# Optimal Budget Aggregation with Single-Peaked Preferences 

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We study the problem of aggregating distributions, such as budget proposals, into a collective distribution. An ideal aggregation mechanism would be Pareto efficient, strategyproof, and fair. Most previous work assumes that agents evaluate budgets according to the $\ell_{1}$ distance to their ideal budget. We investigate and compare different models from the larger class of starshaped utility functions - a multi-dimensional generalization of single-peaked preferences. For the case of two alternatives, we extend existing results by proving that under very general assumptions, the uniform phantom mechanism is the only strategyproof mechanism that satisfies proportionality - a minimal notion of fairness introduced by Freeman et al. (2021). Moving to the case of more than two alternatives, we establish sweeping impossibilities for $\ell_{1}$ and $\ell_{\infty}$ disutilities: no mechanism satisfies efficiency, strategyproofness, and proportionality. We then propose a new kind of star-shaped utilities based on evaluating budgets by the ratios of shares between a given budget and an ideal budget. For these utilities, efficiency, strategyproofness, and fairness become compatible. In particular, we prove that the mechanism that maximizes the Nash product of individual utilities is characterized by group-strategyproofness and a core-based fairness condition.

## 1. Introduction

Social choice theory is concerned with the aggregation of individual preferences into a collective outcome (e.g., Arrow et al., 2002, 2011). An important special case arises when the potential collective outcomes are distributions over a fixed set of alternatives. These distributions may represent how a budget should be divided among public projects in a city or among departments in an organization. Alternatively, they may reflect how time or space ought to be allotted between different types of activities at a social event. This scenario is sometimes referred to as budget aggregation or portioning and falls under the framework of participatory budgeting, which has received increasing interest in recent years (Aziz and Shah, 2021; De Vries et al., 2022).

In order to reason about the agents' satisfaction with the collective outcome, one needs to make some assumptions about their preferences. Importantly, in our setting, the realized outcome is a distribution. Therefore, restricting attention to rankings over alternatives is insufficient, as an agent's most preferred outcome is typically a nondegenerate distribution over the alternatives. This is particularly evident in participatory budgeting problems, where even if an agent has a favorite project, she normally also likes other projects and does not want them to be left completely unfunded. ${ }^{1}$ In this paper, consistent with previous research in this domain, we mainly consider utility models where agents' preferences are completely determined by their favorite distribution: their "peak". This keeps the amount of required information from each agent at a manageable level. We assume that each agent's utility decreases as the actual distribution is further from her peak. Such utility functions are called star-shaped (Border and Jordan, 1983).

For two alternatives, i.e., outcomes on the unit interval [ 0,1$]$, star-shaped utilities are equivalent to single-peaked preferences. There is a rather restrictive class of mechanisms that are strategyproof for all single-peaked preferences, the so-called generalized median rules (Moulin, 1980). Moulin's characterization leaves open the possibility that, for restricted subdomains of single-peaked preferences, other mechanisms than generalized median rules are strategyproof. In Section 4, we obtain characterizations which hold not only for single-peaked, but also for any subdomain of single-peaked preferences, by imposing continuity on the mechanism. Our characterizations refine results by Freeman et al. (2021) and Aziz et al. (2022).

For more than two alternatives, most of the previous work on budget aggregation assumes that preferences are given via the $\ell_{1}$ norm (e.g., Lindner et al., 2008; Goel et al., 2019b; Freeman et al., 2021; Caragiannis et al., 2022; Freeman and Schmidt-Kraepelin, 2024). According to $\ell_{1}$ preferences, agent $i$ 's disutility for a distribution $\mathbf{q}=\left(q_{1}, \ldots, q_{m}\right)$ is given by the $\ell_{1}$ distance $\left\|\mathbf{p}_{i}-\mathbf{q}\right\|_{1}=\sum_{j=1}^{m}\left|p_{i, j}-q_{j}\right|$, where $\mathbf{p}_{i}=\left(p_{i, 1}, \ldots, p_{i, m}\right)$ is the agent's peak. We assume without loss of generality that the sum of all components in a distribution is 1 . Under this utility model, Lindner et al. (2008) and Goel et al. (2019b) showed that a mechanism that maximizes utilitarian welfare (i.e., minimizes the sum of agents' disutilities) is both strategyproof and efficient; however, this mechanism has a tendency to overweight majority preferences. Freeman et al. (2021) proposed a mechanism, called the independent markets mechanism, which satisfies strategyproofness along with a weak fairness notion dubbed proportionality. Proportionality requires that the collective distribution is given by the uniform distribution over the agents' peaks whenever all peaks are degenerate. The independent markets mechanism violates efficiency, and Freeman et al. raised the question whether there are mechanisms that satisfy all three properties simultaneously. In Section 6, we settle this question by proving that no such mechanism exists under $\ell_{1}$ as well as under $\ell_{\infty}$ preferences.

Using $\ell_{1}$ distances to define preferences over distributions has some shortcomings when aiming for a suitable representation of alternatives in the collective distribution.

[^0]For instance, if the agents are deciding the amount of time that should be allotted to three countries at an international conference, an agent with an ideal distribution of $(10 \%, 40 \%, 50 \%)$ would find the outcome $(0 \%, 45 \%, 55 \%)$ to be quite desirable according to the $\ell_{1}$ distance, despite the fact that this outcome leaves the first country completely unrepresented. Moreover, any distance-based metric that aggregates coordinate-wise differences, such as $\ell_{2}$ or $\ell_{\infty}$ (or $\ell_{p}$ for any $p \geq 1$ ) has similar shortcomings. For example, if a citizen believes that the two larger districts deserve $40 \%$ of the city budget each and the two smaller districts $10 \%$ each, then for any $p \geq 1, \ell_{p}$ preferences dictate that she is indifferent between the distributions $(50 \%, 30 \%, 10 \%, 10 \%)$ and $(40 \%, 40 \%, 20 \%, 0 \%)$, since for both distributions, the multiset of coordinate-wise differences is $\{0 \%, 0 \%, 10 \%, 10 \%\}$. But intuitively the latter distribution is worse, as it leaves the last district without any funds. As a consequence, a different type of utility function is necessary to capture the representation of alternatives with respect to the ideal distribution.

We introduce a new class of utility functions to the budget aggregation setting where each agent has an ideal distribution and the agent's utility for a distribution equals the smallest quotient, over all alternatives, that the distribution preserves in comparison to the ideal distribution. Formally, agent $i$ 's utility for a distribution $\mathbf{q}$ is given by $\min _{j} q_{j} / p_{i, j}$, where the minimum is taken over all alternatives $j$ for which $p_{i, j}>0$. We call these utility functions minimal quotient ( $M Q$ ) utilities. MQ utilities bear some resemblance to Leontief utilities as commonly studied in economics, especially in consumer theory, with goods corresponding to alternatives (see, e.g., Nicoló, 2004; Li and Xue, 2013; Goel et al., 2019a). However, for Leontief utilities, the denominators are given by weights associated with different goods rather than an ideal distribution, while the numerators represent the amounts of goods; in particular, the numerators and the denominators do not necessarily share the same normalization. Intuitively, an agent with MQ utilities wants all alternatives to receive as large a fraction as possible of their ideal amounts. As such, these utilities are arguably more suitable than $\ell_{1}$ preferences in applications where the representation of the alternatives is crucial - indeed, for both examples in the previous paragraph, distributions that allocate none of the budget to some alternative are least preferred among all distributions according to MQ utilities. Not surprisingly, mechanisms that provide desirable properties such as strategyproofness with respect to $\ell_{1}$ preferences may fail to do so with MQ utilities. ${ }^{2}$ Therefore, one needs to find different mechanisms when dealing with MQ utilities. In Section 7, we show that maximizing Nash welfare, i.e., the product of agents' utilities, results in a mechanism with several desirable properties. In fact, the impossibility for $\ell_{1}$ preferences established in Section 6 can be turned into a complete characterization for MQ utilities: only the Nash product rule satisfies group-strategyproofness and a natural core-based fairness notion, which strengthens both efficiency and proportionality. Thus, in contrast to $\ell_{1}$ preferences, MQ utilities allow for the efficient, strategyproof, and fair aggregation of budgets via a unique attractive mechanism.

[^1]
## 2. Related work

Any model in which preferences over lotteries are aggregated to a collective lottery, including literature on probabilistic social choice (e.g., Gibbard, 1977) and fair mixing (e.g., Bogomolnaia et al., 2005), can be interpreted in the context of budget aggregation. However, the underlying assumptions on preferences are hardly applicable to budget aggregation because the ideal distributions of agents always include degenerate distributions. Lindner et al. (2008) and Goel et al. (2019b) therefore introduced models in which each agent has a unique ideal distribution and the utilities over all remaining distributions are given by the $\ell_{1}$ distance to this peak. In other words, each agent's preference relation is completely induced by her peak, and the considered mechanisms only need to aggregate individual distributions into a collective distribution. This idea can be traced back to Intriligator (1973) who proposed three simple axioms for this setting, which characterize the rule that returns the average (i.e., the arithmetic mean) of all individual distributions. Intriligator was not concerned with strategyproofness, but it is fairly obvious that the average rule is highly manipulable for almost any reasonable definition of preferences. In this paper, we follow Intriligator's "refreshing change from typical approaches which presume that each individual has a fixed preference order on the alternatives" (Fishburn, 1975) by considering the general class of star-shaped utilities.

Freeman et al. (2021) expanded the idea of Moulin's generalized median rules for two alternatives to strategyproof moving phantom mechanisms for larger numbers of alternatives $m$. Intuitively, the $n+1$ phantoms are not "fixed" like in Moulin's characterization but increase continuously from 0 to 1 over time. For any point in time and any alternative $j$, the mechanism computes the median of $p_{1, j}, \ldots, p_{n, j}$ and the phantom voters. Freeman et al. then showed that there exists a well-defined point in time where the $m$ medians sum up to 1 and thus form a valid distribution. Furthermore, they proved that within this class, maximizing utilitarian welfare is the unique efficient mechanism. A different mechanism in this class, the independent markets mechanism, is inefficient but satisfies a fairness notion they called proportionality: when all voters have degenerate peaks, the collective distribution is the arithmetic mean of these peaks. Freeman et al. observed an "inherent tradeoff between Pareto optimality and proportionality" for strategyproof mechanisms. We prove this tradeoff formally in Theorem 6.1, which shows that all three properties are incompatible.

Proportionality was generalized by Caragiannis et al. (2022), who measured the "disproportionality" of a mechanism as the worst-case $\ell_{1}$ distance between the mechanism outcome and the mean. Similarly, Freeman and Schmidt-Kraepelin (2024) measured disproportionality using the $\ell_{\infty}$ distance. Both papers present variants of moving phantom mechanisms that guarantee low disproportionality. Elkind et al. (2023) defined various axioms for budget aggregation with $\ell_{1}$ disutilities, analyzed the implications between axioms, and determined which axioms are satisfied by common aggregation rules.

Nehring and Puppe $(2022,2023)$ studied a more general preference domain: they assumed only that agents' preferences are convex and separable, or convex and quadratic. In these larger domains, the only strategyproof mechanisms are dictatorships (Nehring
and Puppe, 2007). These authors also studied an aggregation rule that selects all alternatives that are ex-ante Condorcet winners, i.e., Condorcet winners for all preference profiles consistent with the agents' peaks.

Belief aggregation is a setting in which several experts have different beliefs, expressed as probability density functions over a set of potential outcomes. The goal is to construct a single aggregated distribution. Technically, the problem is similar to budget aggregation; however, the utility functions are often different. Varloot and Laraki (2022) assumed that the outcomes are linearly ordered (for example: outcome $j$ is an earthquake of magnitude $j$ ). Then, an expert whose belief is 3 with probability 1 would prefer the outcome 6 with probability 1 to the outcome 9 with probability 1 , even though the $\ell_{1}$ distance is 2 in both cases. These authors suggested preferences based on distance between the cumulative distribution functions, and characterized aggregation rules satisfying appropriate strategyproofness and proportionality axioms.

Brandt et al. (2023) studied donor coordination, where individual monetary contributions by agents are distributed on projects based on the agents' preferences. Assuming Leontief preferences, they proposed the equilibrium distribution rule ( $E D R$ ), which maximizes Nash welfare and distributes the contributions of the donors in such a way that no subset of donors has an incentive to redistribute their contributions. $E D R$ can be interpreted as a budget aggregation mechanism, by setting the contribution of each agent to $1 / n$ (where $n$ is the number of agents, so the total contribution is 1 ) and setting the ideal distribution of an agent to the distribution given by the relative proportions of the Leontief weights. This allows the transfer of positive results concerning efficiency and strategyproofness of $E D R$ from donor coordination to budget aggregation (see Section 7). However, other properties considered by Brandt et al., like contribution-monotonicity and being in equilibrium, are irrelevant in budget aggregation, whereas properties like core fair share and proportionality have not been considered in donor coordination.

## 3. Preliminaries

Let $N=[n]$ be a set of agents and $M=[m]$ be a set of alternatives, where $[k]:=$ $\{1, \ldots, k\}$ for each positive integer $k$. A distribution is a vector $\mathbf{q}=\left(q_{1}, \ldots, q_{m}\right) \in \Delta^{m}$ with nonnegative entries $q_{j}$ such that $\sum_{j \in M} q_{j}=1$; here, $q_{j}$ denotes the fraction of a public resource (e.g., money or time) allocated to alternative $j$. For a set of alternatives $T \subseteq M$, we denote $\mathbf{q}(T):=\sum_{j \in T} q_{j}$.

Each agent $i$ has a utility function $u_{i}$ over distributions; we denote by $\mathcal{U}$ the set of all possible utility functions and leave this set unspecified for now.

The ideal distribution or peak of agent $i$ is denoted by $\mathbf{p}_{i}:=\arg \max _{\mathbf{q} \in \Delta^{m}} u_{i}(\mathbf{q})$, which is assumed to be unique. If not explicitly stated otherwise, $\mathbf{p}_{i}$ completely determines $u_{i}$, and we can identify a profile $P \in \mathcal{P}:=\left(\Delta^{m}\right)^{n}$ with the matrix $\left(p_{i, j}\right)_{i \in N, j \in M}$ containing the peaks $\mathbf{p}_{1}, \ldots, \mathbf{p}_{n}$ as rows. We further assume that "walking" towards an agent's peak strictly increases her utility. This constitutes a generalization of single-peakedness (Black, 1948) and is sometimes referred to as star-shaped preferences (Border and Jordan, 1983). More formally, if agent $i$ has peak $\mathbf{p}_{i}$ and utility function $u_{i}$, then, for any
distribution $\mathbf{q} \neq \mathbf{p}_{i}$ and $\lambda \in(0,1)$,

$$
u_{i}\left(\mathbf{p}_{i}\right)>u_{i}\left(\lambda \mathbf{p}_{i}+(1-\lambda) \mathbf{q}\right)>u_{i}(\mathbf{q}) .
$$

A mechanism $f: \mathcal{P} \rightarrow \Delta^{m}$ aggregates individual distributions into a collective distribution. In the following, we define desirable properties of aggregated distributions and mechanisms.

### 3.1. Properties of distributions

Two important properties of distributions are efficiency and fairness.
Definition 3.1. A distribution $\mathbf{q} \in \Delta^{m}$ satisfies (Pareto) efficiency if there does not exist a distribution $\mathbf{q}^{\prime} \in \Delta^{m}$ such that $u_{i}\left(\mathbf{q}^{\prime}\right) \geq u_{i}(\mathbf{q})$ for all $i \in N$ and $u_{i}\left(\mathbf{q}^{\prime}\right)>u_{i}(\mathbf{q})$ for at least one $i \in N$.

Our first fairness axiom is inspired by the core in cooperative game theory and was transferred to participatory budgeting by Fain et al. (2016) and to fair mixing by Aziz et al. (2020) under the name of core fair share. We slightly adapt the notation to account for the fact that, in the end, we still need to choose a probability distribution $p$ (and not just a partial distribution $\left(\left|N^{\prime}\right| / n\right) p$ ).

Definition 3.2. A distribution $\mathbf{q}$ satisfies core fair share (CFS) if for every group of agents $N^{\prime} \subseteq N$, there is no distribution $\mathbf{q}^{\prime}$ such that the following hold for every $\mathbf{q}^{\prime \prime} \in \Delta^{m}$ :

$$
\begin{array}{lr}
u_{i}\left(\left(\left|N^{\prime}\right| / n\right) \mathbf{q}^{\prime}+\left(1-\left|N^{\prime}\right| / n\right) \mathbf{q}^{\prime \prime}\right) \geq u_{i}(\mathbf{q}) & \text { for all } i \in N^{\prime}, \text { and } \\
u_{i}\left(\left(\left|N^{\prime}\right| / n\right) \mathbf{q}^{\prime}+\left(1-\left|N^{\prime}\right| / n\right) \mathbf{q}^{\prime \prime}\right)>u_{i}(\mathbf{q}) & \text { for at least one } i \in N^{\prime} .
\end{array}
$$

A distribution $\mathbf{q}$ satisfies weak core fair share if we replace the above two conditions with:

$$
u_{i}\left(\left(\left|N^{\prime}\right| / n\right) \mathbf{q}^{\prime}+\left(1-\left|N^{\prime}\right| / n\right) \mathbf{q}^{\prime \prime}\right)>u_{i}(\mathbf{q}) \quad \text { for all } i \in N^{\prime}
$$

Intuitively, if there is a distribution $\mathbf{q}$ that satisfies these inequalities (so CFS is violated), then $N^{\prime}$ can take their share of the decision power $\left(\left|N^{\prime}\right| / n\right)$, and redistribute it via $\mathbf{q}^{\prime}$ so that no member of $N^{\prime}$ loses and at least one member (or, in the case of weak CFS, all members) gains utility compared to $\mathbf{q}$, even if the remaining probability is distributed in the worst possible way $\mathbf{q}^{\prime \prime}$ (e.g., on an alternative that no agent from $N^{\prime}$ values).

It is easy to see that efficiency is the special case of CFS where $N^{\prime}=N$.
Proposition 3.3. Core fair share implies efficiency.
For some utility models, even weak CFS implies efficiency; see Corollary 5.5. We consider another, weaker fairness axiom that is only informative on specific profiles in the next subsection.

### 3.2. Properties of aggregation mechanisms

Definition 3.4. A mechanism $f$ satisfies efficiency (resp., core fair share) if for every profile $P \in \mathcal{P}, f(P)$ satisfies efficiency (resp., core fair share).

The next axioms ensure that agents and alternatives are treated independently of their identities.

Definition 3.5. A mechanism $f$ satisfies anonymity if for every profile $P \in \mathcal{P}$ and permutation $\pi$ of the agents in $P$, it holds that $f(P)=f(\pi \circ P)$.

Definition 3.6. A mechanism $f$ satisfies neutrality if for every profile $P \in \mathcal{P}$ and permutation $\pi$ of the alternatives resulting in profile $P^{\prime}$, it holds that $f\left(P^{\prime}\right)=\pi \circ f(P)$.

As agents report a peak in $\Delta^{m}$, we do not want small perturbations of the peaks arising from uncertainties of the agents about their exact peak or inaccuracies during the aggregation process to have a large influence on the outcome.

Definition 3.7. A mechanism $f$ satisfies continuity if

$$
\forall P \in \mathcal{P} \forall \varepsilon>0 \exists \delta>0 \forall P^{\prime} \in \mathcal{P}:\left\|P-P^{\prime}\right\|_{1}<\delta \Longrightarrow\left\|f(P)-f\left(P^{\prime}\right)\right\|_{1}<\varepsilon .
$$

For simplicity, we define continuity using the $\ell_{1}$ distance, but note that due to the norm equivalence on finite-dimensional vector spaces, this choice has no impact on the generality of our results.

As $\mathcal{P}=\left(\Delta^{m}\right)^{n}$ is compact with respect to the $\ell_{1}$ distance (or other equivalent norms), the Heine-Cantor theorem implies that a continuous mechanism $f$ is also uniformly continuous, i.e.,

$$
\forall \varepsilon>0 \exists \delta>0 \forall P, P^{\prime} \in \mathcal{P}:\left\|P-P^{\prime}\right\|_{1}<\delta \Longrightarrow\left\|f(P)-f\left(P^{\prime}\right)\right\|_{1}<\varepsilon .
$$

This insight will play an important role in the proof of Theorem 7.9.
Another common goal is to prevent agents from misreporting their peaks on purpose.
Definition 3.8. A mechanism $f$ satisfies group-strategyproofness if for all $N^{\prime} \subseteq N$ and all $P, P^{\prime} \in \mathcal{P}$ with $\mathbf{p}_{i}=\mathbf{p}_{i}^{\prime}$ for $i \notin N^{\prime}$, either $u_{i}(f(P))>u_{i}\left(f\left(P^{\prime}\right)\right)$ for at least one $i \in N^{\prime}$ or $u_{i}(f(P))=u_{i}\left(f\left(P^{\prime}\right)\right)$ for all $i \in N^{\prime}$, where $u_{i}$ refers to the utility function of agent $i$ with peak at $\mathbf{p}_{i}$. The mechanism $f$ satisfies strategyproofness if the above statement holds for $\left|N^{\prime}\right|=1$.

Finally, we consider another fairness property called proportionality by Freeman et al. (2021). It restricts the set of outcomes only on profiles where all agents are "singleminded", thus representing a rather weak form of "traditional" proportionality considered, e.g., in fair division.

Definition 3.9. A mechanism $f$ satisfies proportionality if for all $P \in \mathcal{P}$ with $p_{i, j} \in$ $\{0,1\}$ for all $i \in N$ and $j \in M$, it holds that $f(P)_{j}=\sum_{i \in N} p_{i, j} / n$.

The following diagram shows logical relationships between efficiency and the fairness notions we consider.


## 4. The case of two alternatives

For two alternatives $M=\{a, b\}$, the set of outcomes can be identified with the unit interval $[0,1]$, where the endpoints 0 and 1 correspond to allocating the entire budget to alternatives $a$ and $b$, respectively. We denote agent $i$ 's peak $p_{i}$ as a scalar in $[0,1]$ representing her favorite distribution $\left[1-p_{i}: a, p_{i}: b\right]$. In this section, we deviate from our assumption that $\mathcal{U}$ has to contain exactly one utility function per peak in $[0,1]$; we only demand that $\mathcal{U}$ contains at least one utility function per peak. ${ }^{3}$ This generalization is possible because the mechanisms we characterize satisfy strategyproofness and further desirable properties without relying on any knowledge of the agents' utility functions except their peaks.

In this one-dimensional setting, the class of star-shaped utilities coincides with the well-studied class of single-peaked utilities. Denote by $\mathcal{U}^{\mathcal{P} \mathcal{P}}$ the set of all single-peaked utility functions. Under the assumption that $\mathcal{U}=\mathcal{U}^{\mathcal{S P}}$, Moulin (1980) characterized the set of all strategyproof mechanisms as generalized median rules. His proof requires some specific utility functions to be in $\mathcal{U}$. As a consequence, his characterization no longer holds when restricting $\mathcal{U}$, e.g., to $\ell_{1}$ preferences (a strict subset of $\mathcal{U}^{\mathcal{S P}}$ ).

Under this particular restriction, Freeman et al. (2021) noted that Moulin's characterization for $m=2$ still applies when demanding continuity of the aggregation mechanism; this follows from a result of Massó and de Barreda (2011). In the following, we prove that imposing continuity on the mechanism leads to Moulin's characterization, even when considering an arbitrary, possibly non-symmetric ${ }^{4}$, domain $\mathcal{U} \subseteq \mathcal{U}^{\mathcal{S P}}$.

Theorem 4.1. For $m=2$ and arbitrary $\mathcal{U} \subseteq \mathcal{U}^{\mathcal{S P}}$ that contains at least one utility function per peak, a continuous mechanism $f$ satisfies anonymity and strategyproofness if and only if there exist $\alpha_{0} \leq \alpha_{1} \leq \cdots \leq \alpha_{n}$ in $[0,1]$ such that

$$
f(P)=\operatorname{med}\left(p_{1}, \ldots, p_{n}, \alpha_{0}, \ldots, \alpha_{n}\right) .
$$

The proof can be found in Appendix A.
Remark 1. Characterizations of strategyproof mechanisms for certain subdomains of $\mathcal{U}^{\mathcal{S P}}$ are already known (e.g., Border and Jordan, 1983; Berga and Serizawa, 2000). In

[^2]particular, Berga and Serizawa (2000) characterized generalized median rules as the only strategyproof and surjective mechanisms on minimally rich domains. Their result can be used as an intermediate step in an alternative proof of Theorem 4.1.

Freeman et al. (2021) showed that the only distribution of $n+1$ phantoms that ensures proportionality in addition to all axioms from Theorem 4.1 is to distribute the peaks uniformly, i.e., $\alpha_{k}=k / n$ for $k \in\{0, \ldots, n\}$. Aziz et al. (2022) strengthened this result by pointing out that continuity, strategyproofness, and proportionality suffice for characterizing the uniform phantom mechanism for symmetric single-peaked preferences. ${ }^{5}$ We again show that symmetry is not required.
Theorem 4.2. For $m=2$ and arbitrary $\mathcal{U} \subseteq \mathcal{U}^{\mathcal{S P}}$ that contains at least one utility function per peak, the only continuous mechanism that satisfies strategyproofness and proportionality is the uniform phantom mechanism.

The proof can be found in Appendix A.
For $m=2$, an outcome $q$ is efficient for a profile $P$ if and only if $\min _{i \in N} p_{i} \leq q \leq$ $\max _{i \in N} p_{i}$; this property is called range-respect. Obviously, the uniform phantom mechanism is range-respecting. Thus, for only two alternatives, it is possible to aggregate utilities in an efficient, strategyproof, and fair manner, even without knowledge of the specific underlying utility model.

## 5. Utility functions

In this section, we discuss star-shaped generalizations of single-peaked preferences for $m>2$.

## 5.1. $\ell_{p}$ preferences

A natural approach for specifying a utility function based on a single peak is to measure the distance to the peak using some metric $d: \Delta^{m} \times \Delta^{m} \rightarrow \mathbb{R}$. Given an agent with peak $\mathbf{p}_{i}$, her utility for a distribution $\mathbf{q}$ is then defined as $u_{i}(\mathbf{q})=-d\left(\mathbf{p}_{i}, \mathbf{q}\right)$.

Among such models, $\ell_{p}$ norms, given by $\|\mathbf{q}\|_{p}:=\left(\sum_{j \in M}\left|q_{j}\right|^{p}\right)^{1 / p}$ for $p \geq 1$, belong to the most studied utility functions. In particular, the special case of $p=1$ has received considerable attention.

Definition 5.1. An agent $i$ with peak $\mathbf{p}_{i}$ has $\ell_{p}$ preferences if $u_{i}(\mathbf{q})=-\left\|\mathbf{p}_{i}-\mathbf{q}\right\|_{p}$.
We will also sometimes refer to these preferences as $\ell_{p}$ disutilities. In this paper, we focus on $\ell_{1}$ preferences $\left(u_{i}(\mathbf{q})=-\sum_{j \in M}\left|p_{i, j}-q_{j}\right|\right)$ and $\ell_{\infty}$ preferences $\left(u_{i}(\mathbf{q})=\right.$ $\left.-\max _{j \in M}\left|p_{i, j}-q_{j}\right|\right)$. It can be easily shown that $\ell_{p}$ preferences are star-shaped for $p \geq 1$. This does not hold for arbitrary metrics, e.g., consider the trivial metric, $d(x, y)=0$ if $x=y$ and $d(x, y)=1$ otherwise.

[^3]
(a) MQ utilities

(b) $\ell_{1}$ disutilities

Figure 1: Illustration of indifference classes for MQ utilities and $\ell_{1}$ disutilities for 3 alternatives when the ideal distribution is $(0.1,0.4,0.5)$. The vertices of the main triangle represent the degenerate distributions $(1,0,0),(0,1,0)$, and $(0,0,1)$. For each type of utilities, the peak forms an indifference class by itself, and three other indifference classes are displayed.

### 5.2. Minimal quotient (MQ) utilities

In contrast to $\ell_{p}$ preferences, MQ utilities are not based on a metric. In particular, they are not symmetric and also not based on disutilities. As discussed in the introduction, metric-based preferences fail to capture important aspects of certain practical situations, notably the representation of alternatives - this is precisely captured by MQ utilities.

Let $M_{i}:=\left\{j \in M: p_{i, j}>0\right\} ;$ note that $M_{i} \neq \emptyset$. The utility that agent $i$ derives from a distribution $\mathbf{q}$ is given by the $M Q$ utility

$$
u_{i}(\mathbf{q})=\min _{j \in M_{i}} \frac{q_{j}}{p_{i, j}} .
$$

Observe that $0 \leq u_{i}(\mathbf{q}) \leq 1$ for all distributions $\mathbf{q}$. Moreover, $u_{i}(\mathbf{q})=1$ if and only if $\mathbf{q}=\mathbf{p}_{i}$, and $u_{i}(\mathbf{q})=0$ if and only if $q_{j}=0$ for some $j \in M_{i}$. As discussed in Section 1, MQ utilities are based on the assumption that agents want all alternatives to receive as large a fraction of their ideal amounts as possible. The indifference curves of $\ell_{1}$ preferences and MQ utilities are illustrated in Figure 1.

It is possible to refine MQ utilities further by considering the leximin over the quotients - that is, breaking ties in the smallest quotient using the second smallest quotient, and so on. This refinement is discussed in Appendix F.

A useful concept when dealing with MQ utilities is that of critical alternatives.
Definition 5.2 (Critical alternatives). Given a distribution q, we define the set of agent $i$ 's critical alternatives

$$
T_{\mathbf{q}, i}:=\arg \min _{j \in M_{i}} \frac{q_{j}}{p_{i, j}} .
$$

Critical alternatives allow for a characterization of efficient distributions.

Lemma 5.3 (Brandt et al. (2023, Lem. 4.7)). With MQ utilities, a distribution $\boldsymbol{q}$ is efficient if and only if every alternative $j$ with $q_{j}>0$ is critical for some agent.

Furthermore, it turns out that both core fairness notions coincide for MQ utilities.
Proposition 5.4. With $M Q$ utilities, weak core fair share implies core fair share, so these two properties are equivalent.

The proof can be found in Appendix B. Combining Propositions 3.3 and 5.4 results in the following corollary.

Corollary 5.5. With MQ utilities, weak core fair share implies efficiency.
We conclude this section by showing that core fair share is a stronger fairness axiom than proportionality for $\ell_{1}$ and $\ell_{\infty}$ preferences as well as MQ utilities.

Proposition 5.6. With $\ell_{1}$ preferences, $\ell_{\infty}$ preferences, or $M Q$ utilities, weak core fair share implies proportionality.

Proof. Assume that a mechanism $f$ is not proportional for some profile $P \in \mathcal{P}$ with $p_{i, j} \in\{0,1\}$ for all $i \in N, j \in M$. Denote $\mathbf{q}:=f(P)$, and let $N^{\prime} \subseteq N$ be a maximal subset of agents for which proportionality is violated by $\mathbf{q}$ (in particular, all agents in $N^{\prime}$ put probability 1 on the same alternative). Denote by $j^{*}$ the favorite alternative of agents in $N^{\prime}$, so that $p_{i, j^{*}}=1$ for all $i \in N^{\prime}$. Denote $r:=\left|N^{\prime}\right| / n$. Violation of proportionality means that $q_{j^{*}}<r$.

Let $\mathbf{q}^{\prime}$ be the ideal distribution of all agents $i \in N^{\prime}$, i.e., $q_{j^{*}}=1$ and $q_{j}=0$ for all $j \neq j^{\prime}$. We claim that $u_{i}\left(r \mathbf{q}^{\prime}+(1-r) \mathbf{q}^{\prime \prime}\right)>u_{i}(\mathbf{q})$ for all distributions $\mathbf{q}^{\prime \prime} \in \Delta^{m}$.

With $\ell_{1}, \ell_{\infty}$, and MQ preferences, for all $i \in N^{\prime}, u_{i}(\mathbf{q})$ depends only on $q_{j^{*}}$, and it is an increasing function of $q_{j^{*}}$. Specifically, with $\ell_{1}$ preferences $u_{i}(\mathbf{q})=-2\left(1-q_{j^{*}}\right)$, with $\ell_{\infty}$ preferences $u_{i}(\mathbf{q})=-\left(1-q_{j^{*}}\right)$, and with MQ preferences $u_{i}(\mathbf{q})=q_{j^{*}}$. Since $\left(r \mathbf{q}^{\prime}+(1-r) \mathbf{q}^{\prime \prime}\right)_{j^{*}} \geq r>q_{j^{*}}$ for all $\mathbf{q}^{\prime \prime} \in \Delta^{m}$, we have $u_{i}\left(r \mathbf{q}^{\prime}+(1-r) \mathbf{q}^{\prime \prime}\right)>u_{i}(\mathbf{q})$, so $f$ violates weak core fair share.

Note that the proof of Proposition 5.6 does not work for $\ell_{2}$ preferences. For example, suppose $n=m=3$ and some two agents have their peak at ( $1,0,0$ ). A rule that returns $\mathbf{q}=f(P)=(0.64,0.18,0.18)$ violates proportionality but does not violate CFS, as for both $i \in N^{\prime}, u_{i}(\mathbf{q})>-\sqrt{0.2}$, but for $\mathbf{q}^{\prime \prime}=(0,1,0)$, we have $r \mathbf{q}^{\prime}+(1-r) \mathbf{q}^{\prime \prime}=(2 / 3,1 / 3,0)$, which leads to $u_{i}=-\sqrt{2 / 9}<-\sqrt{0.2}$. This indicates that proportionality may not be a very natural notion for such preferences.

## 6. Impossibilities for $\ell_{1}$ and $\ell_{\infty}$ preferences

In this section, we show that efficiency, strategyproofness, and the rather weak fairness condition of proportionality are incompatible when agents have $\ell_{1}$ or $\ell_{\infty}$ preferences.

## 6.1. $\ell_{1}$ preferences

Under $\ell_{1}$ preferences, Freeman et al. (2021) observed that the utilitarian welfare maximizing mechanism is the only efficient mechanism in their class of moving phantom mechanisms. However, maximizing utilitarian welfare violates weak fairness axioms such as proportionality. We prove that this tradeoff between efficiency and fairness is inevitable in the presence of strategyproofness.

Theorem 6.1. With $\ell_{1}$ preferences, no mechanism satisfies efficiency (EFF), strategyproofness (SP), and proportionality ( $P R O P$ ) when $m \geq 3$ and $n \geq 3$.

For the proof of this theorem, the following simple observation is important to keep in mind.

Observation 6.2. With $\ell_{1}$ preferences, if some agent $i$ has $p_{i, j}=1$, then for any distribution $\boldsymbol{q}, d_{i}(\boldsymbol{q})=2-2 q_{j}$, regardless of the distribution on alternatives other than $j$. Therefore, agent $i$ is indifferent if some amount is moved between alternatives other than $j$.

Proof of Theorem 6.1. We start with the case $m=3$ and $n=3$. For $m=3$, we set $M=\{a, b, c\}$ and write $\mathbf{q}=\left(q_{a}, q_{b}, q_{c}\right)$. For simplicity, we number profiles by a superscript $(k)$. We denote the disutility function of agent $i$ in profile $k$, i.e., the $\ell_{1}$ distance to her peak $p_{i}$, by $d_{i}^{(k)}$, and the returned distribution in profile $k$ by $\mathbf{q}^{(k)}$.

Consider first the following two profiles.

$$
\begin{array}{ccc}
\mathrm{a} & \mathrm{~b} & \mathrm{c} \\
1 / 2 & 1 / 2 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\mathbf{q}^{(1)} \geq 1 / 6 & \geq 1 / 2 & \leq 1 / 3
\end{array}
$$

Profile 1


Profile 2

The outcome in Profile 2 must be $(1 / 3,1 / 3,1 / 3)$ by $P R O P$.
We now justify the bounds on the outcome in Profile 1. As agent 1 can manipulate between Profile 1 and Profile 2, $S P$ requires that agent 1 does not gain from either manipulation. This implies

$$
\begin{align*}
& d_{1}^{(1)}\left(\mathbf{q}^{(1)}\right) \leq d_{1}^{(1)}\left(\mathbf{q}^{(2)}\right)=2 / 3 \text { and }  \tag{1}\\
& d_{1}^{(2)}\left(\mathbf{q}^{(1)}\right) \geq d_{1}^{(2)}\left(\mathbf{q}^{(2)}\right)=4 / 3 . \tag{2}
\end{align*}
$$

By (1), $q_{a}^{(1)} \geq 1 / 6$ (implying $\left.q_{b}^{(1)} \leq 5 / 6\right), q_{b}^{(1)} \geq 1 / 6$, and $q_{c}^{(1)} \leq 1 / 3$. $\operatorname{By}(2), q_{a}^{(1)} \leq 1 / 3$, implying $q_{b}^{(1)}+q_{c}^{(1)} \geq 2 / 3$, and thus $q_{b}^{(1)} \geq 1 / 3$.

The left figure on the next page illustrates both inequalities. The $x$-axis corresponds to $q_{a}$ and the $y$-axis to $q_{b}$, so the entire set of possible outcomes is represented by the triangle


$x \geq 0, y \geq 0, x+y \leq 1$. The first inequality $d_{1}^{(1)}\left(\mathbf{q}^{(1)}\right) \leq d_{1}^{(1)}\left(\mathbf{q}^{(2)}\right)$ is illustrated by the bright trapezoid near the blue dot at $(1 / 2,1 / 2,0)$; the second inequality is represented by the bright trapezoid far from the red dot at $(1,0,0) . S P$ requires that $\mathbf{q}^{(1)}$ be inside the intersection of the two trapezoids, i.e., the white region. Hence,

$$
1 / 6 \leq q_{a}^{(1)} \leq 1 / 3, \quad 1 / 3 \leq q_{b}^{(1)} \leq 5 / 6, \quad \text { and } \quad 0 \leq q_{c}^{(1)} \leq 1 / 3
$$

By EFF, we can even show that $q_{b}^{(1)} \geq 1 / 2$. Otherwise, as $q_{a}^{(1)}>0$, some small amount could be moved from $a$ to $b$. Agent 3 is indifferent due to Observation 6.2 and agent 2 strictly gains. Furthermore, this does not change agent 1's disutility as $q_{b}^{(1)}<1 / 2$.

Next, we consider the following two profiles.

|  | a | b |
| :---: | :---: | :---: |
| $\mathbf{q}^{(3)}$ | c |  |
| $1 / 4$ | $3 / 4$ | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |
|  | 0 | $2 / 3$ | $1 / 3$

Profile 3


Profile 4

The outcome in Profile 4 follows from $P R O P$. We now prove that the outcome in Profile 3 must be identical. As agent 1 can manipulate between Profile 3 and Profile 4, $S P$ requires that agent 1 does not gain from either manipulation. This implies that

$$
\begin{align*}
& d_{1}^{(3)}\left(\mathbf{q}^{(3)}\right) \leq d_{1}^{(3)}\left(\mathbf{q}^{(4)}\right)=2 / 3, \text { and }  \tag{3}\\
& d_{1}^{(4)}\left(\mathbf{q}^{(3)}\right) \geq d_{1}^{(4)}\left(\mathbf{q}^{(4)}\right)=2 / 3 . \tag{4}
\end{align*}
$$

By (3), $q_{c}^{(3)} \leq 1 / 3$, implying $q_{a}^{(3)}+q_{b}^{(3)} \geq 2 / 3$. By (4), $q_{b}^{(3)} \leq 2 / 3$. Graphically, $S P$ for agent 1 implies that $\mathbf{q}^{(3)}$ must be in the white region in the right figure on this page.

However, by $E F F$, if $q_{a}^{(3)}>0$ then $q_{b}^{(3)} \geq 3 / 4$. Otherwise, some small amount can be moved from $a$ to $b$. Agent 3 is indifferent due to Observation 6.2 and agent 2 strictly gains. Furthermore, this does not change agent 1's disutility as $q_{b}^{(3)}<3 / 4$. Therefore, $q_{a}^{(3)}=0$ must hold, and the only outcome compatible with $S P$ is $\mathbf{q}^{(3)}=(0,2 / 3,1 / 3)$.

Now that we know $\mathbf{q}^{(3)}$, we consider a manipulation of agent 1 from Profile 3 to Profile 1. $S P$ implies

$$
d_{1}^{(3)}\left(\mathbf{q}^{(1)}\right) \geq d_{1}^{(3)}\left(\mathbf{q}^{(3)}\right)=2 / 3 .
$$

But the bounds we already have for $\mathbf{q}^{(1)}$ imply that $d_{1}^{(3)}\left(\mathbf{q}^{(1)}\right) \leq 2 / 3$ as $q_{a}^{(1)} \geq 1 / 6$ and $q_{b}^{(1)} \geq 1 / 2$. Therefore, $d_{1}^{(3)}\left(\mathbf{q}^{(1)}\right)=2 / 3$ together with $q_{a}^{(1)}=1 / 6$ and $q_{b}^{(1)}=1 / 2$. Hence, $\mathbf{q}^{(1)}=(1 / 6,1 / 2,1 / 3)$.

Finally, we consider the following two profiles.


Profile 5


Profile 6
$\mathbf{q}^{(6)}$ is determined by arguments analogous to those for $\mathbf{q}^{(1)}$, reasoning about agent 3 instead of agent 1 .

We now consider a manipulation of agent 1 from Profile 5 to Profile 6. It follows from $S P$ that $d_{1}^{(5)}\left(\mathbf{q}^{(5)}\right) \leq d_{1}^{(5)}\left(\mathbf{q}^{(6)}\right)=1 / 3$, which implies that $q_{c}^{(5)} \leq 1 / 6$. Similarly, we consider a manipulation of agent 3 from Profile 5 to Profile 1. It follows from $S P$ that $d_{3}^{(5)}\left(\mathbf{q}^{(5)}\right) \leq d_{3}^{(5)}\left(\mathbf{q}^{(1)}\right)=1 / 3$, which implies that $q_{c}^{(5)} \geq 1 / 2-1 / 6=1 / 3$, a contradiction.

Graphically, both inequalities are shown in the figure on the right. The bright region on the right contains the points satisfying the first inequality, and the bright region on the left contains the points satisfying the second inequality. It is evident that the two inequalities cannot be satisfied simultaneously.

This example can be extended to arbitrary numbers of alternatives and agents in the following way.

To increase the number of alternatives, simply
 add alternatives $j^{+}$with $p_{i, j^{+}}=0$ for all agent $i$. These new alternatives do not affect the argument, as $E F F$ ensures that none of them ever receives a positive amount.

Adding agents is more involved, as our proof relies on explicit distributions induced by $P R O P$ and thus depends on the number of agents. However, we note that, throughout the proof, agent 2 always has the same peak, which puts all mass on alternative $b$. Therefore, when adding agents $i^{+}$with $p_{i^{+}, b}=1$, we can run through the exact same proof but with adapted distributions. The generalized proof can be found in Appendix C.

Remark 2. The bounds $m \geq 3$ and $n \geq 3$ in Theorem 6.1 are tight. Indeed, there exists a moving phantoms mechanism that satisfies strategyproofness, proportionality, and range-respect (Freeman et al., 2021, p. 22), and it is known that range-respect and efficiency coincide when $m=2$ or $n=2$ (Elkind et al., 2023, Sec. 8).

The three axioms required for the impossibility are independent. Indeed, all axioms except proportionality are satisfied by the mechanism that maximizes utilitarian welfare (Lindner et al., 2008), all axioms except efficiency are satisfied by the independent markets mechanism (Freeman et al., 2021), and all axioms except strategyproofness are satisfied by mechanisms that are dictatorial on all profiles that are not determined by proportionality.

Freeman et al. (2021, p. 30) posed the question of whether every anonymous, neutral, continuous, and strategyproof mechanism can be represented as a moving phantoms mechanism. While such a characterization could simplify the previous proof, it does not hold in general; see Appendix D.

## 6.2. $\ell_{\infty}$ preferences

$\ell_{1}$ preferences take a special role among $\ell_{p}$ disutilities in terms of efficiency: indifference curves partially move along distributions with a constant sum on "approved" ( $p_{i, x}>$ 0 ) alternatives. As an example, consider $M=\{a, b, c, d\}$ and an agent $i$ with peak $\mathbf{p}_{i}=(1 / 2,1 / 2,0,0)$. With $\ell_{1}$ preferences, she is indifferent between all distributions $\mathbf{q}$ with $q_{a}+q_{b}=1 / 2$. This implies that if we have two agents and the second agent $i^{\prime}$ has $\mathbf{p}_{i^{\prime}}=(0,0,0,1)$, every efficient distribution $\mathbf{q}$ with $q_{a}+q_{b}=1 / 2$ must put $1 / 2$ on alternative $d$. By contrast, when considering, e.g., $\ell_{2}$ preferences, it also matters for agent $i$ how $1 / 2$ is distributed on $c$ and $d$. As a result, more distributions become efficient, which weakens the role of efficiency for a potential impossibility when $p>1$.

We proceed by proving an impossibility for the preference model at the other end of the spectrum: $\ell_{\infty}$ preferences. These preferences behave similarly to $\ell_{1}$ disutilities (Observation 6.2), which is helpful when arguing about efficiency.

Observation 6.3. With $\ell_{\infty}$ preferences, if some agent $i$ has $p_{i, j}=1$, then for any distribution $\boldsymbol{q}, d_{i}(\boldsymbol{q})=1-q_{j}$, regardless of the distribution on alternatives other than $j$. Therefore, agent $i$ is indifferent if some amount is moved between projects other than $j$.

Theorem 6.4. With $\ell_{\infty}$ preferences, no mechanism satisfies efficiency, strategyproofness, and proportionality when $m \geq 3$ and $n \geq 3$.

The proof uses the same profiles as the one for Theorem 6.1, but needs more involved arguments when reasoning about efficiency and extending the argument from $m=3$ to a larger number of alternatives. It can be found in Appendix E.

Remark 3. Theorem 6.4 requires $m \geq 3$, since for $m=2$ all metrics are equivalent (and thus induce the same preferences), and there are mechanisms that satisfy all requirements (see Remark 2). Moreover, $n \geq 3$ is required because, for $m=3$, the $\ell_{\infty}$ and $\ell_{1}$ metrics
are equivalent - the $\ell_{1}$ distance is always twice the $\ell_{\infty}$ distance. Therefore, for $m=3$ and $n=2$, the same mechanisms satisfy all the requirements.

Similar to the impossibility for $\ell_{1}$ preferences, we expect all axioms to be independent. However, to the best of our knowledge, this does not follow from existing results, as $\ell_{\infty}$ preferences have been studied significantly less than $\ell_{1}$.

We conjecture that the incompatibility of efficiency, strategyproofness, and weak fairness conditions holds for $\ell_{p}$ disutilities for any $1 \leq p \leq \infty$, when $m \geq 3$ and $n \geq 3$.

In the next section, we demonstrate that the impossibility does not generalize to arbitrary star-shaped utility functions.

## 7. The Nash product rule

From now on, we assume that utilities are given by MQ utilities. Inspired by the positive results obtained by maximizing the product of utilities in similar contexts (in particular, Brandt et al., 2023), we define the Nash product rule for budget aggregation as follows. For any $P \in \mathcal{P}$,

$$
N A S H(P)=\underset{\mathbf{q} \in \Delta^{m}}{\arg \max } \prod_{i \in N} u_{i}(\mathbf{q}) .
$$

NASH is well-defined as it always returns exactly one distribution (Brandt et al., 2023). The following example illustrates the difference between NASH for MQ utilities and the independent markets mechanism for $\ell_{1}$ utilities.

Example 7.1. Let $m=3$ and $n=2$. Assume that the two agents' ideal distributions are $(4 / 5,1 / 5,0)$ and $(4 / 5,0,1 / 5)$. One can check that $N A S H$ returns the distribution $(2 / 3,1 / 6,1 / 6)$, while the independent markets mechanism returns $(3 / 5,1 / 5,1 / 5)$.

Analogously, due to its application in donor coordination (Brandt et al., 2023), we can also interpret the outcome of $N A S H$ as a Nash equilibrium where each agent $i$ reports a vector $\mathbf{s}_{i} \in \Delta^{m}$ and the outcome is determined by adding up the score $\sum_{i \in N} s_{i, j}$ of each alternative $j$. Here, the strategy set is the set of preferences. This interpretation will be useful for proving certain properties of NASH.

Definition 7.2 (Decomposition). A decomposition of a distribution $\mathbf{q}$ is a vector of nonnegative score vectors $\left(\mathbf{s}_{i}\right)_{i \in N}$ with

$$
\begin{array}{ll}
\sum_{i \in N} s_{i, j}=q_{j} & \text { for all } j \in M ; \\
\sum_{j \in M} s_{i, j}=\frac{1}{n} & \text { for all } i \in N
\end{array}
$$

An alternative characterization of the $N A S H$ outcome uses the notion of critical alternatives

Lemma 7.3 (Brandt et al. (2023, Sec. 4.3)). A distribution $\boldsymbol{q}$ maximizes the Nash product if and only if it has a decomposition $\left(s_{i}\right)_{i \in N}$ such that $s_{i, j}=0$ for every alternative $j \notin T_{\boldsymbol{q}, i}$.

### 7.1. Properties

In this section, we investigate properties of $N A S H$ for budget aggregation.
Anonymity follows immediately from the fact that multiplication is commutative. Neutrality is also straightforward as NASH does not take into account the identities of the alternatives. Another fact to keep in mind is that utilities and efficient outcomes admit a one-to-one correspondence.

Lemma 7.4 (Brandt et al. (2023, Lem. 4.9)). Let $\boldsymbol{q}$ and $\boldsymbol{q}^{\prime}$ be efficient distributions inducing the same utility vector, that is, $u_{i}(\boldsymbol{q})=u_{i}\left(\boldsymbol{q}^{\prime}\right)$ for all $i \in N$. Then, $\boldsymbol{q}=\boldsymbol{q}^{\prime}$.

With $\ell_{1}$ preferences, every efficient distribution $\mathbf{q}$ must be range-respecting (that is, $q_{j}$ must be between $\min _{i} p_{i, j}$ and $\max _{i} p_{i, j}$ for all $\left.j \in M\right)$. However, with MQ utilities, this is not the case. In fact, $N A S H$ is efficient but not range-respecting, as shown in Example 7.1. Intuitively, $N A S H$ prefers to decrease the distribution for alternative 1 below the minimum peak, in order to increase the distribution for other alternatives whose ratio is smaller. Nevertheless, an efficient distribution always satisfies one direction of range-respect.

Lemma 7.5. With $M Q$ utilities, if $\boldsymbol{q}$ is an efficient distribution, then $q_{j} \leq \max _{i} p_{i, j}$ for all $j \in M$. In particular, this is satisfied by NASH.

Proof. Suppose by contradiction that, for some $j \in M$, it holds that $q_{j}>p_{i, j}$ for all $i \in N$. Construct a new distribution $\mathbf{q}^{\prime}$ from $\mathbf{q}$ by removing some amount from $j$ and allocating it equally among the other alternatives, such that $q_{j}>p_{i, j}$ for all $i \in N$ still holds.

Note that, for any agent $i$ with MQ utilities, $u_{i}(\mathbf{q}) \leq 1$ for all $\mathbf{q}$. Therefore, the decrease in $q_{j}$ does not decrease the MQ utility of any agent, as $q_{j} / p_{i, j}>1$. But the increase in allocation to other alternatives must increase the utilities of all agents. Therefore, $\mathbf{q}^{\prime}$ is a Pareto improvement of $\mathbf{q}$, contradicting efficiency.
$N A S H$ also satisfies continuity, which will be important for the axiomatic characterization we give in the next section.

Proposition 7.6. With MQ utilities, NASH is continuous.
Proof. Suppose we are given a sequence of profiles $P^{1}, P^{2}, \ldots$ converging to $P^{*}$, i.e., $\lim _{k \rightarrow \infty} p_{i, j}^{k}=p_{i, j}^{*}$ for every agent $i \in N$ and alternative $j \in M$. Denote by $u_{i}^{k}$ the utility of agent $i$ in profile $P^{k}$, and $u_{i}^{*}$ the utility in profile $P^{*}$. Denote $\mathbf{q}^{k}=\operatorname{NASH}\left(P^{k}\right)$ for every $k \in \mathbb{N}$ and $\mathbf{q}^{*}=N A S H\left(P^{*}\right)$. By boundedness, it suffices to show that every convergent subsequence of $\mathbf{q}^{1}, \mathbf{q}^{2}, \ldots$ converges to $\mathbf{q}^{*}$. Take such a subsequence, which must exist by the Bolzano-Weierstrass theorem, and denote its limit by $\mathbf{q}$. With abuse of notation, we now refer to this subsequence as $\mathbf{q}^{1}, \mathbf{q}^{2}, \ldots$. Our goal is to show that $\mathbf{q}=\mathbf{q}^{*}$.

Case 1: $q_{j}^{*}>0$ for all alternatives $j \in M . \quad$ By definition of $N A S H$ on $P^{k}$, we have

$$
\operatorname{Nash}\left(\mathbf{q}^{*}, P^{k}\right) \leq \operatorname{Nash}\left(\mathbf{q}^{k}, P^{k}\right)
$$

for every $k$, where $\operatorname{Nash}(\mathbf{q}, P)$ denotes the Nash welfare of the outcome $\mathbf{q}$ when the profile is $P$. We take the limit of both sides as $k \rightarrow \infty$.

- For the left-hand side, for every $i \in N$, the utility $u_{i}^{k}\left(\mathbf{q}^{*}\right)$ is a minimum of ratios $q_{j}^{*} / p_{i, j}^{k}$ where all numerators are at least $\varepsilon$, for some $\varepsilon>0$. Since the minimum is always at most 1 , elements with $p_{i, j}^{k}<\varepsilon$ do not affect the minimum and can be ignored. Therefore, the minimum is determined only by ratios with $p_{i, j}^{k} \geq \varepsilon$. In this domain, the ratios are continuous, and their minimum is continuous too. Therefore, $\lim _{k \rightarrow \infty} u_{i}^{k}\left(\mathbf{q}^{*}\right)=u_{i}^{*}\left(\mathbf{q}^{*}\right)$, and the limit of the product at the left-hand side equals $\operatorname{Nash}\left(\mathbf{q}^{*}, P^{*}\right)$.
- For the right-hand side, for some agents $i$, there may be alternatives $j$ for which both $q_{j}^{k}$ and $p_{i, j}^{k}$ approach 0 , so $u_{i}$ may be discontinuous. However, as $p_{i, j}^{*}=0$ for such alternatives, they are removed from the minimum, so the minimum can only be larger. Therefore, $\lim _{k \rightarrow \infty} u_{i}^{k}\left(\mathbf{q}^{k}\right) \leq u_{i}^{*}(\mathbf{q})$, and the limit of the product at the right-hand side is at most $\operatorname{Nash}\left(\mathbf{q}, P^{*}\right)$.

Therefore, we have $\operatorname{Nash}\left(\mathbf{q}^{*}, P^{*}\right) \leq \operatorname{Nash}\left(\mathbf{q}, P^{*}\right)$. By definition and uniqueness of $N A S H$ on $P^{*}$, we get $\mathbf{q}=\mathbf{q}^{*}$.

Case 2: $q_{j}^{*}=0$ for some alternatives $j \in M$. Define $Z=\left\{j \in M: q_{j}^{*}=0\right\}$. As $\mathbf{q}^{*}$ maximizes Nash welfare for $P^{*}$, we have $p_{i, j}^{*}=0$ for every $i \in N$ and $j \in Z$; otherwise, an agent $i$ with $p_{i, j}^{*}>0$ would receive zero utility causing the whole product to become zero. Consequently, $\lim _{k \rightarrow \infty} p_{i, j}^{k}=0$ for every $i \in N, j \in Z$. By Lemma 7.5, the amount allocated to alternatives in $Z$ by $\mathbf{q}^{k}$ also tends to zero for $k \rightarrow \infty$. Hence, $q_{j}^{*}=0$ implies $q_{j}=0$.

Now, we will measure utilities and Nash welfare only with respect to the alternatives outside $Z$. Let $w^{k}$ be the total amount allocated to alternatives outside $Z$ in $\mathbf{q}^{k}$, so we know that $w^{k}$ converges to 1 for $k \rightarrow \infty$. By definition of $N A S H$ on $P^{k}$, we have

$$
\operatorname{Nash}\left(w^{k} \cdot \mathbf{q}^{*}, P^{k}\right) \leq \operatorname{Nash}\left(\mathbf{q}^{k}, P^{k}\right) .
$$

As in Case 1, we take the limit of both sides, the left-hand side equals $\operatorname{Nash}\left(\mathbf{q}^{*}, P^{*}\right)$, and the right-hand side is at most $\operatorname{Nash}\left(\mathbf{q}, P^{*}\right)$. Since in $P^{*}$ all agents assign zero to alternatives in $Z$, these two quantities remain the same even if we take the alternatives in $Z$ back into account for the Nash welfare. Hence, as in Case 1, we get $\operatorname{Nash}\left(\mathbf{q}^{*}, P^{*}\right) \leq$ $\operatorname{Nash}\left(\mathbf{q}, P^{*}\right)$. By definition and uniqueness of $N A S H$ on $P^{*}$, we conclude that $\mathbf{q}=$ $\mathrm{q}^{*}$.

In addition, NASH satisfies efficiency and group-strategyproofness (Brandt et al.,
2023). ${ }^{6}$ We show next that NASH furthermore satisfies core fair share (and thus also proportionality).

Proposition 7.7. With MQ utilities, NASH satisfies core fair share.
Proof. Assume for contradiction that there exists $P \in \mathcal{P}$ such that $\mathbf{q}:=\operatorname{NASH}(P)$ does not satisfy core fair share for some $G \subseteq N$. Then, there exists $\mathbf{q}^{\prime} \in \Delta^{m}$ such that

$$
\begin{array}{lr}
u_{i}\left((|G| / n) \mathbf{q}^{\prime}+(1-|G| / n) \mathbf{q}^{\prime \prime}\right) \geq u_{i}(\mathbf{q}) & \text { for all } i \in G, \text { and } \\
u_{i}\left((|G| / n) \mathbf{q}^{\prime}+(1-|G| / n) \mathbf{q}^{\prime \prime}\right)>u_{i}(\mathbf{q}) & \text { for at least one } i \in G
\end{array}
$$

and every $\mathbf{q}^{\prime \prime} \in \Delta^{m}$.
Let $T_{\mathbf{q}, G}:=\bigcup_{i \in G} T_{\mathbf{q}, i}$ be the set of alternatives critical to at least one agent from $G$. Note that $T_{\mathbf{q}, G}=M$ cannot hold; otherwise, by Lemma 5.3, $\mathbf{q}$ would be efficient not only for $N$ but already for $G$, contradicting that $\mathbf{q}$ does not satisfy core fair share for $G$. Therefore, there exists a distribution $\mathbf{q}^{\prime \prime}$ with $q_{j}^{\prime \prime}=0$ for every $j \in T_{\mathbf{q}, G}$. Choosing such a distribution $\mathbf{q}^{\prime \prime}$ shows that $(|G| / n) q_{j}^{\prime} \geq q_{j}$; otherwise some agent from $G$ for whom $j$ is critical would have a smaller utility. Thus, $\mathbf{q}\left(T_{\mathbf{q}, G}\right):=\sum_{j \in T_{\mathbf{q}, G}} q_{j} \leq(|G| / n)$. $\sum_{j \in T_{\mathrm{q}, G}} q_{j}^{\prime} \leq|G| / n$.
By Lemma 7.3, the NASH distribution can be decomposed in such a way that every agent from $G$ only contributes her share of $1 / n$ to alternatives in $T_{\mathbf{q}, G}$. Thus, $\mathbf{q}\left(T_{\mathbf{q}, G}\right) \geq$ $|G| / n$. All in all, $\mathbf{q}\left(T_{\mathbf{q}, G}\right)=|G| / n$, implying $(|G| / n) q_{j}^{\prime}=q_{j}$ for $j \in T_{\mathbf{q}, G}$.

However, this contradicts $u_{i}\left((|G| / n) \mathbf{q}^{\prime}+(1-|G| / n) \mathbf{q}^{\prime \prime}\right)>u_{i}(\mathbf{q})$ for at least one $i \in$ $G$.

There exists an interesting connection of $N A S H$ to the independent markets mechanism for $\ell_{1}$ preferences, which follows immediately from the mechanisms' axiomatic properties.

Proposition 7.8. With two alternatives, NASH for MQ utilities is equivalent to the uniform phantom mechanism for $\ell_{1}$ preferences.

Proof. For $m=2$, MQ utilities as well as $\ell_{1}$ preferences are subsets of $\mathcal{U}^{\mathcal{S P}}$, each of which contains one utility function per peak. Both mechanisms are continuous, strategyproof, and proportional. Therefore, they need to be equivalent on their respective utility models by Theorem 4.2.

However, this equivalence no longer holds when $m>2$, as shown in Example 7.1.

[^4]
### 7.2. Characterization

Next, we show that $N A S H$ admits an appealing characterization via strategyproofness and fairness.

Theorem 7.9. With MQ utilities, NASH is the only continuous mechanism that satisfies group-strategyproofness and weak core fair share.

Let $f$ be a mechanism satisfying the properties in the theorem statement. The proof is divided into three lemmas and has the following structure. Starting at an arbitrary profile $P$, we first show in Lemma 7.10 that moving to a "key" profile $P^{*}$ cannot change the outcome: $f\left(P^{*}\right)=f(P)$. Then, Lemma 7.11 states that $f\left(P^{*}\right)=\operatorname{NASH}\left(P^{*}\right)$. Finally, Lemma 7.12 proves $\operatorname{NASH}\left(P^{*}\right)=\operatorname{NASH}(P)$, which completes the proof as we then have $f(P)=\operatorname{NASH}(P)$.

Let $\mathbf{q}:=f(P)$ and denote by $P^{*}$ the profile with peaks

$$
p_{i, j}^{*}= \begin{cases}p_{i, j} / \mathbf{p}_{i}\left(T_{\mathbf{q}, i}\right) & \text { for } j \in T_{\mathbf{q}, i} \\ 0 & \text { for } j \notin T_{\mathbf{q}, i},\end{cases}
$$

where $\mathbf{p}_{i}\left(T_{\mathbf{q}, i}\right):=\sum_{j \in T_{\mathbf{q}, i}} p_{i, j}$. That is, in $P^{*}$, each agent moves her peak so that it is nonzero only on alternatives critical for her under $\mathbf{q}$. For example, suppose $\mathbf{p}_{i}=$ $(0.1,0.2,0.3,0.4)$ and $\mathbf{q}=(0.1,0.1,0.6,0.2)$. Then $T_{\mathbf{q}, i}=\{2,4\}, \mathbf{p}_{i}\left(T_{\mathbf{q}, i}\right)=0.6$, and $\mathbf{p}_{i}^{*}=(0,1 / 3,0,2 / 3)$.

By weak core fair share, without loss of generality, we can assume that $q_{j}=0$ if and only if $p_{i, j}=0$ for all $i \in N$. To see this, note that if $q_{j}=0$ for some $j \in M$, then $u_{i}(\mathbf{q})=0$ for all agents with $p_{i, j}>0$, meaning that weak core fair share is violated for each of these agents. Moreover, $p_{i, j}=0$ for all $i \in N$ implies that $q_{j}=0$ for every efficient mechanism, where efficiency follows from Corollary 5.5.

Lemma 7.10. If $f$ is a continuous mechanism satisfying group-strategyproofness and efficiency, then
(a) the outcome does not change, that is, $f\left(P^{*}\right)=f(P)=\boldsymbol{q}$;
(b) the sets of critical alternatives do not change, that is, $T_{\boldsymbol{q}, i}=T_{\boldsymbol{q}, i}^{*}$ for every $i \in N$.

Proof. (a) We move the peak of each agent in turn. For each agent $i$, we change $\mathbf{p}_{i}$ towards $\mathbf{p}_{i}^{*}$ gradually, to some $\widehat{\mathbf{p}}_{i}:=\lambda \mathbf{p}_{i}^{*}+(1-\lambda) \mathbf{p}_{i}$, for some $\lambda \in[0,1]$ to be computed later. Then we proceed along this line until we reach $\mathbf{p}_{i}^{*}$. In particular, for an agent $i$ with $T_{\mathbf{q}, i}=M$, it holds that $\mathbf{p}_{i}=\mathbf{p}_{i}^{*}$. In the above example, $\lambda=0.3$ gives $\widehat{\mathbf{p}}_{i}=$ $(0.07,0.24,0.21,0.48)$. If $\mathbf{p}_{i}=\mathbf{p}_{i}^{*}$, it is clear that the outcome does not change, so assume that $\mathbf{p}_{i} \neq \mathbf{p}_{i}^{*}$. The change from $\mathbf{p}_{i}$ to $\widehat{\mathbf{p}}_{i}$ has a simple structure:

- $\widehat{p}_{i, j}>p_{i, j}$ for all $j \in T_{\mathbf{q}, i}$, and the ratio $\widehat{p}_{i, j} / p_{i, j}=\lambda / \mathbf{p}_{i}\left(T_{\mathbf{q}, i}\right)+(1-\lambda)=: \lambda^{+}$, a constant independent of $j$ (in the example, $\lambda^{+}=1.2$ );
- $\widehat{p}_{i, j}<p_{i, j}$ for all $j \notin T_{\mathbf{q}, i}$, and the ratio $\widehat{p}_{i, j} / p_{i, j}=(1-\lambda)=: \lambda^{-}$, again independent of $j$ (in the example, $\lambda^{-}=0.7$ ).

Now, consider the ratios $q_{j} / p_{i, j}$ versus the ratios $q_{j} / \widehat{p}_{i, j}$. For each $j \in T_{\mathbf{q}, i}$, we have $q_{j} / p_{i, j}>q_{j} / \widehat{p}_{i, j}$ because $\widehat{p}_{i, j}>p_{i, j}$, whereas for each $j \notin T_{\mathbf{q}, i}$, we have $q_{j} / p_{i, j}<q_{j} / \widehat{p}_{i, j}$ because $\widehat{p}_{i, j}<p_{i, j}$. Furthermore, for all $j \in T_{\mathbf{q}, i}$, the ratios $q_{j} / \widehat{p}_{i, j}$ remain equal (as $\widehat{p}_{i, j} / p_{i, j}$ is constant) and smallest when moving from $\mathbf{p}$ to $\widehat{\mathbf{p}}$. This implies that $\widehat{T}_{\mathbf{q}, i}=T_{\mathbf{q}, i}$.

Moreover, the entire ordering of alternatives by the ratio $q_{j} / p_{i, j}$ is identical to the ordering of alternatives by the ratio $q_{j} / \widehat{p}_{i, j}$, as the smallest ratio is divided by $\lambda^{+}>1$ and the other ratios are divided by $\lambda^{-}<1$. In other words, suppose we partition the alternatives into subsets according to the ratio $q_{j} / p_{i, j}$, and denote the subset with the smallest ratio by $T_{\mathbf{q}, i, 1} \equiv T_{\mathbf{q}, i}$, the subset with the second-smallest ratio by $T_{\mathbf{q}, i, 2}$, etc., then $T_{\mathbf{q}, i, r}=\widehat{T}_{\mathbf{q}, i, r}$ for all $r \geq 1$.

Computing $\lambda$. We pick $\lambda$ sufficiently small such that no new alternative becomes critical for $i$. Specifically, set

$$
\varepsilon:=\min _{j \in T_{\mathbf{q}, i}, j^{\prime} \notin T_{\mathbf{q}, i}}\left(q_{j^{\prime}} p_{i, j}-q_{j} p_{i, j^{\prime}}\right) \leq \min _{j \in T_{\mathbf{q}, i}, j^{\prime} \notin T_{\mathbf{q}, i}} \frac{q_{j^{\prime}} p_{i, j}-q_{j} p_{i, j^{\prime}}}{p_{i, j}+p_{i, j^{\prime}}} .
$$

Note that $\varepsilon>0$, as $q_{j^{\prime}} / p_{i, j^{\prime}}>q_{j} / p_{i, j}$, by definition of critical alternatives.
By uniform continuity of $f$, there exists $\delta>0$ such that $\left\|f(P)-f\left(P^{\prime}\right)\right\|_{1}<2 \varepsilon$ for all $P^{\prime}$ with $\left\|P-P^{\prime}\right\|_{1} \leq \delta$. Set

$$
\lambda:=\min \left(1, \frac{\delta}{\left\|\mathbf{p}_{i}-\mathbf{p}_{i}^{*}\right\|_{1}}\right),
$$

and define $\widehat{P}$ as a profile identical to $P$ except that $i$ changes her peak from $\mathbf{p}_{i}$ to $\widehat{\mathbf{p}}_{i}:=\lambda \mathbf{p}_{i}^{*}+(1-\lambda) \mathbf{p}_{i}$. Note that $\|P-\widehat{P}\|_{1}=\lambda\left\|\mathbf{p}_{i}-\mathbf{p}_{i}^{*}\right\|_{1} \leq \delta$, so $\|\mathbf{q}-\widehat{\mathbf{q}}\|_{1}<2 \varepsilon$, where $\mathbf{q}=f(P)$ and $\widehat{\mathbf{q}}=f(\widehat{P})$.

The choice of $\varepsilon$ ensures that $T_{\widehat{\mathbf{q}}, i} \subseteq T_{\mathbf{q}, i}$, as for arbitrary $j \in T_{\mathbf{q}, i}$ and $j^{\prime} \notin T_{\mathbf{q}, i}$ it holds that $\widehat{q}_{j}<q_{j}+\varepsilon$ and $\widehat{q}_{j^{\prime}}>q_{j^{\prime}}-\varepsilon$, so

$$
\begin{aligned}
\frac{\widehat{q}_{j^{\prime}}}{p_{i, j^{\prime}}} & >\frac{q_{j^{\prime}}-\varepsilon}{p_{i, j^{\prime}}} \geq \frac{q_{j^{\prime}}}{p_{i, j^{\prime}}}-\frac{q_{j^{\prime}} p_{i, j}-q_{j} p_{i, j^{\prime}}}{p_{i, j^{\prime}}\left(p_{i, j}+p_{i, j^{\prime}}\right)}=\frac{q_{j^{\prime}} p_{i, j^{\prime}}+q_{j} p_{i, j^{\prime}}}{p_{i, j^{\prime}}\left(p_{i, j}+p_{i, j^{\prime}}\right)}=\frac{q_{j^{\prime}}+q_{j}}{p_{i, j}+p_{i, j^{\prime}}} \\
& =\frac{q_{j} p_{i, j}+q_{j^{\prime}} p_{i, j}}{p_{i, j}\left(p_{i, j}+p_{i, j^{\prime}}\right)}=\frac{q_{j}}{p_{i, j}}+\frac{q_{j^{\prime}} p_{i, j}-q_{j} p_{i, j^{\prime}}}{p_{i, j}\left(p_{i, j}+p_{i, j^{\prime}}\right)} \geq \frac{q_{j}+\varepsilon}{p_{i, j}}>\frac{\widehat{q}_{j}}{p_{i, j}} .
\end{aligned}
$$

So every $j^{\prime}$ which is not critical for $i$ under $\mathbf{q}$ cannot be critical for $i$ under $\widehat{\mathbf{q}}$. Therefore,

$$
T_{\widehat{\mathbf{q}}, i} \subseteq T_{\mathbf{q}, i}=\widehat{T}_{\mathbf{q}, i} .
$$

Proving that the outcome does not change. Consider a manipulation of agent $i$ who manipulates between reporting $\mathbf{p}_{i}$ and $\widehat{\mathbf{p}}_{i}$. Strategyproofness for $i$ implies both $u_{i}(\mathbf{q}) \geq u_{i}(\widehat{\mathbf{q}})$ and $\widehat{u}_{i}(\widehat{\mathbf{q}}) \geq \widehat{u}_{i}(\mathbf{q})$.

The latter condition implies that, for every alternative $j \in T_{\mathbf{q}, i}$,

$$
\frac{q_{j}}{\widehat{p}_{i, j}}=\widehat{u}_{i}(\mathbf{q}) \quad \text { since } j \in T_{\mathbf{q}, i}=\widehat{T}_{\mathbf{q}, i},
$$

$$
\begin{array}{ll}
\leq \widehat{u}_{i}(\widehat{\mathbf{q}}) & \text { by strategyproofness, } \\
\leq \frac{\widehat{q}_{j}}{\widehat{p}_{i, j}} & \text { by the definition of MQ utilities }
\end{array}
$$

So $q_{j} \leq \widehat{q}_{j}$ for each alternative $j \in T_{\mathbf{q}, i}$. Together with $T_{\widehat{\mathbf{q}}, i} \subseteq T_{\mathbf{q}, i}$, this implies $u_{i}(\mathbf{q}) \leq$ $u_{i}(\widehat{\mathbf{q}})$. Therefore, $u_{i}(\mathbf{q})=u_{i}(\widehat{\mathbf{q}})$. Furthermore, if $\widehat{u}_{i}(\widehat{\mathbf{q}})>\widehat{u}_{i}(\mathbf{q})$, then $\widehat{q}_{j}>q_{j}$ for all $j \in \widehat{T}_{\mathbf{q}, i} \supseteq T_{\widehat{\mathbf{q}}, i}$, which means that $u_{i}(\mathbf{q})<u_{i}(\widehat{\mathbf{q}})$, contradicting $u_{i}(\mathbf{q})=u_{i}(\widehat{\mathbf{q}})$. Thus, $\widehat{u}_{i}(\widehat{\mathbf{q}})=\widehat{u}_{i}(\mathbf{q})$.

Moreover, if the utility of some other agent $i^{\prime}$ increases, group-strategyproofness is violated for the pair $\left\{i, i^{\prime}\right\}$, as this pair could profitably manipulate from $\mathbf{q}$ to $\widehat{\mathbf{q}}$. Similarly, if the utility of some other agent $i^{\prime}$ decreases, group-strategyproofness is again violated for the pair $\left\{i, i^{\prime}\right\}$, as this pair could profitably manipulate from $\widehat{\mathbf{q}}$ to $\mathbf{q}$. Thus, $u_{r}(\mathbf{q})=u_{r}(\widehat{\mathbf{q}})$ for all $r \in N$. Since $\mathbf{q}$ is efficient with respect to $P$, so is $\widehat{\mathbf{q}}$. By Lemma 7.4, $\mathbf{q}=\widehat{\mathbf{q}}$.

Applying this argument repeatedly, we get a sequence of profiles $\left(P^{k}\right)$ with $P^{0}=P$ where $\mathbf{p}_{i}^{k}$ lies on the line $\lambda \mathbf{p}_{i}^{*}+(1-\lambda) \mathbf{p}_{i}$ for every $k$. It remains to show that ( $\mathbf{p}^{k}$ ) reaches $\mathbf{p}_{i}^{*}$ after a finite number of steps. For that, consider the expression in the definition of $\varepsilon$ :

$$
\min _{j \in T_{\mathbf{q}, i,}, j^{\prime} \notin T_{\mathbf{q}, i}}\left(q_{j^{\prime}} p_{i, j}-q_{j} p_{i, j^{\prime}}\right) .
$$

As $\mathbf{p}_{i}$ comes closer to $\mathbf{p}_{i}^{*}, p_{i, j}$ increases and $p_{i, j^{\prime}}$ decreases while $\mathbf{q}$ and $T_{\mathbf{q}, i}$ stay the same, so overall the expression increases. Thus, we can take the $\varepsilon$ (and the corresponding $\delta$ ) from the first step for every step. Furthermore, $\left\|P^{k}-P^{k+1}\right\|_{1}=\delta$ (unless $\lambda=1$, but then we have reached $\mathbf{p}_{i}^{*}$ ) implying that we reach $\mathbf{p}_{i}^{*}$ after at most $\left\lceil\left\|\mathbf{p}_{i}-\mathbf{p}_{i}^{*}\right\|_{1} / \delta\right\rceil$ steps; as we move on a line of length $\left\|P^{k}-P^{k^{\prime}}\right\|_{1}=\sum_{\ell=k}^{k^{\prime}-1}\left\|P^{\ell}-P^{\ell+1}\right\|_{1}$ for $k^{\prime} \geq k$.

After the first agent has reached her desired peak $\mathbf{p}_{i}^{*}$, we turn to the next agent and repeat the procedure. In that way, we eventually arrive at $P^{*}$.
(b) To see that $T_{\mathbf{q}, i}^{*}=T_{\mathbf{q}, i}$ for all $i \in N$, note that for every non-critical project $j \notin T_{\mathbf{q}, i}$ we have $p_{i, j}^{*}=0$, so $j \notin T_{\mathbf{q}, i}^{*}$. Furthermore, for any critical project $j \in T_{\mathbf{q}, i}$ and any other $j^{\prime} \in T_{\mathbf{q}, i}$,

$$
\frac{q_{j}}{p_{i, j}^{*}}=\frac{q_{j} \cdot \mathbf{p}_{i}\left(T_{\mathbf{q}, i}\right)}{p_{i, j}}=\frac{q_{j^{\prime}} \cdot \mathbf{p}_{i}\left(T_{\mathbf{q}, i}\right)}{p_{i, j^{\prime}}}=\frac{q_{j^{\prime}}}{p_{i, j^{\prime}}^{*}},
$$

so $j \in T_{\mathbf{q}, i}^{*}$. Therefore, $T_{\mathbf{q}, i}=T_{\mathbf{q}, i}^{*}$.
Lemma 7.11. Let $P^{*}$ be a profile and $\boldsymbol{q}$ be a distribution in which every agent values every non-critical alternative at $0\left(j \notin T_{q, i}^{*}\right.$ implies $p_{i, j}^{*}=0$ for any agent $\left.i\right)$. If $\boldsymbol{q}$ satisfies weak core fair share, then $\boldsymbol{q}=\operatorname{NASH}\left(P^{*}\right)$.

Proof. Let $P^{*}$ be an arbitrary profile and let $\mathbf{q} \neq \operatorname{NASH}\left(P^{*}\right)$ be a distribution such that $j \notin T_{\mathbf{q}, i}^{*}$ implies $p_{i, j}^{*}=0$. In particular, $\mathbf{q}$ does not maximize Nash welfare. By the proof of Lemma 4.10 of Brandt et al. (2023), there exists a group $N^{-}$of agents such that the total amount given to alternatives critical for some agent from $N^{-}$is less than $\left|N^{-}\right| / n$, that is,

$$
\begin{equation*}
\mathbf{q}\left(T_{\mathbf{q}, N_{-}}^{*}\right)<\frac{\left|N_{-}\right|}{n}, \tag{5}
\end{equation*}
$$

where $T_{\mathbf{q}, N_{-}}^{*}:=\bigcup_{i \in N_{-}} T_{\mathbf{q}, i}^{*}$. We will now show that weak core fair share is violated for this $N^{-}$. This is clear if $\mathbf{q}\left(T_{\mathbf{q}, N_{-}}^{*}\right)=0$, so assume that $\mathbf{q}\left(T_{\mathbf{q}_{, ~}, N_{-}}^{*}\right)>0$.

Define a new distribution in which only alternatives in $T_{\mathbf{q}, N_{-}}^{*}$ are funded:

$$
\mathbf{q}^{\prime}:= \begin{cases}q_{j} / \mathbf{q}\left(T_{\mathbf{q}, N_{-}}^{*}\right) & \text { for } j \in T_{\mathbf{q}, N_{-}}^{*} \\ 0 & \text { for } j \notin T_{\mathbf{q}, N_{-}}^{*} .\end{cases}
$$

For every $i \in N_{-}$, as $p_{i, j}^{*}=0$ for $j \notin T_{\mathbf{q}, N_{-}}^{*} \supseteq T_{\mathbf{q}, i}^{*}$, the utility $u_{i}^{*}\left(\mathbf{q}^{\prime}\right)$ equals $u_{i}^{*}(\mathbf{q}) / \mathbf{q}\left(T_{\mathbf{q}, N_{-}}^{*}\right)$, which is larger than $u_{i}^{*}(\mathbf{q}) /\left(\left|N_{-}\right| / n\right)$ by (5). Therefore, the utility $u_{i}^{*}\left(\left(\left|N_{-}\right| / n\right) \mathbf{q}^{\prime}+\left(1-\left|N_{-}\right| / n\right) \mathbf{q}^{\prime \prime}\right)$ is at least $\left(\left|N_{-}\right| / n\right) u_{i}^{*}\left(\mathbf{q}^{\prime}\right)>u_{i}^{*}(\mathbf{q})$ for every $\mathbf{q}^{\prime \prime} \in \Delta^{m}$, contradicting weak core fair share for $N_{-}$.

Lemma 7.12. Let $P^{*}$ and $P$ be profiles where $T_{\boldsymbol{q}, i}^{*}=T_{\boldsymbol{q}, i}$ for $\boldsymbol{q}=\operatorname{NASH}\left(P^{*}\right)$. Then, $\operatorname{NASH}(P)=\boldsymbol{q}$.

Proof. As $\mathbf{q}$ maximizes Nash welfare in $P^{*}$, by Lemma 7.3 there exists a decomposition $\left(\mathbf{s}_{i}\right)_{i \in N}$ such that $s_{i, j}=0$ for every $i \in N$ and $j \notin T_{\mathbf{q}, i}^{*}$. Due to $T_{\mathbf{q}, i}^{*}=T_{\mathbf{q}, i}$, the same decomposition proves that $\mathbf{q}$ also maximizes Nash welfare in $P$ by Lemma 7.3, thus $\operatorname{NASH}(P)=q$.

Proof of Theorem 7.9. Let $P$ be an arbitrary profile, and $P^{*}$ a modified profile defined as in Lemma 7.10. Then,

$$
f(P) \stackrel{\text { Lemma }}{=} 7.10 f\left(P^{*}\right)^{\text {Lemma }} \stackrel{7.11}{=} \operatorname{NASH}\left(P^{*}\right)^{\text {Lemma }} \stackrel{7.12}{=} \text { NASH }(P),
$$

where Lemma 7.12 uses the fact that the sets of critical alternatives under $\mathbf{q}$ did not change when moving from $P$ to $P^{*}$.

The condition of group-strategyproofness is used only in Lemma 7.10. If we assume that agents have Leximin-MQ preferences (that is, subject to maximizing the smallest ratio, they maximize the second smallest ratio, etc.), then this condition can be weakened to ordinary strategyproofness; the proof is given in Appendix F.

Theorem 7.13. With Leximin-MQ preferences, NASH is the only continuous mechanism that satisfies strategyproofness and weak core fair share.

As to the independence of the axioms, it is easy to see that weak core fair share is required for Theorem 7.9 since any constant mechanism satisfies the remaining axioms. The necessity of group-strategyproofness can be shown by slightly perturbing the outcome of NASH when $m=2$. We conjecture that continuity is also required for the characterization.

## 8. Conclusion

Aggregating individual distributions into a collective distribution constitutes an important problem in social choice. Our work shows that understanding how agents form their preferences has crucial implications on the possibility of optimal aggregation mechanisms. When agents' utilities are based on metrics such as $\ell_{1}$ and $\ell_{\infty}$, no rule simultaneously guarantees strategyproofness, efficiency, and proportionality. However, when agents' utilities are non-metric and based on quotients (MQ utilities), the Nash product rule guarantees group-strategyproofness and weak core fair share, which implies both efficiency and proportionality. Moreover, the Nash product rule is characterized by these properties and continuity. To the best of our knowledge, this is the first characterization of the Nash product rule based on strategyproofness.

The Nash product rule satisfies further desirable properties such as reinforcement and participation. The former states that when aggregating distributions for two disjoint electorates results in the same distribution, then the mechanism should return the same distribution for the union of both electorates. The latter requires that agents are never better off by not participating in the aggregation mechanism. Both statements follow trivially from the definition of the Nash product rule and hold for arbitrary utility models with nonnegative utilities.

It would be interesting to identify other sensible utility models for which the Nash product rule is a most attractive aggregation mechanism, and to pinpoint conditions within the domain of star-shaped preferences that cause impossibilities similar to the ones for $\ell_{1}$ and $\ell_{\infty}$ preferences.

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## A. Proofs of Theorems 4.1 and 4.2

Theorem 4.1. For $m=2$ and arbitrary $\mathcal{U} \subseteq \mathcal{U}^{\mathcal{S P}}$ that contains at least one utility function per peak, a continuous mechanism $f$ satisfies anonymity and strategyproofness if and only if there exist $\alpha_{0} \leq \alpha_{1} \leq \cdots \leq \alpha_{n}$ in $[0,1]$ such that

$$
f(P)=\operatorname{med}\left(p_{1}, \ldots, p_{n}, \alpha_{0}, \ldots, \alpha_{n}\right)
$$

Proof. The proof closely follows the lines of Moulin's original proof, but instead of exploiting the richness of the domain of single-peaked preferences, continuity is applied. The "if" direction is obvious. For the "only if" direction, we use induction on the number of agents.

Base case $n=1$. By the extreme value theorem, $f$ attains a maximum and a minimum on the compact set $[0,1]$; thus, set

$$
\begin{aligned}
& \alpha_{0}=\min _{p \in[0,1]} f(p), \\
& \alpha_{1}=\max _{p \in[0,1]} f(p) .
\end{aligned}
$$

Note that $0 \leq \alpha_{0} \leq \alpha_{1} \leq 1$. Furthermore, by the intermediate value theorem, for every $q \in\left[\alpha_{0}, \alpha_{1}\right]$, there exists $p \in[0,1]$ such that $f(p)=q$.

Let the peak of agent 1 be at $p_{1}$, and consider three cases.

Case 1: $p_{1} \in\left[\alpha_{0}, \alpha_{1}\right]$. Then, by strategyproofness, $f\left(p_{1}\right)=p_{1}$; otherwise agent 1 could beneficially misreport her peak as some $p$ with $f(p)=p_{1}$.

Case 2: $p_{1}<\alpha_{0}$. As we already proved that $f\left(\alpha_{0}\right)=\alpha_{0}$, the agent can get a utility of $u_{1}\left(\alpha_{0}\right)$ by reporting $\alpha_{0}$. Therefore, by strategyproofness, $f\left(p_{1}\right)$ must be at least as good for the agent as $\alpha_{0}$.

Note that $\arg \max _{p \in[0,1]} u_{1}(f(p))=\alpha_{0}$ is the unique maximum, by the definition of $\mathcal{U}^{\mathcal{S P}}$. Therefore, $f\left(p_{1}\right)=\alpha_{0}$.

Case 3: $p_{1}>\alpha_{1}$. By analogous arguments as in Case 2, $f\left(p_{1}\right)=\alpha_{1}$.

Combining all three cases, $f\left(p_{1}\right)$ is the median of $p_{1}, \alpha_{0}$, and $\alpha_{1}$, proving the statement for $n=1$.

Induction step $n \rightarrow n+1$. Assume that the statement holds for $n$ or fewer agents, and suppose we are given $n+1$ agents with peaks $p_{1}, \ldots, p_{n+1}$ and an anonymous, continuous, and strategyproof mechanism $f$.

Fix the peak of agent $n+1$ and consider the mechanism $f_{N}^{p_{n+1}}:\left(p_{1}, \ldots, p_{n}\right) \mapsto$ $f\left(p_{1}, \ldots, p_{n}, p_{n+1}\right)$. It is anonymous, continuous, and strategyproof for agents 1 to $n$, as $f$ is anonymous, continuous, and strategyproof by assumption. By the induction hypothesis, there exist $\alpha_{0}^{p_{n+1}}, \ldots, \alpha_{n}^{p_{n+1}}$ such that $f_{N}^{p_{n+1}}\left(p_{1}, \ldots, p_{n}\right)=$ $\operatorname{med}\left(p_{1}, \ldots, p_{n}, \alpha_{0}^{p_{n+1}}, \ldots, \alpha_{n}^{p_{n+1}}\right)$. As this is true for every $p_{n+1} \in[0,1]$, we can interpret $\alpha_{0}, \ldots \alpha_{n}$ as functions with $\alpha_{k}: p_{n+1} \mapsto \alpha_{k}^{p_{n+1}}$ for $k \in\{0, \ldots, n\}$. Consequently,

$$
f\left(p_{1}, \ldots, p_{n}, p_{n+1}\right)=\operatorname{med}\left(p_{1}, \ldots, p_{n}, \alpha_{0}\left(p_{n+1}\right), \ldots, \alpha_{n}\left(p_{n+1}\right)\right),
$$

where the $\alpha_{k}$ 's can be chosen such that $\alpha_{0}\left(p_{n+1}\right) \leq \alpha_{1}\left(p_{n+1}\right) \leq \cdots \leq \alpha_{n}\left(p_{n+1}\right)$ for all $p_{n+1}$.

Next, we show that each of these functions $\alpha_{k}$ can be represented as the median of $p_{n+1}$ and two constants. To this end, we prove that each $\alpha_{k}$, seen as a mechanism for the single agent $n+1$, is a continuous and strategyproof mechanism, and apply the base case.

For any $k \in\{0, \ldots, n\}$, fix the peaks of agents $1, \ldots, n-k$ at 0 , and the peaks of agents $n-k+1, \ldots, n$ at 1 . Note that

$$
f\left(p_{1}, \ldots, p_{n}, p_{n+1}\right)=\operatorname{med}(\underbrace{0, \ldots, 0}_{n-k}, \underbrace{1, \ldots, 1}_{k}, \alpha_{0}\left(p_{n+1}\right), \ldots, \alpha_{n}\left(p_{n+1}\right))=\alpha_{k}\left(p_{n+1}\right),
$$

as there are $(n-k)+k$ values below $\alpha_{k}\left(p_{n+1}\right)$ and $k+(n-k)$ values above $\alpha_{k}\left(p_{n+1}\right)$.
As $f$ is continuous and strategyproof by assumption, $\alpha_{k}\left(p_{n+1}\right)$ is continuous in $p_{n+1}$ and strategyproof with respect to manipulations of agent $n+1$. By the base case, there exist constants $c_{k}, d_{k} \in[0,1]$ such that $\alpha_{k}\left(p_{n+1}\right)=\operatorname{med}\left(p_{n+1}, c_{k}, d_{k}\right)$. Therefore,

$$
f\left(p_{1}, \ldots, p_{n+1}\right)=\operatorname{med}\left(p_{1}, \ldots, p_{n}, \operatorname{med}\left(p_{n+1}, c_{0}, d_{0}\right), \ldots, \operatorname{med}\left(p_{n+1}, c_{n}, d_{n}\right)\right) .
$$

From this point, the remainder of Moulin's proof works as-is, since it does not make use of any specific single-peaked utilities later on.

Theorem 4.2. For $m=2$ and arbitrary $\mathcal{U} \subseteq \mathcal{U}^{\mathcal{S P}}$ that contains at least one utility function per peak, the only continuous mechanism that satisfies strategyproofness and proportionality is the uniform phantom mechanism.

Proof. At a high level, the proof works as follows. In case the chosen distribution equals the outcome of the uniform phantom mechanism, we are done. Otherwise, step by step, each agent on the left side of the chosen distribution moves her peak closer and closer to 0 and each agent on the right side moves to 1 . Continuity and strategyproofness imply that the chosen distribution cannot change in the process. Finally, for all agents with peaks at the chosen distribution, move their peaks to the alternative that is not
"separated" from the peak by the outcome of the uniform phantom mechanism. In the process, the chosen distribution can only move further away from the outcome of the uniform phantom mechanism, contradicting proportionality when all agents have reached an alternative as their peak.

In detail, for $m=2$, let $f$ be a continuous mechanism that satisfies strategyproofness and proportionality, and let $g$ be the uniform phantom mechanism. Let $P$ a profile for which $f(P) \neq g(P)$. Assume that $f(P)<g(P)$; the case $f(P)>g(P)$ can be handled analogously. Denote $q:=f(P)$.

Partition the set of agents into four groups: $N=N^{01} \cup N^{-} \cup N^{=} \cup N^{+}$, where $N^{01}=\left\{i \in N: p_{i} \in\{0,1\}\right\}, N^{-}=\left\{i \in N \backslash N^{01}: p_{i}<q\right\}, N^{=}=\left\{i \in N \backslash N^{01}: p_{i}=q\right\}$, and $N^{+}=\left\{i \in N \backslash N^{01}: p_{i}>q\right\}$. Our overall goal is to "move" all agents to $N^{01}$ while keeping the chosen distribution different from $g(P)$.

Take any agent $i \in N^{-}$, and consider the function $F:[0,1] \rightarrow[0,1]$ defined by $F(p):=f\left(p, P_{-i}\right)$. Since $f$ is continuous, so is $F$. Note that $F\left(p_{i}\right)=f(P)=q>p_{i}$. We now prove that $F(p)=q$ also for all $p<p_{i}$. The proof is by contradiction:

- If $q<F(p)$, then $p<p_{i}<F\left(p_{i}\right)<F(p)$, so agent $i$ with peak at $p$ could beneficially manipulate from $p$ to $p_{i}$, contradicting strategyproofness.
- If $F(p)<q$, then by the intermediate value theorem, there exists $p^{\prime} \in\left[p, p_{i}\right)$ with $F\left(p^{\prime}\right)=\max \left\{p_{i}, F(p)\right\}$, as $F$ is continuous and $p_{i}<F\left(p_{i}\right)$. Thus, agent $i$ with peak at $p_{i}$ can beneficially change her reported peak to $p^{\prime}$ since $p_{i} \leq F\left(p^{\prime}\right)<F\left(p_{i}\right)=q$.

In particular, $F(0)=q=f(P)$. Denote the profile where agent $i$ changed her peak to 0 by $P^{\{i\}}$; then $f\left(P^{\{i\}}\right)=F(0)=f(P)$. The same argument applies to all other agents from $N^{-}$, so $f\left(P^{N^{-}}\right)=f(P)$, where $P^{N^{-}}$denotes the profile resulting from $P$ after all agents in $N^{-}$moved their peak to 0 . Also, $g\left(P^{N^{-}}\right)=g(P)$, as all agents from $N^{-}$were also on the left side of $g(P)$ due to $f(P)<g(P)$, so moving them further left does not change the distribution returned by $g$.

For agents $i \in N^{+}$and the same function $F$, one can show analogously that $F(p)=q$ for all $p \geq p_{i}$, and the outcome remains $q$ when $i$ moves her peak to 1 . Therefore, $f\left(P^{N^{-} \cup N^{+}}\right)=f(P)$, where $P^{N^{-} \cup N^{+}}$denotes the profile resulting from $P$ after all agents in $N^{-}$moved their peak to 0 and all agents in $N^{+}$moved their peak to 1 . Also, $g\left(P^{N^{-} \cup N^{+}}\right) \geq g\left(P^{N^{-}}\right)$as moving peaks to the right can only increase the median returned by $g$. Therefore, $f\left(P^{N^{-} \cup N^{+}}\right)<g\left(P^{N^{-} \cup N^{+}}\right)$still holds.

We now consider an agent $i \in N^{=}$, for whom $p_{i}=q=f(P)<g(P) \leq g\left(P^{N^{-} \cup N^{+}}\right)$.
Again, define $F^{\prime}(p):=f\left(p, P_{-i}^{N_{-}^{-} \cup N^{+}}\right)$, which is continuous as $f$ is continuous. By strategyproofness, $F^{\prime}(p) \leq q$ for $p \leq p_{i}$, since $F^{\prime}(p)>q$ would imply $p<p_{i}=F^{\prime}\left(p_{i}\right)<$ $F^{\prime}(p)$, so agent $i$ with peak at $p$ could beneficially manipulate from $p$ to $p_{i}$. Thus, with $P^{N^{-} \cup N^{+} \cup\{i\}}$ denoting the profile where agent $i$ moved her peak to $0, f\left(P^{N^{-} \cup N^{+} \cup\{i\}}\right) \leq$ $f\left(P^{N^{-} \cup N^{+}}\right)<g\left(P^{N^{-} \cup N^{+}}\right)=g\left(P^{N^{-} \cup N^{+} \cup\{i\}}\right)$. If $f\left(P^{N^{-} \cup N^{+} \cup\{i\}}\right)=f\left(P^{N^{-} \cup N^{+}}\right)$, repeat the procedure with the next agent from $N^{=}$. If $f\left(P^{N^{-} \cup N^{+} \cup\{i\}}\right)<f\left(P^{N^{-} \cup N^{+}}\right)$, all remaining agents from $N^{=}$now have their peak on the right side of $f\left(P^{N^{-} \cup N^{+} \cup\{i\}}\right)$ and can move their peak to 1 without changing the chosen distribution. Again, the outcome from $g$ can only move to the right or stay fixed.

In the end, all agents have their peaks at 0 or 1 but $f\left(P^{N^{-} \cup N^{+} \cup N^{=}}\right)<g\left(P^{N^{-} \cup N^{+} \cup N^{=}}\right)$, where $P^{N^{-} \cup N^{+} \cup N^{=}}$denotes the profile after all agents in $N^{-} \cup N^{+} \cup N^{=}$have moved their peaks. However, this contradicts proportionality of $f$.

## B. Proof of Proposition 5.4

Proposition 5.4. With $M Q$ utilities, weak core fair share implies core fair share, so these two properties are equivalent.

Proof. Assume that $\mathbf{q}$ violates core fair share for some set of agents $N^{\prime} \subseteq N$. For brevity, denote $r:=\left|N^{\prime}\right| / n$. So there is a distribution $\mathbf{q}^{\prime}$ for which the following hold for every $\mathbf{q}^{\prime \prime} \in \Delta^{m}:$

$$
\begin{array}{rr}
u_{i}\left(r \mathbf{q}^{\prime}+(1-r) \mathbf{q}^{\prime \prime}\right) \geq u_{i}(\mathbf{q}) & \text { for all } i \in N^{\prime}, \text { and }  \tag{6}\\
u_{i^{\prime}}\left(r \mathbf{q}^{\prime}+(1-r) \mathbf{q}^{\prime \prime}\right)>u_{i^{\prime}}(\mathbf{q}) & \text { for at least one } i^{\prime} \in N^{\prime} .
\end{array}
$$

The latter condition implies that every alternative $j \in T_{\mathbf{q}, i^{\prime}}$ is allocated strictly more in $\left(r \mathbf{q}^{\prime}+(1-r) \mathbf{q}^{\prime \prime}\right)$ than in $\mathbf{q}$. In particular, $T_{\mathbf{q}, i^{\prime}} \neq M$. As $\mathbf{q}^{\prime \prime}$ might allocate nothing to alternatives in $T_{\mathbf{q}, i^{\prime}}$, this implies that $r q_{j}^{\prime}>q_{j}$ for all $j \in T_{\mathbf{q}, i^{\prime}}$.

We now construct a new distribution $\mathbf{q}^{*}$ from $\mathbf{q}^{\prime}$, by taking a small amount $m \varepsilon$ from some $j_{0} \in T_{\mathbf{q}, i^{\prime}}$, such that $r q_{j_{0}}^{*}>q_{j_{0}}$ still holds, and then adding $\varepsilon$ to every $j \in M$. Now we have

$$
\begin{aligned}
r q_{j 0}^{*} & >q_{j 0}, \\
r q_{j}^{*} & >r q_{j}^{\prime}
\end{aligned} \quad \text { for all } j \neq j_{0} .
$$

Therefore, for every distribution $\mathbf{q}^{\prime \prime}$,

$$
\begin{aligned}
& r q_{j_{0}^{*}}^{*}+(1-r) q_{j_{0}}^{\prime \prime}>q_{j_{0}}, \\
& r q_{j}^{*}+(1-r) q_{j}^{\prime \prime}>r q_{j}^{\prime}+(1-r) q_{j}^{\prime \prime} \quad \text { for all } j \neq j_{0} .
\end{aligned}
$$

Denoting $\mathbf{q}^{+}:=r \mathbf{q}^{*}+(1-r) \mathbf{q}^{\prime \prime}$, the above becomes:

$$
\begin{align*}
& q_{j_{0}}^{+}>q_{j_{0}},  \tag{7}\\
& q_{j}^{+}>r q_{j}^{\prime}+(1-r) q_{j}^{\prime \prime} \quad \text { for all } j \neq j_{0}
\end{align*}
$$

We claim that $u_{i}\left(\mathbf{q}^{+}\right)>u_{i}(\mathbf{q})$ for all $i \in N^{\prime}$. Indeed, for all $i \in N^{\prime}$, if $j_{0} \in M_{i}$ :

$$
\begin{aligned}
u_{i}\left(\mathbf{q}^{+}\right) & =\min _{j \in M_{i}} \frac{q_{j}^{+}}{p_{i, j}} & & \text { (by definition of MQ utilities) } \\
& =\min \left(\frac{q_{j_{0}}^{+}}{p_{i, j_{0}}}, \min _{j \in M_{i}, j \neq j_{0}} \frac{q_{j}^{+}}{p_{i, j}}\right) & & \text { (by min properties) } \\
& >\min \left(\frac{q_{j_{0}}}{p_{i, j_{0}}}, \min _{j \in M_{i}, j \neq j_{0}} \frac{r q_{j}^{\prime}+(1-r) q_{j}^{\prime \prime}}{p_{i, j}}\right) & & (\text { by }(7))
\end{aligned}
$$

$$
\begin{array}{ll}
\geq \min \left(\frac{q_{j_{0}}}{p_{i, j_{0}}}, \min _{j \in M_{i}} \frac{r q_{j}^{\prime}+(1-r) q_{j}^{\prime \prime}}{p_{i, j}}\right) & \\
=\operatorname{(by} \min \text { properties) } \\
=\min \left(\frac{q_{j_{0}}}{p_{i, j_{0}}}, u_{i}\left(r \mathbf{q}^{\prime}+(1-r) \mathbf{q}^{\prime \prime}\right)\right) & \\
\geq \operatorname{lby} \text { definition of MQ utilities) } \\
\geq \min \left(\frac{q_{j_{0}}}{p_{i, j_{0}}}, u_{i}(\mathbf{q})\right) & \text { (by (6), as } \left.i \in N^{\prime}\right) \\
=u_{i}(\mathbf{q}) & \text { (by definition of MQ utilities). }
\end{array}
$$

If $j_{0} \notin M_{i}$, we can repeat a similar argument to arrive again at $u_{i}\left(\mathbf{q}^{+}\right)>u_{i}(\mathbf{q})$. To sum up, for every distribution $\mathbf{q}^{\prime \prime}$, we have $u_{i}\left(r \mathbf{q}^{*}+(1-r) \mathbf{q}^{\prime \prime}\right)>u_{i}(\mathbf{q})$ for all $i \in N^{\prime}$. Hence, the distribution $\mathbf{q}^{*}$ shows that $\mathbf{q}$ violates weak core fair share.

## C. Impossibility of efficiency, strategyproofness, and proportionality for $\ell_{1}$ preferences and arbitrary $n \geq 3$

In this section, we present the proof of Theorem 6.1 for arbitrary $n \geq 3$ but still fixed $m=3$.

Proof of Theorem 6.1 continued. Again, we set $M=\{a, b, c\}$ and write $\mathbf{q}=\left(q_{a}, q_{b}, q_{c}\right)$.
Note that in contrast to the case $n=3$, we now also need to denote the number of agents with certain peaks in a profile.

Consider first the following two profiles.

$$
\begin{aligned}
& \begin{array}{rcccrccc}
\text { \# agents } & \mathrm{a} & \mathrm{~b} & \mathrm{c} & \text { \# agents } & \mathrm{a} & \mathrm{~b} & \mathrm{c} \\
1 & 3 / 2 n & (2 n-3) / 2 n & 0 & 1 & 1 & 0 & 0 \\
n-2 & 0 & 1 & 0 & n-2 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\cline { 6 - 8 } & \mathbf{q}^{(1)} & \geq 1 / 2 n & \geq(2 n-3) / 2 n & \leq 1 / n & \mathbf{q}^{(2)} & 1 / n & (n-2) / n \\
\end{array} \\
& \text { Profile } 1 \\
& \text { Profile } 2
\end{aligned}
$$

The outcome in Profile 2 must be $(1 / n,(n-2) / n, 1 / n)$ by PROP. We now justify the bounds on the outcome in Profile 1. As agent 1 can manipulate between Profile 1 and Profile 2, $S P$ requires that agent 1 does not gain from either manipulation. This implies that

$$
\begin{align*}
& d_{1}^{(1)}\left(\mathbf{q}^{(1)}\right) \leq d_{1}^{(1)}\left(\mathbf{q}^{(2)}\right)=2 / n,  \tag{8}\\
& d_{1}^{(2)}\left(\mathbf{q}^{(1)}\right) \geq d_{1}^{(2)}\left(\mathbf{q}^{(2)}\right)=(2 n-2) / n . \tag{9}
\end{align*}
$$

By (8), $q_{a}^{(1)} \geq 1 / 2 n$ (implying $\left.q_{b}^{(1)} \leq(2 n-1) / 2 n\right), q_{b}^{(1)} \geq(2 n-5) / 2 n$, and $q_{c}^{(1)} \leq 1 / n$. By (9), $q_{a}^{(1)} \leq 1 / n$, implying $q_{b}^{(1)}+q_{c}^{(1)} \geq(n-1) / n$, and thus $q_{b}^{(1)} \geq(n-2) / n$.

By $E F F$, we can even show that $q_{b}^{(1)} \geq(2 n-3) / 2 n$. Otherwise, as $q_{a}^{(1)}>0$, some small amount could be moved from $a$ to $b$. Agent $n$ is indifferent due to Observation
6.2 and agents $2, \ldots, n-1$ strictly gain. Furthermore, this does not change agent 1 's disutility as $q_{b}^{(1)}<(2 n-3) / 2 n$.

Profile 3
Next, we consider Profiles 3 and 4. The outcome in Profile 4 follows from $P R O P$. We now prove that the outcome in Profile 3 must be the same. As agent 1 can manipulate between Profile 3 and Profile $4, S P$ requires that agent 1 does not gain from either manipulation. This implies that

$$
\begin{align*}
& d_{1}^{(3)}\left(\mathbf{q}^{(3)}\right) \leq d_{1}^{(3)}\left(\mathbf{q}^{(4)}\right)=2 / n  \tag{10}\\
& d_{1}^{(4)}\left(\mathbf{q}^{(3)}\right) \geq d_{1}^{(4)}\left(\mathbf{q}^{(4)}\right)=2 / n \tag{11}
\end{align*}
$$

By $(10), q_{c}^{(3)} \leq 1 / n$, implying $q_{a}^{(3)}+q_{b}^{(3)} \geq(n-1) / n$. By $(11), q_{b}^{(3)} \leq(n-1) / n$.
However, by $E F F$, if $q_{a}^{(3)}>0$ then $q_{b}^{(3)} \geq n /(n+1)$. Otherwise, some small amount can be moved from $a$ to $b$. Agent $n$ is indifferent due to Observation 6.2 and agents $2, \ldots, n-1$ strictly gain. Furthermore, this does not change agent 1's disutility as $q_{b}^{(3)}<n /(n+1)$. Therefore, $q_{a}^{(3)}=0$ must hold, and the only outcome compatible with $S P$ is $\mathbf{q}^{(3)}=(0,(n-1) / n, 1 / n)$.

Now that we know $\mathbf{q}^{(3)}$, we consider a manipulation of agent 1 from Profile 3 to Profile 1. $S P$ implies

$$
d_{1}^{(3)}\left(\mathbf{q}^{(1)}\right) \geq d_{1}^{(3)}\left(\mathbf{q}^{(3)}\right)=2 / n
$$

But the bounds we already have for $\mathbf{q}^{(1)}$ imply that $d_{1}^{(3)}\left(\mathbf{q}^{(1)}\right) \leq 2 / n$ as $q_{a}^{(1)} \geq 1 / 2 n$ and $q_{b}^{(1)} \geq(2 n-3) / 2 n$. Therefore, $d_{1}^{(3)}\left(\mathbf{q}^{(1)}\right)=2 / n$ together with $q_{a}^{(1)}=1 / 2 n$ and $q_{b}^{(1)}=(2 n-3) / 2 n$. Hence, $\mathbf{q}^{(1)}=(1 / 2 n,(2 n-3) / 2 n, 1 / n)$.

Finally, we consider the following two profiles.

$$
\begin{array}{rcccrccc}
\text { \# agents } & \mathrm{a} & \mathrm{~b} & \mathrm{c} & \text { \# agents } & \mathrm{a} & \mathrm{~b} & \mathrm{c} \\
1 & 3 / 2 n & (2 n-3) / 2 n & 0 & 1 & 1 & 0 & 0 \\
n-2 & 0 & 1 & 0 & n-2 & 0 & 1 & 0 \\
1 & 0 & (2 n-3) / 2 n & 3 / 2 n & 1 & 0 & (2 n-3) / 2 n & 3 / 2 n \\
\cline { 6 - 8 } & & & \mathbf{q}^{(6)} & 1 / n & (2 n-3) / 2 n & 1 / 2 n
\end{array}
$$

Profile 5
Profile 6
$\mathbf{q}^{(6)}$ is determined by arguments analogous to those for $\mathbf{q}^{(1)}$, reasoning about agent $n$ instead of agent 1 .

We now consider a manipulation of agent 1 from Profile 5 to Profile 6. It follows from $S P$ that

$$
d_{1}^{(5)}\left(\mathbf{q}^{(5)}\right) \leq d_{1}^{(5)}\left(\mathbf{q}^{(6)}\right)=1 / n,
$$

which implies that $q_{c}^{(5)} \leq 1 / 2 n$. Similarly, we consider a manipulation of agent $n$ from Profile 5 to Profile 1. It follows from $S P$ that

$$
d_{3}^{(5)}\left(\mathbf{q}^{(5)}\right) \leq d_{3}^{(5)}\left(\mathbf{q}^{(1)}\right)=1 / n,
$$

which implies that $q_{c}^{(5)} \geq 3 / 2 n-1 / 2 n=1 / n$, a contradiction.

## D. Not every anonymous, neutral, continuous, and strategyproof rule is moving phantoms

Proposition D.1. With $\ell_{1}$ preferences, not every anonymous, neutral, continuous, and strategyproof mechanism can be represented as a moving phantoms mechanism, for any $n \geq 1$ and $m \geq 3$ and for any number of phantom functions.

Proof. We first prove the claim for $n=1$ and $m=3$. Consider the mechanism which, in general, returns the agent's peak but cannot put more than 0.9 on an alternative. If (without loss of generality) $p_{1,1}>0.9$, the mechanism returns $q_{1}=0.9, q_{2}=p_{1,2}+\left(p_{1,1}-\right.$ $0.9) / 2$, and $q_{3}=p_{1,3}+\left(p_{1,1}-0.9\right) / 2$. Since this outcome minimizes the $\ell_{1}$ distance of agent 1 among all "legal" distributions, the mechanism is strategyproof.

Anonymity is trivially satisfied as there is only one agent. For neutrality, if $p_{1, j}=p_{1, k}$, then both alternatives receive the same probability share. In particular, if $p_{1, l}>0.9$ for the third alternative $l$, then $j$ and $k$ both receive $p_{1, j}+\left(p_{1, l}-0.9\right) / 2$, and the distribution of the surplus does not depend on the identity of the alternatives. For continuity, the only "critical" points are those where $p_{1, j}>0.9$ approaches 0.9 from above. For such peaks, $q_{j}=0.9$ is constant and the "surplus" $p_{1, j}-0.9$ is distributed on the other two alternatives in a continuous manner. Thus, the mechanism also satisfies continuity.

Suppose by contradiction that the above mechanism can be represented as a moving phantoms mechanism with phantom functions $\mathbf{h}$; let $k$ be the number of phantoms. Let $\mathbf{p}_{1}=(0.91,0.08,0.01)$. Given this profile, the mechanism returns $\mathbf{q}=(0.9,0.085,0.015)$. This implies that, for some $t \in[0,1], 0.085$ is the median of 0.08 and $h_{1}(t), \ldots, h_{k}(t)$, so the number of phantoms larger than or equal to 0.085 should be at least $k / 2+1$ (for even $k$ ) or $(k+3) / 2$ (for odd $k) .{ }^{7}$ By similar considerations, since 0.015 is the median of 0.01 and $h_{1}(t), \ldots, h_{k}(t)$, the number of phantoms smaller than or equal to 0.015 should be at least $k / 2$ (for even $k$ ) or $(k-1) / 2$ (for odd $k$ ). These two observations are contradictory as there are only $k$ phantoms in total.

[^5]A similar construction also works for larger $m$. Moreover, since the considered properties do not relate instances with different $n$, such a construction can be extended to a rule for arbitrary $n$ by using this mechanism when $n=1$ and moving phantoms when $n \geq 2$.

Note that the proof of Proposition D. 1 does not assume continuity or any other property of the phantom functions.

After constructing this counterexample, we learned that de Berg et al. (2024) independently came up with a similar construction and are working on a more natural extension to larger $n$, that does not coincide with a moving phantoms mechanism for $n>1$.

## E. Impossibility of efficiency, strategyproofness and proportionality for $\ell_{\infty}$ preferences and $n, m \geq 3$

In this section, we present the proof of Theorem 6.4 for arbitrary $n \geq 3$ and $m \geq 3$. We start by fixing $m=3$ and considering $n \geq 3$.

Lemma E.1. With $\ell_{\infty}$ preferences, no mechanism satisfies efficiency (EFF), strategyproofness (SP), and proportionality ( $P R O P$ ) when $m=3$ and $n \geq 3$.

Proof. We use the same notation as in the proof of Theorem 6.1.

| \# agents | a | b | c | \# agents | a | b | c |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | $3 / 2 n$ | $(2 n-3) / 2 n$ | 0 | 1 | 1 | 0 | 0 |
| $n-2$ | 0 | 1 | 0 | $n-2$ | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| $\mathbf{q}^{(1)}$ | $\geq 1 / 2 n$ | $\geq(2 n-3) / 2 n$ | $\leq 1 / n$ | $\mathbf{q}^{(2)}$ | $1 / n$ | $(n-2) / n$ | $1 / n$ |
| Profile 1 Profile 2 |  |  |  |  |  |  |  |

Consider first Profiles 1 and 2. The outcome in Profile 2 must be $(1 / n,(n-2) / n, 1 / n)$ by $\operatorname{PROP}$. We now justify the bounds on the outcome in Profile 1. As agent 1 can manipulate between Profile 1 and Profile 2, $S P$ requires that agent 1 does not gain from either manipulation. This implies that

$$
\begin{align*}
& d_{1}^{(1)}\left(\mathbf{q}^{(1)}\right) \leq d_{1}^{(1)}\left(\mathbf{q}^{(2)}\right)=1 / n,  \tag{12}\\
& d_{1}^{(2)}\left(\mathbf{q}^{(1)}\right) \geq d_{1}^{(2)}\left(\mathbf{q}^{(2)}\right)=(n-1) / n . \tag{13}
\end{align*}
$$

By (12), $q_{a}^{(1)} \geq 1 / 2 n$ (implying $\left.q_{b}^{(1)} \leq(2 n-1) / 2 n\right), q_{b}^{(1)} \geq(2 n-5) / 2 n$, and $q_{c}^{(1)} \leq 1 / n$. By (13), $q_{a}^{(1)} \leq 1 / n$, implying $q_{b}^{(1)}+q_{c}^{(1)} \geq(n-1) / n$, and thus $q_{b}^{(1)} \geq(n-2) / n$.

By $E F F$, we can even show that $q_{b}^{(1)} \geq(2 n-3) / 2 n$. Otherwise, $q_{b}^{(1)}<(2 n-3) / 2 n$ and $q_{c}^{(1)}+q_{a}^{(1)}>3 / 2 n$, and some small amount can be moved from $a$ to $b$. Agent $n$ is indifferent due to Observation 6.3 and agents $2, \ldots, n-1$ strictly gain. Furthermore,

| \# agents | a | b | c | \# agents | a | b | c |
| ---: | :---: | :---: | :---: | ---: | :---: | :---: | :---: |
| 1 | $1 /(n+1)$ | $n /(n+1)$ | 0 | 1 | 0 | 1 | 0 |
| $n-2$ | 0 | 1 | 0 | $n-2$ | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
|  | $\mathbf{q}^{(3)}$ |  | $\mathbf{q}^{(4)}$ | 0 | $(n-1) / n$ | $1 / n$ |  |

Profile 3
Profile 4
this does not increase agent 1's disutility as $d_{1}^{(1)}\left(\mathbf{q}^{(1)}\right) \geq q_{c}^{(1)}>3 / 2 n-q_{a}^{(1)}$ and $q_{b}^{(1)}<$ $(2 n-3) / 2 n$. Hence, $q_{b}^{(1)} \geq(2 n-3) / 2 n$.

Assume for contradiction that $q_{c}^{(1)} \leq 3 / 4 n$.
Consider a manipulation of agent 1 from Profile 3 to Profile 1. Note that $d_{1}^{(3)}\left(\mathbf{q}^{(1)}\right) \leq$ $3 / 4 n$ with the bounds established for $\mathbf{q}^{(1)}$. By $S P$ for agent $1, q_{c}^{(3)} \leq 3 / 4 n$.

By $E F F, q_{b}^{(3)} \geq n /(n+1)$. Otherwise, $q_{b}^{(3)}<n /(n+1)$ and $q_{a}^{(3)}>1 /(n+1)-q_{c}^{(3)}$, and some small amount can be moved from $a$ to $b$. Agent $n$ is indifferent due to Observation 6.3 and agents $2, \ldots, n-1$ strictly gain. Furthermore, this does not increase agent 1's disutility as $d_{1}^{(3)}\left(\mathbf{q}^{(3)}\right) \geq q_{c}^{(3)}>1 /(n+1)-q_{a}^{(3)}$ and $q_{b}^{(3)}<n /(n+1)$. Hence, $q_{b}^{(3)} \geq n /(n+1)$. However, as $n /(n+1)>(n-1) / n$, this contradicts $S P$ for agent 1 manipulating from Profile 4 to Profile 3 , where $\mathbf{q}^{(4)}=(0,(n-1) / n, 1 / n)$ follows from PROP.

$$
\begin{aligned}
& \begin{array}{rcccrccc}
\text { \# agents } & \mathrm{a} & \mathrm{~b} & \mathrm{c} & \text { \# agents } & \mathrm{a} & \mathrm{~b} & \mathrm{c} \\
1 & 1 & 0 & 0 & 1 & 3 / 2 n & (2 n-3) / 2 n & 0 \\
n-2 & 0 & 1 & 0 & n-2 & 0 & 1 & 0 \\
1 & 0 & (2 n-3) / 2 n & 3 / 2 n & 1 & 0 & (2 n-3) / 2 n & 3 / 2 n \\
& \mathbf{q}^{(5)} & \leq 1 / n & \geq(2 n-3) / 2 n \geq 1 / 2 n & \mathbf{q}^{(6)} & & &
\end{array} \\
& \text { Profile } 5 \\
& \text { Profile } 6
\end{aligned}
$$

Therefore, $q_{c}^{(1)}>3 / 4 n$ has to hold. By analogous arguments with reversed roles of agents 1 and $n$, the same bounds for $\mathbf{q}^{(5)}$ as well as $q_{a}^{(5)}>3 / 4 n$ hold.

Since $q_{b}^{(1)} \geq(2 n-3) / 2 n$, we must have $q_{a}^{(1)}<3 / 4 n$. Consider a manipulation of agent $n$ from Profile 6 to Profile 1. Note that $d_{n}^{(6)}\left(\mathbf{q}^{(1)}\right)<3 / 4 n$, as $q_{b}^{(1)} \geq(2 n-3) / 2 n$ and $q_{c}^{(1)}>3 / 4 n$. By $S P$ for agent $n$, we have $d_{n}^{(6)}\left(\mathbf{q}^{(6)}\right) \leq d_{n}^{(6)}\left(\mathbf{q}^{(1)}\right)<3 / 4 n$, and thus $q_{a}^{(6)}<3 / 4 n$. Finally, consider a manipulation of agent 1 from Profile 6 to Profile 5. Analogously, $d_{1}^{(6)}\left(\mathbf{q}^{(6)}\right) \leq d_{1}^{(6)}\left(\mathbf{q}^{(5)}\right)<3 / 4 n$, which implies $q_{a}^{(6)}>3 / 2 n-3 / 4 n=3 / 4 n$, a contradiction.

We finish the proof of Theorem 6.4 by showing that this argument remains valid under the addition of alternatives $j^{+}$with $p_{i, j^{+}}=0$ for all agents $i \in N$.

Proof of Theorem 6.4. We prove that no efficient mechanism puts positive probability
on new alternatives $j^{+}$with $p_{i, j^{+}}=0$ for all agents $i \in N$ in any of the six profiles used for the proof of Lemma E.1. Together with Lemma E.1, this completes the proof of Theorem 6.4.
$P R O P$ directly implies that adding such an alternative $j^{+}$to Profiles 2 and 4 does not change the distribution.

Next, consider Profile 1. If $\left|(2 n-3) / 2 n-q_{b}^{(1)}\right|<d_{1}^{(1)}\left(\mathbf{q}^{(1)}\right)$, or $\left|(2 n-3) / 2 n-q_{b}^{(1)}\right|=$ $d_{1}^{(1)}\left(\mathbf{q}^{(1)}\right)$ and $q_{b}^{(1)}<(2 n-3) / 2 n$, moving some amount of probability from $j^{+}$to $b$ cannot increase agent 1 's disutility, does not change agent $n$ 's disutility by Observation 6.3, and decreases the disutilities of all other agents. Therefore, such a redistribution would correspond to a Pareto improvement. Note that $\left|(2 n-3) / 2 n-q_{b}^{(1)}\right|=d_{1}^{(1)}\left(\mathbf{q}^{(1)}\right)$ and $q_{b}^{(1)}=(2 n-3) / 2 n$ cannot hold simultaneously, as $q_{j^{+}}^{(1)}>0$. Hence, the only remaining case we need to consider is $\left|(2 n-3) / 2 n-q_{b}^{(1)}\right|=d_{1}^{(1)}\left(\mathbf{q}^{(1)}\right)$ and $q_{b}^{(1)}>(2 n-3) / 2 n$. This implies $3 / 2 n-q_{a}^{(1)} \leq d_{1}^{(1)}\left(\mathbf{q}^{(1)}\right)$ and thus, $q_{a}^{(1)}+q_{b}^{(1)} \geq 3 / 2 n-d_{1}^{(1)}\left(\mathbf{q}^{(1)}\right)+(2 n-3) / 2 n+$ $d_{1}^{(1)}\left(\mathbf{q}^{(1)}\right)=1$, contradicting $q_{j^{+}}^{(1)}>0$. The argument for Profile 5 works analogously.

For Profile 3, moving some amount of probability from $j^{+}$to $b$ can only potentially increase the disutility of one agent, namely agent 1 , if $q_{b}^{(3)} \geq n /(n+1)$ and $d_{1}^{(3)}\left(\mathbf{q}^{(3)}\right)=$ $q_{b}^{(3)}-n /(n+1)$. But then, $1 /(n+1)-q_{a}^{(3)} \leq d_{1}^{(3)}\left(\mathbf{q}^{(3)}\right)$ and again, $q_{a}^{(3)}+q_{b}^{(3)}=1$, contradicting $q_{j+}^{(3)}>0$.

Finally, for Profile 6 , moving some amount of probability from $j^{+}$to $b$ can only potentially increase the disutilities of two agents, namely agents 1 and $n$, if $q_{b}^{(6)} \geq$ $(2 n-3) / 2 n$ and $d_{k}^{(6)}\left(\mathbf{q}^{(6)}\right)=q_{b}^{(6)}-(2 n-3) / 2 n$ holds for at least one $k \in\{1, n\}$, without loss of generality for $k=1$. But then, $3 / 2 n-q_{a}^{(6)} \leq d_{1}^{(6)}\left(\mathbf{q}^{(6)}\right)$ and again, $q_{a}^{(6)}+q_{b}^{(6)}=1$, contradicting $q_{j^{+}}^{(6)}>0$.

## F. Leximin-MQ preferences

MQ utilities, as defined in Section 5.2, assume that agents rank distributions only by the smallest ratio, $\min _{j \in M_{i}} q_{j} / p_{i, j}$. In this section, we assume that agents rank distributions with the same smallest ratio by the second-smallest ratio, and distributions with the same smallest and second-smallest ratio by the third-smallest ratio, and so on. We call these preferences Leximin-MQ preferences. We denote the strict Leximin-MQ preferences of each agent $i$ by $\succ_{i}^{l e x}$, and the weak relation by $\succeq_{i}^{l e x}$. When we want to emphasize that the leximin relation uses a specific peak $\mathbf{p}_{i}$, we write $\succ_{\mathbf{p}_{i}}^{l e x}$ and $\succeq_{\mathbf{p}_{i}}^{l e x}$.

We still define the $N A S H$ rule based on the minimum ratio only, which we continue to denote by $u_{i}(\mathbf{q})$. Therefore, the NASH distribution remains a continuous function of the peaks (even though the Leximin-MQ preferences are not continuous). However, the change of preferences may potentially affect some properties of the rule. In particular, $\mathbf{q} \succeq_{i}^{l e x} \mathbf{q}^{\prime}$ implies $u_{i}(\mathbf{q}) \geq u_{i}\left(\mathbf{q}^{\prime}\right)$, but for the strict relation the opposite direction is true: $u_{i}(\mathbf{q})>u_{i}\left(\mathbf{q}^{\prime}\right)$ implies $\mathbf{q} \succ_{i}^{l e x} \mathbf{q}^{\prime}$. Therefore, properties defined by the weak relation only, such as strategyproofness and weak core fair share, are stronger with Leximin-MQ
preferences than with MQ utilities. However, properties defined by both the weak and the strict relations, such as group-strategyproofness, core fair share, and efficiency, are not a-priori stronger or weaker with Leximin-MQ preferences than with MQ utilities.

First, we claim that Lemma 5.3 still holds, where the critical alternatives are defined as in Definition 5.2 (based on the minimum ratio only).

Lemma F.1. With Leximin-MQ preferences, an outcome $\boldsymbol{q}$ is efficient if and only if every alternative $j$ with $q_{j}>0$ is critical for some agent.
Proof sketch. $\Rightarrow$ : Suppose that some alternative $j$ with $q_{j}>0$ is not critical for any agent. We can construct a new outcome $\mathbf{q}^{\prime}$ by removing a sufficiently small amount from $j$ and distributing it equally among all other alternatives. This increases $\min _{j} q_{j} / p_{i, j}$ for all agents, and thus makes the new distribution strictly leximin-better for all agents. Hence, $\mathbf{q}$ is not efficient.
$\Leftarrow$ : Suppose that every alternative $j$ with $q_{j}>0$ is critical for some agent. Let $\mathbf{q}^{\prime}$ be any outcome different than $\mathbf{q}$, and let $y$ be an alternative with $q_{y}^{\prime}<q_{y}$. As $q_{y}>0$, the assumption implies that $y$ is critical to some agent; denote one such agent by $i_{y}$. Then

$$
\begin{aligned}
\min _{j} \frac{q_{j}^{\prime}}{p_{i_{y}, j}} \leq \frac{q_{y}^{\prime}}{p_{i_{y}, y}} & <\frac{q_{y}}{p_{i_{y}, y}} \\
& \left.=\min _{j} \frac{q_{j}}{p_{i_{y}, j}}, \quad \quad \text { (as } y \text { is critical for } i_{y} \text { under } \mathbf{q}\right)
\end{aligned}
$$

so $\mathbf{q}^{\prime}$ is leximin-worse for $i$ than $\mathbf{q}$. Hence, $\mathbf{q}^{\prime}$ does not Pareto-dominate $\mathbf{q}$. This holds for all $\mathbf{q}^{\prime}$, which implies that $\mathbf{q}$ is efficient.

Lemma F. 1 implies that $\mathbf{q}$ is efficient for Leximin-MQ preferences if and only if it is efficient for the corresponding MQ utilities. In particular, NASH remains efficient. Moreover, NASH is still neutral and Lemma 7.5 (efficiency implies one-sided rangerespect) remains valid as well.

Next, we show that NASH remains group-strategyproof too. We need a lemma.
Lemma F.2. Let $\boldsymbol{q}^{\prime}$ and $\boldsymbol{q}^{\prime \prime}$ be two distributions, and $i \in N$ an agent. If $\boldsymbol{q}^{\prime \prime} \succeq_{i}^{l e x} \boldsymbol{q}^{\prime}$, then every alternative in $T_{\boldsymbol{q}^{\prime}, i}$ receives at least as much in $\boldsymbol{q}^{\prime \prime}$ as in $\boldsymbol{q}^{\prime}$, that is, $q_{y}^{\prime \prime} \geq q_{y}^{\prime}$ for all $y \in T_{q^{\prime}, i}$.
Proof. For every alternative $y \in T_{\mathbf{q}^{\prime}, i}$ :

$$
\begin{aligned}
q_{i}^{\prime} & =p_{i, y} \cdot u_{i}\left(\mathbf{q}^{\prime}\right) & & \left(\text { as } y \text { is critical for } i \text { under } \mathbf{q}^{\prime}\right) \\
& \leq p_{i, y} \cdot u_{i}\left(\mathbf{q}^{\prime \prime}\right) & & \left(\text { since } \mathbf{q}^{\prime \prime} \succeq_{i}^{l e x} \mathbf{q}^{\prime} \text { implies } u_{i}\left(\mathbf{q}^{\prime \prime}\right) \geq u_{i}\left(\mathbf{q}^{\prime}\right)\right) \\
& =p_{i, y} \cdot \min _{j \in M_{i}} \frac{q_{j}^{\prime \prime}}{p_{i, j}} & & \left(\text { by definition of } u_{i}\right) \\
& \leq p_{i, y} \cdot \frac{q_{y}^{\prime \prime}}{p_{i, y}} & & \left(\text { since } y \in T_{\mathbf{q}^{\prime}, i} \subseteq M_{i}\right) \\
& =q_{y}^{\prime \prime}, & &
\end{aligned}
$$

completing the proof.

Theorem F.3. With Leximin-MQ preferences, NASH is group-strategyproof.
Proof. Assume for contradiction that there exist profiles $P$ and $P^{\prime}$ with $N A S H$ distributions $\mathbf{q} \neq \mathbf{q}^{\prime}$ respectively, and an inclusion-maximal group of agents $G \subseteq N$ which do not lose from the manipulation from $P$ to $P^{\prime}$. Let $T_{\mathbf{q}, G}:=\bigcup_{i \in G} T_{\mathbf{q}, i}$ be the set of alternatives critical to at least one agent from $G$ under $\mathbf{q}$. As no agent from $G$ loses from the manipulation, Lemma F. 2 implies that $q_{x}^{\prime} \geq q_{x}$ for all $x \in T_{\mathbf{q}, G}$.

As $\mathbf{q}^{\prime} \neq \mathbf{q}$, there is an alternative $y \in M$ for which $q_{y}^{\prime}>q_{y}$. Denote $B:=T_{\mathbf{q}, G} \cup\{y\}$ (it is possible that $y \in T_{\mathbf{q}, G}$ ). Then, $\mathbf{q}^{\prime}(B)>\mathbf{q}(B)$.

We now consider the decompositions of $\mathbf{q}$ and $\mathbf{q}^{\prime}$ guaranteed to exist by Lemma 7.3. Since $\mathbf{q}^{\prime}(B)>\mathbf{q}(B)$, there exists an agent $j \in N$ who contributes more to $B$ in the decomposition of $\mathbf{q}^{\prime}$ than in the decomposition of $\mathbf{q}$. This implies that, in the decomposition of $\mathbf{q}$, agent $j$ contributes some of her share of $1 / n$ to alternatives not in $B$. It follows that $T_{\mathbf{q}, j} \nsubseteq B$, so $j \notin G$, and thus $u_{j}=u_{j}^{\prime}$ (as $j$ is not a part of the manipulating group).

In the decomposition of $\mathbf{q}^{\prime}$, agent $j$ must contribute a positive amount to some alternative $x \in B$, which means that $x$ is critical for $j$ under $\mathbf{q}^{\prime}$. Since $u_{j}=u_{j}^{\prime}$, we have $u_{j}\left(\mathbf{q}^{\prime}\right)=u_{j}^{\prime}\left(\mathbf{q}^{\prime}\right)=q_{x}^{\prime} / p_{j, x}^{\prime} \geq q_{x} / p_{j, x}^{\prime} \geq u_{j}(\mathbf{q})$. Therefore, all agents in $G \cup\{j\}$ do not lose from the manipulation, which contradicts the maximality of $G$.

We now extend Proposition 7.7 to Leximin-MQ preferences.
Proposition F.4. With Leximin- $M Q$ preferences, $N A S H$ satisfies core fair share.
Proof. Assume for contradiction that there exists $P \in \mathcal{P}$ such that $\mathbf{q}:=N A S H(P)$ does not satisfy core fair share for some $G \subseteq N$. Then, there exists $\mathbf{q}^{\prime} \in \Delta^{m}$ such that, for every $\mathbf{q}^{\prime \prime} \in \Delta^{m}$,

$$
\begin{array}{lr}
(|G| / n) \mathbf{q}^{\prime}+(1-|G| / n) \mathbf{q}^{\prime \prime} \succeq_{i}^{l e x} \mathbf{q} & \text { for all } i \in G, \text { and } \\
(|G| / n) \mathbf{q}^{\prime}+(1-|G| / n) \mathbf{q}^{\prime \prime} \succ_{i}^{l e x} \mathbf{q} & \text { for at least one } i \in G .
\end{array}
$$

Let $T_{\mathbf{q}, G}:=\bigcup_{i \in G} T_{\mathbf{q}, i}$ be the set of alternatives critical to at least one agent from $G$.
Note that $T_{\mathbf{q}, G}=M$ cannot hold. Otherwise, by Lemma F.1, $\mathbf{q}$ would be efficient not only for $N$ but already for $G$, contradicting that $\mathbf{q}$ does not satisfy core fair share for $G$. Therefore, there exists a distribution $\mathbf{q}^{\prime \prime}$ with $q_{j}^{\prime \prime}=0$ for every $j \in T_{\mathbf{q}, G}$. Choosing such a distribution $\mathbf{q}^{\prime \prime}$ shows that $(|G| / n) q_{j}^{\prime} \geq q_{j}$; otherwise some agent from $G$ for whom $j$ is critical would have a smaller utility. Thus, $\mathbf{q}\left(T_{\mathbf{q}, G}\right):=\sum_{j \in T_{\mathbf{q}, G}} q_{j} \leq(|G| / n)$. $\sum_{j \in T_{\mathbf{q}, G}} q_{j}^{\prime} \leq|G| / n$.

By Lemma 7.3, the $N A S H$ distribution can be decomposed in such a way that every agent from $G$ only contributes her share of $1 / n$ to alternatives in $T_{\mathbf{q}, G}$. Thus, $\mathbf{q}\left(T_{\mathbf{q}, G}\right) \geq$ $|G| / n$. All in all, $\mathbf{q}\left(T_{\mathbf{q}, G}\right)=|G| / n$ and $(|G| / n) q_{j}^{\prime}=q_{j}$ for every $j \in T_{\mathbf{q}, G}$. But this also implies that $(|G| / n) \cdot \mathbf{q}^{\prime}\left(T_{\mathbf{q}, G}\right)=|G| / n$, so $\mathbf{q}^{\prime}\left(T_{\mathbf{q}, G}\right)=1$. This means that $\mathbf{q}^{\prime}$ only allocates to alternatives in $T_{\mathbf{q}, G}$. As the allocation to alternatives in $T_{\mathbf{q}, G}$ is the same in $\mathbf{q}$ and $(|G| / n) \mathbf{q}^{\prime}$, no agent in $G$ can have a better leximin vector in $(|G| / n) \mathbf{q}^{\prime}$ than in q.

We now consider the characterization (Theorem 7.9) for Leximin-MQ preferences. It turns out that group-strategyproofness can be weakened to strategyproofness.

Theorem 7.13. With Leximin-MQ preferences, NASH is the only continuous mechanism that satisfies strategyproofness and weak core fair share.

As in the proof of Theorem 7.9, to show the statement, we would like to change $P$ gradually to $P^{*}$, where each agent's peak puts 0 on non-critical alternatives. However, in order to exploit the fact that Leximin-MQ preferences constitute a refinement of MQ utilities, which allows us to weaken group-strategyproofness to strategyproofness, we need to adapt the proof. We first show that $f$ coincides with NASH on all strictlypositive profiles, that is, all profiles $P \in \mathcal{P}^{+}$, where $\mathcal{P}^{+}:=\left\{P \in \mathcal{P}: p_{i, j}>0\right.$ for all $i \in$ $N, j \in M\}$.

Lemma F.5. If $f$ is a continuous mechanism satisfying strategyproofness, then when moving from $P$ to $P^{*}$,
(a) the outcome does not change, that is, $f\left(P^{*}\right)=f(P)=\boldsymbol{q}$ for $P \in \mathcal{P}^{+}$;
(b) the sets of critical alternatives do not change, that is, $T_{\boldsymbol{q}, i}=T_{\boldsymbol{q}, i}^{*}$ for every $i \in N$.

Proof. We first show the statement for a slightly perturbed $\tilde{P}_{\tilde{\varepsilon}}^{*}$ with $\tilde{\mathbf{p}}_{i}^{*}=(1-\tilde{\varepsilon}) \mathbf{p}_{i}^{*}+\tilde{\varepsilon}_{\mathbf{p}_{i}}$ and arbitrary small but fixed $\tilde{\varepsilon}>0$. Note that $P^{*}$ may not be in $\mathcal{P}^{+}$, but $\tilde{P}_{\tilde{\varepsilon}}^{*}$ is always in $\mathcal{P}^{+}$. Note also that, for an agent $i$ with $T_{\mathbf{q}, i}=M$, it holds that $\mathbf{p}_{i}=\tilde{\mathbf{p}}_{i}^{*}=\mathbf{p}_{i}^{*}$.

Again, we move the peak of each agent in turn. For each agent $i$, we change $\mathbf{p}_{i}$ towards $\tilde{\mathbf{p}}_{i}^{*}$ gradually, to some $\widehat{\mathbf{p}}_{i}:=\lambda \tilde{\mathbf{p}}_{i}^{*}+(1-\lambda) \mathbf{p}_{i}$, for some $\lambda \in[0,1]$ to be computed later. Then we proceed along this line until we reach $\lambda=1$ and $\tilde{\mathbf{p}}_{i}^{*}$.

Given the outcome $\mathbf{q}=f(P)$, we partition the alternatives of each agent $i$ into critical classes, i.e., subsets with the same ratio $q_{j} / p_{i, j}$. Here we use the fact that $P \in \mathcal{P}^{+}$, so $p_{i, j}>0$ for all $i, j$. Denote the subset with the smallest ratio by $T_{\mathbf{q}, i, 1} \equiv T_{\mathbf{q}, i}$, the subset with the second-smallest ratio by $T_{\mathbf{q}, i, 2}$, and so on, up to $T_{\mathbf{q}, i, w}$, where $w$ is the number of different ratios. Also, for $r \in[w]$, denote $T_{\mathbf{q}, i, \leq r}:=T_{\mathbf{q}, i, 1} \cup \cdots \cup T_{\mathbf{q}, i, r}$, and define $T_{\mathbf{q}, i,>r}$ analogously.

As $\widehat{\mathbf{p}}_{i}$ lies along the line between $\mathbf{p}_{i}$ and $\mathbf{p}_{i}^{*}$, the change from $\mathbf{p}_{i}$ to $\widehat{\mathbf{p}}_{i}$ has a simple structure:

- $\widehat{p}_{i, j}>p_{i, j}$ for all $j \in T_{\mathbf{q}, i, 1}$, and the ratio $\widehat{p}_{i, j} / p_{i, j}=\lambda(1-\tilde{\varepsilon}) / \mathbf{p}_{i}\left(T_{\mathbf{q}, i, 1}\right)+\lambda \tilde{\varepsilon}+(1-$ $\lambda)=: \lambda^{+}$, a constant independent of $j$;
- $\widehat{p}_{i, j}<p_{i, j}$ for all $j \in T_{\mathbf{q}, i,>1}$, and the ratio $\widehat{p}_{i, j} / p_{i, j}=\lambda \tilde{\varepsilon}+(1-\lambda)=: \lambda^{-}$, again independent of $j$.

Computing $\lambda$. We pick $\lambda$ sufficiently small such that no new alternative becomes critical for $i$, and moreover, critical classes do not mix, i.e., $q_{j}^{\prime} / p_{i, j^{\prime}}>q_{j} / p_{i, j}$ implies $\widehat{q}_{j^{\prime}} / p_{i, j^{\prime}}>$ $\widehat{q}_{j} / p_{i, j}$ for all $j, j^{\prime} \in M$. Specifically, set

$$
\varepsilon:=\min _{j \in T_{\mathbf{q}, i, r}, j^{\prime} \in T_{\mathbf{q}, i, s}} \frac{q_{j^{\prime}} p_{i, j}-q_{j} p_{i, j^{\prime}}}{p_{i, j}+p_{i, j^{\prime}}}
$$

where the minimum is taken over all $r, s \in[w]$ and $s>r$. Note that $\varepsilon>0$, as $q_{j^{\prime}} / p_{i, j^{\prime}}>$ $q_{j} / p_{i, j}$, by definition of critical classes.

By uniform continuity of $f$, there exists $\delta>0$ such that $\left\|f(P)-f\left(P^{\prime}\right)\right\|_{1}<2 \varepsilon$ for all $P^{\prime}$ with $\left\|P-P^{\prime}\right\|_{1} \leq \delta$. Set

$$
\lambda:=\min \left(1, \frac{\delta}{\left\|\mathbf{p}_{i}-\tilde{\mathbf{p}}_{i}^{*}\right\|_{1}}\right),
$$

and define $\widehat{P}$ as a profile identical to $P$ except that $i$ changes her peak from $\mathbf{p}_{i}$ to $\widehat{\mathbf{p}}_{i}:=\lambda \tilde{\mathbf{p}}_{i}^{*}+(1-\lambda) \mathbf{p}_{i}$. Note that $\|P-\widehat{P}\|_{1}=\lambda\left\|\mathbf{p}_{i}-\tilde{\mathbf{p}}_{i}^{*}\right\|_{1} \leq \delta$, so $\|\mathbf{q}-\widehat{\mathbf{q}}\|_{1}<2 \varepsilon$, where $\mathbf{q}=f(P)$ and $\widehat{\mathbf{q}}=f(\widehat{P})$.

The choice of $\varepsilon$ ensures that for arbitrary $r, s \in[w]$ with $s>r, j \in T_{\mathbf{q}, i, r}$, and $j^{\prime} \in T_{\mathbf{q}, i, s}$,

$$
\begin{aligned}
\frac{\widehat{q}_{j^{\prime}}}{p_{i, j^{\prime}}} & >\frac{q_{j^{\prime}}-\varepsilon}{p_{i, j^{\prime}}} \geq \frac{q_{j^{\prime}}}{p_{i, j^{\prime}}}-\frac{q_{j^{\prime}} p_{i, j}-q_{j} p_{i, j^{\prime}}}{p_{i, j^{\prime}}\left(p_{i, j}+p_{i, j^{\prime}}\right)}=\frac{q_{j^{\prime}} p_{i, j^{\prime}}+q_{j} p_{i, j^{\prime}}}{p_{i, j^{\prime}}\left(p_{i, j}+p_{i, j^{\prime}}\right)}=\frac{q_{j^{\prime}}+q_{j}}{p_{i, j}+p_{i, j^{\prime}}} \\
& =\frac{q_{j} p_{i, j}+q_{j^{\prime}} p_{i, j}}{p_{i, j}\left(p_{i, j}+p_{i, j^{\prime}}\right)}=\frac{q_{j}}{p_{i, j}}+\frac{q_{j^{\prime}} p_{i, j}-q_{j} p_{i, j^{\prime}}}{p_{i, j}\left(p_{i, j}+p_{i, j^{\prime}}\right)} \geq \frac{q_{j}+\varepsilon}{p_{i, j}}>\frac{\widehat{q}_{j}}{p_{i, j}} .
\end{aligned}
$$

Therefore, we have $T_{\mathbf{q}, i, r}=\widehat{T}_{\mathbf{q}, i, r}$ for all $r \in[w]$.
Proving that the outcome does not change. Consider a manipulation of agent $i$ who manipulates between reporting $\mathbf{p}_{i}$ and $\widehat{\mathbf{p}}_{i}$. Strategyproofness for agent $i$ implies both $\mathbf{q} \succeq_{\mathbf{p}_{i}}^{l e x} \widehat{\mathbf{q}}$ and $\widehat{\mathbf{q}} \succeq_{\widehat{\mathbf{p}}_{i}}^{l e x} \mathbf{q}$.

We now prove, by induction on $r$, that $q_{j}=\widehat{q}_{j}$ for all $j \in T_{\mathbf{q}, i, r}$. For the base case $r=1$, consider the alternatives in $T_{\mathbf{q}, i, 1}$.

- As all alternatives in $T_{\mathbf{q}, i, 1}$ are at the bottom of the ordering by $q_{j} / p_{i, j}$ (by definition) as well as by $\widehat{q}_{j} / p_{i, j}$ (by the choice of $\varepsilon$ ), the relation $\mathbf{q} \succeq_{\mathbf{p}_{i}}^{l e x} \widehat{\mathbf{q}}$ implies the same relation among the sub-vectors corresponding to the alternatives in $T_{\mathbf{q}, i, 1}$, that is,

$$
\begin{equation*}
\left[q_{j} \mid j \in T_{\mathbf{q}, i, 1}\right] \succeq_{\mathbf{p}_{i}}^{l e x}\left[\widehat{q}_{j} \mid j \in T_{\mathbf{q}, i, 1}\right] . \tag{14}
\end{equation*}
$$

- Similarly, all alternatives in $\widehat{T}_{\mathbf{q}, i, 1}=T_{\mathbf{q}, i, 1}$ are at the bottom of the ordering by $q_{j} / \widehat{p}_{i, j}$ by construction. Therefore, the relation $\widehat{\mathbf{q}} \succeq_{\mathbf{p}_{i}}^{l e x} \mathbf{q}$ implies

$$
\begin{equation*}
\left[\widehat{q}_{j} \mid j \in T_{\mathbf{q}, i, 1}\right] \succeq_{\widehat{\mathbf{p}}_{i}}^{l e x}\left[q_{j} \mid j \in T_{\mathbf{q}, i, 1}\right] . \tag{15}
\end{equation*}
$$

- But since $\widehat{p}_{i, j}$ differs from $p_{i, j}$ by a constant factor $\lambda^{+}$for all $j \in T_{\mathbf{q}, i, 1}$, (15) implies the same inequality with $\succeq_{\mathbf{p}_{i}}^{l e x}$ instead of $\succeq_{\mathbf{p}_{i}}^{l e x}$. Combining this with (14), we get

$$
\left[\widehat{q}_{j} \mid j \in T_{\mathbf{q}, i, 1}\right] \simeq_{\mathbf{p}_{i}}^{l e x}\left[q_{j} \mid j \in T_{\mathbf{q}, i, 1}\right] .
$$

As $q_{j} / p_{i, j}$ is constant for $j \in T_{\mathbf{q}, i, 1}$, lexicographic equivalence with respect to $\mathbf{p}_{i}$ implies $q_{j} / p_{i, j}=\widehat{q}_{j} / p_{i, j}$ for all $j \in T_{\mathbf{q}, i, 1}$. Thus, $q_{j}=\widehat{q}_{j}$ must hold for all $j \in T_{\mathbf{q}, i, 1}$.

For the induction step, assume that $q_{j}=\widehat{q}_{j}$ holds for all $j \in T_{\mathbf{q}, i, \leq r}$, for some $r \in[w-1]$. Next, consider the alternatives in $T_{\mathbf{q}, i, r+1}=\widehat{T}_{\mathbf{q}, i, r+1}$.

- As $q_{j}=\widehat{q}_{j}$ holds for all other alternatives with smaller ratios, the relation $\mathbf{q} \succeq \succeq_{\mathbf{p}_{i}}^{l} \widehat{\mathbf{q}}$ implies the same relation for the subset $T_{\mathbf{q}, i, r+1}$, that is, $\left[q_{j} \mid j \in T_{\mathbf{q}, i, r+1}\right] \succeq_{\mathbf{p}_{i}}^{l e x}\left[\widehat{q}_{j} \mid\right.$ $\left.j \in T_{\mathbf{q}, i, r+1}\right]$.
- Similarly, the relation $\widehat{\mathbf{q}} \succeq_{\mathbf{p}_{i}}^{l e x} \mathbf{q}$ implies $\left[\widehat{q}_{j} \mid j \in T_{\mathbf{q}, i, r+1}\right] \succeq_{\widehat{\mathbf{p}}_{i}}^{l e x}\left[q_{j} \mid j \in T_{\mathbf{q}, i, r+1}\right]$.
- But since $\widehat{p}_{i, j}$ differs from $p_{i, j}$ by a constant factor $\lambda^{-}$for all $j \in T_{\mathbf{q}, i, r+1}, \widehat{\mathbf{q}} \succeq_{\mathbf{p}_{i}}^{l e x} \mathbf{q}$ implies $\widehat{\mathbf{q}} \succeq_{\mathbf{p}_{i}}^{l e x} \mathbf{q}$, so all in all $\left[\widehat{q}_{j} \mid j \in T_{\mathbf{q}, i, r+1}\right] \simeq_{\mathbf{p}_{i}}^{l e x}\left[q_{j} \mid j \in T_{\mathbf{q}, i, r+1}\right]$ must hold. This means that $q_{j}=\widehat{q}_{j}$ must hold for all $j \in T_{\mathbf{q}, i, r+1}$.

Therefore, $\widehat{\mathbf{q}}=\mathbf{q}$.
Applying this argument repeatedly, we get a sequence of profiles $\left(P^{k}\right)$ with $P^{0}=P$ where $\mathbf{p}_{i}^{k}$ lies on the line $\lambda \tilde{\mathbf{p}}_{i}^{*}+(1-\lambda) \mathbf{p}_{i}$ for every $k$. It remains to show that ( $\mathbf{p}^{k}$ ) reaches $\tilde{\mathbf{p}}_{i}^{*}$ after a finite number of steps. For that, consider the expression in the definition of $\varepsilon$ :

$$
\min _{j \in T_{\mathbf{q}, i, r}, j^{\prime} \in T_{\mathbf{q}, i, s}} \frac{q_{j^{\prime}} p_{i, j}-q_{j} p_{i, j^{\prime}}}{p_{i, j}+p_{i, j^{\prime}}} .
$$

For $r=1$, as $\mathbf{p}_{i}$ comes closer to $\tilde{\mathbf{p}}_{i}^{*}, p_{i, j}$ increases and $p_{i, j^{\prime}}$ decreases while $\mathbf{q}$ and the critical classes stay the same, so overall the expression increases. For $s>r>1$, note that $p_{i, j}$ and $p_{i, j^{\prime}}$ both decrease by the same factor $\lambda^{-}$while $\mathbf{q}$ and the critical classes stay the same. Thus, we can take the $\varepsilon$ (and the corresponding $\delta$ ) from the first step for every step. Furthermore, $\left\|P^{k}-P^{k+1}\right\|_{1}=\delta$ (unless $\lambda=1$, but then we have reached $\left.\tilde{\mathbf{p}}_{i}^{*}\right)$ implying that we reach $\tilde{\mathbf{p}}_{i}^{*}$ after at most $\left\lceil\left\|\mathbf{p}_{i}-\tilde{\mathbf{p}}_{i}^{*}\right\|_{1} / \delta\right\rceil$ steps; as we move on a line of length $\left\|P^{k}-P^{k^{\prime}}\right\|_{1}=\sum_{\ell=k}^{k^{\prime}-1}\left\|P^{\ell}-P^{\ell+1}\right\|_{1}$ for $k^{\prime} \geq k$.

After the first agent has reached her desired peak $\tilde{\mathbf{p}}_{i}^{*}$, we turn to the next agent and repeat the procedure. In that way, we eventually arrive at $\tilde{P}_{\tilde{\varepsilon}}^{*}$.

As $\tilde{\varepsilon}$ was chosen arbitrarily and we have $\lim _{\tilde{\varepsilon} \rightarrow 0} P_{\tilde{\varepsilon}}^{*}=P^{*}$ for arbitrary $P \in \mathcal{P}^{+}$, continuity of $f$ implies $f(P)=f\left(P^{*}\right)$.

Statement (b) now follows analogously to the one in Lemma 7.10.
Proof of Theorem 7.13. As Lemmas 7.11 and 7.12 still hold for Leximin-MQ preferences, $f(P)=N A S H(P)$ for all $P \in \mathcal{P}^{+}$. Noting that $\mathcal{P}^{+}$is dense in $\mathcal{P}$ (and $f$ and NASH are continuous), $f$ has to coincide with $N A S H$ on all profiles in $\mathcal{P}$.


[^0]:    ${ }^{1}$ This is in contrast to probabilistic social choice (see, e.g., Brandt, 2017), in which the final outcome is a single alternative picked at random from the distribution, so typically the agents' most-preferred distributions are degenerate.

[^1]:    ${ }^{2}$ Concretely, suppose that there are $m=3$ alternatives and $n=2$ agents with ideal distributions $\mathbf{p}_{1}=(0.8,0.2,0)$ and $\mathbf{p}_{2}=(0.8,0,0.2)$, respectively. The independent markets mechanism of Freeman et al. (2021) returns the distribution $\mathbf{q}=(0.6,0.2,0.2)$. However, if agent 1 reports $\mathbf{p}_{1}^{\prime}=(0.82,0.18,0)$ instead, the mechanism returns $\mathbf{q}^{\prime}=(0.62,0.18,0.2)$, which the agent prefers to $\mathbf{q}$ under MQ utilities.

[^2]:    ${ }^{3}$ The "at least one" requirement is needed to allow agents to misreport their peak to any other peak in $[0,1]$. Note that we still require our mechanism to be tops-only, i.e., it only depends on the peaks.
    ${ }^{4}$ A domain $\mathcal{U}$ is symmetric if for all $u_{i} \in \mathcal{U}$ and $p_{i}, q, q^{\prime} \in[0,1], u_{i}(q)>u_{i}\left(q^{\prime}\right)$ if and only if $\left|p_{i}-q\right|<$ $\left|p_{i}-q^{\prime}\right|$.

[^3]:    ${ }^{5}$ Note that proportionality already contains some form of anonymity: when all agents have peaks at $[1: a]$ or $[1: b]$, proportionality requires picking a specific distribution which is independent of the agents' identities.

[^4]:    ${ }^{6}$ Our notion of group-strategyproofness is slightly different as agents do not have variable contributions. Specifically, we only need to consider manipulations of the individual peaks. However, this is a subset of all manipulations considered by Brandt et al. (2023), so $N A S H$ also satisfies our notion of groupstrategyproofness.

[^5]:    ${ }^{7}$ We assume that when $k$ is odd, the median of $k+1$ elements is the $((k+1) / 2)$-th element.

