Optimal Budget Aggregation with Star-Shaped Preferences

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Abstract. We study the problem of aggregating distributions, such as budget proposals, into a collective distribution. An ideal aggregation mechanism would be Pareto efficient, strategyproof, and fair. Most previous work assumes that agents evaluate budgets according to the ℓ_1 distance to their ideal budget. We investigate and compare different models from the larger class of *star-shaped utility functions*—a multi-dimensional generalization of single-peaked preferences. For the case of two alternatives, we extend existing results by proving that under very general assumptions, the *uniform phantom mechanism* is the only strategyproof mechanism that satisfies proportionality—a minimal notion of fairness introduced by Freeman et al. [21]. Moving to the case of more than two alternatives, we establish sweeping impossibilities for ℓ_1 and ℓ_∞ disutilities: no mechanism satisfies efficiency, strategyproofness, and proportionality. We then propose a new kind of star-shaped utilities based on evaluating budgets by the *ratios* of shares between a given budget and an ideal budget. For these utilities, efficiency, strategyproofness, and fairness become compatible. In particular, we prove that the mechanism that maximizes the Nash product of individual utilities is characterized by group-strategyproofness and a core-based fairness condition.

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1. Introduction Social choice theory is concerned with the aggregation of individual preferences into a collective outcome [e.g., 1, 2]. An important special case arises when the potential collective outcomes are *distributions* over a fixed set of alternatives. These distributions may represent how a budget should be divided among public projects in a city or among departments in an organization. Alternatively, they may reflect how time or space ought to be allotted between different types of activities at a social event. This scenario is sometimes referred to as *budget aggregation* or *portioning* and falls under the framework of *participatory budgeting*, which has received increasing interest in recent years [6, 18].

In order to reason about the agents' satisfaction with the collective outcome, one needs to make some assumptions about their preferences. Importantly, in our setting, the realized outcome is a distribution. Therefore, restricting attention to rankings over alternatives is insufficient, as an agent's most preferred outcome is typically a non-degenerate distribution over the alternatives. This is particularly evident in participatory budgeting problems, where even if an agent has a favorite project, she normally also likes other projects and does not want them to be left completely unfunded. This is in contrast to probabilistic social choice [see, e.g., 14], where the final outcome is a single alternative picked at random from the distribution, so typically the agents' most-preferred distributions are degenerate.

In this paper, consistent with previous research in this domain, we mainly consider utility models where agents' preferences are completely determined by their favorite distribution: their "peak". This keeps the amount of required information from each agent at a manageable level. We assume that each agent's utility decreases as the actual distribution moves away from her peak (see the formal definition in Section 3). Such utility functions are called *star-shaped* [12].

For two alternatives, i.e., outcomes on the unit interval $[0, 1]$, star-shaped utilities are equivalent to *single-peaked preferences*. According to a famous characterization by Moulin [32], there is a rather restrictive class of mechanisms that are strategyproof for all single-peaked preferences, the so-called *generalized median* rules. Moulin's characterization leaves open the possibility that, for restricted subdomains of single-peaked preferences, other mechanisms than generalized median rules are strategyproof. In Section 4, we obtain characterizations of continuous mechanisms which hold not only for single-peaked, but also for any subdomain of single-peaked preferences. Our characterizations refine results by Freeman et al. [21] and Aziz et al. [5].

For more than two alternatives, most of the previous work on budget aggregation assumes that preferences are given via the ℓ_1 norm [e.g., 16, 21, 22, 25, 30]. According to ℓ_1 preferences, agent i's *dis*utility for a distribution $\mathbf{q} = (q_1, \ldots, q_m)$ over m alternatives is given by the ℓ_1 distance $\|\mathbf{p}_i - \mathbf{q}\|_1 = \sum_{j=1}^{m} |p_{i,j} - q_j|$, where $\mathbf{p}_i = (p_{i,1}, \ldots, p_{i,m})$ is the agent's peak. We assume without loss of generality that the sum of all components in a distribution is 1. Under this utility model, Lindner et al. [30] and Goel et al. [25] showed that a mechanism that maximizes utilitarian welfare (i.e., minimizes the sum of agents' disutilities) is both strategyproof and efficient; however, this mechanism has a tendency to overweight majority preferences. Freeman et al. [21] proposed a mechanism, called the *independent markets mechanism*, which satisfies strategyproofness along with a weak fairness notion dubbed *proportionality*. Proportionality requires that the collective distribution is given by the uniform distribution over the agents' peaks whenever all peaks are degenerate. The independent markets mechanism violates efficiency, and Freeman et al. raised the question whether there are mechanisms that satisfy all three properties simultaneously. In Section 6, we settle this question by proving that no such mechanism exists under ℓ_1 as well as under ℓ_∞ preferences.

Using ℓ_1 distances to define preferences over distributions has some shortcomings when aiming for a suitable representation of alternatives in the collective distribution. For instance, if the agents are deciding the amount of time that should be allotted to three countries at an international conference, an agent with an ideal distribution of $(10\%, 40\%, 50\%)$ would find the outcome $(0\%, 45\%, 55\%)$ to be quite desirable according to the ℓ_1 distance, despite the fact that this outcome leaves the first country completely unrepresented. Moreover, any distance-based metric that aggregates coordinate-wise differences, such as ℓ_2 or ℓ_∞ (or ℓ_p for any $p \geq 1$) has similar shortcomings. For example, if a citizen believes that the two larger districts deserve 40% of the city budget each and the two smaller districts 10% each, then for any $p \geq 1$, ℓ_p preferences dictate that she is indifferent between the distributions $(50\%, 30\%, 10\%, 10\%)$ and $(40\%, 40\%, 20\%, 0\%)$, since for both distributions, the multiset of coordinate-wise differences is $\{0\%, 0\%, 10\%, 10\%\}$. But intuitively, the latter distribution is worse, as it leaves the last district without any funds. As a consequence, a different type of utility function is necessary to capture the representation of alternatives with respect to the ideal distribution.

We introduce a new class of utility functions to the budget aggregation setting where each agent has an ideal distribution and the agent's utility for a distribution equals the *smallest quotient*, over all alternatives, that the distribution preserves in comparison to the ideal distribution. Formally, agent i's utility for a distribution $\mathbf{q}$ is given by $\min_j q_j/p_{i,j}$, where the minimum is taken over all alternatives j for which $p_{i,j} > 0$. These utility functions are a special case of Leontief utility functions as commonly studied in economics, especially in consumer theory, with goods corresponding to alternatives [see, e.g., 24, 28, 34]. In our setting, the total amount of goods is fixed, which allows us to interpret the relative distribution of Leontief weights as an ideal distribution. Agents then want all alternatives to receive as large a fraction as possible of their ideal amounts. We will refer to these utility functions as Leontief utility functions in the following. Leontief utilities are arguably more suitable than ℓ_1 preferences in applications where the representation of the alternatives is crucial—indeed, for both examples in the previous paragraph, distributions that allocate none of the budget to some alternative are least preferred among all distributions according to Leontief utilities. Not surprisingly, mechanisms that provide desirable properties such as strategyproofness with respect to ℓ_1 preferences may fail to do so with Leontief utilities.[1] Therefore, one needs to find different mechanisms when dealing with Leontief utilities. In Section 7, we show that maximizing Nash welfare, i.e., the product of agents' utilities, results in a mechanism with several desirable properties. In fact, the impossibility for ℓ_1 preferences established in Section 6 can be turned into a complete characterization for Leontief utilities: only the Nash product rule satisfies group-strategyproofness and a natural core-based fairness notion, which strengthens both efficiency and proportionality. Thus, in contrast to ℓ_1 preferences, Leontief utilities allow for the efficient, strategyproof, and fair aggregation of budgets via a unique attractive mechanism.

2. Related work

Various streams of research have investigated the strategyproof aggregation of preferences when the space of alternatives is a subset of multi-dimensional Euclidean space.

Convex preferences. Zhou [37] proved an impossibility theorem for convex spaces of alternatives that is reminiscent of the Gibbard-Satterthwaite impossibility. Here, the space of alternatives is an arbitrary subset of Euclidean space, and the admissible domain contains all preference relations that are convex and continuous. He then showed that every strategyproof mechanism whose image has dimension at least 2 is dictatorial.[2]

The negative consequences of this and related theorems [e.g., 7, 8, 33] strongly hinge on the richness of the domain of preferences. More restricted domains of preferences allow for more positive results [see, e.g., 21].

Linear preferences. Any model in which preferences over the set of all lotteries Δ^m are aggregated to a collective lottery, including literature on probabilistic social choice [e.g., 23] and fair mixing [e.g., 11], can be interpreted in the context of budget aggregation. However, the underlying assumptions of linear preferences (vNM utilities) are hardly applicable to budget aggregation because there is always a degenerate ideal distribution.

Peak-based preferences. To the best of our knowledge, Intriligator [26] was the first to consider a model in which each agent has a unique ideal distribution, and the considered mechanisms only need to aggregate individual distributions into a collective distribution. He proposed three simple axioms for this setting, which characterize the rule that returns the average (i.e., the arithmetic mean) of all individual distributions. Intriligator was not concerned with strategyproofness, and

[1] Concretely, suppose that there are $n = 2$ agents and $m = 3$ alternatives. The agents' ideal distributions are $\mathbf{p}_1 = (0.8, 0.2, 0)$ and $\mathbf{p}_2 = (0.8, 0, 0.2)$, respectively. The independent markets mechanism of Freeman et al. [21] returns the distribution $\mathbf{q} = (0.6, 0.2, 0.2)$. However, if the first agent reports $\mathbf{p}'_1 = (0.82, 0.18, 0)$ instead, the mechanism returns $\mathbf{q}' = (0.62, 0.18, 0.2)$, which the agent prefers to $\mathbf{q}$ under Leontief utilities.

[2] Zhou also showed that this result already holds for the subdomain of quadratic preferences.

therefore did not consider agents' preferences over distributions, but it is fairly obvious that the average rule is highly manipulable for almost any reasonable definition of preferences.

Border and Jordan [12] studied separable star-shaped preferences when the space of alternatives is the m-dimensional Euclidean space $\mathbb{R}^m$. They focused on *quadratic preferences*, a generalization of ℓ_2-based preferences, and show that strategyproof mechanisms can be decomposed into one-dimensional mechanisms. However, their results do not carry over to budget aggregation because the space of alternatives is the full Euclidean space.

ℓ_1-based preferences. Lindner et al. [30] initiated the study of strategyproof mechanisms for the space of all distributions Δ^m when preferences are based on the ℓ_1-norm [see also 25, 29]. They showed that the utilitarian rule (i.e., minimizing the sum of ℓ_1 distances) satisfies strategyproofness and efficiency when breaking ties appropriately.

Freeman et al. [21] expanded the idea of Moulin's generalized median rules for two alternatives to strategyproof *moving phantom mechanisms* for larger numbers of alternatives m. Intuitively, the $n + 1$ phantoms are not "fixed" like in Moulin's characterization but increase continuously from 0 to 1 over time. For any point in time and any alternative j, the mechanism computes the median of $p_{1,j}, \ldots, p_{n,j}$ and the phantom voters. Freeman et al. then showed that there exists a well-defined point in time where the m medians sum up to 1 and thus form a valid distribution. Furthermore, they proved that within this class, maximizing utilitarian welfare is the unique efficient mechanism. A different mechanism in this class, the *independent markets* mechanism, is inefficient but satisfies a fairness notion they called *proportionality*: when all voters have degenerate peaks, the collective distribution is the arithmetic mean of these peaks. Freeman et al. observed an "inherent tradeoff between Pareto optimality and proportionality" for strategyproof mechanisms. We prove this tradeoff formally in Theorem 3, which shows that all three properties are incompatible. Proportionality was generalized by Caragiannis et al. [16], who measured the "disproportionality" of a mechanism as the worst-case ℓ_1 distance between the mechanism outcome and the mean. Similarly, Freeman and Schmidt-Kraepelin [22] measured disproportionality using the ℓ_∞ distance. Both papers present variants of moving phantom mechanisms that guarantee low disproportionality. Elkind et al. [19] defined various axioms for budget aggregation with ℓ_1 disutilities, analyzed the implications between axioms, and determined which axioms are satisfied by common aggregation rules.

Belief aggregation. Belief aggregation is a setting in which several experts have different beliefs, expressed as probability density functions over a set of potential outcomes. The goal is to construct a single aggregated distribution. Technically, the problem is similar to budget aggregation; however, the utility functions are often different. Varloot and Laraki [36] assumed that the outcomes are linearly ordered (for example: outcome j is an earthquake of magnitude j). Then, an expert whose belief is 3 with probability 1 would prefer the outcome 6 with probability 1 to the outcome 9 with probability 1, even though the ℓ_1 distance is 2 in both cases. Varloot and Laraki suggested preferences based on distance between the *cumulative* distribution functions, and characterized aggregation rules satisfying appropriate strategyproofness and proportionality axioms.

Donor Coordination. Brandt et al. [15] studied *donor coordination*, where individual monetary contributions by agents are distributed on projects based on the agents' preferences. Assuming Leontief preferences, they proposed the *equilibrium distribution rule (EDR)*, which maximizes Nash welfare and distributes the contributions of the donors in such a way that no subset of donors has an incentive to redistribute their contributions. EDR can be interpreted as a budget aggregation mechanism, by setting the contribution of each agent to $1/n$ (where n is the number of agents, so the total contribution is 1) and setting the ideal distribution of an agent to the distribution given by the relative proportions of the Leontief weights. This allows the transfer of positive results concerning efficiency and strategyproofness of EDR from donor coordination to budget aggregation (see Section 7). However, other properties considered by Brandt et al., like contribution-monotonicity and being in equilibrium, are irrelevant in budget aggregation, whereas properties like core fair share and proportionality have not been considered in donor coordination.

3. Preliminaries

Let $N = [n]$ be a set of agents and $M = [m]$ be a set of alternatives, where $[k] := \{1, \dots, k\}$ for each positive integer k. We denote by Δ^m the standard simplex with m vertices, that is, the set of vectors $\mathbf{q} = (q_1, \dots, q_m)$ with nonnegative entries q_j such that $\sum_{j \in M} q_j = 1$. Every element $\mathbf{q} \in \Delta^m$ is called a *distribution*; here, q_j denotes the fraction of a public resource (e.g., money or time) allocated to alternative j. The support of distribution $\mathbf{q}$ is given by $\operatorname{supp}(\mathbf{q}) := \{j \in M \colon \mathbf{q}_j > 0\}$. For a set of alternatives $T \subseteq M$, we denote $\mathbf{q}(T) := \sum_{j \in T} q_j$.

Each agent i has a utility function u_i over distributions; we denote by $\mathcal{U}$ the set of all possible utility functions and leave this set unspecified for now.

The *ideal distribution* or *peak* of agent i is denoted by $\mathbf{p}_i := \arg\max_{\mathbf{q} \in \Delta^m} u_i(\mathbf{q})$, which is assumed to be unique. We further assume that "walking" towards an agent's peak strictly increases her utility. Formally, if agent i has peak $\mathbf{p}_i$ and utility function u_i, then, for any distribution $\mathbf{q} \neq \mathbf{p}_i$, the agent prefers to $\mathbf{q}$ any distribution on the line between $\mathbf{q}$ and $\mathbf{p}_i$. That is, for all $\lambda \in (0, 1)$,

$$u_i(\mathbf{p}_i) > u_i(\lambda \mathbf{p}_i + (1 - \lambda)\mathbf{q}) > u_i(\mathbf{q}). \tag{1}$$

This constitutes a generalization of single-peakedness [10] and is referred to as *star-shaped* or star-convex preferences [e.g., 12].[3]

The utility functions considered in this paper belong to a subclass of star-shaped preferences that we refer to as *peak-linear*. A utility function is peak-linear if for any distribution $\mathbf{q}$ and $\lambda \in [0, 1]$,

$$u_i(\lambda \mathbf{p}_i + (1 - \lambda)\mathbf{q}) = \lambda u_i(\mathbf{p}_i) + (1 - \lambda) u_i(\mathbf{q}). \tag{2}$$

Every distribution $\mathbf{q} \in \Delta^m$ lies on a line between $\mathbf{p}$ and some point $\mathbf{q}^B$ on the boundary of Δ^m, that is, there is a unique λ such that $\mathbf{q} = \lambda \mathbf{p}_i + (1 - \lambda)\mathbf{q}^B$. Therefore, peak-linear preferences are completely characterized by an agent's peak $\mathbf{p}_i$ and by how much utility she assigns to these boundary distributions $\mathbf{q}^B$ (in fact, it is sufficient to know how much utility she assigns to boundary distributions with $\operatorname{supp}(\mathbf{p}_i) \nsubseteq \operatorname{supp}(\mathbf{q}^B)$).

Unless explicitly stated otherwise, we further assume that $\mathbf{p}_i$ completely determines u_i, so we can identify a profile $P \in \mathcal{P} := (\Delta^m)^n$ with the matrix $(p_{i,j})_{i \in N, j \in M}$ containing the peaks $\mathbf{p}_1, \dots, \mathbf{p}_n$ as rows.

A *mechanism* $f : \mathcal{P} \to \Delta^m$ aggregates individual distributions into a collective distribution. In the following, we define desirable properties of aggregated distributions and mechanisms.

[3] Braga de Freitas et al. [13] use a weaker definition to star-shaped preferences: for every distribution $\mathbf{q}$, the set of distributions that are weakly-preferred to $\mathbf{q}$ is a star domain. This definition is implied by the definition we use, as for any distribution $\mathbf{q}$, if $\mathbf{q}'$ is preferred to $\mathbf{q}$, then all distributions on the line from $\mathbf{p}$ to $\mathbf{q}'$ are also preferred to $\mathbf{q}$, so the set of distributions weakly-preferred to $\mathbf{q}$ is a star domain with respect to $\mathbf{p}$.

3.1. Properties of distributions Two important properties of distributions are efficiency and fairness.

Definition 1. A distribution $\mathbf{q} \in \Delta^m$ satisfies *(Pareto) efficiency* if there does not exist a distribution $\mathbf{q}' \in \Delta^m$ such that $u_i(\mathbf{q}') \geq u_i(\mathbf{q})$ for all $i \in N$ and $u_i(\mathbf{q}') > u_i(\mathbf{q})$ for at least one $i \in N$.

The following is a weaker efficiency property [see, e.g., 19, 21]:

Definition 2. A distribution $\mathbf{q} \in \Delta^m$ is *range-respecting* for profile P if $\min_{i \in N} \mathbf{p}_{ij} \leq q_j \leq \max_{i \in N} \mathbf{p}_{ij}$ for all $j \in M$.

Our first fairness axiom is inspired by the core in cooperative game theory and was transferred to participatory budgeting by Fain et al. [20] and to fair mixing by Aziz et al. [4] under the name of core fair share. We slightly adapt the notation to account for the fact that, in the end, we still need to choose a probability distribution p (and not just a partial distribution $(|N'|/n)p$). In fact, this leads us back to the original definition of the core due to Aumann [3] and Scarf [35].

Definition 3. A distribution $\mathbf{q}$ satisfies *core fair share (CFS)* if for every group of agents $N' \subseteq N$, there is no distribution $\mathbf{q}'$ such that the following hold for every $\mathbf{q}'' \in \Delta^m$:

$$u_i(((|N'|/n)\mathbf{q}' + (1 - |N'|/n)\mathbf{q}'') \geq u_i(\mathbf{q}) \qquad \text{for all } i \in N', \text{ and}$$
$$u_i(((|N'|/n)\mathbf{q}' + (1 - |N'|/n)\mathbf{q}'') > u_i(\mathbf{q}) \qquad \text{for at least one } i \in N'.$$

A distribution $\mathbf{q}$ satisfies *weak core fair share* if we replace the above two conditions with:

$$u_i(((|N'|/n)\mathbf{q}' + (1 - |N'|/n)\mathbf{q}'') > u_i(\mathbf{q}) \qquad \text{for all } i \in N'.$$

Intuitively, if there is a distribution $\mathbf{q}'$ that satisfies these inequalities (so CFS is violated), then N' can take their share of the decision power ($|N'|/n$), and redistribute it via $\mathbf{q}'$ so that no member of N' loses and at least one member (or, in the case of weak CFS, all members) gains utility compared to $\mathbf{q}$, even if the remaining probability is distributed in the worst possible way $\mathbf{q}''$ (e.g., on an alternative that no agent from N' values).

It is easy to see that efficiency is the special case of CFS where $N' = N$.

Proposition 1. *Core fair share implies efficiency.*

For some utility models, even weak CFS implies efficiency; see Corollary 1. We consider another, weaker fairness axiom that is only informative on specific profiles in the next subsection.

3.2. Properties of aggregation mechanisms

Definition 4. A mechanism f satisfies efficiency (resp., core fair share) if for every profile $P \in \mathcal{P}$, $f(P)$ satisfies efficiency (resp., core fair share).

The next axioms ensure that agents and alternatives are treated independently of their identities.

Definition 5. A mechanism f satisfies *anonymity* if for every profile $P \in \mathcal{P}$ and permutation π of the agents in P, it holds that $f(P) = f(\pi \circ P)$.

Definition 6. A mechanism f satisfies *neutrality* if for every profile $P \in \mathcal{P}$ and permutation π of the alternatives resulting in profile P', it holds that $f(P') = \pi \circ f(P)$.

As agents report a peak in Δ^m, we do not want small perturbations of the peaks arising from uncertainties of the agents about their exact peak or inaccuracies during the aggregation process to have a large influence on the outcome.

Definition 7. A mechanism f satisfies *continuity* if

$$\forall P \in \mathcal{P} : \forall \varepsilon > 0 : \exists \delta > 0 : \forall P' \in \mathcal{P} : \|P - P'\|_1 < \delta \implies \|f(P) - f(P')\|_1 < \varepsilon.$$

For simplicity, we define continuity using the ℓ_1 distance, but note that due to the norm equivalence on finite-dimensional vector spaces, our results are the same for every other norm-based distance.

As $\mathcal{P} = (\Delta^m)^n$ is compact with respect to the ℓ_1 distance (or other equivalent norms), the Heine-Cantor theorem implies that a continuous mechanism f is also *uniformly continuous*, i.e.,

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall P, P' \in \mathcal{P} : \|P - P'\|_1 < \delta \implies \|f(P) - f(P')\|_1 < \varepsilon.$$

This insight will play an important role in the proof of Theorem 5.

Another common goal is to prevent agents from misreporting their peaks on purpose.

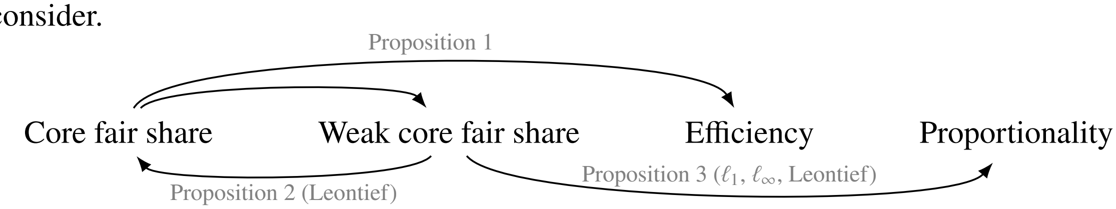
Definition 8. A mechanism f satisfies *group-strategyproofness* if for all $N' \subseteq N$ and all $P, P' \in \mathcal{P}$ with $\mathbf{p}_i = \mathbf{p}'_i$ for $i \notin N'$, either $u_i(f(P)) > u_i(f(P'))$ for at least one $i \in N'$ or $u_i(f(P)) = u_i(f(P'))$ for all $i \in N'$, where u_i refers to the utility function of agent i with peak at $\mathbf{p}_i$. The mechanism f satisfies *strategyproofness* if the above statement holds for $|N'| = 1$.

Finally, we consider another fairness property called *proportionality* by Freeman et al. [21]. It restricts the set of outcomes only on profiles where all agents are "single-minded", thus representing a rather weak form of "traditional" proportionality considered, e.g., in fair division.

Definition 9. A profile $P \in \mathcal{P}$ is called *single-minded* if $p_{i,j} \in \{0, 1\}$ for all $i \in N$ and $j \in M$.

Definition 10. A mechanism f satisfies *proportionality* if for all single-minded profiles $P \in \mathcal{P}$, it holds that $f(P)_j = \sum_{i \in N} p_{i,j}/n$.

The following diagram shows logical relationships between efficiency and the fairness notions we consider.



4. The case of two alternatives For two alternatives $M = \{a, b\}$, the set of outcomes can be identified with the unit interval $[0, 1]$, where the endpoints 0 and 1 correspond to allocating the entire budget to alternatives a and b, respectively. The class of star-shaped utilities coincides with the well-studied class of *single-peaked utilities*. Denote by $\mathcal{U}^{SP}$ the set of all single-peaked utility functions. We denote agent i's peak p_i as a scalar in $[0, 1]$ representing her favorite distribution $[1 - p_i : a,\ p_i : b]$.

In this section, we relax our assumption that $\mathcal{U}$ has to contain *exactly* one utility function per peak in $[0, 1]$; we only demand that $\mathcal{U}$ contains *at least* one utility function per peak and implicitly assume that property from now on for all $\mathcal{U} \subseteq \mathcal{U}^{SP}$. The "at least one" requirement is needed to allow agents to misreport their peak to any other peak in $[0, 1]$. Note that we still require our mechanisms to be *tops-only*, i.e., depend only on the agents' peaks. This generalization is possible because the mechanisms we characterize satisfy strategyproofness and further desirable properties without relying on any knowledge of the agents' utility functions except their peaks.

Under the assumption that $\mathcal{U} = \mathcal{U}^{SP}$, Moulin [32] characterized the set of all strategyproof mechanisms as generalized median rules. This characterization assumes that the rules have to handle *all* profiles in $\mathcal{U}^{SP}$. As a consequence, it no longer holds when restricting $\mathcal{U}$ to a strict subset of $\mathcal{U}^{SP}$. In principle, allowing rules to handle only a subset of $\mathcal{U}^{SP}$ may enable a greater selection of strategyproof rules. This possibility has been studied in some later works [5, 9, 12, 21, 31], which extended Moulin's result for some alternative axioms and to some specific subdomains of $\mathcal{U}^{SP}$.

We substantially generalize these results by proving that Moulin's characterization holds for *any* subdomain $\mathcal{U} \subseteq \mathcal{U}^{SP}$, when assuming continuity of the mechanism in addition.

We start with a characterization for a single agent; this will be used in further characterizations.

LEMMA 1. *For $m = 2$ and $n = 1$, a mechanism f on a domain $\mathcal{U} \subseteq \mathcal{U}^{SP}$ is continuous and strategyproof if and only if there exist $\alpha_0 \leq \alpha_1$ in $[0, 1]$ such that*

$$f(p) = \text{med}(p, \alpha_0, \alpha_1).$$

Proof. The "if" direction is obvious; we focus on the "only if".

Let f be a continuous strategyproof mechanism for $n = 1$ and any m. Let $S := f(\Delta^m)$ be the image of f. By continuity, S is a closed set. By strategyproofness, for all $\mathbf{p} \in \Delta^m$ we have $f(\mathbf{p}) \in \arg\max_{\mathbf{q} \in S} u(\mathbf{q})$. Moreover, S must be convex, since if a segment with endpoints in S is not contained in S, then some internal point of this segment would be a discontinuity point of arg max.

For $n = 1$ and $m = 2$, this boils down to S being a closed interval, $S = [\alpha_0, \alpha_1]$ for some $\alpha_0 \leq \alpha_1$ in $[0, 1]$, and f being a function that maps each p to the point nearest to p in $[\alpha_0, \alpha_1]$. This is equivalent to $f(p) = \text{med}(p, \alpha_0, \alpha_1)$. □

Continuity is essential for the characterization. For example, the following discontinuous mechanism is strategyproof:

$$f(p) := \begin{cases} 0 & p < 0.5; \\ 1 & p \geq 0.5. \end{cases}$$

We now move on to mechanisms for any number of agents. The following lemma shows that, for $m = 2$, any continuous strategyproof mechanism is completely determined by its outcomes on single-minded profiles.

LEMMA 2. *For $m = 2$ and arbitrary domain $\mathcal{U} \subseteq \mathcal{U}^{SP}$, if two continuous and strategyproof mechanisms yield the same distribution for all single-minded profiles, then they yield the same distribution for all profiles.*

Proof. Let f and g be two continuous strategyproof mechanisms. Let P a profile for which $f(P) \neq g(P)$. We prove that there is a single-minded profile P' for which $f(P') \neq g(P')$.

At a high level, the proof works as follows. Step by step, each agent with peak on the left side of $f(P)$ moves her peak closer and closer to 0 and each agent with peak on the right side moves to 1. Continuity and strategyproofness imply that $f(P)$ cannot change in the process. Finally, for all agents with peaks at $f(P)$, move their peaks to the alternative that is not "separated" from the peak by $g(P)$. In the process, $f(P)$ can only move further away from $g(P)$.

In detail, assume that $f(P) < g(P)$; the case $f(P) > g(P)$ can be handled analogously. Denote $q := f(P)$.

Partition the set of agents into four groups: $N = N^{01} \cup N^- \cup N^= \cup N^+$, where $N^{01} = \{i \in N : p_i \in \{0,1\}\}$, $N^- = \{i \in N \setminus N^{01} : p_i < q\}$, $N^= = \{i \in N \setminus N^{01} : p_i = q\}$, and $N^+ = \{i \in N \setminus N^{01} : p_i > q\}$. Our overall goal is to "move" all agents to N^{01} while keeping the chosen distribution different from $g(P)$.

Take any agent $i \in N^-$, and consider the function $F : [0,1] \to [0,1]$ defined by $F(p) := f(p, P_{-i})$. Since f is continuous and strategyproof, so is F, as a mechanism for a single agent. Hence, by Lemma 1, $F(p) = \text{med}(p, \alpha_0, \alpha_1)$ for some constants $\alpha_0 \le \alpha_1$. Note that $F(p_i) = f(P) > p_i$ as $i \in N^-$, so $p_i < \text{med}(p_i, \alpha_0, \alpha_1)$. The median properties imply that $F(p) = F(p_i)$ also for all $p < p_i$. In particular, $F(0) = F(p_i) = f(P)$.

Denote the profile where agent i changed her peak to 0 by $P^{\{i\}}$; then $f(P^{\{i\}}) = F(0) = f(P)$. The same argument applies to all other agents from N^-, so $f(P^{N^-}) = f(P)$, where P^{N^-} denotes the profile resulting from P after all agents in N^- moved their peak to 0. Also, $g(P^{N^-}) = g(P)$, as all agents from N^- were also on the left side of $g(P)$ due to $f(P) < g(P)$, so moving them further left does not change the distribution returned by g.

For an agent $i \in N^+$, define $F(p) := f(p, P^{N^-}_{-i})$. One can show analogously that $F(p) = F(P_i) = f(P^{N^-}) = f(P)$ for all $p \ge p_i$, so the outcome remains $f(P)$ when i moves her peak to 1. Therefore, $f(P^{N^- \cup N^+}) = f(P)$, where $P^{N^- \cup N^+}$ denotes the profile resulting from P after all agents in N^- moved their peak to 0 and all agents in N^+ moved their peak to 1. Also, $g(P^{N^- \cup N^+}) \ge g(P^{N^-})$ as moving peaks to the right can only increase the median returned by g. Therefore, $f(P^{N^- \cup N^+}) < g(P^{N^- \cup N^+})$ still holds.

We now consider an agent $i \in N^=$, for whom $p_i = f(P) < g(P) \le g(P^{N^- \cup N^+})$. Define $F'(p) := f(p, P^{N^- \cup N^+}_{-i})$. As it is continuous and strategyproof, Lemma 1 implies that $F'(p) = \text{med}(p, \alpha_0, \alpha_1)$ for some constants $\alpha_0 \le \alpha_1$. As the median is a weakly monotone function of its arguments, $F'(p) \le F'(p_i)$ for $p \le p_i$. Thus, with $P^{N^- \cup N^+ \cup \{i\}}$ denoting the profile where agent i moved her peak to 0, $f(P^{N^- \cup N^+ \cup \{i\}}) \le f(P^{N^- \cup N^+}) < g(P^{N^- \cup N^+}) = g(P^{N^- \cup N^+ \cup \{i\}})$. If $f(P^{N^- \cup N^+ \cup \{i\}}) = f(P^{N^- \cup N^+})$, repeat the procedure with the next agent from $N^=$. If $f(P^{N^- \cup N^+ \cup \{i\}}) < f(P^{N^- \cup N^+})$, all remaining agents from $N^=$ now have their peak on the right side of $f(P^{N^- \cup N^+ \cup \{i\}})$ and can move their peak to 1 without changing the chosen distribution, as in the case $i \in N^+$. Again, the outcome from g can only move to the right or stay fixed.

Let $P^{N^- \cup N^+ \cup N^=}$ denote the profile after all agents in $N^- \cup N^+ \cup N^=$ have moved their peaks. This profile is single-minded, as all agents have their peaks at 0 or 1, and $f(P^{N^- \cup N^+ \cup N^=}) < g(P^{N^- \cup N^+ \cup N^=})$, as required. □

Remark 1. Border and Jordan [12] considered a property called *uncompromisingness*, which states that the outcome cannot change when agents from N^- and N^+ move their peaks to 0 and 1, respectively (i.e., there is no compromise with agents who express extreme preferences). They showed that uncompromisingness implies continuity. By contrast, we assume continuity and obtain uncompromisingness. As a result, all our characterizations hold if we replace continuity with uncompromisingness.

Using Lemma 2, we can now prove several characterizations.

4.1. Characterizing generalized median rules The following theorem generalizes Moulin's proof from $\mathcal{U}^{SP}$ to any domain $\mathcal{U} \subseteq \mathcal{U}^{SP}$ when requiring continuity in addition.

THEOREM 1. *For $m = 2$ and arbitrary domain $\mathcal{U} \subseteq \mathcal{U}^{SP}$, a continuous mechanism f satisfies anonymity and strategyproofness if and only if there exist $\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n$ in $[0, 1]$ such that*

$$f(P) = \text{med}(p_1, \ldots, p_n, \alpha_0, \ldots, \alpha_n).$$

Proof. The "if" direction is obvious; we focus on the "only if".

For any $k \in \{0, \ldots, n\}$, let P_k be a single-minded profile in which some k agents have their peak at 1 and the other $n - k$ have their peak at 0. Let $\alpha_k := f(P_k)$; due to the anonymity of f, it holds that α_k does not depend on the selection of P_k.

Since f is strategyproof, $\alpha_k \le \alpha_{k+1}$ for all $k \in \{0, \ldots, n-1\}$; otherwise, in profile P_k, some agent with peak at 0 could gain from reporting a peak at 1.

Let $g(P) := \text{med}(p_1, \ldots, p_n, \alpha_0, \ldots, \alpha_n)$. Then for any $k \in \{0, \ldots, n\}$, $g(P_k) = \alpha_k$, as n arguments of the median are at most α_k and n arguments are at least α_k. This means that f and g agree on all single-minded profiles. By Lemma 2, $f \equiv g$. □

It is possible to obtain two additional characterizations in a similar way:

1. A continuous mechanism f satisfies anonymity, strategyproofness, and efficiency if and only if there exist $\alpha_1 \le \cdots \le \alpha_{n-1}$ in $[0, 1]$ such that

$$f(P) = \text{med}(p_1, \ldots, p_n, \alpha_1, \ldots, \alpha_{n-1}).$$

2. A continuous mechanism f satisfies strategyproofness if and only if there exist 2^n constants, $\alpha_G \in [0, 1]$ for all $G \subseteq N$, such that

$$f(P) = \max_{G \subseteq N} \min(\alpha_G, \min_{i \in G} p_i).$$

4.2. Characterizing the uniform phantom rule The *uniform phantom rule* is a special case of the generalized median rule in which the peaks are distributed uniformly in $[0, 1]$, that is, $\alpha_k = k/n$ for $k \in \{0, \ldots, n\}$. It is range-respecting, as at least $n + 1$ arguments to the median (namely, $\alpha_0, p_1, \ldots, p_n$) are at most $\max_i p_i$, and at least $n + 1$ arguments to the median (namely, $\alpha_n, p_1, \ldots, p_n$) are at least $\min_i p_i$.

Freeman et al. [21] showed that the uniform phantom rule is the only mechanism that ensures proportionality in addition to all axioms from Theorem 1. Aziz et al. [5] strengthened this result by pointing out that continuity, strategyproofness, and proportionality suffice for characterizing the uniform phantom mechanism for symmetric single-peaked preferences. Note that proportionality already contains some form of anonymity: when all agents have peaks at $[1 : a]$ or $[1 : b]$, proportionality requires picking a specific distribution that is independent of the agents' identities.

Recently, Jennings et al. [27] showed that the uniform phantom rule is the unique mechanism that satisfies strategyproof and proportionality, among all rules defined on $\mathcal{U}^{SP}$. Again, we present a characterization that holds for every subset of $\mathcal{U}^{SP}$, whether symmetric or not.

THEOREM 2. *For $m = 2$ and arbitrary domain $\mathcal{U} \subseteq \mathcal{U}^{SP}$, the only continuous mechanism that satisfies strategyproofness and proportionality is the uniform phantom rule.*

Proof. Let g be the uniform phantom mechanism. Then g is proportional. Since proportionality completely specifies the outcomes for all single-minded profiles, any proportional mechanism agrees with g on all single-minded profiles. By Lemma 2, any such mechanism must equal g. □

For $m = 2$, an outcome is efficient for a profile P if and only if it is range-respecting, which means that the uniform phantom rule is efficient. Thus, for only two alternatives, there is a unique way to aggregate utilities in an efficient, strategyproof, and fair manner, even without knowledge of the specific underlying utility model.

5. Utility functions for three or more alternatives In this and the following sections, we move to the multi-dimensional setting and consider domains with $m \geq 3$ alternatives. We initially discuss some typical star-shaped utility functions, for which we will present results in the next section.

5.1. ℓ_p preferences A natural approach for specifying a utility function based on a single peak is to measure the distance to the peak using some metric $d : \Delta^m \times \Delta^m \to \mathbb{R}_{\geq 0}$. Given an agent with peak $\mathbf{p}_i$, her utility for a distribution $\mathbf{q}$ is then defined as $u_i(\mathbf{q}) = -d(\mathbf{p}_i, \mathbf{q})$.

Among such models, ℓ_p norms, given by $\|\mathbf{q}\|_p := \left(\sum_{j\in M} |q_j|^p\right)^{1/p}$ for $p \geq 1$, are the most studied utility functions. In particular, the special case of $p = 1$ has received considerable attention.

Definition 11. An agent i with peak $\mathbf{p}_i$ has *ℓ_p preferences* if $u_i(\mathbf{q}) = -\|\mathbf{p}_i - \mathbf{q}\|_p$.

We will also sometimes refer to these preferences as ℓ_p disutilities. In this paper, we focus on ℓ_1 preferences ($u_i(\mathbf{q}) = -\sum_{j\in M} |p_{i,j} - q_j|$) and ℓ_∞ preferences ($u_i(\mathbf{q}) = -\max_{j\in M} |p_{i,j} - q_j|$). It can be easily shown that ℓ_p preferences are star-shaped for $p \geq 1$. This does not hold for arbitrary metrics, e.g., consider the *trivial metric*, $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise. Moreover, ℓ_p preferences are peak-linear. The utility of every point in Δ^m can be computed using (2) with $u_i(\mathbf{p}_i) = 0$ and $u_i(\mathbf{q}) = -\|p_i - q\|_p$ for all distributions $\mathbf{q}$ on the boundary of Δ^m.

5.2. Leontief utilities In contrast to ℓ_p preferences, Leontief utilities are not based on a metric. In particular, they are not symmetric and also not based on disutilities. As discussed in the introduction, metric-based preferences fail to capture important aspects of certain practical situations, notably the need to guarantee that all alternatives are adequately represented. This requirement is captured by Leontief utilities.

Let $M_i := \{j \in M : p_{i,j} > 0\}$ be the set of alternatives to which i wants to allocate a positive amount; note that $M_i \neq \emptyset$. The *Leontief utility* that agent i derives from a distribution $\mathbf{q}$ is given by

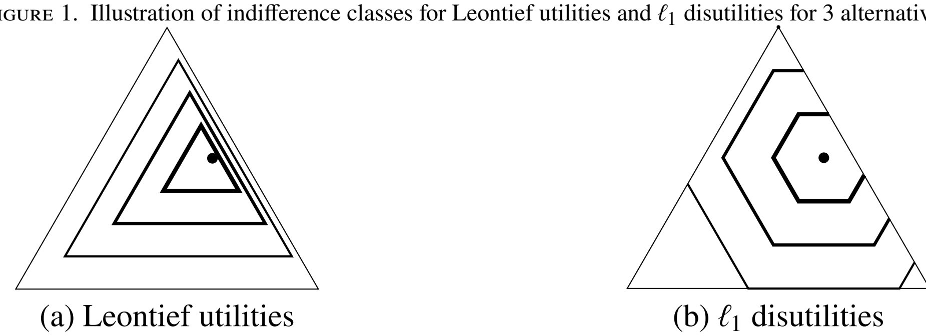
$$u_i(\mathbf{q}) = \min_{j\in M_i} \frac{q_j}{p_{i,j}}.$$

Observe that $0 \leq u_i(\mathbf{q}) \leq 1$ for all distributions $\mathbf{q}$. Moreover, $u_i(\mathbf{q}) = 1$ if and only if $\mathbf{q} = \mathbf{p}_i$, and $u_i(\mathbf{q}) = 0$ if and only if $q_j = 0$ for some $j \in M_i$. As discussed in Section 1, Leontief utilities are based on the assumption that agents want all alternatives to receive as large a fraction of their ideal amounts as possible. The indifference curves of ℓ_1 preferences and Leontief utilities are illustrated in Figure 1.

Leontief utility functions are peak-linear with $u_i(\mathbf{p}_i) = 1$ and $u_i(\mathbf{q}) = 0$ for all boundary distributions $\mathbf{q}$. In fact, as explained in the preliminaries, Leontief utilities are characterized by these properties: they are the only peak-linear utilities that assign utility 1 to the peak and utility 0 to all boundary distributions.

It is possible to refine Leontief utilities further by considering the *leximin* over the quotients—that is, breaking ties in the smallest quotient using the second smallest quotient, and so on. This refinement is discussed in Appendix E.

FIGURE 1. Illustration of indifference classes for Leontief utilities and ℓ_1 disutilities for 3 alternatives.



Note. The ideal distribution $(0.1, 0.4, 0.5)$ is represented by the black point. The main triangle is the simplex of distributions among three alternatives. Its vertices represent the degenerate distributions $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. For each type of utilities, the peak forms an indifference class by itself, and three other indifference classes are displayed with different line widths.

A useful concept when dealing with Leontief utilities is that of *critical alternatives*.

DEFINITION 12. Given a distribution $\mathbf{q}$, we define the set of agent i's *critical alternatives*

$$T_{\mathbf{q},i} := \arg\min_{j \in M_i} \frac{q_j}{p_{i,j}}.$$

Critical alternatives allow for a characterization of efficient distributions.

LEMMA 3 **(Brandt et al. [15, Lem. 4.8])**. *With Leontief utilities, a distribution $\mathbf{q}$ is efficient if and only if every alternative j with $q_j > 0$ is critical for some agent.*

Furthermore, it turns out that both core fairness notions coincide for Leontief utilities.

PROPOSITION 2. *With Leontief utilities, weak core fair share and core fair share are equivalent.*

The proof can be found in Appendix A. Combining Propositions 1 and 2 results in the following corollary.

COROLLARY 1. *With Leontief utilities, weak core fair share implies efficiency.*

We now give a sufficient condition and another necessary condition for CFS with Leontief utilities. For any subset of agents $G \subseteq N$, let $T_{\mathbf{q},G} := \bigcup_{i \in G} T_{\mathbf{q},i}$ denote the set of alternatives critical to at least one agent from G.

LEMMA 4. *With Leontief utilities, if $\mathbf{q}(T_{\mathbf{q},G}) \geq |G|/n$ for all subsets $G \subseteq N$, then $\mathbf{q}$ satisfies weak core fair share, hence also core fair share.*

Proof. Assume for contradiction that $\mathbf{q}$ violates weak CFS for some $G \subseteq N$. Then, there exists $\mathbf{q}' \in \Delta^m$ such that for every $\mathbf{q}'' \in \Delta^m$,

$$u_i((|G|/n)\mathbf{q}' + (1 - |G|/n)\mathbf{q}'') > u_i(\mathbf{q}) \qquad \text{for all } i \in G.$$

Note that $T_{\mathbf{q},G} = M$ cannot hold; otherwise, by Lemma 3, $\mathbf{q}$ would be efficient not only for N but already for G, contradicting that $\mathbf{q}$ does not satisfy core fair share for G. Therefore, there exists a distribution $\mathbf{q}''$ with $q''_j = 0$ for every $j \in T_{\mathbf{q},G}$. Choosing such a distribution $\mathbf{q}''$ in the above inequality shows that $(|G|/n)q'_j > q_j$ for all $j \in T_{\mathbf{q},G}$. Thus, $\mathbf{q}(T_{\mathbf{q},G}) := \sum_{j \in T_{\mathbf{q},G}} q_j < (|G|/n) \cdot \sum_{j \in T_{\mathbf{q},G}} q'_j \leq |G|/n$. □

The opposite direction of Lemma 4 does not hold even for $n = 2$ and $m = 2$. For example, suppose $\mathbf{p}_1 = (1/2, 1/2)$, $\mathbf{p}_2 = (1/3, 2/3)$, and $\mathbf{q} = (1/3, 2/3)$. Then, $\mathbf{q}$ satisfies CFS as the utility of each agent is at least $1/2$, but $\mathbf{q}(T_{\mathbf{q},1}) = 1/3 < 1/2$.

LEMMA 5. *With Leontief utilities, if* $\mathbf{q}$ *satisfies weak core fair share, then* $q_j = 0$ *if and only if* $p_{i,j} = 0$ *for all* $i \in N$.

Proof. If $q_j = 0$ for some $j \in M$, then $u_i(\mathbf{q}) = 0$ for all agents with $p_{i,j} > 0$, meaning that weak core fair share is violated for each of these agents.

Conversely, $p_{i,j} = 0$ for all $i \in N$ implies that $q_j = 0$ for every efficient mechanism, where efficiency follows from Corollary 1. □

5.3. Weak core fair share and proportionality We conclude this section by showing that core fair share is a stronger fairness axiom than proportionality for ℓ_1 and ℓ_∞ preferences as well as Leontief utilities.

PROPOSITION 3. *With* ℓ_1 *preferences,* ℓ_∞ *preferences, or Leontief utilities, weak core fair share implies proportionality.*

Proof. Assume that a mechanism f is not proportional for some single-minded profile $P \in \mathcal{P}$. Denote $\mathbf{q} := f(P)$, and let $N' \subseteq N$ be a maximal subset of agents where all agents in N' allocate 1 to the same alternative j^* and proportionality is violated, i.e., $q_{j^*} < r$ for $r := |N'|/n$.

Let $\mathbf{q}'$ be the peak of all agents $i \in N'$, i.e., $q'_{j^*} = 1$ and $q'_j = 0$ for all $j \neq j^*$. We claim that $u_i(r\mathbf{q}' + (1-r)\mathbf{q}'') > u_i(\mathbf{q})$ for all distributions $\mathbf{q}'' \in \Delta^m$.

With ℓ_1, ℓ_∞, and Leontief preferences, for all $i \in N'$, $u_i(\mathbf{q})$ depends only on q_{j^*}, and it is an increasing function of q_{j^*}. Specifically, with ℓ_1 preferences $u_i(\mathbf{q}) = -2(1-q_{j^*})$, with ℓ_∞ preferences $u_i(\mathbf{q}) = -(1-q_{j^*})$, and with Leontief preferences $u_i(\mathbf{q}) = q_{j^*}$.

Since $(r\mathbf{q}' + (1-r)\mathbf{q}'')_{j^*} \geq r > q_{j^*}$ for all $\mathbf{q}'' \in \Delta^m$, we have $u_i(r\mathbf{q}' + (1-r)\mathbf{q}'') > u_i(\mathbf{q})$, so f violates weak core fair share. □

As the proof shows, CFS implies a property even stronger than proportionality: we do not need that $p_{i,j} \in \{0,1\}$ for all $i \in N$ but the guarantee is for every single-minded agent group, independently of the other agents' preferences.

Note that the proof of Proposition 3 does not work for ℓ_2 preferences even for core fair share. For example, suppose $n = m = 3$ and some two agents have their peak at $(1,0,0)$. A rule that returns $\mathbf{q} = f(P) = (0.64, 0.18, 0.18)$ violates proportionality but does not violate CFS, as for both $i \in N'$, $u_i(\mathbf{q}) > -\sqrt{0.2}$, but for $\mathbf{q}'' = (0,1,0)$, we have $r\mathbf{q}' + (1-r)\mathbf{q}'' = (2/3, 1/3, 0)$, which leads to $u_i = -\sqrt{2/9} < -\sqrt{0.2}$. In fact, proportionality does not seem to be a very natural notion for such preferences, as the proportionality guarantee for single-minded agents, who put 1 on alternative j, concerns only the distribution on alternative j, whereas ℓ_2 agents care also about the distribution on alternatives other than j.

6. Impossibilities for ℓ_1 and ℓ_∞ preferences In this section, we show that efficiency, strategyproofness, and the rather weak fairness condition of proportionality are incompatible when agents have ℓ_1 or ℓ_∞ preferences.

6.1. ℓ_1 preferences Under ℓ_1 preferences, Freeman et al. [21] observed that the utilitarian welfare maximizing mechanism is the only efficient mechanism in their class of moving phantom mechanisms. However, maximizing utilitarian welfare violates weak fairness axioms such as proportionality. We prove that this tradeoff between efficiency and fairness is inevitable in the presence of strategyproofness.

THEOREM 3. *With ℓ_1 preferences, no mechanism satisfies efficiency, strategyproofness, and proportionality when $m \geq 3$ and $n \geq 3$.*

For the proof of this theorem, we consider disutilities d_i (the ℓ_1 distance to an agent's peak p_i) instead of utilities, i.e., $d_i(\mathbf{q}) = \|\mathbf{p}_i - \mathbf{q}\|_1$. It is also important to keep the following simple observations in mind.

Observation 1 *With ℓ_1 preferences, if some agent i has $p_{i,j} = 1$, then for any distribution $\mathbf{q}$, $d_i(\mathbf{q}) = 2 - 2q_j$, regardless of the distribution on alternatives other than j. Therefore, agent i is indifferent if some amount is moved between alternatives other than j.*

Observation 2 *With ℓ_1 preferences, $d_i(\mathbf{q}) \geq 2 \cdot |p_{i,j} - q_j|$ for all $j \in M$ and $i \in N$.*

PROOF OF THEOREM 3. We start with the case $m = 3$ and $n = 3$. For $m = 3$, we set $M = \{a, b, c\}$ and write $\mathbf{q} = (q_a, q_b, q_c)$. For simplicity, we number profiles by a superscript (k). We denote the disutility function of agent i in profile k by $d_i^{(k)}$, and the returned distribution in profile k by $\mathbf{q}^{(k)}$. Sometimes, we cannot determine $\mathbf{q}^{(k)}$ completely. In these cases, we give lower or upper bounds on the entries of $q^{(k)}$.

Consider first the following two profiles. The outcome in Profile 2 must be $(1/3, 1/3, 1/3)$ by proportionality. For Profile 1, the last row of the table gives lower bounds on $\mathbf{q}_a^{(1)}, \mathbf{q}_b^{(1)}$ and an upper bound on $\mathbf{q}_c^{(1)}$, which we justify below.

Profile 1

	a	b	c
	1/2	1/2	0
	0	1	0
	0	0	1
$\mathbf{q}^{(1)}$	$\geq 1/6$	$\geq 1/2$	$\leq 1/3$

Profile 2

	a	b	c
	1	0	0
	0	1	0
	0	0	1
$\mathbf{q}^{(2)}$	1/3	1/3	1/3

As Agent 1 can manipulate between Profile 1 and Profile 2, strategyproofness requires that Agent 1 does not gain from either manipulation. This implies

$$d_1^{(1)}(\mathbf{q}^{(1)}) \leq d_1^{(1)}(\mathbf{q}^{(2)}) = 2/3 \text{ and} \tag{3}$$

$$d_1^{(2)}(\mathbf{q}^{(1)}) \geq d_1^{(2)}(\mathbf{q}^{(2)}) = 4/3. \tag{4}$$

By (3) and Observation 2, $q_a^{(1)} \geq 1/6$ (implying $q_b^{(1)} \leq 5/6$), $q_b^{(1)} \geq 1/6$, and $q_c^{(1)} \leq 1/3$. By (4) and Observation 1, $q_a^{(1)} \leq 1/3$, implying $q_b^{(1)} + q_c^{(1)} \geq 2/3$, and thus $q_b^{(1)} \geq 1/3$.

The left figure below illustrates both inequalities. The blue area corresponds to the set of distributions $\mathbf{q}^{(1)}$ with $d_1^{(1)}(\mathbf{q}^{(1)}) \leq 2/3$ whereas the red area consists of all distributions $\mathbf{q}^{(1)}$ satisfying (4). Strategyproofness requires $\mathbf{q}^{(1)}$ to be inside the intersection of the two areas, i.e., the purple region. Hence,

$$1/6 \leq q_a^{(1)} \leq 1/3, \qquad 1/3 \leq q_b^{(1)} \leq 5/6, \qquad \text{and} \qquad 0 \leq q_c^{(1)} \leq 1/3.$$

By efficiency, we can even show that $q_b^{(1)} \geq 1/2$. Otherwise, as $q_a^{(1)} > 0$, some small amount could be moved from a to b. Agent 3 is indifferent due to Observation 1 and agent 2 strictly gains. Furthermore, this does not change agent 1's disutility as $q_b^{(1)} < 1/2$.

Next, we consider the following two profiles.

Profile 3

	a	b	c
	1/4	3/4	0
	0	1	0
	0	0	1
$\mathbf{q}^{(3)}$	0	2/3	1/3

Profile 4

	a	b	c
	0	1	0
	0	1	0
	0	0	1
$\mathbf{q}^{(4)}$	0	2/3	1/3

The outcome in Profile 4 follows from proportionality. We now prove that the outcome in Profile 3 must be identical. As Agent 1 can manipulate between Profile 3 and Profile 4, strategyproofness requires that Agent 1 does not gain from either manipulation. This implies that

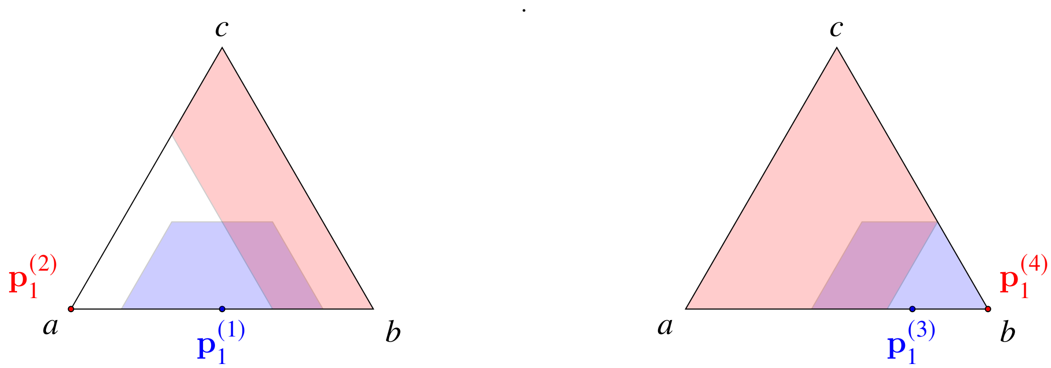
$$d_1^{(3)}(\mathbf{q}^{(3)}) \leq d_1^{(3)}(\mathbf{q}^{(4)}) = 2/3, \text{ and} \tag{5}$$
$$d_1^{(4)}(\mathbf{q}^{(3)}) \geq d_1^{(4)}(\mathbf{q}^{(4)}) = 2/3. \tag{6}$$

By (5), $q_c^{(3)} \leq 1/3$, implying $q_a^{(3)} + q_b^{(3)} \geq 2/3$. By (6), $q_b^{(3)} \leq 2/3$. Graphically, strategyproofness for Agent 1 implies that $\mathbf{q}^{(3)}$ must be in the purple region in the right figure on the previous page.

However, by efficiency, if $q_a^{(3)} > 0$ then $q_b^{(3)} \geq 3/4$. Otherwise, some small amount can be moved from a to b. Agent 3 is indifferent due to Observation 1 and Agent 2 strictly gains. Furthermore, this does not change agent 1's disutility as $q_b^{(3)} < 3/4$. Therefore, $q_a^{(3)} = 0$ must hold, and the only outcome compatible with strategyproofness is $\mathbf{q}^{(3)} = (0, 2/3, 1/3)$.

Now that we know $\mathbf{q}^{(3)}$, we consider a manipulation of Agent 1 from Profile 3 to Profile 1. Strategyproofness implies

$$d_1^{(3)}(\mathbf{q}^{(1)}) \geq d_1^{(3)}(\mathbf{q}^{(3)}) = 2/3.$$



(a) Inequalities (3) and (4)

(b) Inequalities (5) and (6)

But the bounds we already have for $\mathbf{q}^{(1)}$ imply that $d_1^{(3)}(\mathbf{q}^{(1)}) \leq 2/3$ as $q_a^{(1)} \geq 1/6$ and $q_b^{(1)} \geq 1/2$. Therefore, $d_1^{(3)}(\mathbf{q}^{(1)}) = 2/3$ together with $q_a^{(1)} = 1/6$ and $q_b^{(1)} = 1/2$. Hence, $\mathbf{q}^{(1)} = (1/6, 1/2, 1/3)$.

Finally, we consider Profiles 5 and 6.

Profile 5

	a	b	c
	1/2	1/2	0
	0	1	0
	0	1/2	1/2
$\mathbf{q}^{(5)}$			

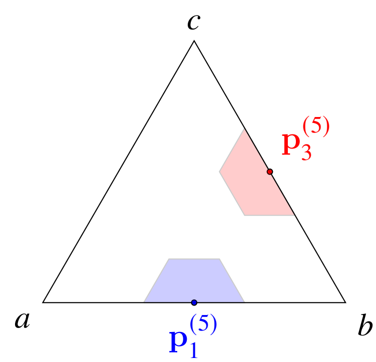
Profile 6

	a	b	c
	1	0	0
	0	1	0
	0	1/2	1/2
$\mathbf{q}^{(6)}$	1/3	1/2	1/6

The distribution $\mathbf{q}^{(6)}$ is determined by arguments analogous to those for $\mathbf{q}^{(1)}$, reasoning about Agent 3 instead of Agent 1.

We now consider a manipulation of Agent 1 from Profile 5 to Profile 6. It follows from strategyproofness that $d_1^{(5)}(\mathbf{q}^{(5)}) \leq d_1^{(5)}(\mathbf{q}^{(6)}) = 1/3$, which implies that $q_c^{(5)} \leq 1/6$. Similarly, we consider a manipulation of Agent 3 from Profile 5 to Profile 1. It follows from strategyproofness that $d_3^{(5)}(\mathbf{q}^{(5)}) \leq d_3^{(5)}(\mathbf{q}^{(1)}) = 1/3$, which implies that $q_c^{(5)} \geq 1/2 - 1/6 = 1/3$, a contradiction.

Graphically, both inequalities are shown in the figure on the right. The blue area on the right contains the points satisfying the first inequality, and the red area on the left contains the points satisfying the second inequality. It is evident that the two inequalities cannot be satisfied simultaneously.



This example can be extended to arbitrary numbers of alternatives and agents in the following way.

To increase the number of alternatives, simply add alternatives j^+ with $p_{i,j^+} = 0$ for all agents i. These new alternatives do not affect the argument, as efficiency implies range-respect, which ensures that none of them ever receives a positive amount.

Adding agents is more involved, as our proof relies on explicit distributions induced by proportionality and thus depends on the number of agents. However, we note that, throughout the proof, Agent 2 always has the same peak, which puts all mass on alternative b. Therefore, when adding agents i^+ with $p_{i^+,b} = 1$, we can run through the exact same proof but with adapted distributions. The generalized proof can be found in Appendix B. □

REMARK 2. The bounds $m \geq 3$ and $n \geq 3$ in Theorem 3 are tight. Indeed, there exists a moving phantoms mechanism that satisfies strategyproofness, proportionality, and range-respect [21, p. 22], and it is known that range-respect and efficiency coincide when $m = 2$ or $n = 2$ [19, Sec. 8].

The three axioms required for the impossibility are independent. Indeed, efficiency and strategyproofness (without proportionality) are satisfied by the mechanism that maximizes utilitarian welfare [30]; strategyproofness and proportionality (without efficiency) are satisfied by the independent markets mechanism [21]; and proportionality and efficiency (without strategyproofness) are satisfied by a natural generalization of the maximum Nash welfare mechanism studied by Aziz et al. [5].

Freeman et al. [21, p. 30] posed the question of whether every anonymous, neutral, continuous, and strategyproof mechanism can be represented as a moving phantoms mechanism. While such a characterization might have the potential to simplify the previous proof, it does not hold in general; see Appendix C.

6.2. ℓ_∞ preferences ℓ_1 preferences take a special role among ℓ_p disutilities in terms of efficiency: indifference curves partially move along distributions with a constant sum on "approved" ($p_{i,j} > 0$) alternatives. As an example, consider $M = \{a, b, c, d\}$ and an agent i with peak $\mathbf{p}_i = (1/2, 1/2, 0, 0)$. With ℓ_1 preferences, she is indifferent between all distributions $\mathbf{q}$ with $q_a + q_b = 1/2$. This implies that if we have two agents and the second agent i' has $\mathbf{p}_{i'} = (0, 0, 0, 1)$, every efficient distribution $\mathbf{q}$ with $q_a + q_b = 1/2$ (equivalently $q_c + q_d = 1/2$) must put 0 on alternative c and $1/2$ on alternative d. By contrast, when considering, e.g., ℓ_2 preferences, it also matters for agent i *how* $1/2$ is distributed on c and d. As a result, more distributions become efficient, which weakens the role of efficiency for a potential impossibility when $p > 1$.

We proceed by proving an impossibility for the preference model at the other end of the spectrum: ℓ_∞ preferences. These preferences behave similarly to ℓ_1 disutilities (Observation 1), which is helpful when arguing about efficiency.

Observation 3 *With ℓ_∞ preferences, if some agent i has $p_{i,j} = 1$, then for any distribution $\mathbf{q}$, $d_i(\mathbf{q}) = 1 - q_j$, regardless of the distribution on alternatives other than j. Therefore, agent i is indifferent if some amount is moved between alternatives other than j.*

THEOREM 4. *With ℓ_∞ preferences, no mechanism satisfies efficiency, strategyproofness, and proportionality when $m \geq 3$ and $n \geq 3$.*

The proof of Theorem 4 can be found in Appendix D. It uses the same profiles as the one for Theorem 3, but needs more involved arguments when reasoning about efficiency and extending the argument to $m > 3$ alternatives. The reason is that, with ℓ_∞ disutilities (in contrast to ℓ_1), efficiency does not imply range-respect, and an efficient distribution might allocate a positive amount to an alternative to which all agents allocate 0. For example, let $m = 4$ and $n = 2$ with peaks $(1/2, 1/4, 1/4, 0)$ and $(1/4, 1/2, 1/4, 0)$. Then, $(3/8, 3/8, 1/8, 1/8)$ is efficient, as the maximal distance is $1/8$ for both agents and if Agent 1 is better off in distribution $\mathbf{q}$, this means $q_1 > 3/8$ which decreases Agent 2's utility.

REMARK 3. Theorem 4 requires $m \geq 3$, since for $m = 2$ all metrics are equivalent (and thus induce the same preferences), and there are mechanisms that satisfy all requirements (see Remark 2). Moreover, $n \geq 3$ is required because, for $m = 3$, the ℓ_∞ and ℓ_1 metrics are equivalent—the ℓ_1 distance is always twice the ℓ_∞ distance. Therefore, for $m = 3$ and $n = 2$, the same mechanisms satisfy all the requirements.

Similar to the impossibility for ℓ_1 preferences, we expect all axioms to be independent. However, to the best of our knowledge, this does not follow from existing results, as ℓ_∞ preferences have been studied significantly less than ℓ_1.

We conjecture that the incompatibility of efficiency, strategyproofness, and weak fairness conditions holds for ℓ_p disutilities for any $1 \leq p \leq \infty$, when $m \geq 3$ and $n \geq 3$.

In the next section, we demonstrate that the impossibility does not generalize to arbitrary peak-linear utility functions.

7. The Nash product rule From now on, we assume that utilities are given by Leontief utilities. Inspired by the positive results obtained by maximizing the product of utilities in similar contexts [in particular, 15], we define the *Nash product rule* for budget aggregation as follows. For any $P \in \mathcal{P}$,

$$NASH[P] = \arg\max_{\mathbf{q}\in\Delta^m} \prod_{i\in N} u_i(\mathbf{q}).$$

NASH is well-defined as it always returns exactly one distribution [15]. The following example illustrates the difference between *NASH* for Leontief utilities and the independent markets mechanism for ℓ_1 utilities.

EXAMPLE 1. Let $m = 3$ and $n = 2$. Assume that the two agents' ideal distributions are $(4/5, 1/5, 0)$ and $(4/5, 0, 1/5)$. One can check that the independent markets mechanism returns $(3/5, 1/5, 1/5)$, which gives each agent a utility of $3/4$, while *NASH* returns the distribution $(2/3, 1/6, 1/6)$, which gives each agent a utility of $5/6$.

Interestingly, *NASH* and independent markets coincide for $m = 2$; see Proposition 6.

Analogously to its application in donor coordination [15], we can also interpret the outcome of *NASH* as a Nash equilibrium where each agent i reports a vector $\mathbf{s}_i \in \Delta^m$ and the outcome is determined by adding up the score $\sum_{i\in N} s_{i,j}$ of each alternative j. Here, the strategy set is the set of preferences. This interpretation will be useful for proving certain properties of *NASH*.

DEFINITION 13 (DECOMPOSITION). A *decomposition* of a distribution $\mathbf{q}$ is a vector of nonnegative score vectors $(\mathbf{s}_i)_{i\in N}$ with

$$\sum_{i\in N} s_{i,j} = q_j \qquad \text{for all } j \in M;$$
$$\sum_{j\in M} s_{i,j} = \frac{1}{n} \qquad \text{for all } i \in N.$$

An alternative characterization of the *NASH* outcome uses the notion of critical alternatives.

LEMMA 6 **(Brandt et al. [15, Sec. 4.3])**. *A distribution* $\mathbf{q}$ *maximizes the Nash product if and only if it has a decomposition* $(\mathbf{s}_i)_{i\in N}$ *such that* $s_{i,j} = 0$ *for every alternative* $j \notin T_{\mathbf{q},i}$.

7.1. Properties In this section, we investigate properties of *NASH* for budget aggregation.

Anonymity follows immediately from the fact that multiplication is commutative. Neutrality is also straightforward as *NASH* does not take into account the identities of the alternatives. Another fact to keep in mind is that utilities and efficient outcomes admit a one-to-one correspondence.

LEMMA 7 **(Brandt et al. [15, Lem. 4.10])**. *Let* $\mathbf{q}$ *and* $\mathbf{q}'$ *be efficient distributions inducing the same utility vector, that is,* $u_i(\mathbf{q}) = u_i(\mathbf{q}')$ *for all* $i \in N$. *Then,* $\mathbf{q} = \mathbf{q}'$.

With ℓ_1 preferences, every efficient distribution $\mathbf{q}$ must be range-respecting (that is, q_j must be between $\min_i p_{i,j}$ and $\max_i p_{i,j}$ for all $j \in M$). However, with Leontief utilities, this is not the case. In fact, *NASH* is efficient but not range-respecting, as shown in Example 1. Intuitively, *NASH* prefers to decrease the distribution for alternative 1 below the minimum peak, in order to increase the distribution for other alternatives whose ratio is smaller. Nevertheless, an efficient distribution always satisfies one direction of range-respect.

Lemma 8. *With Leontief utilities, if* $\mathbf{q}$ *is an efficient distribution, then* $q_j \leq \max_i p_{i,j}$ *for all* $j \in M$*. In particular, this holds for any distribution returned by NASH.*

Proof. Suppose by contradiction that, for some $j \in M$, it holds that $q_j > p_{i,j}$ for all $i \in N$. Construct a new distribution $\mathbf{q}'$ from $\mathbf{q}$ by removing some amount from j and allocating it equally among the other alternatives, such that $q_j > p_{i,j}$ for all $i \in N$ still holds.

Note that, for any agent i with Leontief utilities, $u_i(\mathbf{q}) \leq 1$ for all $\mathbf{q}$. Therefore, the decrease in q_j does not decrease the Leontief utility of any agent, as $q_j/p_{i,j} > 1$. But the increase in allocation to other alternatives must increase the utilities of all agents. Therefore, $\mathbf{q}'$ is a Pareto improvement of $\mathbf{q}$, contradicting efficiency. □

NASH also satisfies continuity, which will be important for the axiomatic characterization we give in the next section.

Proposition 4. *With Leontief utilities, NASH is continuous.*

Proof. Suppose we are given a sequence of profiles P^1, P^2, $\ldots$ converging to P^*, i.e., $\lim_{k\to\infty} p^k_{i,j} = p^*_{i,j}$ for every agent $i \in N$ and alternative $j \in M$. Denote by u^k_i the utility of agent i in profile P^k, and u^*_i the utility in profile P^*. Denote $\mathbf{q}^k = \mathit{NASH}[P^k]$ for every $k \in \mathbb{N}$ and $\mathbf{q}^* = \mathit{NASH}[P^*]$. By boundedness, it suffices to show that every convergent subsequence of $\mathbf{q}^1, \mathbf{q}^2, \ldots$ converges to $\mathbf{q}^*$. Take such a subsequence, which must exist by the Bolzano-Weierstrass theorem, and denote its limit by $\mathbf{q}^L$. With abuse of notation, we now refer to this subsequence as $\mathbf{q}^1, \mathbf{q}^2, \ldots$. Our goal is to show that $\mathbf{q}^L = \mathbf{q}^*$.

Case 1: $q^*_j > 0$ *for all alternatives* $j \in M$. Denote by $\mathit{NASH}[\mathbf{q}, P]$ the Nash welfare of the outcome $\mathbf{q}$ when the profile is P. By definition of *NASH* on P^k, we have

$$\mathit{NASH}[\mathbf{q}^*, P^k] \leq \mathit{NASH}[\mathbf{q}^k, P^k]$$

for every k. We take the limit of both sides as $k \to \infty$.

- The left-hand side is a product of utilities $u^k_i(\mathbf{q}^*)$, where the distribution is fixed and only the utility functions change. Each utility is a minimum of ratios $q^*_j/p^k_{i,j}$ where all numerators are at least ε, for some $\varepsilon > 0$. Since the minimum is always at most 1, elements with $p^k_{i,j} < \varepsilon$ do not affect the minimum and can be ignored. Therefore, the minimum is determined only by ratios with $p^k_{i,j} \geq \varepsilon$. In this domain, the ratios are continuous functions of $p^k_{i,j}$, and their minimum is continuous too. Therefore, $\lim_{k\to\infty} u^k_i(\mathbf{q}^*) = u^*_i(\mathbf{q}^*)$, so the limit of the product at the left-hand side equals $\mathit{NASH}[\mathbf{q}^*, P^*]$.
- The right-hand side is a product of utilities $u^k_i(\mathbf{q}^k)$, where both the distribution and the utility functions change. There may be agents i and alternatives j for which both q^k_j and $p^k_{i,j}$ approach 0 as $k \to \infty$, so the limit of $u^k_i(\mathbf{q}^k)$ may differ from $u^*_i(\mathbf{q}^L)$. For example, if $\mathbf{p}^k_i = (2/k, 1 - 2/k)$ and $\mathbf{q}^k = (1/k, 1 - 1/k)$ then $u^k_i(\mathbf{q}^k) = 1/2$ for all k, but $\mathbf{q}^L = (0, 1)$ so $u^*_i(\mathbf{q}^L) = 1$. However, any alternative j for which $p^*_{i,j} = \lim_{k\to\infty} p^k_{i,j} = 0$ is removed from the minimum, so the minimum at the limit profile can only be larger than the limit of minima. Therefore, $\lim_{k\to\infty} u^k_i(\mathbf{q}^k) \leq u^*_i(\mathbf{q}^L)$, and the limit of the product at the right-hand side is at most $\mathit{NASH}[\mathbf{q}^L, P^*]$.

Therefore, we have $\mathit{NASH}[\mathbf{q}^*, P^*] \leq \mathit{NASH}[\mathbf{q}^L, P^*]$. By definition and uniqueness of *NASH* on P^*, we get $\mathbf{q}^L = \mathbf{q}^*$.

Case 2: $q^*_j = 0$ *for some alternatives* $j \in M$. Define $Z = \{j \in M : q^*_j = 0\}$. As $\mathbf{q}^*$ maximizes Nash welfare for P^*, we have $p^*_{i,j} = 0$ for every $i \in N$ and $j \in Z$; otherwise, an agent i with $p^*_{i,j} > 0$ would receive zero utility causing the whole product to become zero. Consequently, $\lim_{k\to\infty} p^k_{i,j} = 0$ for every $i \in N$, $j \in Z$. By Lemma 8, the amount allocated to alternatives in Z by $\mathbf{q}^k$ also tends to zero for $k \to \infty$. Hence, $q^*_j = 0$ implies $q^L_j = 0$.

Now, we will measure utilities and Nash welfare only with respect to the alternatives outside Z. Let w^k be the total amount allocated to alternatives outside Z in $\mathbf{q}^k$, so we know that w^k converges to 1 for $k \to \infty$. By definition of *NASH* on P^k, we have

$$NASH[w^k \cdot \mathbf{q}^*, P^k] \le NASH[\mathbf{q}^k, P^k].$$

As in Case 1, we take the limit of both sides, the left-hand side equals $NASH[\mathbf{q}^*, P^*]$, and the right-hand side is at most $NASH[\mathbf{q}^L, P^*]$. Since in P^* all agents assign zero to alternatives in Z, these two quantities remain the same even if we take the alternatives in Z back into account for the Nash welfare. Hence, as in Case 1, we get $NASH[\mathbf{q}^*, P^*] \le NASH[\mathbf{q}^L, P^*]$. By definition and uniqueness of *NASH* on P^*, we conclude that $\mathbf{q}^L = \mathbf{q}^*$. □

In addition, *NASH* satisfies efficiency and group-strategyproofness [15].[4] We show next that *NASH* furthermore satisfies core fair share (and thus also proportionality).

Proposition 5. *With Leontief utilities, NASH satisfies core fair share.*

Proof. Let G be any subset of agents. By Lemma 6, the *NASH* distribution can be decomposed in such a way that every agent from G only contributes her share of $1/n$ to alternatives in $T_{\mathbf{q},G}$. Thus, $\mathbf{q}(T_{\mathbf{q},G}) \ge |G|/n$. By Lemma 4, *NASH* satisfies core fair share. □

There exists an interesting connection of *NASH* to the independent markets mechanism for ℓ_1 preferences, which follows immediately from the mechanisms' axiomatic properties.

Proposition 6. *With two alternatives, NASH for Leontief utilities is equivalent to the uniform phantom mechanism for ℓ_1 preferences.*

Proof. For $m = 2$, Leontief utilities as well as ℓ_1 preferences are subsets of $\mathcal{U}^{SP}$, each of which contains one utility function per peak. Both mechanisms are continuous, strategyproof, and proportional. Therefore, they need to be equivalent on their respective utility models by Theorem 2. □

However, this equivalence no longer holds when $m > 2$, as shown in Example 1.

7.2. Characterization Next, we show that *NASH* admits an appealing characterization via strategyproofness and fairness.

Theorem 5. *With Leontief utilities, NASH is the only continuous mechanism that satisfies group-strategyproofness and weak core fair share.*

Let f be a mechanism satisfying the properties in the theorem statement. The proof is divided into three lemmas and has the following structure. Starting at an arbitrary profile P, we first show in Lemma 9 that moving to a "key" profile P^* cannot change the outcome: $f(P^*) = f(P)$. Then, Lemma 10 states that $f(P^*) = NASH[P^*]$. Finally, Lemma 11 proves $NASH[P^*] = NASH[P]$, which completes the proof as we then have $f(P) = NASH[P]$.

Let $\mathbf{q} := f(P)$. By weak core fair share and Lemma 5, $q_j = 0$ if and only if $p_{i,j} = 0$ for all $i \in N$.

[4] Our notion of group-strategyproofness is slightly different as agents do not have variable contributions. Specifically, we only need to consider manipulations of the individual peaks. However, this is a subset of all manipulations considered by Brandt et al. [15], so *NASH* also satisfies our notion of group-strategyproofness.

Denote by P^* the profile with peaks

$$p^*_{i,j} = \begin{cases} p_{i,j}/\mathbf{p}_i(T_{\mathbf{q},i}) & \text{for } j \in T_{\mathbf{q},i}; \\ 0 & \text{for } j \notin T_{\mathbf{q},i}, \end{cases}$$

where $\mathbf{p}_i(T_{\mathbf{q},i}) := \sum_{j \in T_{\mathbf{q},i}} p_{i,j}$. That is, in P^*, each agent moves her peak so that it is nonzero only on alternatives critical for her under $\mathbf{q}$. For example, suppose $\mathbf{p}_i = (0.1, 0.2, 0.3, 0.4)$ and $\mathbf{q} = (0.1, 0.1, 0.6, 0.2)$. Then $T_{\mathbf{q},i} = \{2, 4\}$, $\mathbf{p}_i(T_{\mathbf{q},i}) = 0.6$, and $\mathbf{p}^*_i = (0, 1/3, 0, 2/3)$. Note that, for an agent i with $T_{\mathbf{q},i} = M$, it holds that $\mathbf{p}_i = \mathbf{p}^*_i$.

LEMMA 9. *With Leontief utilities, if f is a continuous mechanism satisfying group-strategyproofness and efficiency, then*
(a) the outcome does not change, that is, $f(P^*) = f(P) = \mathbf{q}$;
(b) the sets of critical alternatives do not change, that is, $T_{\mathbf{q},i} = T^*_{\mathbf{q},i}$ for every $i \in N$.

Proof. (a) We move the peak of each agent in turn. For each agent i, we change $\mathbf{p}_i$ towards $\mathbf{p}^*_i$ gradually, to some $\widehat{\mathbf{p}}_i := \lambda \mathbf{p}^*_i + (1-\lambda)\mathbf{p}_i$, for some $\lambda \in [0, 1]$ to be computed later. Then we proceed along this line until we reach $\mathbf{p}^*_i$. In the above example, $\lambda = 0.3$ gives $\widehat{\mathbf{p}}_i = (0.07, 0.24, 0.21, 0.48)$. If $\mathbf{p}_i = \mathbf{p}^*_i$, it is clear that the outcome does not change, so assume that $\mathbf{p}_i \neq \mathbf{p}^*_i$. The change from $\mathbf{p}_i$ to $\widehat{\mathbf{p}}_i$ has a simple structure:

- $\widehat{p}_{i,j} > p_{i,j}$ for all $j \in T_{\mathbf{q},i}$, and the ratio $\widehat{p}_{i,j}/p_{i,j} = \lambda/\mathbf{p}_i(T_{\mathbf{q},i}) + (1-\lambda) =: \lambda^+$, a constant independent of j (in the example, $\lambda^+ = 1.2$);
- $\widehat{p}_{i,j} < p_{i,j}$ for all $j \notin T_{\mathbf{q},i}$, and the ratio $\widehat{p}_{i,j}/p_{i,j} = (1-\lambda) =: \lambda^-$, again independent of j (in the example, $\lambda^- = 0.7$).

Now, consider the ratios $q_j/p_{i,j}$ versus the ratios $q_j/\widehat{p}_{i,j}$. For each $j \in T_{\mathbf{q},i}$, we have $q_j/p_{i,j} > q_j/\widehat{p}_{i,j}$ because $\widehat{p}_{i,j} > p_{i,j}$, whereas for each $j \notin T_{\mathbf{q},i}$, we have $q_j/p_{i,j} < q_j/\widehat{p}_{i,j}$ because $\widehat{p}_{i,j} < p_{i,j}$. Furthermore, for all $j \in T_{\mathbf{q},i}$, the ratios $q_j/\widehat{p}_{i,j}$ remain equal (as $\widehat{p}_{i,j}/p_{i,j}$ is constant) when moving from $\mathbf{p}$ to $\widehat{\mathbf{p}}$, and they remain the smallest ratios for agent i. This implies that $\widehat{T}_{\mathbf{q},i} = T_{\mathbf{q},i}$.

Moreover, the entire ordering of alternatives by the ratio $q_j/p_{i,j}$ is identical to the ordering of alternatives by the ratio $q_j/\widehat{p}_{i,j}$, as the smallest ratio is divided by $\lambda^+ > 1$ and the other ratios are divided by $\lambda^- < 1$. In other words, suppose we partition the alternatives into subsets according to the ratio $q_j/p_{i,j}$, and denote the subset with the smallest ratio by $T_{\mathbf{q},i,1} \equiv T_{\mathbf{q},i}$, the subset with the second-smallest ratio by $T_{\mathbf{q},i,2}$, etc., then $T_{\mathbf{q},i,r} = \widehat{T}_{\mathbf{q},i,r}$ for all $r \geq 1$.

Computing λ. We pick λ sufficiently small such that no new alternative becomes critical for i in the new distribution yielded by f. Specifically, set

$$\varepsilon := \min_{j \in T_{\mathbf{q},i},\, j' \notin T_{\mathbf{q},i}} (q_{j'} p_{i,j} - q_j p_{i,j'}) \leq \min_{j \in T_{\mathbf{q},i},\, j' \notin T_{\mathbf{q},i}} \frac{q_{j'} p_{i,j} - q_j p_{i,j'}}{p_{i,j} + p_{i,j'}}.$$

Note that $\varepsilon > 0$, as $q_{j'}/p_{i,j'} > q_j/p_{i,j}$, by definition of critical alternatives.

By uniform continuity of f, there exists $\delta > 0$ such that $\|f(P) - f(P')\|_1 < 2\varepsilon$ for all P' with $\|P - P'\|_1 \leq \delta$. Set

$$\lambda := \min\left(1, \frac{\delta}{\|\mathbf{p}_i - \mathbf{p}^*_i\|_1}\right),$$

and define $\widehat{P}$ as a profile identical to P except that i changes her peak from $\mathbf{p}_i$ to $\widehat{\mathbf{p}}_i := \lambda \mathbf{p}^*_i + (1-\lambda)\mathbf{p}_i$. Note that $\|P - \widehat{P}\|_1 = \lambda \|\mathbf{p}_i - \mathbf{p}^*_i\|_1 \leq \delta$, so $\|\mathbf{q} - \widehat{\mathbf{q}}\|_1 < 2\varepsilon$, where $\mathbf{q} = f(P)$ and $\widehat{\mathbf{q}} = f(\widehat{P})$.

The choice of ε ensures that $T_{\widehat{\mathbf{q}},i} \subseteq T_{\mathbf{q},i}$, as for arbitrary $j \in T_{\mathbf{q},i}$ and $j' \notin T_{\mathbf{q},i}$ it holds that $\widehat{q}_j < q_j + \varepsilon$ and $\widehat{q}_{j'} > q_{j'} - \varepsilon$, so

$$\begin{aligned}\frac{\widehat{q}_{j'}}{p_{i,j'}} > \frac{q_{j'} - \varepsilon}{p_{i,j'}} \geq \frac{q_{j'}}{p_{i,j'}} - \frac{q_{j'}p_{i,j} - q_j p_{i,j'}}{p_{i,j'}(p_{i,j} + p_{i,j'})} = \frac{q_{j'}p_{i,j'} + q_j p_{i,j'}}{p_{i,j'}(p_{i,j} + p_{i,j'})} = \frac{q_{j'} + q_j}{p_{i,j} + p_{i,j'}} \\ = \frac{q_j p_{i,j} + q_{j'} p_{i,j}}{p_{i,j}(p_{i,j} + p_{i,j'})} = \frac{q_j}{p_{i,j}} + \frac{q_{j'}p_{i,j} - q_j p_{i,j'}}{p_{i,j}(p_{i,j} + p_{i,j'})} \geq \frac{q_j + \varepsilon}{p_{i,j}} > \frac{\widehat{q}_j}{p_{i,j}}.\end{aligned}$$

So every j' which is not critical for i under $\mathbf{q}$ cannot be critical for i under $\widehat{\mathbf{q}}$. Therefore,

$$T_{\widehat{\mathbf{q}},i} \subseteq T_{\mathbf{q},i} = \widehat{T}_{\mathbf{q},i}.$$

Proving that the outcome does not change. Consider a manipulation of agent i who manipulates between reporting $\mathbf{p}_i$ and $\widehat{\mathbf{p}}_i$. Strategyproofness for i implies both $u_i(\mathbf{q}) \geq u_i(\widehat{\mathbf{q}})$ and $\widehat{u}_i(\widehat{\mathbf{q}}) \geq \widehat{u}_i(\mathbf{q})$.

The latter condition implies that, for every alternative $j \in T_{\mathbf{q},i}$,

$$\begin{aligned}\frac{q_j}{\widehat{p}_{i,j}} &= \widehat{u}_i(\mathbf{q}) && \text{since } j \in T_{\mathbf{q},i} = \widehat{T}_{\mathbf{q},i}, \\ &\leq \widehat{u}_i(\widehat{\mathbf{q}}) && \text{by strategyproofness,} \\ &\leq \frac{\widehat{q}_j}{\widehat{p}_{i,j}} && \text{by the definition of Leontief utilities.}\end{aligned}$$

So $q_j \leq \widehat{q}_j$ for each alternative $j \in T_{\mathbf{q},i}$. Together with $T_{\widehat{\mathbf{q}},i} \subseteq T_{\mathbf{q},i}$, this implies $u_i(\mathbf{q}) \leq u_i(\widehat{\mathbf{q}})$. Therefore, $u_i(\mathbf{q}) = u_i(\widehat{\mathbf{q}})$. Furthermore, if $\widehat{u}_i(\widehat{\mathbf{q}}) > \widehat{u}_i(\mathbf{q})$, then $\widehat{q}_j > q_j$ for all $j \in \widehat{T}_{\mathbf{q},i} \supseteq T_{\widehat{\mathbf{q}},i}$, which means that $u_i(\mathbf{q}) < u_i(\widehat{\mathbf{q}})$, contradicting $u_i(\mathbf{q}) = u_i(\widehat{\mathbf{q}})$. Thus, $\widehat{u}_i(\widehat{\mathbf{q}}) = \widehat{u}_i(\mathbf{q})$.

Moreover, if the utility of some other agent i' increases, group-strategyproofness is violated for the pair $\{i, i'\}$, as this pair could profitably manipulate from $\mathbf{q}$ to $\widehat{\mathbf{q}}$. Similarly, if the utility of some other agent i' decreases, group-strategyproofness is again violated for the pair $\{i, i'\}$, as this pair could profitably manipulate from $\widehat{\mathbf{q}}$ to $\mathbf{q}$. Thus, $u_r(\mathbf{q}) = u_r(\widehat{\mathbf{q}})$ for all $r \in N$. Since $\mathbf{q}$ is efficient with respect to P, so is $\widehat{\mathbf{q}}$. By Lemma 7, $\mathbf{q} = \widehat{\mathbf{q}}$.

Applying this argument repeatedly, we get a sequence of profiles (P^k) with $P^0 = P$ where $\mathbf{p}_i^k$ lies on the line $\lambda \mathbf{p}_i^* + (1 - \lambda)\mathbf{p}_i$ for every k. It remains to show that $(\mathbf{p}^k)$ reaches $\mathbf{p}_i^*$ after a finite number of steps. For that, consider the expression in the definition of ε:

$$\min_{j \in T_{\mathbf{q},i},\, j' \notin T_{\mathbf{q},i}} (q_{j'} p_{i,j} - q_j p_{i,j'}).$$

As $\mathbf{p}_i$ comes closer to $\mathbf{p}_i^*$, $p_{i,j}$ increases and $p_{i,j'}$ decreases while $\mathbf{q}$ and $T_{\mathbf{q},i}$ stay the same, so overall the expression increases. Thus, we can take the ε (and the corresponding δ) from the first step for every step. Furthermore, $\|P^k - P^{k+1}\|_1 = \delta$ (unless $\lambda = 1$, but then we have reached $\mathbf{p}_i^*$) implying that we reach $\mathbf{p}_i^*$ after at most $\lceil \|\mathbf{p}_i - \mathbf{p}_i^*\|_1/\delta \rceil$ steps; as we move on a line of length $\|P^k - P^{k'}\|_1 = \sum_{\ell=k}^{k'-1} \|P^\ell - P^{\ell+1}\|_1$ for $k' \geq k$.

After the first agent has reached her desired peak $\mathbf{p}_i^*$, we turn to the next agent and repeat the procedure. In that way, we eventually arrive at P^*.

(b) To see that $T^*_{\mathbf{q},i} = T_{\mathbf{q},i}$ for all $i \in N$, note that for every non-critical alternative $j \notin T_{\mathbf{q},i}$ we have $p^*_{i,j} = 0$, so $j \notin T^*_{\mathbf{q},i}$. Furthermore, for any critical alternative $j \in T_{\mathbf{q},i}$ and any other $j' \in T_{\mathbf{q},i}$,

$$\frac{q_j}{p^*_{i,j}} = \frac{q_j \cdot \mathbf{p}_i(T_{\mathbf{q},i})}{p_{i,j}} = \frac{q_{j'} \cdot \mathbf{p}_i(T_{\mathbf{q},i})}{p_{i,j'}} = \frac{q_{j'}}{p^*_{i,j'}},$$

so $j \in T^*_{\mathbf{q},i}$. Therefore, $T_{\mathbf{q},i} = T^*_{\mathbf{q},i}$. □

Lemma 10. *Let P^* be a profile and $\mathbf{q}$ be a distribution in which every agent values every non-critical alternative at 0 ($j \notin T^*_{\mathbf{q},i}$ implies $p^*_{i,j} = 0$ for any agent i). If $\mathbf{q}$ satisfies weak core fair share, then $\mathbf{q} = NASH(P^*)$.*

Proof. Let P^* be an arbitrary profile and let $\mathbf{q} \neq NASH(P^*)$ be a distribution such that $j \notin T^*_{\mathbf{q},i}$ implies $p^*_{i,j} = 0$. In particular, $\mathbf{q}$ does not maximize Nash welfare. By Lemma 4.12 of Brandt et al. [15], there exists a group N^- of agents such that the total amount given to alternatives critical for some agent from N^- is less than $|N^-|/n$. That is,

$$\mathbf{q}(T^*_{\mathbf{q},N_-}) < \frac{|N_-|}{n}, \tag{7}$$

where $T^*_{\mathbf{q},N_-} := \bigcup_{i \in N_-} T^*_{\mathbf{q},i}$.

We will now show that weak core fair share is violated for this N^-. This is clear if $\mathbf{q}(T^*_{\mathbf{q},N_-}) = 0$, so assume that $\mathbf{q}(T^*_{\mathbf{q},N_-}) > 0$.

Define a new distribution in which only alternatives in $T^*_{\mathbf{q},N_-}$ are funded:

$$\mathbf{q}' := \begin{cases} q_j/\mathbf{q}(T^*_{\mathbf{q},N_-}) & \text{for } j \in T^*_{\mathbf{q},N_-}; \\ 0 & \text{for } j \notin T^*_{\mathbf{q},N_-}. \end{cases}$$

For every $i \in N_-$, as $p^*_{i,j} = 0$ for $j \notin T^*_{\mathbf{q},N_-} \supseteq T^*_{\mathbf{q},i}$, the utility $u_i^*(\mathbf{q}')$ equals $u_i^*(\mathbf{q})/\mathbf{q}(T^*_{\mathbf{q},N_-})$, which is larger than $u_i^*(\mathbf{q})/(|N_-|/n)$ by (7). Therefore, the utility $u_i^*\left((|N_-|/n)\mathbf{q}' + (1 - |N_-|/n)\,\mathbf{q}''\right)$ is at least $(|N_-|/n)u_i^*(\mathbf{q}') > u_i^*(\mathbf{q})$ for every $\mathbf{q}'' \in \Delta^m$, contradicting weak core fair share for N_-. □

Lemma 11. *Let P^* and P be profiles where $T^*_{\mathbf{q},i} = T_{\mathbf{q},i}$ for $\mathbf{q} = NASH[P^*]$ and all $i \in N$. Then, $NASH[P] = \mathbf{q}$.*

Proof. As $\mathbf{q}$ maximizes Nash welfare in P^*, by Lemma 6 there exists a decomposition $(\mathbf{s}_i)_{i \in N}$ such that $s_{i,j} = 0$ for every $i \in N$ and $j \notin T^*_{\mathbf{q},i}$. Due to $T^*_{\mathbf{q},i} = T_{\mathbf{q},i}$, the same decomposition proves that $\mathbf{q}$ also maximizes Nash welfare in P by Lemma 6, thus $NASH[P] = \mathbf{q}$. □

PROOF OF THEOREM 5. Let P be an arbitrary profile, and P^* a modified profile defined as in Lemma 9. Then,

$$f(P) \overset{Lemma\ 9}{=} f(P^*) \overset{Lemma\ 10}{=} NASH[P^*] \overset{Lemma\ 11}{=} NASH[P],$$

where Lemma 11 uses the fact that the sets of critical alternatives under $\mathbf{q}$ did not change when moving from P to P^*. □

The condition of group-strategyproofness is used only in Lemma 9. If we assume that agents have *Leximin-Leontief preferences* (that is, subject to maximizing the smallest ratio, they maximize the second smallest ratio, etc.), then this condition can be weakened to ordinary strategyproofness; the proof is given in Appendix E.

THEOREM 6. *With Leximin-Leontief preferences, NASH is the only continuous mechanism that satisfies strategyproofness and weak core fair share.*

As to the independence of the axioms, it is easy to see that weak core fair share is required for Theorem 5 since any constant mechanism satisfies continuity and group-strategyproofness. The necessity of (group-)strategyproofness can be shown by slightly perturbing the outcome of *NASH*. For example, consider $n = m = 2$ and the two peaks at $1/4$ and $3/4$, respectively. *NASH* returns $\mathbf{q} = (1/2, 1/2)$ with $u_i(\mathbf{q}) = 2/3$. However, core fair share only guarantees utility $1/2$ for agent i. So any distribution which puts at least $3/8$ on both alternatives satisfies core fair share. This "gap" can be used to slightly change the outcome without violating CFS. Defining such changes in a continuous way (note that if both agents have the same peak, that change is 0) should result in a different continuous mechanism satisfying CFS.

We conjecture that continuity is required for the characterization as well.

8. Conclusion

Aggregating individual distributions into a collective distribution constitutes an important problem in social choice theory. Our work shows that understanding how agents form their preferences has crucial implications on the possibility of optimal aggregation mechanisms.

When agents' utilities are based on metrics such as ℓ_1 and ℓ_∞, no rule simultaneously guarantees strategyproofness, efficiency, and proportionality. However, when agents' utilities are non-metric and based on quotients (Leontief utilities), the Nash product rule guarantees group-strategyproofness and weak core fair share, which implies both efficiency and proportionality. Moreover, this rule is characterized by group-strategyproofness, weak core fair share, and continuity.

The Nash product rule satisfies further desirable properties such as *reinforcement* and *participation*. The former states that when aggregating distributions for two disjoint electorates results in the same distribution, then the mechanism should return the same distribution for the union of both electorates. The latter requires that agents are never better off by not participating in the aggregation mechanism. Both statements follow trivially from the definition of the Nash product rule and hold for arbitrary utility models with nonnegative utilities.

It would be interesting to identify other sensible utility models for which the Nash product rule is a most attractive aggregation mechanism, and to pinpoint domain conditions that cause impossibilities similar to Theorems 3 and 4. Some concrete open questions are:

- Does Theorem 5 also hold when weakening group-strategyproofness to strategyproofness? Is continuity required for the characterization of the Nash product rule?
- Are there classes of peak-linear or star-shaped utility functions, other than Leontief, for which mechanisms satisfying core fair share and strategyproofness exist? In particular, for ℓ_p preferences with $1 \le p \le \infty$ such as ℓ_2, are there strategyproof mechanisms that satisfy weak core fair share?

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Appendix A: Proof of Proposition 2

PROPOSITION 2. *With Leontief utilities, weak core fair share and core fair share are equivalent.*

Proof. Assume that $\mathbf{q}$ violates core fair share for some set of agents $N' \subseteq N$. For brevity, denote $r := |N'|/n$. So there is a distribution $\mathbf{q}'$ for which the following hold for every $\mathbf{q}'' \in \Delta^m$:

$$\begin{aligned} u_i(r\mathbf{q}' + (1-r)\mathbf{q}'') &\geq u_i(\mathbf{q}) && \text{for all } i \in N', \text{ and} \\ u_{i'}(r\mathbf{q}' + (1-r)\mathbf{q}'') &> u_{i'}(\mathbf{q}) && \text{for at least one } i' \in N'. \end{aligned} \tag{8}$$

The latter condition implies that every alternative $j \in T_{\mathbf{q},i'}$ is allocated strictly more in $(r\mathbf{q}' + (1-r)\mathbf{q}'')$ than in $\mathbf{q}$. In particular, $T_{\mathbf{q},i'} \neq M$. As $\mathbf{q}''$ might allocate nothing to alternatives in $T_{\mathbf{q},i'}$, this implies that $rq'_j > q_j$ for all $j \in T_{\mathbf{q},i'}$.

We now construct a new distribution $\mathbf{q}^*$ from $\mathbf{q}'$, by taking a small amount $m\varepsilon$ from some $j_0 \in T_{\mathbf{q},i'}$, such that $rq^*_{j_0} > q_{j_0}$ still holds, and then adding ε to every $j \in M$. Now we have

$$\begin{aligned} rq^*_{j_0} &> q_{j_0}, \\ rq^*_j &> rq'_j && \text{for all } j \neq j_0. \end{aligned}$$

Therefore, for every distribution $\mathbf{q}''$,

$$\begin{aligned} rq^*_{j_0} + (1-r)q''_{j_0} &> q_{j_0}, \\ rq^*_j + (1-r)q''_j &> rq'_j + (1-r)q''_j \qquad \text{for all } j \neq j_0. \end{aligned}$$

Denoting $\mathbf{q}^+ := r\mathbf{q}^* + (1-r)\mathbf{q}''$, the above becomes:

$$\begin{aligned} q^+_{j_0} &> q_{j_0}, \\ q^+_j &> rq'_j + (1-r)q''_j \qquad \text{for all } j \neq j_0. \end{aligned} \tag{9}$$

We claim that $u_i(\mathbf{q}^+) > u_i(\mathbf{q})$ for all $i \in N'$. Indeed, for all $i \in N'$, if $j_0 \in M_i$:

$$\begin{aligned} u_i(\mathbf{q}^+) &= \min_{j\in M_i} \frac{q^+_j}{p_{i,j}} && \text{(by definition of Leontief utilities)} \\ &= \min\left(\frac{q^+_{j_0}}{p_{i,j_0}},\ \min_{j\in M_i, j\neq j_0} \frac{q^+_j}{p_{i,j}}\right) && \text{(by min properties)} \\ &> \min\left(\frac{q_{j_0}}{p_{i,j_0}},\ \min_{j\in M_i, j\neq j_0} \frac{rq'_j + (1-r)q''_j}{p_{i,j}}\right) && \text{(by (9))} \\ &\geq \min\left(\frac{q_{j_0}}{p_{i,j_0}},\ \min_{j\in M_i} \frac{rq'_j + (1-r)q''_j}{p_{i,j}}\right) && \text{(by min properties)} \\ &= \min\left(\frac{q_{j_0}}{p_{i,j_0}},\ u_i(r\mathbf{q}' + (1-r)\mathbf{q}'')\right) && \text{(by definition of Leontief utilities)} \\ &\geq \min\left(\frac{q_{j_0}}{p_{i,j_0}},\ u_i(\mathbf{q})\right) && \text{(by (8), as } i \in N') \\ &= u_i(\mathbf{q}) && \text{(by definition of Leontief utilities).} \end{aligned}$$

If $j_0 \notin M_i$, we can repeat a similar argument to arrive again at $u_i(\mathbf{q}^+) > u_i(\mathbf{q})$. To sum up, for every distribution $\mathbf{q}''$, we have $u_i(r\mathbf{q}^* + (1-r)\mathbf{q}'') > u_i(\mathbf{q})$ for all $i \in N'$. Hence, the distribution $\mathbf{q}^*$ shows that $\mathbf{q}$ violates weak core fair share. □

Appendix B: Impossibility of efficiency, strategyproofness, and proportionality for ℓ_1 preferences and arbitrary $n \geq 3$ In this section, we present the proof of Theorem 3 for arbitrary $n \geq 3$ but still fixed $m = 3$. Again, we set $M = \{a, b, c\}$ and write $\mathbf{q} = (q_a, q_b, q_c)$. Note that in contrast to the case $n = 3$, we now also need to denote the number of agents with certain peaks in a profile.

Consider first the following two profiles.

Profile 1

# agents	a	b	c
1	$3/2n$	$(2n-3)/2n$	0
$n-2$	0	1	0
1	0	0	1
$\mathbf{q}^{(1)}$	$\geq 1/2n$	$\geq (2n-3)/2n$	$\leq 1/n$

Profile 2

# agents	a	b	c
1	1	0	0
$n-2$	0	1	0
1	0	0	1
$\mathbf{q}^{(2)}$	$1/n$	$(n-2)/n$	$1/n$

The outcome in Profile 2 must be $(1/n, (n-2)/n, 1/n)$ by proportionality. We now justify the bounds on the outcome in Profile 1. As Agent 1 can manipulate between Profile 1 and Profile 2, strategyproofness requires that Agent 1 does not gain from either manipulation. This implies that

$$d_1^{(1)}(\mathbf{q}^{(1)}) \leq d_1^{(1)}(\mathbf{q}^{(2)}) = 2/n, \tag{10}$$

$$d_1^{(2)}(\mathbf{q}^{(1)}) \geq d_1^{(2)}(\mathbf{q}^{(2)}) = (2n-2)/n. \tag{11}$$

By (10), $q_a^{(1)} \geq 1/2n$ (implying $q_b^{(1)} \leq (2n-1)/2n$), $q_b^{(1)} \geq (2n-5)/2n$, and $q_c^{(1)} \leq 1/n$. By (11), $q_a^{(1)} \leq 1/n$, implying $q_b^{(1)} + q_c^{(1)} \geq (n-1)/n$, and thus $q_b^{(1)} \geq (n-2)/n$.

By efficiency, we can even show that $q_b^{(1)} \geq (2n-3)/2n$. Otherwise, as $q_a^{(1)} > 0$, some small amount could be moved from a to b. Agent n is indifferent due to Observation 1 and agents $2, \ldots, n-1$ strictly gain. Furthermore, this does not change agent 1's disutility as $q_b^{(1)} < (2n-3)/2n$.

Profile 3

# agents	a	b	c
1	$1/(n+1)$	$n/(n+1)$	0
$n-2$	0	1	0
1	0	0	1
$\mathbf{q}^{(3)}$	0	$(n-1)/n$	$1/n$

Profile 4

# agents	a	b	c
1	0	1	0
$n-2$	0	1	0
1	0	0	1
$\mathbf{q}^{(4)}$	0	$(n-1)/n$	$1/n$

Next, we consider Profiles 3 and 4. The outcome in Profile 4 follows from proportionality. We now prove that the outcome in Profile 3 must be the same. As Agent 1 can manipulate between Profile 3 and Profile 4, strategyproofness requires that Agent 1 does not gain from either manipulation. This implies that

$$d_1^{(3)}(\mathbf{q}^{(3)}) \leq d_1^{(3)}(\mathbf{q}^{(4)}) = 2/n, \tag{12}$$

$$d_1^{(4)}(\mathbf{q}^{(3)}) \geq d_1^{(4)}(\mathbf{q}^{(4)}) = 2/n. \tag{13}$$

By (12), $q_c^{(3)} \leq 1/n$, implying $q_a^{(3)} + q_b^{(3)} \geq (n-1)/n$. By (13), $q_b^{(3)} \leq (n-1)/n$.

However, by efficiency, if $q_a^{(3)} > 0$ then $q_b^{(3)} \geq n/(n+1)$. Otherwise, some small amount can be moved from a to b. Agent n is indifferent due to Observation 1 and Agents $2, \ldots, n-1$ strictly gain. Furthermore, this does not change Agent 1's disutility as $q_b^{(3)} < n/(n+1)$. Therefore, $q_a^{(3)} = 0$ must hold, and the only outcome compatible with strategyproofness is $\mathbf{q}^{(3)} = (0, (n-1)/n, 1/n)$.

Now that we know $\mathbf{q}^{(3)}$, we consider a manipulation of Agent 1 from Profile 3 to Profile 1. Strategyproofness implies

$$d_1^{(3)}(\mathbf{q}^{(1)}) \geq d_1^{(3)}(\mathbf{q}^{(3)}) = 2/n.$$

But the bounds we already have for $\mathbf{q}^{(1)}$ imply that $d_1^{(3)}(\mathbf{q}^{(1)}) \leq 2/n$ as $q_a^{(1)} \geq 1/2n$ and $q_b^{(1)} \geq (2n-3)/2n$. Therefore, $d_1^{(3)}(\mathbf{q}^{(1)}) = 2/n$ together with $q_a^{(1)} = 1/2n$ and $q_b^{(1)} = (2n-3)/2n$. Hence, $\mathbf{q}^{(1)} = (1/2n, (2n-3)/2n, 1/n)$.

Finally, we consider the following two profiles.

Profile 5

# agents	a	b	c
1	$3/2n$	$(2n-3)/2n$	0
$n-2$	0	1	0
1	0	$(2n-3)/2n$	$3/2n$
$\mathbf{q}^{(5)}$			

Profile 6

# agents	a	b	c
1	1	0	0
$n-2$	0	1	0
1	0	$(2n-3)/2n$	$3/2n$
$\mathbf{q}^{(6)}$	$1/n$	$(2n-3)/2n$	$1/2n$

The distribution $\mathbf{q}^{(6)}$ is determined by arguments analogous to those for $\mathbf{q}^{(1)}$, reasoning about Agent n instead of Agent 1.

We now consider a manipulation of Agent 1 from Profile 5 to Profile 6. It follows from strategyproofness that

$$d_1^{(5)}(\mathbf{q}^{(5)}) \leq d_1^{(5)}(\mathbf{q}^{(6)}) = 1/n,$$

which implies that $q_c^{(5)} \leq 1/2n$. Similarly, we consider a manipulation of Agent n from Profile 5 to Profile 1. It follows from strategyproofness that

$$d_3^{(5)}(\mathbf{q}^{(5)}) \leq d_3^{(5)}(\mathbf{q}^{(1)}) = 1/n,$$

which implies that $q_c^{(5)} \geq 3/2n - 1/2n = 1/n$, a contradiction. □

Appendix C: Not every anonymous, neutral, continuous, and strategyproof rule is moving phantoms For a formal definition of moving phantom mechanisms, see Definition 6 of Freeman et al. [21].

Proposition 7. *With ℓ_1 preferences, not every anonymous, neutral, continuous, and strategyproof mechanism can be represented as a moving phantoms mechanism, for any $n \geq 1$ and $m \geq 3$ and for any number of phantom functions.*

Proof. We first prove the claim for $n = 1$ and $m = 3$. Consider the mechanism which, in general, returns the agent's peak but cannot put more than 0.9 on an alternative. If (without loss of generality) $p_{1,1} > 0.9$, the mechanism returns $q_1 = 0.9$, $q_2 = p_{1,2} + (p_{1,1} - 0.9)/2$, and $q_3 = p_{1,3} + (p_{1,1} - 0.9)/2$.

Since this outcome minimizes the ℓ_1 distance of agent 1 among all "legal" distributions, the mechanism is strategyproof.

Anonymity is trivially satisfied as there is only one agent. For neutrality, if $p_{1,j} = p_{1,k}$, then both alternatives receive the same probability share. In particular, if $p_{1,l} > 0.9$ for the third alternative l, then j and k both receive $p_{1,j} + (p_{1,l} - 0.9)/2$, and the distribution of the surplus does not depend on the identity of the alternatives. For continuity, the only "critical" points are those where $p_{1,j} > 0.9$ approaches 0.9 from above. For such peaks, $q_j = 0.9$ is constant and the "surplus" $p_{1,j} - 0.9$ is distributed on the other two alternatives in a continuous manner. Thus, the mechanism also satisfies continuity.

Suppose by contradiction that the above mechanism can be represented as a moving phantoms mechanism with phantom functions $\mathbf{h}$; let k be the number of phantoms. Let $\mathbf{p}_1 = (0.91, 0.08, 0.01)$. Given this profile, the mechanism returns $\mathbf{q} = (0.9, 0.085, 0.015)$. This implies that, for some $t \in [0, 1]$, 0.085 is the median of 0.08 and $h_1(t), \ldots, h_k(t)$, so the number of phantoms larger than or equal to 0.085 should be at least $k/2 + 1$ (for even k) or $(k + 3)/2$ (for odd k).[5] By similar considerations, since 0.015 is the median of 0.01 and $h_1(t), \ldots, h_k(t)$, the number of phantoms smaller than or equal to 0.015 should be at least $k/2$ (for even k) or $(k - 1)/2$ (for odd k). These two observations are contradictory as there are only k phantoms in total.

A similar construction also works for larger m. Moreover, since the considered properties do not relate instances with different n, such a construction can be extended to a rule for arbitrary n by using this mechanism when $n = 1$ and moving phantoms when $n \geq 2$. □

Note that the proof of Proposition 7 does not assume continuity or any other property of the phantom functions.

After constructing this counterexample, we learned that de Berg et al. [17] independently came up with a similar construction with a more natural extension to larger n, that does not coincide with a moving phantoms mechanism for $n > 1$.

Appendix D: Impossibility of efficiency, strategyproofness and proportionality for ℓ_∞ preferences and $n, m \geq 3$ In this section, we present the proof of Theorem 4 for arbitrary $n \geq 3$ and $m \geq 3$. We start by fixing $m = 3$ and considering $n \geq 3$.

LEMMA 12. *With ℓ_∞ preferences, no mechanism satisfies efficiency, strategyproofness, and proportionality when $m = 3$ and $n \geq 3$.*

Proof. We use the same notation as in the proof of Theorem 3.

Profile 1

# agents	a	b	c
1	$3/2n$	$(2n-3)/2n$	0
$n-2$	0	1	0
1	0	0	1
$\mathbf{q}^{(1)}$	$\geq 1/2n$	$\geq (2n-3)/2n$	$\leq 1/n$

Profile 2

# agents	a	b	c
1	1	0	0
$n-2$	0	1	0
1	0	0	1
$\mathbf{q}^{(2)}$	$1/n$	$(n-2)/n$	$1/n$

Consider first Profiles 1 and 2. The outcome in Profile 2 must be $(1/n, (n-2)/n, 1/n)$ by proportionality. We now justify the bounds on the outcome in Profile 1. As Agent 1 can manipulate

[5] We assume that when k is odd, the median of $k + 1$ elements is the $((k + 1)/2)$-th element.

between Profile 1 and Profile 2, strategyproofness requires that Agent 1 does not gain from either manipulation. This implies that

$$d_1^{(1)}(\mathbf{q}^{(1)}) \le d_1^{(1)}(\mathbf{q}^{(2)}) = 1/n, \tag{14}$$

$$d_1^{(2)}(\mathbf{q}^{(1)}) \ge d_1^{(2)}(\mathbf{q}^{(2)}) = (n-1)/n. \tag{15}$$

By (14), $q_a^{(1)} \ge 1/2n$ (implying $q_b^{(1)} \le (2n-1)/2n$), $q_b^{(1)} \ge (2n-5)/2n$, and $q_c^{(1)} \le 1/n$. By (15), $q_a^{(1)} \le 1/n$, implying $q_b^{(1)} + q_c^{(1)} \ge (n-1)/n$, and thus $q_b^{(1)} \ge (n-2)/n$.

By efficiency, we can even show that $q_b^{(1)} \ge (2n-3)/2n$. Otherwise, $q_b^{(1)} < (2n-3)/2n$ and $q_c^{(1)} + q_a^{(1)} > 3/2n$, and some small amount can be moved from a to b. Agent n is indifferent due to Observation 3 and Agents $2, \dots, n-1$ strictly gain. Furthermore, this does not increase Agent 1's disutility as $d_1^{(1)}(\mathbf{q}^{(1)}) \ge q_c^{(1)} > 3/2n - q_a^{(1)}$ and $q_b^{(1)} < (2n-3)/2n$. Hence, $q_b^{(1)} \ge (2n-3)/2n$.

Profile 3

# agents	a	b	c
1	$1/(n+1)$	$n/(n+1)$	0
$n-2$	0	1	0
1	0	0	1
$\mathbf{q}^{(3)}$			

Profile 4

# agents	a	b	c
1	0	1	0
$n-2$	0	1	0
1	0	0	1
$\mathbf{q}^{(4)}$	0	$(n-1)/n$	$1/n$

Assume for contradiction that $q_c^{(1)} \le 3/4n$.

Consider a manipulation of Agent 1 from Profile 3 to Profile 1. Note that $d_1^{(3)}(\mathbf{q}^{(1)}) \le 3/4n$ with the bounds established for $\mathbf{q}^{(1)}$. By strategyproofness for Agent 1, $q_c^{(3)} \le 3/4n$.

By efficiency, $q_b^{(3)} \ge n/(n+1)$. Otherwise, $q_b^{(3)} < n/(n+1)$ and $q_a^{(3)} > 1/(n+1) - q_c^{(3)}$, and some small amount can be moved from a to b. Agent n is indifferent due to Observation 3 and Agents $2, \dots, n-1$ strictly gain. Furthermore, this does not increase agent 1's disutility as $d_1^{(3)}(\mathbf{q}^{(3)}) \ge q_c^{(3)} > 1/(n+1) - q_a^{(3)}$ and $q_b^{(3)} < n/(n+1)$. Hence, $q_b^{(3)} \ge n/(n+1)$. However, as $n/(n+1) > (n-1)/n$, this contradicts strategyproofness for Agent 1 manipulating from Profile 4 to Profile 3, where $\mathbf{q}^{(4)} = (0, (n-1)/n, 1/n)$ follows from proportionality.

Profile 5

# agents	a	b	c
1	1	0	0
$n-2$	0	1	0
1	0	$(2n-3)/2n$	$3/2n$
$\mathbf{q}^{(5)}$	$\le 1/n$	$\ge (2n-3)/2n$	$\ge 1/2n$

Profile 6

# agents	a	b	c
1	$3/2n$	$(2n-3)/2n$	0
$n-2$	0	1	0
1	0	$(2n-3)/2n$	$3/2n$
$\mathbf{q}^{(6)}$			

Therefore, $q_c^{(1)} > 3/4n$ has to hold. By analogous arguments with reversed roles of Agents 1 and n, the same bounds for $\mathbf{q}^{(5)}$ as well as $q_a^{(5)} > 3/4n$ hold.

Since $q_b^{(1)} \ge (2n-3)/2n$, we must have $q_a^{(1)} < 3/4n$. Consider a manipulation of Agent n from Profile 6 to Profile 1. Note that $d_n^{(6)}(\mathbf{q}^{(1)}) < 3/4n$, as $q_b^{(1)} \ge (2n-3)/2n$ and $q_c^{(1)} > 3/4n$. By strategyproofness for agent n, we have $d_n^{(6)}(\mathbf{q}^{(6)}) \le d_n^{(6)}(\mathbf{q}^{(1)}) < 3/4n$, and thus $q_a^{(6)} < 3/4n$. Finally, consider a manipulation of Agent 1 from Profile 6 to Profile 5. Analogously, $d_1^{(6)}(\mathbf{q}^{(6)}) \le d_1^{(6)}(\mathbf{q}^{(5)}) < 3/4n$, which implies $q_a^{(6)} > 3/2n - 3/4n = 3/4n$, a contradiction. □

We finish the proof of Theorem 4 by showing that this argument remains valid under the addition of alternatives j^+ with $p_{i,j^+} = 0$ for all agents $i \in N$.

Proof of Theorem 4. We prove that no efficient mechanism puts positive probability on new alternatives j^+ with $p_{i,j^+} = 0$ for all agents $i \in N$ in any of the six profiles used for the proof of Lemma 12. Together with Lemma 12, this completes the proof of Theorem 4.

Proportionality directly implies that adding such an alternative j^+ to Profiles 2 and 4 does not change the distribution.

Next, consider Profile 1. If $|(2n-3)/2n - q_b^{(1)}| < d_1^{(1)}(\mathbf{q}^{(1)})$, or $|(2n-3)/2n - q_b^{(1)}| = d_1^{(1)}(\mathbf{q}^{(1)})$ and $q_b^{(1)} < (2n-3)/2n$, moving some amount of probability from j^+ to b cannot increase Agent 1's disutility, does not change Agent n's disutility by Observation 3, and decreases the disutilities of all other agents. Therefore, such a redistribution would correspond to a Pareto improvement. Note that $|(2n-3)/2n - q_b^{(1)}| = d_1^{(1)}(\mathbf{q}^{(1)})$ and $q_b^{(1)} = (2n-3)/2n$ cannot hold simultaneously, as $q_{j^+}^{(1)} > 0$. Hence, the only remaining case we need to consider is $|(2n-3)/2n - q_b^{(1)}| = d_1^{(1)}(\mathbf{q}^{(1)})$ and $q_b^{(1)} > (2n-3)/2n$. This implies $3/2n - q_a^{(1)} \leq d_1^{(1)}(\mathbf{q}^{(1)})$ and thus, $q_a^{(1)} + q_b^{(1)} \geq 3/2n - d_1^{(1)}(\mathbf{q}^{(1)}) + (2n-3)/2n + d_1^{(1)}(\mathbf{q}^{(1)}) = 1$, contradicting $q_{j^+}^{(1)} > 0$. The argument for Profile 5 works analogously.

For Profile 3, moving some amount of probability from j^+ to b can only potentially increase the disutility of one agent, namely Agent 1, if $q_b^{(3)} \geq n/(n+1)$ and $d_1^{(3)}(\mathbf{q}^{(3)}) = q_b^{(3)} - n/(n+1)$. But then, $1/(n+1) - q_a^{(3)} \leq d_1^{(3)}(\mathbf{q}^{(3)})$ and again, $q_a^{(3)} + q_b^{(3)} = 1$, contradicting $q_{j^+}^{(3)} > 0$.

Finally, for Profile 6, moving some amount of probability from j^+ to b can only potentially increase the disutilities of two agents, namely Agents 1 and n, if $q_b^{(6)} \geq (2n-3)/2n$ and $d_k^{(6)}(\mathbf{q}^{(6)}) = q_b^{(6)} - (2n-3)/2n$ holds for at least one $k \in \{1, n\}$, without loss of generality for $k = 1$. But then, $3/2n - q_a^{(6)} \leq d_1^{(6)}(\mathbf{q}^{(6)})$ and again, $q_a^{(6)} + q_b^{(6)} = 1$, contradicting $q_{j^+}^{(6)} > 0$. □

Appendix E: Leximin-Leontief preferences Leontief utilities, as defined in Section 5.2, assume that agents rank distributions only by the smallest ratio, $\min_{j \in M_i} q_j / p_{i,j}$. In this section, we assume that agents rank distributions with the same smallest ratio by the second-smallest ratio, and distributions with the same smallest and second-smallest ratio by the third-smallest ratio, and so on. We call these preferences *Leximin-Leontief preferences*. We denote the strict Leximin-Leontief preferences of each agent i by $\succ_i^{lex}$, and the weak relation by $\succeq_i^{lex}$. When we want to emphasize that the leximin relation uses a specific peak $\mathbf{p}_i$, we write $\succ_{\mathbf{p}_i}^{lex}$ and $\succeq_{\mathbf{p}_i}^{lex}$.

We still define the *NASH* rule based on the minimum ratio only, which we continue to denote by $u_i(\mathbf{q})$. Therefore, the *NASH* distribution remains a continuous function of the peaks (even though the Leximin-Leontief preferences are not continuous). However, the change of preferences may potentially affect some properties of the rule. In particular, $\mathbf{q} \succeq_i^{lex} \mathbf{q}'$ implies $u_i(\mathbf{q}) \geq u_i(\mathbf{q}')$, but for the strict relation the opposite direction is true: $u_i(\mathbf{q}) > u_i(\mathbf{q}')$ implies $\mathbf{q} \succ_i^{lex} \mathbf{q}'$. Therefore, properties defined by the weak relation only, such as strategyproofness and weak core fair share, are stronger with Leximin-Leontief preferences than with Leontief utilities. However, properties defined by both the weak and the strict relations, such as group-strategyproofness, core fair share, and efficiency, are not a priori stronger or weaker with Leximin-Leontief preferences than with Leontief utilities.

First, we claim that Lemma 3 still holds, where the critical alternatives are defined as in Definition 12 (based on the minimum ratio only).

Lemma 13. *With Leximin-Leontief preferences, an outcome* $\mathbf{q}$ *is efficient if and only if every alternative* j *with* $q_j > 0$ *is critical for some agent.*

Proof sketch of Lemma 13. ⇒: Suppose that some alternative j with $q_j > 0$ is not critical for any agent. We can construct a new outcome $\mathbf{q}'$ by removing a sufficiently small amount from j and distributing it equally among all other alternatives. This increases $\min_j q_j/p_{i,j}$ for all agents, and thus makes the new distribution strictly leximin-better for all agents. Hence, $\mathbf{q}$ is not efficient.

⇐: Suppose that every alternative j with $q_j > 0$ is critical for some agent. Let $\mathbf{q}'$ be any outcome different than $\mathbf{q}$, and let y be an alternative with $q'_y < q_y$. As $q_y > 0$, the assumption implies that y is critical to some agent; denote one such agent by i_y. Then

$$\begin{aligned}\min_j \frac{q'_j}{p_{i_y,j}} \leq \frac{q'_y}{p_{i_y,y}} < \frac{q_y}{p_{i_y,y}}&\\ = \min_j \frac{q_j}{p_{i_y,j}},& \qquad \text{(as } y \text{ is critical for } i_y \text{ under } \mathbf{q})\end{aligned}$$

so $\mathbf{q}'$ is leximin-worse for i than $\mathbf{q}$. Hence, $\mathbf{q}'$ does not Pareto-dominate $\mathbf{q}$. This holds for all $\mathbf{q}'$, which implies that $\mathbf{q}$ is efficient. □

Lemma 13 implies that $\mathbf{q}$ is efficient for Leximin-Leontief preferences if and only if it is efficient for the corresponding Leontief utilities. In particular, *NASH* remains efficient. Moreover, *NASH* is still neutral and Lemma 8 (efficiency implies one-sided range-respect) remains valid as well.

Next, we show that *NASH* remains group-strategyproof too. We need a lemma.

Lemma 14. *Let $\mathbf{q}'$ and $\mathbf{q}''$ be two distributions, and $i \in N$ an agent. If $\mathbf{q}'' \succeq_i^{lex} \mathbf{q}'$, then every alternative in $T_{\mathbf{q}',i}$ receives at least as much in $\mathbf{q}''$ as in $\mathbf{q}'$, that is, $q''_y \geq q'_y$ for all $y \in T_{\mathbf{q}',i}$.*

Proof. For every alternative $y \in T_{\mathbf{q}',i}$:

$$\begin{aligned}
q'_i &= p_{i,y} \cdot u_i(\mathbf{q}') && \text{(as } y \text{ is critical for } i \text{ under } \mathbf{q}')\\
&\leq p_{i,y} \cdot u_i(\mathbf{q}'') && \text{(since } \mathbf{q}'' \succeq_i^{lex} \mathbf{q}' \text{ implies } u_i(\mathbf{q}'') \geq u_i(\mathbf{q}'))\\
&= p_{i,y} \cdot \min_{j \in M_i} \frac{q''_j}{p_{i,j}} && \text{(by definition of } u_i)\\
&\leq p_{i,y} \cdot \frac{q''_y}{p_{i,y}} && \text{(since } y \in T_{\mathbf{q}',i} \subseteq M_i)\\
&= q''_y,
\end{aligned}$$

completing the proof. □

Theorem 7. *With Leximin-Leontief preferences, NASH is group-strategyproof.*

Proof. Assume for contradiction that there exist profiles P and P' with *NASH* distributions $\mathbf{q} \neq \mathbf{q}'$ respectively, and an inclusion-maximal group of agents $G \subseteq N$ which do not lose from the manipulation from P to P'. Let $T_{\mathbf{q},G} := \bigcup_{i \in G} T_{\mathbf{q},i}$ be the set of alternatives critical to at least one agent from G under $\mathbf{q}$. As no agent from G loses from the manipulation, Lemma 14 implies that $q'_x \geq q_x$ for all $x \in T_{\mathbf{q},G}$.

As $\mathbf{q}' \neq \mathbf{q}$, there is an alternative $y \in M$ for which $q'_y > q_y$. Denote $B := T_{\mathbf{q},G} \cup \{y\}$ (it is possible that $y \in T_{\mathbf{q},G}$). Then, $\mathbf{q}'(B) > \mathbf{q}(B)$.

We now consider the decompositions of $\mathbf{q}$ and $\mathbf{q}'$ guaranteed to exist by Lemma 6. Since $\mathbf{q}'(B) > \mathbf{q}(B)$, there exists an agent $j \in N$ who contributes more to B in the decomposition of $\mathbf{q}'$ than in the decomposition of $\mathbf{q}$. This implies that, in the decomposition of $\mathbf{q}$, agent j contributes some of her share of $1/n$ to alternatives not in B. It follows that $T_{\mathbf{q},j} \not\subseteq B$, so $j \notin G$, and thus $u_j = u'_j$ (as j is not a part of the manipulating group).

In the decomposition of $\mathbf{q}'$, agent j must contribute a positive amount to some alternative $x \in B$, which means that x is critical for j under $\mathbf{q}'$. Since $u_j = u'_j$, we have $u_j(\mathbf{q}') = u'_j(\mathbf{q}') = q'_x/p'_{j,x} \geq q_x/p'_{j,x} \geq u_j(\mathbf{q})$. Therefore, all agents in $G \cup \{j\}$ do not lose from the manipulation, which contradicts the maximality of G. □

We now extend Proposition 5 to Leximin-Leontief preferences.

Proposition 8. *With Leximin-Leontief preferences, NASH satisfies core fair share.*

Proof. Assume for contradiction that there exists $P \in \mathcal{P}$ such that $\mathbf{q} := NASH[P]$ does not satisfy core fair share for some $G \subseteq N$. Then, there exists $\mathbf{q}' \in \Delta^m$ such that, for every $\mathbf{q}'' \in \Delta^m$,

$$(|G|/n)\mathbf{q}' + (1 - |G|/n)\mathbf{q}'' \succeq_i^{lex} \mathbf{q} \qquad \text{for all } i \in G, \text{ and}$$
$$(|G|/n)\mathbf{q}' + (1 - |G|/n)\mathbf{q}'' \succ_i^{lex} \mathbf{q} \qquad \text{for at least one } i \in G.$$

Let $T_{\mathbf{q},G} := \bigcup_{i \in G} T_{\mathbf{q},i}$ be the set of alternatives critical to at least one agent from G.

Note that $T_{\mathbf{q},G} = M$ cannot hold. Otherwise, by Lemma 13, $\mathbf{q}$ would be efficient not only for N but already for G, contradicting that $\mathbf{q}$ does not satisfy core fair share for G. Therefore, there exists a distribution $\mathbf{q}''$ with $q''_j = 0$ for every $j \in T_{\mathbf{q},G}$. Choosing such a distribution $\mathbf{q}''$ shows that $(|G|/n)q'_j \geq q_j$; otherwise some agent from G for whom j is critical would have a smaller utility. Thus, $\mathbf{q}(T_{\mathbf{q},G}) := \sum_{j \in T_{\mathbf{q},G}} q_j \leq (|G|/n) \cdot \sum_{j \in T_{\mathbf{q},G}} q'_j \leq |G|/n$.

By Lemma 6, the *NASH* distribution can be decomposed in such a way that every agent from G only contributes her share of $1/n$ to alternatives in $T_{\mathbf{q},G}$. Thus, $\mathbf{q}(T_{\mathbf{q},G}) \geq |G|/n$. All in all, $\mathbf{q}(T_{\mathbf{q},G}) = |G|/n$ and $(|G|/n)q'_j = q_j$ for every $j \in T_{\mathbf{q},G}$. But this also implies that $(|G|/n) \cdot \mathbf{q}'(T_{\mathbf{q},G}) = |G|/n$, so $\mathbf{q}'(T_{\mathbf{q},G}) = 1$. This means that $\mathbf{q}'$ only allocates to alternatives in $T_{\mathbf{q},G}$. As the allocation to alternatives in $T_{\mathbf{q},G}$ is the same in $\mathbf{q}$ and $(|G|/n)\mathbf{q}'$, no agent in G can have a better leximin vector in $(|G|/n)\mathbf{q}'$ than in $\mathbf{q}$. □

We now consider the characterization (Theorem 5) for Leximin-Leontief preferences. It turns out that group-strategyproofness can be weakened to strategyproofness.

As in the proof of Theorem 5, to show the statement, we would like to change P gradually to P^*, where each agent's peak puts 0 on non-critical alternatives. However, in order to exploit the fact that Leximin-Leontief preferences constitute a refinement of Leontief utilities, which allows us to weaken group-strategyproofness to strategyproofness, we need to adapt the proof. We first show that f coincides with *NASH* on all *strictly-positive profiles*, that is, all profiles $P \in \mathcal{P}^+$, where $\mathcal{P}^+ := \{P \in \mathcal{P} : p_{i,j} > 0 \text{ for all } i \in N, j \in M\}$.

Lemma 15. *If f is a continuous mechanism satisfying strategyproofness, then when moving from P to P^*,*

(a) the outcome does not change, that is, $f(P^*) = f(P) = \mathbf{q}$ for $P \in \mathcal{P}^+$;

(b) the sets of critical alternatives do not change, that is, $T_{\mathbf{q},i} = T^*_{\mathbf{q},i}$ for every $i \in N$.

Proof. We first show the statement for a slightly perturbed $\tilde{P}^*_{\tilde{\varepsilon}}$ with $\tilde{\mathbf{p}}^*_i = (1 - \tilde{\varepsilon})\mathbf{p}^*_i + \tilde{\varepsilon}\mathbf{p}_i$ and arbitrary small but fixed $\tilde{\varepsilon} > 0$. Note that P^* may not be in $\mathcal{P}^+$, but $\tilde{P}^*_{\tilde{\varepsilon}}$ is always in $\mathcal{P}^+$. Note also that, for an agent i with $T_{\mathbf{q},i} = M$, it holds that $\mathbf{p}_i = \tilde{\mathbf{p}}^*_i = \mathbf{p}^*_i$.

Again, we move the peak of each agent in turn. For each agent i, we change $\mathbf{p}_i$ towards $\tilde{\mathbf{p}}_i^*$ gradually, to some $\widehat{\mathbf{p}}_i := \lambda\tilde{\mathbf{p}}_i^* + (1-\lambda)\mathbf{p}_i$, for some $\lambda \in [0,1]$ to be computed later. Then we proceed along this line until we reach $\lambda = 1$ and $\tilde{\mathbf{p}}_i^*$.

Given the outcome $\mathbf{q} = f(P)$, we partition the alternatives of each agent i into *critical classes*, i.e., subsets with the same ratio $q_j/p_{i,j}$. Here we use the fact that $P \in \mathcal{P}^+$, so $p_{i,j} > 0$ for all i, j. Denote the subset with the smallest ratio by $T_{\mathbf{q},i,1} \equiv T_{\mathbf{q},i}$, the subset with the second-smallest ratio by $T_{\mathbf{q},i,2}$, and so on, up to $T_{\mathbf{q},i,w}$, where w is the number of different ratios. Also, for $r \in [w]$, denote $T_{\mathbf{q},i,\leq r} := T_{\mathbf{q},i,1} \cup \cdots \cup T_{\mathbf{q},i,r}$, and define $T_{\mathbf{q},i,>r}$ analogously.

As $\widehat{\mathbf{p}}_i$ lies along the line between $\mathbf{p}_i$ and $\mathbf{p}_i^*$, the change from $\mathbf{p}_i$ to $\widehat{\mathbf{p}}_i$ has a simple structure:

- $\widehat{p}_{i,j} > p_{i,j}$ for all $j \in T_{\mathbf{q},i,1}$, and the ratio $\widehat{p}_{i,j}/p_{i,j} = \lambda(1-\tilde{\varepsilon})/\mathbf{p}_i(T_{\mathbf{q},i,1}) + \lambda\tilde{\varepsilon} + (1-\lambda) =: \lambda^+$, a constant independent of j;
- $\widehat{p}_{i,j} < p_{i,j}$ for all $j \in T_{\mathbf{q},i,>1}$, and the ratio $\widehat{p}_{i,j}/p_{i,j} = \lambda\tilde{\varepsilon} + (1-\lambda) =: \lambda^-$, again independent of j.

Computing λ. We pick λ sufficiently small such that no new alternative becomes critical for i, and moreover, critical classes do not mix, i.e., $q'_j/p_{i,j'} > q_j/p_{i,j}$ implies $\widehat{q}_{j'}/p_{i,j'} > \widehat{q}_j/p_{i,j}$ for all $j, j' \in M$. Specifically, set

$$\varepsilon := \min_{j \in T_{\mathbf{q},i,r},\, j' \in T_{\mathbf{q},i,s}} \frac{q_{j'}p_{i,j} - q_j p_{i,j'}}{p_{i,j} + p_{i,j'}}$$

where the minimum is taken over all $r, s \in [w]$ and $s > r$. Note that $\varepsilon > 0$, as $q_{j'}/p_{i,j'} > q_j/p_{i,j}$, by definition of critical classes.

By uniform continuity of f, there exists $\delta > 0$ such that $\|f(P) - f(P')\|_1 < 2\varepsilon$ for all P' with $\|P - P'\|_1 \leq \delta$. Set

$$\lambda := \min\left(1, \frac{\delta}{\|\mathbf{p}_i - \tilde{\mathbf{p}}_i^*\|_1}\right),$$

and define $\widehat{P}$ as a profile identical to P except that i changes her peak from $\mathbf{p}_i$ to $\widehat{\mathbf{p}}_i := \lambda\tilde{\mathbf{p}}_i^* + (1-\lambda)\mathbf{p}_i$. Note that $\|P - \widehat{P}\|_1 = \lambda\|\mathbf{p}_i - \tilde{\mathbf{p}}_i^*\|_1 \leq \delta$, so $\|\mathbf{q} - \widehat{\mathbf{q}}\|_1 < 2\varepsilon$, where $\mathbf{q} = f(P)$ and $\widehat{\mathbf{q}} = f(\widehat{P})$.

The choice of ε ensures that for arbitrary $r, s \in [w]$ with $s > r$, $j \in T_{\mathbf{q},i,r}$, and $j' \in T_{\mathbf{q},i,s}$,

$$\begin{aligned}\frac{\widehat{q}_{j'}}{p_{i,j'}} &> \frac{q_{j'} - \varepsilon}{p_{i,j'}} \geq \frac{q_{j'}}{p_{i,j'}} - \frac{q_{j'}p_{i,j} - q_j p_{i,j'}}{p_{i,j'}(p_{i,j} + p_{i,j'})} = \frac{q_{j'}p_{i,j'} + q_j p_{i,j'}}{p_{i,j'}(p_{i,j} + p_{i,j'})} = \frac{q_{j'} + q_j}{p_{i,j} + p_{i,j'}} \\ &= \frac{q_j p_{i,j} + q_{j'} p_{i,j}}{p_{i,j}(p_{i,j} + p_{i,j'})} = \frac{q_j}{p_{i,j}} + \frac{q_{j'}p_{i,j} - q_j p_{i,j'}}{p_{i,j}(p_{i,j} + p_{i,j'})} \geq \frac{q_j + \varepsilon}{p_{i,j}} > \frac{\widehat{q}_j}{p_{i,j}}.\end{aligned}$$

Therefore, we have $T_{\mathbf{q},i,r} = \widehat{T}_{\mathbf{q},i,r}$ for all $r \in [w]$.

Proving that the outcome does not change.

Consider a manipulation of agent i who manipulates between reporting $\mathbf{p}_i$ and $\widehat{\mathbf{p}}_i$. Strategyproofness for agent i implies both $\mathbf{q} \succeq^{lex}_{\mathbf{p}_i} \widehat{\mathbf{q}}$ and $\widehat{\mathbf{q}} \succeq^{lex}_{\widehat{\mathbf{p}}_i} \mathbf{q}$.

We now prove, by induction on r, that $q_j = \widehat{q}_j$ for all $j \in T_{\mathbf{q},i,r}$. For the base case $r = 1$, consider the alternatives in $T_{\mathbf{q},i,1}$.

- As all alternatives in $T_{\mathbf{q},i,1}$ are at the bottom of the ordering by $q_j/p_{i,j}$ (by definition) as well as by $\widehat{q}_j/p_{i,j}$ (by the choice of ε), the relation $\mathbf{q} \succeq^{lex}_{\mathbf{p}_i} \widehat{\mathbf{q}}$ implies the same relation among the sub-vectors corresponding to the alternatives in $T_{\mathbf{q},i,1}$, that is,

$$[q_j \mid j \in T_{\mathbf{q},i,1}] \succeq^{lex}_{\mathbf{p}_i} [\widehat{q}_j \mid j \in T_{\mathbf{q},i,1}]. \tag{16}$$

• Similarly, all alternatives in $\widehat{T}_{\mathbf{q},i,1} = T_{\mathbf{q},i,1}$ are at the bottom of the ordering by $q_j/\widehat{p}_{i,j}$ by construction. Therefore, the relation $\widehat{\mathbf{q}} \succeq^{lex}_{\widehat{\mathbf{p}}_i} \mathbf{q}$ implies

$$[\widehat{q}_j \mid j \in T_{\mathbf{q},i,1}] \succeq^{lex}_{\widehat{\mathbf{p}}_i} [q_j \mid j \in T_{\mathbf{q},i,1}]. \tag{17}$$

• But since $\widehat{p}_{i,j}$ differs from $p_{i,j}$ by a constant factor λ^+ for all $j \in T_{\mathbf{q},i,1}$, (17) implies the same inequality with $\succeq^{lex}_{\mathbf{p}_i}$ instead of $\succeq^{lex}_{\widehat{\mathbf{p}}_i}$. Combining this with (16), we get

$$[\widehat{q}_j \mid j \in T_{\mathbf{q},i,1}] \simeq^{lex}_{\mathbf{p}_i} [q_j \mid j \in T_{\mathbf{q},i,1}].$$

As $q_j/p_{i,j}$ is constant for $j \in T_{\mathbf{q},i,1}$, lexicographic equivalence with respect to $\mathbf{p}_i$ implies $q_j/p_{i,j} = \widehat{q}_j/p_{i,j}$ for all $j \in T_{\mathbf{q},i,1}$. Thus, $q_j = \widehat{q}_j$ must hold for all $j \in T_{\mathbf{q},i,1}$.

For the induction step, assume that $q_j = \widehat{q}_j$ holds for all $j \in T_{\mathbf{q},i,\le r}$, for some $r \in [w-1]$. Next, consider the alternatives in $T_{\mathbf{q},i,r+1} = \widehat{T}_{\mathbf{q},i,r+1}$.

• As $q_j = \widehat{q}_j$ holds for all other alternatives with smaller ratios, the relation $\mathbf{q} \succeq^{lex}_{\mathbf{p}_i} \widehat{\mathbf{q}}$ implies the same relation for the subset $T_{\mathbf{q},i,r+1}$, that is, $[q_j \mid j \in T_{\mathbf{q},i,r+1}] \succeq^{lex}_{\mathbf{p}_i} [\widehat{q}_j \mid j \in T_{\mathbf{q},i,r+1}]$.

• Similarly, the relation $\widehat{\mathbf{q}} \succeq^{lex}_{\widehat{\mathbf{p}}_i} \mathbf{q}$ implies $[\widehat{q}_j \mid j \in T_{\mathbf{q},i,r+1}] \succeq^{lex}_{\widehat{\mathbf{p}}_i} [q_j \mid j \in T_{\mathbf{q},i,r+1}]$.

• But since $\widehat{p}_{i,j}$ differs from $p_{i,j}$ by a constant factor λ^- for all $j \in T_{\mathbf{q},i,r+1}$, $\widehat{\mathbf{q}} \succeq^{lex}_{\widehat{\mathbf{p}}_i} \mathbf{q}$ implies $\widehat{\mathbf{q}} \succeq^{lex}_{\mathbf{p}_i} \mathbf{q}$, so all in all $[\widehat{q}_j \mid j \in T_{\mathbf{q},i,r+1}] \simeq^{lex}_{\mathbf{p}_i} [q_j \mid j \in T_{\mathbf{q},i,r+1}]$ must hold. This means that $q_j = \widehat{q}_j$ must hold for all $j \in T_{\mathbf{q},i,r+1}$.

Therefore, $\widehat{\mathbf{q}} = \mathbf{q}$.

Applying this argument repeatedly, we get a sequence of profiles (P^k) with $P^0 = P$ where $\mathbf{p}_i^k$ lies on the line $\lambda\tilde{\mathbf{p}}_i^* + (1-\lambda)\mathbf{p}_i$ for every k. It remains to show that $(\mathbf{p}^k)$ reaches $\tilde{\mathbf{p}}_i^*$ after a finite number of steps. For that, consider the expression in the definition of ε:

$$\min_{j \in T_{\mathbf{q},i,r},\, j' \in T_{\mathbf{q},i,s}} \frac{q_{j'} p_{i,j} - q_j p_{i,j'}}{p_{i,j} + p_{i,j'}}.$$

For $r = 1$, as $\mathbf{p}_i$ comes closer to $\tilde{\mathbf{p}}_i^*$, $p_{i,j}$ increases and $p_{i,j'}$ decreases while $\mathbf{q}$ and the critical classes stay the same, so overall the expression increases. For $s > r > 1$, note that $p_{i,j}$ and $p_{i,j'}$ both decrease by the same factor λ^- while $\mathbf{q}$ and the critical classes stay the same. Thus, we can take the ε (and the corresponding δ) from the first step for every step. Furthermore, $\|P^k - P^{k+1}\|_1 = \delta$ (unless $\lambda = 1$, but then we have reached $\tilde{\mathbf{p}}_i^*$) implying that we reach $\tilde{\mathbf{p}}_i^*$ after at most $\lceil \|\mathbf{p}_i - \tilde{\mathbf{p}}_i^*\|_1/\delta \rceil$ steps; as we move on a line of length $\|P^k - P^{k'}\|_1 = \sum_{\ell=k}^{k'-1} \|P^\ell - P^{\ell+1}\|_1$ for $k' \ge k$.

After the first agent has reached her desired peak $\tilde{\mathbf{p}}_i^*$, we turn to the next agent and repeat the procedure. In that way, we eventually arrive at $\tilde{P}^*_{\tilde{\varepsilon}}$.

As $\tilde{\varepsilon}$ was chosen arbitrarily and we have $\lim_{\tilde{\varepsilon}\to 0} P^*_{\tilde{\varepsilon}} = P^*$ for arbitrary $P \in \mathcal{P}^+$, continuity of f implies $f(P) = f(P^*)$.

Statement (b) now follows analogously to the one in Lemma 9. □

Proof of Theorem 6. As Lemmas 10 and 11 still hold for Leximin-Leontief preferences, $f(P) = NASH[P]$ for all $P \in \mathcal{P}^+$. Noting that $\mathcal{P}^+$ is dense in $\mathcal{P}$ (and f and $NASH$ are continuous), f has to coincide with $NASH$ on all profiles in $\mathcal{P}$. □