) with probability 1 and thus, it is the dictatorial SDS of this voter. This contradicts the assumption that \( f \) is non-dictatorial, which implies that \( f \) is 1-unanimous.

Next, we use Lemma 14 to prove Lemma 3. Note that we use Lemma 11 in the proof because it is straightforward to adapt this lemma to the case for even \( n \) if we use group-strategyproofness instead of strategyproofness.

**Lemma 3.** Assume \( n \geq 3 \) and \( m \geq 3 \). A group-strategyproof and non-imposing SDS on a super Condorcet domain is non-dictatorial if it is Condorcet-consistent.

**Proof.** Consider an arbitrary super Condorcet domain \( D \) for \( n \geq 3 \) and suppose that \( f \) is a group-strategyproof and non-imposing SDS on \( D \). First, note that if \( f \) is also Condorcet-consistent, it is obviously non-dictatorial. The reason for this is that \( f(R, x) = 1 \) holds for every preference profile \( R \in D' \) in which only \( n - 1 \) voters prefer \( x \) the most. Consequently, no voter \( i \in N \) is a dictator for \( f \) as there is a profile in which voter \( i \)'s best alternative gets probability 0.

For the inverse direction, suppose that \( f \) is group-strategyproof, non-imposing, and non-dictatorial. First, note that Lemma 14 shows that \( f \) is 1-unanimous. Next, it is straightforward to adapt the proof of Lemma 11 to derive that \( f(R, x) = 1 \) for all preference profiles \( R \in D' \) and alternatives \( x \in A \) such that more than half of the voters report \( x \) as their favorite alternative in \( R \). Based on this insight, it is easy to show by contradiction that \( f \) is Condorcet-consistent: suppose that there is a profile \( R \in D' \) with Condorcet winner \( c \) but \( f(R, c) \neq 1 \). Hence, there is an alternative \( d \in A \setminus \{c\} \) with \( f(R, d) > 0 \). This means in particular that
\[ f(R, U(R_c, c)) < 1 \] for all voters \( i \in N \) with \( cR_id \). As a consequence, these voters can group-manipulate by making \( c \) into their best alternative. This step results in a profile \( R' \) in which more than half of the voters top-rank \( c \) and hence, our previous observation implies that \( f(R', c) = 1 \). Thus, \( f(R, U(R_c, c)) < f(R', U(R_c, c)) \) for all \( i \in N \) with \( cR_id \), which shows that \( f \) is group-manipulable. This means that the initial assumption is wrong and every group-strategyproof, non-imposing, and non-dictatorial SDS on a super Condorcet domain is Condorcet-consistent.

**Proof of Theorem 3**

Finally, we use our previous insights to prove Theorem 3.

**Theorem 3.** Assume \( m \geq 3 \). An SDS on the Condorcet domain is group-strategyproof and non-imposing iff it is a dictatorial SDS or the Condorcet rule. An SDS on a strict super Condorcet domain is group-strategyproof and non-imposing iff it is a dictatorial SDS.

**Proof.** The theorem consists of two independent characterizations of group-strategyproof and non-imposing SDSs for the Condorcet domain and for strict super Condorcet domains, respectively. We prove these results separately.

**Claim 1:** An SDS on the Condorcet domain is group-strategyproof and non-imposing iff it is a dictatorial SDS or the Condorcet rule.

We first show the direction from left to right and thus, consider a group-strategyproof and non-imposing SDS \( f \) on the Condorcet domain. We need to show that \( f \) is the Condorcet rule or a dictatorial SDS. First, note that if \( n \geq 3 \), this follows immediately from Lemma 3 because the Condorcet rule is the only Condorcet-consistent SDS on the Condorcet domain. Hence, this lemma shows that \( f \) is either dictatorial if it is not Condorcet-consistent, or the Condorcet rule. Moreover, if \( n \leq 2 \), our claim follows from the fact that \( f \) is Pareto-optimal (see Lemma 12). In more detail, if \( n = 1 \), the best alternative \( x \) of the single voter always Pareto-dominates every other alternative and thus, \( f(R, x) = 1 \). If \( n = 2 \), a profile \( R \) is only in the Condorcet domain if the two voters agree on the best alternative \( x \). Hence, alternative \( x \) Pareto-dominates every other alternative, which means again that \( f(R, x) = 1 \). In both cases, the single Pareto-optimal alternatives is also the Condorcet winner, which means that \( f \) is the Condorcet rule (which is equivalent to every dictatorship in this case).

For the other direction, we need to show that all dictatorial SDSs and the Condorcet rule are group-strategyproof and non-imposing. First, note that all these SDSs are by definition non-imposing on the Condorcet domain because they are even deterministic. Next, every dictatorial SDS \( d_i \) is group-strategyproof because only a single voter \( i \) can change the outcome. However, this voter has no incentive to manipulate as his best alternative is already chosen with probability 1, which means that any other outcome makes this voter worse. Thus, every dictatorial SDS is group-strategyproof.

Finally, we show that the Condorcet rule is group-strategyproof. Assume for contradiction that this is not the case, i.e., there are preference profiles \( R, R' \in D_C \) and a set of voters \( I \subseteq N \) such that \( R_j = R_j' \) for all \( j \in N \setminus I \) and \( COND(R) \not\subseteq_i COND(R') \) for all voters \( i \in I \). If the Condorcet winner in \( R \) and \( R' \) is the same, this is clearly no manipulation since \( COND(R) = COND(R') \). Hence, assume that the Condorcet winner \( c \) of \( R \) is not the same alternative as the Condorcet winner \( c' \) of \( R' \). This means that \( g_{R}(c, c') > 0 \) and \( g_{R'}(c', c) > 0 \), which is only possible if there is a manipulator \( i \in I \) with \( cR_i c' \). However, this voter prefers \( COND(R) \) to \( COND(R') \), contradicting that \( COND(R) \not\subseteq_i COND(R') \) for all \( j \in I \). Thus, the Condorcet rule is group-strategyproof on the Condorcet domain.

**Claim 2:** An SDS on a strict super Condorcet domain is group-strategyproof and non-imposing iff it is dictatorial.

First, note that there is not strict super Condorcet domain if \( n \leq 2 \). In more detail, if \( n = 1 \), every profile has a Condorcet winner, which means that there is no super Condorcet domain with \( D \neq D_C \). Furthermore, if \( n = 2 \), the best alternative \( x \) of the first voter satisfies in every preference profile \( R \in R^2 \) that \( g_R(x, y) \geq 0 \) for all \( y \in A \setminus \{x\} \) and thus, there cannot be a profile in which every alternative strictly loses the majority comparison to another alternative. Hence, we suppose in the sequel that \( n \geq 3 \) and consider an arbitrary strict super Condorcet domain \( D \). Observe first that the same arguments as for Claim 1 show that every dictatorship is group-strategyproof and non-imposing on \( D \). Thus, we focus on proving that every group-strategyproof and non-imposing SDS on \( D \) is dictatorial. For this, we show that there is no group-strategyproof, non-imposing, and Condorcet-consistent SDS on \( D \) because Lemma 3 shows then that only dictatorial SDSs can be group-strategyproof and non-imposing on \( D \).

Therefore, suppose for contradiction that \( f \) is a group-strategyproof and Condorcet-consistent SDS on \( D \). Subsequently, we use the fact that \( D \) is a strict super Condorcet domain: there is profile \( R \in D \) such that every alternative \( x \in A \) strictly loses the pairwise comparison against another alternative \( y \in A \setminus \{x\} \), i.e., such that \( g_R(x, y) > 0 \). Next, consider an alternative \( x \in A \) such that \( f(R, x) > 0 \); such an alternative must exist because \( \sum_{x \in A} f(R, x) = 1 \).

We show that the voters \( I = \{i \in N : yRx\} \) can group-manipulate by reporting \( y \) as their favorite alternative. In particular, this manipulation leads to a profile \( R' \in D_C \) since \( g_R(y, x) > 0 \) implies that \( |I| > \frac{n}{2} \). In more detail, \( y \) is the Condorcet winner in \( R' \) and thus, Condorcet-consistency requires that \( f(R, y) = 1 \). However, it holds that \( f(R, U(R_i, y)) < 1 = f(R', U(R_i, y)) \) for all \( i \in I \) because \( x \notin U(R_i, y) \) and \( f(R, x) > 0 \). Hence, this step is indeed a group-manipulation for the voters \( i \in I \), contradicting that \( f \) is group-strategyproof. Therefore, there is no group-strategyproof and Condorcet-consistent SDS on \( D \) and consequently, Lemma 3 entails that only dictatorial SDSs can satisfy group-strategyproofness and non-imposition on strict super Condorcet domains.

\( \square \)