Balanced Donor Coordination

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Charity is typically done either by individual donors, who donate money to the charities that they support, or by centralized organizations such as governments or municipalities, which collect the individual contributions and distribute them among a set of charities. On the one hand, individual charity respects the will of the donors but may be inefficient due to a lack of coordination. On the other hand, centralized charity is potentially more efficient but may ignore the will of individual donors. We present a mechanism that combines the advantages of both methods by distributing the contribution of each donor in an efficient way such that no subset of donors has an incentive to redistribute their donations. Assuming Leontief utilities (i.e., each donor is interested in maximizing an individually weighted minimum of all contributions across the charities), our mechanism is group-strategyproof, preference-monotonic, contribution-monotonic, maximizes Nash welfare, and can be computed using convex programming.

1 INTRODUCTION

Private charity, given by individual donors to underprivileged people in their vicinity, has existed long before institutionalized charity via municipal or governmental organizations. Its main advantage is transparency—the donors know exactly where their money goes to, which may increase their willingness to donate. A major disadvantage of private charity is the lack of coordination: donors may donate to certain people or charities without knowing that these recipients have already received ample money from other donors. Centralized charity via governments or municipalities is potentially more efficient but, if not done carefully, may disrespect the will of the donors.

As an example, consider the following scenario involving two donors and four charities. The first donor is willing to contribute \$900 and supports charities A, B, and C, whereas the second donor is willing to contribute \$100 and supports charities C and D.

- A central organization may collect the contributions of the donors and divide them equally among the four charities, so that each charity receives \$250. While this outcome is the most balanced possible for the charities, it goes against the will of the first donor, since \$150 of her contribution is used to support charity D.
- By contrast, without any coordination, each donor may split her individual contribution equally between the charities that she approves. As a result, charities A and B receive \$300 each, charity C receives \$350, and charity D receives \$50. However, if the second donor knew that charity C would already receive \$300 from the first donor, she would probably prefer to donate more to charity D, for which she is the only contributor.
- Our suggested mechanism would give \$300 to each of charities A, B, and C, and \$100 to charity D. This distribution can be implemented in such a way that the contribution of each donor only goes to charities that the donor approves, and subject to that, the donations are divided as equally as possible. The distribution can also be understood as recommendations to the individual donors: the first donor should distribute her contribution uniformly over charities A, B, and C whereas the second donor should transfer all her contribution to charity D.

Evaluating and comparing donor coordination mechanisms requires some assumptions on the utility functions of the donors. In this paper, we assume that each donor's utility is given by the smallest amount of money allocated to one of the donor's approved charities. For example, for the distribution $(300, 300, 300, 100)$, the first agent's utility is 300 and the second agent's utility is 100.

More generally, our model allows donors to attribute different values than merely 1 and 0 (which indicate approval and disapproval, respectively) to different charities. If a donor i values a project x at $v_{i,x}$, then i's utility from a distribution δ equals $\min_x \delta(x)/v_{i,x}$, where the minimum is taken over all projects x for which $v_{i,x} > 0$. Such utilities are known as *Leontief utilities* and are often studied in resource allocation problems [e.g., Bei et al., 2022, Brânzei et al., 2016, Goel et al., 2019, Li and Xue, 2013, Nicoló, 2004, Parkes et al., 2015, Yin et al., 2021]. Whenever $v_{i,x} \in \{0, 1\}$ for all agents i and projects x, we refer to this as (Leontief) utility functions with *binary weights*.

Given the contribution and utility function of each donor, our goal is to distribute the money among the charities in a way that respects the individual donors' preferences. The idea of "respecting the donors' preferences" is captured by the notion of an *equilibrium distribution*. We say that a distribution is *in equilibrium* if it can be implemented by telling each donor how to distribute her contribution among the charities, such that the prescribed distribution maximizes the donor's utility given that the distributions of the other donors remain fixed. One can check that, in the above example, the unique equilibrium distribution is $(300, 300, 300, 100)$.

A priori, it is not clear that an equilibrium distribution (in pure strategies) always exists. Our first main result is that each profile admits a *unique* equilibrium distribution. Moreover, we prove that the unique equilibrium distribution coincides with the unique distribution that maximizes the product of individual utilities weighted by their contributions (*Nash welfare*), which implies that it is Pareto efficient, and can be computed via convex programming.

In our example, the equilibrium distribution $(300, 300, 300, 100)$ also maximizes the minimum utility of all agents (*egalitarian welfare*) subject to each donor only contributing to her approved charities. We show that this is true in general when weights are binary, and extends to an infinite class of welfare measures "in between" Nash welfare and egalitarian welfare. Moreover, for the case of binary weights, we show that the equilibrium distribution coincides with the distribution that allocates individual contributions to approved projects such that the minimum contribution to projects is maximized lexicographically. This allows for simpler computation via linear programming. Further, we propose a simple spending dynamics for binary weights that is based on best responses and converges to the equilibrium distribution.

Based on existence and uniqueness, we can define the *equilibrium distribution rule (EDR)*—the mechanism that returns the unique equilibrium distribution of a given profile. Our second main result is that *EDR* exhibits remarkable axiomatic properties:

- *Group-strategyproofness*: agents and coalitions thereof are never better off by misrepresenting their preferences, and are strictly better off by contributing more money,
- *Preference-monotonicity*: the amount donated to a project can only increase when agents increase their valuation for the project, and
- *Contribution-monotonicity*: the amount donated to a project can only increase when agents increase their contributions.

Our results can be applied not only to private charity, but also to donation programs—prominent examples include *AmazonSmile* and *cinque per mille* by the Italian Revenue Agency.[1] In these programs, participants can redirect a portion of their payments (purchase price and income tax, respectively) to charitable organizations of their choice.[2] Note that, in contrast to private charity, participants of donation programs do not have the option of taking their money out of the system, which means that the important issue lies in finding a desirable distribution of the contributions rather than in incentivizing the participants to take part in the donation exercise in the first place.

[1] https://smile.amazon.com, https://www.agenziaentrate.gov.it/portale/web/guest/elenchi-versione-precedenti

[2] For AmazonSmile and cinque per mille, each participant can choose only one charitable organization. However, as Brandl et al. [2022] argued, permitting them to indicate support for multiple organizations can increase efficiency of the process.

Another example of a potential application is the transmission of signals in a network. Consider a directed graph and a set of agents where each agent intends to transmit a signal along an individual path in the graph. Agents are able to invest in the "transmission quality" of each edge. Their utilities are given by the quality of the signal at the last vertex on their path, which equals the minimal transmission quality of an edge along that path.

The remainder of this paper is structured as follows. After discussing related work in Section 2, we formally introduce our model in Section 3. Section 4 lays the foundation for the proposed distribution rule by showing existence and uniqueness of equilibrium distributions as well as characterizing Pareto efficient distributions in our setting. Subsequently, we define the *equilibrium distribution rule* as the rule that always returns the equilibrium distribution and examine it axiomatically in Section 5. The special case of Leontief utilities with binary weights is covered in Section 6. Binary weights allow for alternative characterizations of *EDR* that enable its computation via linear programming. Furthermore, we justify *EDR* from a welfare point of view and present a simple best response dynamics that converges to the equilibrium distribution. Finally, we conclude in Section 7 and point out open problems and further directions for future research.

2 RELATED WORK

The work most closely related to ours is that of Brandl et al. [2022, 2021] who initiated the axiomatic study of donor coordination mechanisms. In their model, the utility of each donor is defined as the weighted *sum* of contributions to projects, where the weights correspond to the agent's inherent utilities for a unit of contribution to each project. Under this assumption, the only efficient distribution in the introductory example is to allocate the entire donation of $1000 to charity *C*, since this distribution gives the highest possible utility, 1000, to all donors. However, this distribution leaves charities *A*, *B*, and *D* with no money at all, which may not be what the donors intended. With sum-based utilities, as studied by Brandl et al., charities are perfect *substitutes*: when a donor assigns the same utility to several charities, she is completely indifferent to how money is distributed among these charities. By contrast, in our model of *minimum-based* utilities, charities are perfect *complements*: donors want their money to be evenly distributed among charities they like equally much. Preferences over charities can be expressed by setting weights for Leontief utility functions. It can be argued that this assumption better reflects the spirit of charity by not leaving anyone behind. The modified definition of utility functions critically affects the nature of elementary concepts such as efficiency or strategyproofness and fundamentally changes the landscape of attractive mechanisms.

The main result by Brandl et al. [2022] shows that, in their model of sum-based utilities, the Nash product rule incentivizes agents to contribute their entire budget, even when attractive outside options are available. On the other hand, the Nash product rule fails to be strategyproof [Aziz et al., 2019] and violates simple monotonicity conditions [Brandl et al., 2021]. In fact, a sweeping impossibility by Brandl et al. [2021] shows that, even in the simple case of binary valuations, no distribution rule that spends money to at least one approved project of each agent can simultaneously satisfy efficiency and strategyproofness. This confirms a conjecture by Bogomolnaia et al. [2005] and demonstrates the severe limitations of donor coordination with sum-based utilities. Interestingly, as we show in this paper, Leontief utilities allow for much more positive results. The rule we propose satisfies virtually all desirable properties one could think of.

Originating from the *Nash bargaining solution* [Nash, 1950], the Nash product rule can be interpreted as a tradeoff between maximizing utilitarian and egalitarian welfare, a recurring idea when it comes to finding efficient *and* fair solutions. When allocating divisible private goods to agents with additive valuations, the Nash product rule returns the set of all *competitive equilibria from equal incomes* [Eisenberg and Gale, 1959]; thus, it results in an efficient and *envy-free* allocation [Foley, 1967]. For the case of indivisible private goods, Caragiannis et al. [2019] showed that maximizing Nash welfare returns an allocation that is not only efficient but also satisfies *envy-freeness up to one good*, and Suksompong [2023] and Yuen and Suksompong [2023] obtained characterizations of the Nash product rule using the latter axiom. The Nash product rule is also a sensible mechanism in our context and, as shown in Section 4, its outcome is completely characterized by another concept due to Nash [1950]: when defining a game in which the players' strategies are redistributions of their individual contributions, there is a unique Nash equilibrium which coincides with the distribution maximizing Nash welfare.

A natural special case of our model is that of Leontief utilities with *binary weights*, where agents only approve or disapprove projects and the utility of each agent is given by the minimal amount transferred to any of her approved projects. Under the assumption that agents only contribute to projects they approve and that all individual contributions are equal, this can be interpreted as a (many-to-many) matching problem on a bipartite graph where agents (and their contributions) need to be assigned to projects with unlimited capacity. Bogomolnaia and Moulin [2004] proposed a solution to such matching problems that maximizes egalitarian welfare of the projects (rather than the agents). The reasons for the intriguing connection between these two types of egalitarianism are addressed in Section 6. These authors also showed that their solution constitutes a competitive equilibrium from equal incomes (from the project holders' point of view).

A problem remotely related to the setting we study in this paper is that of *private provision of public goods* [e.g., Bergstrom et al., 1986, Samuelson, 1954]. In this stream of research, each agent decides on how much money she wants to contribute to funding a public good. Typically, this leads to under-provision of the public good in equilibrium, resulting in inefficient outcomes. In our model, we assume that agents have already set aside a budget to support public projects, either voluntarily or compulsorily (as part of their taxes or payments to a company). The inefficiency that we are worried about is an inefficient allocation among different public goods. As a result, the problem we study has the flavor of both social choice and fair division.

Finally, in *participatory budgeting* [e.g., Aziz and Shah, 2021], it is typically assumed that project funding is discrete, that is, each project should be either fully funded or not at all (e.g., constructing a new bridge) and money is owned by a central authority rather than the agents. Aziz and Ganguly [2021] studied a donor coordination version where the budget belongs to the agents, but still considered discrete project funding.

3 THE MODEL

Let N be a set of *n agents*. Each agent i contributes an amount $C_i \geq 0$. For every subset of agents $N' \subseteq N$, we denote $C_{N'} := \sum_{i \in N'} C_i$. The sum of all contributions, C_N, is called the *endowment*.

Further, consider a set A of m potential recipients of the contributions, which we call *projects*. A *distribution* is a function δ assigning a nonnegative real number to each project, such that $\sum_{x \in A} \delta(x) = C_N$. The support $\{x : \delta(x) > 0\}$ of δ is denoted by $\text{supp}(\delta)$, and the space of all possible distributions is denoted by $\Delta(C_N)$. For a subset of projects $A' \subseteq A$, we define $\delta(A') := \sum_{x \in A'} \delta(x)$ as the total amount allocated to projects in A'.

For every $i \in N$ and $x \in A$, there is a real number $v_{i,x} \geq 0$ that represents the value of project x to agent i. We assume that each agent i has at least one project x for which $v_{i,x} > 0$. For every agent $i \in N$, we define $A_i := \{x : v_{i,x} > 0\}$ as the set of projects to which i attributes a positive value.

The utility that agent i derives from distribution δ is denoted by $u_i(\delta)$ and is given by the Leontief utility function[3]

$$u_i(\delta) = \min_{x \in A_i} \frac{\delta(x)}{v_{i,x}}.$$

Note that, for every project $x \in A$ and every agent $i \in N$,

$$\delta(x) \geq v_{i,x} \cdot u_i(\delta).$$

If all $v_{i,x}$ are in $\{0, 1\}$, we refer to Leontief utilities with *binary* weights. A *profile* P consists of $\{C_i\}_{i \in N}$ and $\{v_{i,x}\}_{i \in N, x \in A}$. Throughout this paper, agents with contribution zero do not have any influence on the outcome and can thus be treated as agents who choose not to participate in the mechanism.

A *distribution rule* f maps every profile to a distribution $\Delta(C_N)$ of the total endowment C_N.

4 EQUILIBRIUM DISTRIBUTIONS

Donor coordination differs from other participatory budgeting problems in that the budget is initially owned by the agents. This makes it all the more important that agents are able to observe the distribution of their individual contribution. We formalize this intuition using the notion of a decomposition.

Definition 4.1 (Decomposition). A *decomposition* of a distribution δ is a vector of distributions $(\delta_i)_{i \in N}$ with

$$\sum_{i \in N} \delta_i(x) = \delta(x) \qquad \text{for all } x \in A; \tag{1}$$

$$\sum_{x \in A} \delta_i(x) = C_i \qquad \text{for all } i \in N. \tag{2}$$

We aim for a decomposition in which no agent can increase her utility by changing δ_i, given C_i and the distributions δ_j for $j \neq i$. In other words, we look for a pure strategy Nash equilibrium of the game in which the strategy space of each agent i is the set of δ_i satisfying (2).

Definition 4.2 (Equilibrium distribution). A distribution δ is *in equilibrium* if it admits a decomposition $(\delta_i)_{i \in N}$ such that, for every agent i and for every alternative distribution δ'_i satisfying $\sum_{x \in A} \delta'_i(x) = C_i$,

$$u_i(\delta) \geq u_i(\delta - \delta_i + \delta'_i).$$

A priori, it is not clear whether an equilibrium distribution always exists. The present section is devoted to proving the following theorem.

THEOREM 4.3. *Every profile admits a unique equilibrium distribution. This distribution is Pareto efficient and can be computed via convex programming.*

As a consequence, we can define the *equilibrium distribution rule* as the distribution rule that selects for each profile its unique equilibrium distribution. In Section 5, we prove that this rule satisfies desirable strategic and monotonicity properties.

[3] The case of *sum-based*, rather than *min-based*, utility functions $u_i(\delta) = \sum_{x \in A} v_{i,x} \cdot \delta(x)$ is discussed in Section 2.

4.1 Critical projects

We start by characterizing equilibrium distributions based on *critical projects*.

Given a distribution δ, we define the set of agent i's *critical projects*

$$T_{\delta,i} := \arg\min_{x \in A_i} \frac{\delta(x)}{v_{i,x}}.$$

Each project $x \in T_{\delta,i}$ is critical for agent i in the sense that the utility of i would decrease if the amount allocated to x were to decrease. Every agent has at least one critical project. For every agent i and project x such that either $v_{i,x} > 0$ or $\delta(x) > 0$, the following implications hold:

$$\begin{aligned} x \in T_{\delta,i} &\iff \delta(x) = v_{i,x} \cdot u_i(\delta); \\ x \notin T_{\delta,i} &\iff \delta(x) > v_{i,x} \cdot u_i(\delta). \end{aligned} \tag{3}$$

We prove below that a distribution is in equilibrium if and only if each agent contributes only to her critical projects.

Lemma 4.4. *[Characterization of equilibrium distributions] A distribution δ is in equilibrium if and only if it has a decomposition $(\delta_i)_{i\in N}$ such that $\delta_i(x) = 0$ for every project $x \notin T_{\delta,i}$. Equivalently, it has a decomposition satisfying the following, instead of* (2)*:*

$$\sum_{x \in T_{\delta,i}} \delta_i(x) = C_i \qquad \text{for all } i \in N. \tag{4}$$

Proof. $\Rightarrow$: Suppose that, in every decomposition of δ, some agent i contributes to a project $y \notin T_{\delta,i}$. Fix a decomposition $(\delta_i)_{i\in N}$ of δ. Since $\delta(y) > 0$, by (3), $\delta(y) > v_{i,y} \cdot u_i(\delta)$. Agent i can reduce a small amount from $\delta_i(y)$ and distribute it equally among all projects in $T_{\delta,i}$; this strictly increases the Leontief utility of i. Therefore, δ is not an equilibrium distribution.

$\Leftarrow$: Suppose δ has a decomposition in which each agent i only contributes to projects in $T_{\delta,i}$. In every other strategy of agent i, she must contribute less money to at least one such project, $y \in T_{\delta,i}$. Since $\delta(y) > 0$, by (3), the original distribution to project y was $\delta(y) = v_{i,y} \cdot u_i(\delta)$, so the new distribution to y is less than $v_{i,y} \cdot u_i(\delta)$. Therefore, the utility of agent i is smaller than $u_i(\delta)$ and the deviation is not beneficial. □

Corollary 4.5. *In an equilibrium distribution, every project that receives a positive amount of contribution is critical for at least one agent.*

Remark 1. Lemma 4.4 implies that an equilibrium distribution satisfies an even stronger equilibrium property: no *group of agents* can deviate without making at least one of its members worse off. This is because any deviation decreases the contribution to a critical project of at least one group member. This equilibrium notion is slightly stronger than *strong equilibrium* by Aumann [1959].

4.2 Efficiency

One of the main objectives of a centralized distribution rule is economic efficiency.

Definition 4.6 (Efficiency). Given a profile P, a distribution $\delta \in \Delta(C_N)$ is *(Pareto) efficient* if there does not exist another distribution $\delta' \in \Delta(C_N)$ that *(Pareto) dominates* δ, i.e., $u_i(\delta') \geq u_i(\delta)$ for all $i \in N$ and $u_i(\delta') > u_i(\delta)$ for at least one $i \in N$. A distribution rule is efficient if it returns an efficient distribution for every profile P.

The following lemma characterizes efficient distributions of an arbitrary profile.

Lemma 4.7. *[Characterization of efficient distributions] A distribution δ is efficient if and only if every project $x \in \operatorname{supp}(\delta)$ is critical for some agent.*

Proof. $\Rightarrow$: Suppose that some project $x \in \text{supp}(\delta)$ is not critical for any agent. Since $\delta(x) > 0$, by (3), $\delta(x) > v_{i,x} \cdot u_i(\delta)$ for all agents $i \in N$. Denote

$$D := \delta(x) - \max_{i \in N} \left(v_{i,x} \cdot u_i(\delta)\right)$$

where our assumptions imply that $D > 0$. Construct a new distribution δ' by removing $D/2$ from project x and distributing it equally among all other projects. We claim that $u_i(\delta') > u_i(\delta)$ for every agent $i \in N$. Indeed, if $v_{i,x} = 0$ then u_i does not decrease by the removal from $\delta(x)$, and strictly increases by the addition to all other projects. Otherwise,

$$u_i(\delta') = \min\left(\frac{\delta'(x)}{v_{i,x}}, \min_{y \in A_i \setminus x} \frac{\delta'(y)}{v_{i,y}}\right).$$

Both terms are larger than $u_i(\delta)$:

- The former term is $(\delta(x) - D/2)/v_{i,x} > (\delta(x) - D)/v_{i,x} = (\max_{j \in N}[v_{j,x} \cdot u_j(\delta)])/v_{i,x} \geq u_i(\delta)$ by construction.
- For the latter term, the fact that $u_i(\delta) < \delta(x)/v_{i,x}$ implies that $u_i(\delta) = \min_{y \in A_i \setminus x} \left(\delta(y)/v_{i,y}\right)$, and $\min_{y \in A_i \setminus x} \left(\delta'(y)/v_{i,y}\right)$ is strictly larger than that since each project $y \in A \setminus x$ receives additional funding in δ'.

Hence, δ is not efficient.

$\Leftarrow$: Suppose that every project $x \in \text{supp}(\delta)$ is critical for some agent. Let δ' be any distribution different than δ. Since the sum of both distributions is the same (C_N), there exists a project $y \in \text{supp}(\delta)$ with $\delta'(y) < \delta(y)$. Let $i_y \in N$ be an agent for whom y is critical in δ. Then the utility of i_y is strictly smaller in δ':

$$\begin{aligned} u_{i_y}(\delta') &\leq \frac{\delta'(y)}{v_{i_y,y}} && \text{(by definition of Leontief utilities)} \\ &< \frac{\delta(y)}{v_{i_y,y}} && (\delta'(y) < \delta(y) \text{ by definition of } y \text{, and } v_{i_y,y} > 0 \text{ by definition of } i_y) \\ &= u_{i_y}(\delta) && \text{(by (3), since } y \text{ is critical for } i_y \text{ in } \delta) \end{aligned}$$

so δ' does not dominate δ. Hence, δ is efficient. □

Despite this characterization, the set of efficient distributions is not convex,[4] similar to the case of sum-based utilities [see Bogomolnaia et al., 2005].

Combining Corollary 4.5 with Lemma 4.7 gives the following implication.

Corollary 4.8. *Every equilibrium distribution is efficient.*

The following lemma shows that every efficient utility vector is generated by at most one distribution.

Lemma 4.9. *Let δ and δ' be efficient distributions inducing the same utility vector, that is, $u_i(\delta) = u_i(\delta')$ for all $i \in N$. Then, $\delta = \delta'$.*

Proof. By Lemma 4.7, for each $x \in \text{supp}(\delta)$ there is an agent for whom x is critical. Denote one such agent by i_x. Then,

$$\begin{aligned} \delta(x) &= v_{i_x,x} \cdot u_{i_x}(\delta) && \text{(by (3), since } x \text{ is critical for } i_x) \\ &= v_{i_x,x} \cdot u_{i_x}(\delta') && \text{(by the lemma assumption)} \\ &\leq \delta'(x) && \text{(by definition of Leontief utilities).} \end{aligned}$$

[4] Consider an example with three projects $\{a, b, c\}$ and two agents with $v_{1,c} = v_{2,a} = 0$ and $v_{i,x} = 1$ otherwise, and $C_1 = C_2 = 1$. Then, $\delta = (1, 1, 0)$ and $\delta' = (0, 1, 1)$ are both efficient distributions, but not $0.5\,\delta + 0.5\,\delta' = (0.5, 1, 0.5)$.

The same inequality $\delta(x) \leq \delta'(x)$ trivially holds also for all $x \notin \text{supp}(\delta)$. Since both distributions sum up to C_N, this implies $\delta = \delta'$. □

Consequently, an efficient distribution rule can be defined as mapping a profile to a utility vector.

4.3 Existence, uniqueness, and computation

One common way to obtain an efficient distribution is to maximize a welfare function. Formally, for any strictly-increasing function g on $\mathbb{R}_{\geq 0}$, we say that a distribution δ is *g-welfare-maximizing* if it maximizes the weighted sum $\sum_{i \in N} C_i \cdot g(u_i(\delta))$. Clearly, any such distribution is efficient. Whenever g is strictly concave, there is a unique g-welfare-maximizing distribution; the proof is straightforward and is given in Lemma B.2 in Appendix B.

We focus on the special case in which g is the log function. The *Nash welfare* of a distribution δ is defined as the sum of logarithms of the agents' utilities, weighted by their contributions:

$$Nash(\delta) := \sum_{i \in N} C_i \cdot \log u_i(\delta).$$

The *Nash rule* selects a distribution δ that maximizes $Nash(\cdot)$ or, equivalently, the weighted product of the agents' utilities $\prod_{i \in N} u_i^{C_i}$ (with the convention $0 \log 0 = 0$ and $0^0 = 1$). The Nash rule has strong fairness guarantees in various settings (see Section 2). As we will see, this is also the case in our model.

The following two lemmas show that a distribution is in equilibrium if and only if it maximizes Nash welfare.

Lemma 4.10. *Every distribution that maximizes Nash welfare is in equilibrium.*

Proof. Let δ be an efficient distribution that is not in equilibrium. We will prove that δ can be modified in a way that increases Nash welfare.

Let $(\delta_i)_{i \in N}$ be any decomposition of δ. Based on this decomposition, construct a directed graph G where nodes correspond to agents, and there is an arc $i \longrightarrow j$ iff $\delta_i(T_{\delta,j}) > 0$, that is, agent i contributes to a critical project of j. We call the arc $i \longrightarrow j$ *strong* if $\delta_i(T_{\delta,j} \setminus T_{\delta,i}) > 0$, that is, agent i "wastes" some of her contribution on a project that is critical for j but not for i. Otherwise, we call the arc $i \longrightarrow j$ *weak*.

Since δ is not an equilibrium, by Lemma 4.4, there is an agent, say agent 1, who contributes to a project $x \notin T_{\delta,1}$. Since δ is efficient, by Lemma 4.7, x is critical to some other agent, say agent 2, so G contains a strong arc $1 \longrightarrow 2$.

If the strong arc is a part of a directed cycle, then we can move a sufficiently small amount ε along the cycle without changing δ. In detail, suppose w.l.o.g. that the cycle is $1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow k \longrightarrow 1$, where the involved projects are $x_1 \in T_{\delta,1},\ x_2 \in T_{\delta,2} \setminus T_{\delta,1},\ x_3 \in T_{\delta,3},\ x_4 \in T_{\delta,4},\ \ldots,\ x_k \in T_{\delta,k}$. We assume that x_2 is in $T_{\delta,2} \setminus T_{\delta,1}$ since the arc $1 \longrightarrow 2$ is strong; in particular, x_2 must be different than x_1. The other arcs may be strong or weak, and some of the x_i may coincide. For every $i \in \{1, \ldots, k-1\}$, move a small amount $\varepsilon > 0$ from $\delta_i(x_{i+1})$ to $\delta_i(x_i)$; move the same ε from $\delta_k(x_1)$ to $\delta_k(x_k)$. Note that the decomposition changes, but the total δ remains the same. Increase ε until one arc of the cycle disappears, or the strong arc becomes weak. Repeat this cycle-removal procedure until all strong arcs are not part of any directed cycle. This process is guaranteed to terminate since in each cycle removal, either the respective strong arc becomes weak or the cycle it is part of is removed. Furthermore, no new (strong) arcs are created as agents do not contribute to additional projects, and the overall distribution δ together with the set of critical projects does not change.

Let G be the graph of the resulting decomposition. Since the total distribution is still δ, which is not in equilibrium, G still has at least one strong arc, say $j \longrightarrow k$. Let N_+ be the set of agents accessible from k via a directed path (where $k \in N_+$), and let $N_- := N \setminus N_+$. Since $j \longrightarrow k$ is not part of any directed cycle, $j \in N_-$. Due to the strong arc $j \to k$, agents of N_- waste some of their own contributions on critical projects of N_+, that are not critical for themselves. Moreover, their own critical projects do not receive any donations from agents of N_+, since they are not accessible from N_+. In contrast, the agents in N_+ spend all their contributions on their own critical projects, that are not critical projects of agents outside N_+. In addition, they receive some donations from agents of N_-. Therefore, denoting by $T_{\delta,N'}$ the set of projects that are critical for at least one agent in N' under δ for any given $N' \subseteq N$,

$$\delta(T_{\delta,N_+} \setminus T_{\delta,N_-}) > C_{N_+}; \tag{5}$$

$$\delta(T_{\delta,N_-}) < C_{N_-}. \tag{6}$$

If $\delta(T_{\delta,N_-}) = 0$, then $Nash(\delta) = -\infty$ and δ is clearly not Nash-optimal, so we may assume that $\delta(T_{\delta,N_-}) > 0$. We construct a new distribution δ' in the following way.

- Remove a small amount ε from $\delta(T_{\delta,N_+} \setminus T_{\delta,N_-})$, such that each project loses proportionally to its current distribution. That is, for each project $x \in T_{\delta,N_+} \setminus T_{\delta,N_-}$, the new distribution is $\delta'(x) := \delta(x) \cdot [1 - \varepsilon/\delta(T_{\delta,N_+} \setminus T_{\delta,N_-})]$.
- Add this ε to $\delta(T_{\delta,N_-})$ such that each project gains proportionally to its current distribution. That is, for each project $y \in T_{\delta,N_-}$, the new distribution is $\delta'(y) := \delta(y) \cdot [1 + \varepsilon/\delta(T_{\delta,N_-})]$.

Choose ε sufficiently small such that the sets of critical projects of agents in N_- do not change (that is, no new projects become critical for them). The effect on the agents' utilities is as follows:

- The utility of each agent $i \in N_+$ may decrease by a factor of up to $[1 - \varepsilon/\delta(T_{\delta,N_+} \setminus T_{\delta,N_-})]$. Therefore, the contribution to Nash welfare changes by at least $\Delta_{N_+}(\varepsilon) := C_{N_+} \cdot \log[1 - \varepsilon/\delta(T_{\delta,N_+} \setminus T_{\delta,N_-})]$. We have $\lim_{\varepsilon\to 0} \Delta_{N_+}(\varepsilon)/\varepsilon = -C_{N_+}/\delta(T_{\delta,N_+} \setminus T_{\delta,N_-})$, which is larger than -1 by inequality (5).
- The utility of each agent $i \in N_-$ increases by a factor of $[1 + \varepsilon/\delta(T_{\delta,N_-})]$. Therefore, the contribution to Nash welfare increases by $\Delta_{N_-}(\varepsilon) := C_{N_-} \cdot \log[1 + \varepsilon/\delta(T_{\delta,N_-})]$. We have $\lim_{\varepsilon\to 0} \Delta_{N_-}(\varepsilon)/\varepsilon = C_{N_-}/\delta(T_{\delta,N_-})$, which is larger than 1 by inequality (6).

The overall difference in Nash welfare is $\Delta(\varepsilon) := \Delta_{N_+}(\varepsilon) + \Delta_{N_-}(\varepsilon)$, and we have $\lim_{\varepsilon\to 0} \Delta(\varepsilon)/\varepsilon > -1 + 1 = 0$, so $\Delta(\varepsilon) > 0$ for sufficiently small ε. Therefore, $Nash(\delta') > Nash(\delta)$, so δ was not Nash-optimal, completing the proof. □

Lemma 4.11. *Every equilibrium distribution maximizes Nash welfare.*

Proof. Let δ^* be an equilibrium distribution and let $(\delta^*_i)_{i\in N}$ be one of its decompositions according to Lemma 4.4. Then, letting $N_{\delta^*,x} = \{i \mid x \in T_{\delta^*,i}\}$, for every distribution δ with $Nash(\delta) \neq -\infty$ we have

$$\begin{aligned}
Nash(\delta) &= \sum_{i\in N} C_i \log(u_i(\delta)) \\
&= \sum_{i\in N} \left(\sum_{x\in T_{\delta^*,i}} \delta^*_i(x) \right) \log(u_i(\delta)) && \text{(by (4))} \\
&\leq \sum_{i\in N} \sum_{x\in T_{\delta^*,i}} \delta^*_i(x) \cdot \log(\delta(x)/v_{i,x}) \\
&= \sum_{x\in A} \left(\sum_{i\in N_{\delta^*,x}} \delta^*_i(x) \cdot \log(\delta(x)/v_{i,x}) \right)
\end{aligned}$$

$$= \sum_{x \in A} \left(\sum_{i \in N_{\delta^*,x}} \delta_i^*(x) \right) \log(\delta(x)) - \sum_{x \in A} \left(\sum_{i \in N_{\delta^*,x}} \delta_i^*(x) \log(v_{i,x}) \right)$$

$$= \sum_{x \in A} \delta^*(x) \log(\delta(x)) - \sum_{x \in A} \left(\sum_{i \in N_{\delta^*,x}} \delta_i^*(x) \log(v_{i,x}) \right) \qquad \text{(by (1))}$$

where the inequality holds because $u_i(\delta) \leq \delta(x)/v_{i,x}$ for all $x \in T_{\delta^*,i}$. We claim that, for every fixed δ^*, the latter expression is maximized for $\delta = \delta^*$. The second term is independent of δ. As for the first term $\sum_{x \in A} \delta^*(x) \log(\delta(x))$, consider the optimization problem of maximizing $\sum_{x \in A} \delta^*(x) \log(\delta(x))$ subject to $\sum_{x \in A} \delta(x) = \sum_{x \in A} \delta^*(x)$ (note that δ^* is a constant in this problem). Its Lagrangian is

$$\sum_{x \in A} \delta^*(x) \log(\delta(x)) + \lambda \cdot \left(\sum_{x \in A} \delta^*(x) - \sum_{x \in A} \delta(x) \right)$$

Setting the derivative with respect to $\delta(x)$ to 0 gives $\delta^*(x)/\delta(x) = \lambda$ for all $x \in A$. Since $\sum_{x \in A} \delta(x) = \sum_{x \in A} \delta^*(x)$, we must have $\lambda = 1$, so $\delta = \delta^*$. This means that

$$Nash(\delta) \leq \sum_{x \in A} \delta^*(x) \log(\delta^*(x)) - \mathrm{const}(\delta^*).$$

For $Nash(\delta^*)$, the same derivation holds, but the inequality becomes an equality, since in equilibrium, $\delta_i^*(x) > 0$ only if $u_i(\delta^*) = \delta^*(x)/v_{i,x}$. Therefore,

$$Nash(\delta) \leq Nash(\delta^*),$$

so δ^* is Nash-optimal. □

Since the log function is strictly concave, Lemma B.2 in Appendix B implies that there is a unique distribution that maximizes Nash welfare. Therefore, Lemmas 4.10 and 4.11 imply that there is a unique equilibrium distribution, and it is efficient, as claimed in Theorem 4.3.

Since the equilibrium distribution maximizes a weighted sum of logarithms, it can be approximated arbitrarily well by considering the corresponding convex optimization problem. With sum-based utilities, Brandl et al. [2022] show that it is impossible to compute the Nash-optimal distribution exactly even for binary valuations, since it may involve irrational numbers. Interestingly, for Leontief utilities the Nash-optimal distribution is rational whenever the agents' valuations and contributions are rational; see Appendix E for a proof.

In the case of binary weights, the equilibrium distribution can be computed exactly using a polynomial number of linear programs; see Section 6. We do not know whether the same is true for non-binary weights.

5 THE EQUILIBRIUM DISTRIBUTION RULE

Based on Theorem 4.3, we define the *equilibrium distribution rule (EDR)* as the distribution rule that, for each profile, returns the unique equilibrium distribution for this profile. In this section, we investigate the axiomatic properties of *EDR*.

5.1 Strategyproofness

A distribution rule is *group-strategyproof* if no coalition of agents can gain utility by misreporting their valuations or contributing less. This incentivizes truthful reports and allows for a correct estimation of agents' utilities under different distributions. Furthermore, a group-strategyproof rule ensures that every agent donates the maximal possible contribution, thereby guaranteeing maximal gains from coordination.

Definition 5.1 (Group-Strategyproofness). (a) Given a distribution rule f, a profile P, and a group $G \subseteq N$, a profile P' is called a *manipulation of P by G* if $C'_G \leq C_G$ (the contribution of G may decrease), and the valuations of agents in G may change, while the contributions and valuations of all agents in $N \setminus G$ remain the same. Such a manipulation is called *successful* if $u_j(f(P')) \geq u_j(f(P))$ for all $j \in G$ and $u_i(f(P')) > u_i(f(P))$ for at least one $i \in G$, where $\mathbf{u}$ refers to the utilities in P.

(b) A distribution rule f is *group-strategyproof* if in any profile, no group of agents has a successful manipulation.

Remark 2. We use the convention that agents that do not value any project are ignored. This is why reporting valuation 0 for every project can be interpreted as reporting contribution 0, a permitted manipulation.

In the following, we prove that *EDR* is group-strategyproof. We will use the following lemma.

Lemma 5.2. *Let δ^1 and δ^2 be two distributions, and $i \in N$ an agent.*
(a) If $u_i(\delta^2) \geq u_i(\delta^1)$, then every project in $T_{\delta^1,i}$ receives at least as much funding in δ^2, that is: $\delta^2(y) \geq \delta^1(y)$ for all $y \in T_{\delta^1,i}$.
(b) Similarly, if $u_i(\delta^2) > u_i(\delta^1)$, then $\delta^2(y) > \delta^1(y)$ for all $y \in T_{\delta^1,i}$.

Proof. For (a), for every project $y \in T_{\delta^1,i}$:

$$
\begin{aligned}
\delta^1(y) &= v_{i,y} \cdot u_i(\delta^1) && \text{(by (3), as } y \text{ is critical for } i \text{ in } \delta^1) \\
&\leq v_{i,y} \cdot u_i(\delta^2) && \text{(by assumption)} \\
&= v_{i,y} \cdot \min_{x \in A} \frac{\delta^2(x)}{v_{i,x}} && \text{(by definition of Leontief utilities)} \\
&\leq v_{i,y} \cdot \min_{x \in T_{\delta^1,i}} \frac{\delta^2(x)}{v_{i,x}} && \text{(since } T_{\delta^1,i} \subseteq A) \\
&\leq v_{i,y} \cdot \frac{\delta^2(y)}{v_{i,y}} && \text{(since } y \in T_{\delta^1,i}) \\
&= \delta^2(y).
\end{aligned}
$$

For (b), the first inequality becomes strict. □

Theorem 5.3. *EDR is group-strategyproof.*

Proof. Suppose by contradiction that some group of agents has a successful manipulation, and let $G \subseteq N$ be an inclusion-maximal such group. For an arbitrary profile P, denote by P' the profile after a successful manipulation by G and by δ^P and $\delta^{P'}$ the respective equilibrium distributions. Since the manipulation succeeds, $u_j(\delta^{P'}) \geq u_j(\delta^P)$ for all $j \in G$ and $u_i(\delta^{P'}) > u_i(\delta^P)$ for at least one $i \in G$. By Lemma 5.2, $\delta^{P'}(x) \geq \delta^P(x)$ for every project x that belongs to $T_{\delta^P,j}$ for some $j \in G$, and $\delta^{P'}(x) > \delta^P(x)$ for every project x in $T_{\delta^P,i}$. This implies

$$
\delta^{P'}\left(\bigcup_{j \in G} T_{\delta^P,j}\right) > \delta^P\left(\bigcup_{j \in G} T_{\delta^P,j}\right). \tag{7}
$$

We write both equilibrium distributions as decompositions $\delta^P = \sum_{k \in N} \delta^P_k$ and $\delta^{P'} = \sum_{k \in N} \delta^{P'}_k$ satisfying Lemma 4.4. Since $C'_G \leq C_G$, inequality (7) above must hold for the individual distribution of at least one agent $k \in N \setminus G$, that is,

$$
\delta^{P'}_k \left(\cup_{j \in G} T_{\delta^P,j}\right) > \delta^P_k \left(\cup_{j \in G} T_{\delta^P,j}\right).
$$

In particular, we must have $T_{\delta^P,k} \not\subseteq \cup_{j\in G} T_{\delta^P,j}$, as otherwise $\delta_k^P(T_{\delta^P,k}) = \delta_k^P\left(\cup_{j\in G} T_{\delta^P,j}\right) = C_k$ (δ^P is the equilibrium distribution), and agent k could not have spent more on projects from $\cup_{j\in G} T_{\delta^P,j}$ in P' since $C'_k = C_k$. Consequently, at least one project $x_G \in \cup_{j\in G} T_{\delta^P,j}$ has $\delta_k^{P'}(x_G) > \delta_k^P(x_G)$. By Lemma 4.4, x_G must be critical for k in $\delta^{P'}$. Therefore,

$$\begin{aligned}
v_{k,x_G} \cdot u_k(\delta^{P'}) = \delta^{P'}(x_G) & \qquad \text{(by (3), as } x_G \text{ is critical for } k \text{ in } \delta^{P'}) \\
\geq v_{j,x_G} \cdot u_j(\delta^{P'}) & \qquad \text{(for some } j \in G \text{ with } x_G \in T_{\delta^P,j}\text{, by Leontief utilities)} \\
\geq v_{j,x_G} \cdot u_j(\delta^{P}) & \qquad \text{(as no agent in } G \text{ loses from the manipulation)} \\
= \delta^{P}(x_G) & \qquad \text{(by (3), as } x_G \in T_{\delta^P,j}) \\
\geq v_{k,x_G} \cdot u_k(\delta^{P}) & \qquad \text{(by Leontief utilities),}
\end{aligned}$$

so agent k's utility is not decreased by the group's manipulation. Consequently, k could be added to G—contradicting the maximality of G.

We conclude that no group of agents has a successful manipulation and thus *EDR* is group-strategyproof. □

In fact, the above proof shows that if the total contribution C_G decreases, then the utility of at least one agent in G has to *strictly* decrease under *EDR* since $\sum_{i\in G} \delta_i^{P'}\left(\bigcup_{j\in G} T_{\delta^P,j}\right) < \sum_{i\in G} \delta_i^{P}\left(\bigcup_{j\in G} T_{\delta^P,j}\right)$ and the above argument applies. In particular, an agent receives *strictly* more utility when she increases her contribution.

Theorem 5.4. *Under EDR, agents are strictly better off by increasing their contribution.*

Remark 3. In the context of sum-based utilities, Brandl et al. [2021] have proposed an even stronger participation axiom called *contribution incentive-compatibility*. This axiom allows agents who do not participate in the mechanism to receive additional utility by spending her saved contribution independently. Unfortunately, in our setting, this property is incompatible with efficiency and also with strategyproofness, even for binary weights. For more details, we refer to Appendix C.

5.2 Preference-monotonicity

An important property for project holders is *preference-monotonicity*, which requires that for every agent i and project $x \in A$, $\delta(x)$ weakly increases when $v_{i,x}$ increases. In other words, a project can only receive more donations when becoming more popular, which, for example, incentivizes advertising projects.

Definition 5.5 (Preference-monotonicity). A distribution rule f satisfies *preference-monotonicity* if for every two profiles P and P' which are identical except that $v'_{i,x} > v_{i,x}$ for one agent i and one project x, we have $f(P')(x) \geq f(P)(x)$.

Theorem 5.6. *EDR satisfies preference-monotonicity.*

Proof. Let P be a profile and P' a modified profile where one agent i increases her valuation for one project x (that is, $v'_{i,x} > v_{i,x}$ and $v'_{i,y} = v_{i,y}$ for all $y \in A \setminus x$). Let δ^P and $\delta^{P'}$ be the respective equilibrium distributions. We need to show that $\delta^{P'}(x) \geq \delta^P(x)$.

Let u_i and u'_i be agent i's utilities in the two profiles. By definition of Leontief utilities, $u'_i(\delta) = \min(u_i(\delta), \delta(x)/v'_{i,x})$ for any distribution δ. In particular, $u'_i(\delta^P) = \min(u_i(\delta^P), \delta^P(x)/v'_{i,x})$. We consider two cases, depending on which of the two expressions within the minimum is larger.

<u>Case 1</u>: $u_i(\delta^P) < \delta^P(x)/v'_{i,x}$. Then $u'_i(\delta^P) = u_i(\delta^P)$, and all projects in $T_{\delta^P,i}$ remain critical for i in the new profile. Therefore, by Lemma 4.4, δ^P is still an equilibrium distribution for P'. By uniqueness of the equilibrium distribution, $\delta^{P'}(x) = \delta^P(x)$.

Case 2: $u_i(\delta^P) \geq \delta^P(x)/v'_{i,x}$. By definition of Leontief utilities,

$$\delta^{P'}(x)/v'_{i,x} \geq u'_i(\delta^{P'}).$$

By strategyproofness (Theorem 5.3),

$$u'_i(\delta^{P'}) \;\geq\; u'_i(\delta^P).$$

By definition of Leontief utilities,

$$\begin{aligned} u'_i(\delta^P) &= \min(u_i(\delta^P),\ \delta^P(x)/v'_{i,x}) \\ &= \delta^P(x)/v'_{i,x}, \end{aligned}$$

since by assumption $u_i(\delta^P) \geq \delta^P(x)/v'_{i,x}$ Combining these three inequalities yields $\delta^{P'}(x) \geq \delta^P(x)$, as desired. □

For binary sum-based utilities, strategyproofness implies preference-monotonicity [Brandl et al., 2021]. This does not hold for binary Leontief utilities, even when demanding that contributions are only allocated to projects approved by at least one agent. An example is provided in Appendix D.

5.3 Contribution-monotonicity

For some applications, it would be desirable if increased contributions do not result in the redistribution of funds that have already been allocated. For example, if agents arrive over time or increase their contributions over time, ideally the mechanism only needs to take care of the additional contributions. This would allow a deployment of the mechanism as an ongoing process in which donations arrive over time and projects can make immediate use of the donations they receive. We formalize this property in the following definition.

Definition 5.7 (Contribution-monotonicity). A distribution rule f satisfies *contribution-monotonicity* if for every two profiles P and P' where P' can be obtained from P by increasing the contribution of one agent (possibly from 0), $f(P')(x) \geq f(P)(x)$ for all projects $x \in A$.

Theorem 5.8. *EDR satisfies contribution-monotonicity.*

Proof. Let P and P' be profiles as in Definition 5.7. We have $C'_i \geq C_i$ for all $i \in N$.

Let δ and δ' be the two equilibrium distributions corresponding to profiles P and P'. Fix a decomposition of δ and of δ' into individual distributions satisfying Lemma 4.4.

Let A^- , $A^=$, and A^+ be the sets of all projects $x \in A$ with $\delta'(x) < \delta(x)$, $\delta'(x) = \delta(x)$, and $\delta'(x) > \delta(x)$, respectively. Assume for contradiction that A^- is not empty. Thus, $\sum_{i\in N} \delta'_i(A^-) < \sum_{i\in N} \delta_i(A^-)$, and $\sum_{i\in N} \delta'_i(A^= \cup A^+) > \sum_{i\in N} \delta_i(A^= \cup A^+)$. Consequently, there has to exist an agent $i \in N$ and a project $y \in A^-$ with $\delta'_i(y) < \delta_i(y)$. But $\delta'_i(A) = C'_i \geq C_i = \delta_i(A)$, so there has to be a project $z \in A^= \cup A^+$ with $\delta'_i(z) > \delta_i(z) \geq 0$. By Lemma 4.4, projects z and y are critical for i under δ' and δ, respectively. This, in particular, implies that $v_{i,z} > 0$ and $v_{i,y} > 0$. Therefore,

$$\frac{\delta'(z)}{v_{i,z}} \leq \frac{\delta'(y)}{v_{i,y}} < \frac{\delta(y)}{v_{i,y}} \leq \frac{\delta(z)}{v_{i,z}},$$

where the first and last inequalities follow from the definition of critical projects. This implies $\delta'(z) < \delta(z)$, a contradiction. □

Remark 4. Theorem 5.8 yields an alternative proof of the uniqueness of equilibrium distributions, which does not use the equivalence with Nash optimality. If δ and δ' are equilibrium distributions for the same profile, then both $\delta'(x) \geq \delta(x)$ and $\delta(x) \geq \delta'(x)$ must hold for every project $x \in A$, which implies $\delta' = \delta$.

6 LEONTIEF UTILITIES WITH BINARY WEIGHTS

In this section, we consider the special case of having binary weights, i.e., $v_{i,x} \in \{0, 1\}$ for all agents $i \in N$ and projects $x \in A$. Equivalently, each agent i has a non-empty set of *approved projects* $A_i \subseteq A$ and her utility from a distribution δ is

$$u_i(\delta) = \min_{x \in A_i} \delta(x).$$

For each project $x \in A$, we denote by $N_x \subseteq N$ the set of agents who approve project x. For a subset of agents $N' \subseteq N$, we denote $A_{N'} := \bigcup_{i \in N'} A_i$ as the set of projects approved by at least one member of N'. Note that, for every project $x \in A$ and every agent $i \in N_x$,

$$\delta(x) \geq u_i(\delta). \tag{8}$$

Binary weights allow for further insights into the structure of the equilibrium distribution, which in turn yield new interpretations and additional properties of *EDR*.

For sum-based utilities with binary weights, a distribution is in equilibrium if and only if each agent contributes only to projects she approves. Brandl et al. [2021] refer to this axiom as *decomposability*.

Definition 6.1 (Decomposable distribution). Given a profile with binary weights ($v_{i,x} \in \{0, 1\}$), a distribution δ is *decomposable* if it has a decomposition $(\delta_i)_{i \in N}$ such that $\delta_i(x) = 0$ for every project $x \notin A_i$. Equivalently, it has a decomposition satisfying the following, instead of (2):

$$\sum_{x \in A_i} \delta_i(x) = C_i \qquad \text{for all } i \in N.$$

The equivalence of decomposable distributions and equilibrium distributions no longer holds with Leontief utilities: there are decomposable distributions that are not in equilibrium even when there is only one agent.

Example 6.2. There is a single agent with $C_1 = 2$, $A = \{a, b\}$, with valuations $v_{1,a} = 1$ and $v_{1,b} = 1$. The distribution $\delta = (2, 0)$ is decomposable, but it is not in equilibrium, since the single agent is better off by the equilibrium distribution $\delta' = (1, 1)$.

Nevertheless, decomposability can be used to establish two alternative interpretations of *EDR* for binary weights.

6.1 Egalitarianism for projects

Motivated by the example from the introduction, we aim at a rule which distributes money on the projects as equally as possible while still respecting the preferences of the donors. One rule that comes to mind selects a distribution that, among all decomposable distributions, maximizes the smallest amount allocated to a project. Subject to this, it maximizes the second-smallest allocation to a project, and so on. We define it formally using the *leximin* relation.

Definition 6.3. Given two vectors $\mathbf{x}, \mathbf{y}$ of the same size, we say that $\mathbf{x}$ *is leximin-higher than* $\mathbf{y}$ (denoted $\mathbf{x} \succ_{lex} \mathbf{y}$) if the smallest value in $\mathbf{x}$ is larger than the smallest value in $\mathbf{y}$; or the smallest values are equal, and the second-smallest value in $\mathbf{x}$ is larger than the second-smallest value in $\mathbf{y}$; and so on. $\mathbf{x} \succeq_{lex} \mathbf{y}$ means that either $\mathbf{x} \succ_{lex} \mathbf{y}$ or the multiset of values in $\mathbf{x}$ is the same as that in $\mathbf{y}$.

Definition 6.4. The *project egalitarian rule* selects a distribution δ^* that, among all decomposable distributions, maximizes the distribution vector by the leximin order, that is: $\delta^* \succeq_{lex} \delta$ for every decomposable distribution δ.

The leximin order on the closed and convex set of decomposable distributions is connected, every two vectors are comparable, and there exists a unique maximal element (otherwise, any convex combination of two different maximal elements would be leximin-higher than the maximal elements). Therefore, the project egalitarian rule selects a unique distribution and is well-defined. We prove below that the returned distribution is the equilibrium distribution, resulting in an alternative characterization of *EDR* for binary weights.

Theorem 6.5. *With binary weights, the project egalitarian rule and EDR are equivalent.*

Proof. By uniqueness of the equilibrium distribution (Theorem 4.3), it is sufficient to show that every project egalitarian distribution is in equilibrium. Let δ^{PEG} be a decomposable project egalitarian distribution, with decomposition $\delta^{PEG} = \sum_{i \in N} \delta_i^{PEG}$. Suppose for contradiction that δ^{PEG} is not in equilibrium. By Lemma 4.4, there is an agent $i \in N$ who contributes to a non-critical project $x \in A_i$, that is, $\delta_i^{PEG}(x) > 0$ and $\delta^{PEG}(x) > u_i(\delta^{PEG})$. Let $y \in A_i$ be a critical project of agent i, that is, $\delta^{PEG}(y) = u_i(\delta^{PEG})$.

If agent i now moves $1/2(\delta^{PEG}(x) - \delta^{PEG}(y))$ from x to y, the resulting distribution is still decomposable, as both x and y are in A_i. It is leximin-higher than δ^{PEG}, contradicting the leximin-maximality of δ^{PEG}. □

Remarkably, this new interpretation of *EDR* ignores the Leontief utilities of the agents and does not directly take into account the different contributions. Instead, they enter indirectly through the constraints induced by decomposability.

Theorem 6.5 implies that *EDR* can be computed by solving the following program, with variables δ_x for all $x \in A$ and $\delta_{i,x}$ for all $i \in N, x \in A$:

$$\begin{aligned} &\operatorname{lex\,max\,min}\{\delta_x\}_{x \in A} && \text{subject to} \\ &\delta_x = \sum_{i \in N} \delta_{i,x} && \forall x \in A \\ &\sum_{x \in A_i} \delta_{i,x} = C_i && \forall i \in N \\ &\delta_{i,x} \geq 0, \quad \delta_x \geq 0 && \forall i \in N, \quad \forall x \in A_i \end{aligned}$$

where "lex max min" refers to finding a solution vector that is maximal in the leximin order subject to the constraints, and the second constraint represents decomposability. It is well-known that such leximin optimization with k objectives and linear constraints can be solved by a sequence of poly(k) linear programs.[5]

Corollary 6.6. *With binary weights, the equilibrium distribution can be computed by solving at most $|A|$ linear programs.*

The *EDR* distribution can be computed efficiently by solving a sequence of linear programs, as explained next.

[5] Ogryczak et al. [2005] showed that every leximin optimization problem with k objectives has an equivalent *lexicographic* optimization problem, denoted (32) in their paper, with $k^2 + k$ additional variables and k^2 additional constraints. In a lexicographic optimization problem, the objectives have a fixed priority order. The goal is to maximize the most important objective, then subject to this, maximize the second most important objective, and so on. A lexicographic optimization problem can be solved by a simple sequential algorithm using at most k linear programs (Algorithm 1 in their work). For the special case of a convex optimization problem, Ogryczak et al. [2005] presented Algorithm 4, which solves the problem using at most k linear programs without additional variables and constraints.

6.2 Egalitarianism for agents

While *EDR* is egalitarian from the point of view of the projects, one could also consider a rule that is egalitarian from the point of view of the agents. The *conditional egalitarian rule* aims to balance the agents' utilities without disregarding their approvals. It selects a decomposable distribution that, among all decomposable distributions, maximizes the utility vector by the leximin order, that is: $\mathbf{u}(\delta^{CEG}) \succeq_{lex} \mathbf{u}(\delta)$ for every decomposable distribution δ.

THEOREM 6.7. *With binary weights, the conditional egalitarian rule and EDR are equivalent.*

PROOF. By uniqueness of the equilibrium distribution (Theorem 4.3), it is sufficient to show that every conditional egalitarian distribution is in equilibrium. Let δ^{CEG} be a conditional egalitarian distribution with decomposition $\delta^{CEG} = \sum_{i\in N} \delta_i^{CEG}$. Suppose for contradiction that δ^{CEG} is not in equilibrium. Then, some agent $i \in N$ contributes to a non-critical project $x \in A_i$, that is, $\delta_i^{CEG}(x) > 0$ and $\delta^{CEG}(x) > u_i(\delta^{CEG})$.

Let $D := \min\left(\delta_i^{CEG}(x),\ \delta^{CEG}(x) - u_i(\delta^{CEG})\right)$; our assumptions imply that $D > 0$. Construct a new distribution δ' from δ^{CEG} by changing only δ_i^{CEG}: remove D from project x, and add $D/|A_i|$ to every project in A_i (including x). The utility of i increases by $D/|A_i|$, since:

- $\delta'(x) = \delta^{CEG}(x) - D + D/|A_i| \geq u_i(\delta^{CEG}) + D/|A_i|$ by definition of D;
- $\delta'(y) = \delta^{CEG}(y) + D/|A_i| \geq u_i(\delta^{CEG}) + D/|A_i|$ for all $y \in A_i \setminus x$, by (8) with equality for $y \in T_{\delta^{CEG},i}$.
- So $u_i(\delta') = \min(\delta'(x), \min_{y\in A_i\setminus x} \delta'(y)) = u_i(\delta^{CEG}) + D/|A_i| > u_i(\delta^{CEG})$.

Moreover, if the utility of some agent j decreases—that is, $u_j(\delta') < u_j(\delta^{CEG})$—then this must be because of the decrease in the distribution to x, so x must be a critical project for agent j in δ', i.e., $u_j(\delta') = \delta'(x) \geq u_i(\delta') > u_i(\delta^{CEG})$.

Thus, moving from δ^{CEG} to δ', the number of agents with utility larger than $u_i(\delta^{CEG})$ strictly increases, and the utility of each agent with utility at most $u_i(\delta^{CEG})$ in δ^{CEG} does not decrease. Therefore, $\mathbf{u}(\delta') \succ_{lex} \mathbf{u}(\delta^{CEG})$. Since δ' is decomposable, this contradicts the optimality of δ^{CEG}. □

Theorem 6.7 implies that the equilibrium distribution can be computed by solving the following program, with variables u_i for all $i \in N$ and $\delta_{i,x}$ for all $i \in N, x \in A_i$.

$$
\begin{aligned}
\text{lex max} \min\{u_i\}_{i\in N} & \qquad \text{subject to} \\
u_i \leq \delta_{i,x} & \qquad \forall i \in N, \quad \forall x \in A_i \\
\sum_{x\in A_i} \delta_{i,x} = C_i & \qquad \forall i \in N \\
\delta_{i,x} \geq 0, \quad u_i \geq 0 & \qquad \forall i \in N, \quad \forall x \in A_i
\end{aligned}
$$

Using the algorithms in the works by Ogryczak et al. [2005] and Ogryczak and Śliwiński [2006], this program can be solved using at most $|N|$ linear programs.

Thus, we have three algorithms for computing the equilibrium distribution in the case of binary weights: one requires at most $|A|$ linear programs; one requires at most $|N|$ linear programs; and one requires a single convex (non-linear) program. It would be interesting to check which of these algorithms is most efficient in practice.

Note that, for general Leontief utilities, equilibrium distributions do not necessarily maximize the leximin vector of either the projects or the agents.

Example 6.8 (For general Leontief utilities, EDR, the conditional egalitarian rule, and the project egalitarian rule are different from one another). There are three projects (x, y, z) and two agents, both of whom contribute 30. The values of agent 1 are $(1, 2, 0)$ and the values of agent 2 are $(0, 1, 1)$.

The project egalitarian rule returns the leximin-maximal distribution for projects (subject to decomposability), which is $(20, 20, 20)$ with decomposition $(20, 10, 0)$, $(0, 10, 20)$. It is not in equilibrium, since agent 1 contributes to project x, which is not critical.

The conditional egalitarian rule returns the leximin-maximal distribution for agents (subject to decomposability), which is $(15, 30, 15)$, with utility vector $(15, 15)$ and decomposition $(15, 15, 0)$, $(0, 15, 15)$. It is not in equilibrium, since agent 2 contributes to project y, which is not critical.

To compute the equilibrium distribution, we can guess that x, y are critical for agent 1 and y, z are critical for agent 2, and solve the system of four equations: $\delta(x) = u_1$; $\delta(y) = 2u_1 = u_2$; $\delta(z) = u_2$; $\delta(x) + \delta(y) + \delta(z) = 60$. The solution is $(12, 24, 24)$; agent 1 contributes $(12, 18, 0)$ and has utility 12, while agent 2 contributes $(0, 6, 24)$ and has utility 24. One can verify that this distribution is indeed in equilibrium, so it is the equilibrium distribution.

6.3 Welfare functions maximized by *EDR*

Based on the observation that *EDR* coincides with both the Nash product rule and the conditional egalitarian rule for binary weights, a natural question to ask is which other welfare notions are maximized by *EDR* subject to decomposability.

For this, we take a closer look at g-welfare (see Section 4.3 and Appendix B), but this time subject to decomposability. Clearly, every g-welfare-maximizing distribution is efficient. Below we prove that efficiency remains even if we maximize among the decomposable distributions.

Lemma 6.9. *Let g be any strictly-increasing function, and let δ be a distribution that maximizes the g-welfare among all decomposable distributions. Then δ is unique and efficient.*

Proof sketch. Suppose by contradiction that δ is not efficient. By Lemma 4.7, there is a project $x \in \text{supp}(\delta)$ which is not critical for any agent. Then, one agent who contributes to x would be able to shift a small amount uniformly to the set of her critical projects such that x is still not critical for any agent. The resulting distribution is still decomposable, and Pareto dominates δ, contradicting the maximality of δ in g-welfare.

Uniqueness is proved similarly to Lemma B.2, using the fact that the set of decomposable distributions is convex, i.e., mixing decomposable distributions results in another decomposable distribution. □

Note that uniqueness holds only within the set of decomposable distributions; there might exist non-decomposable distributions with the same g-welfare, as shown in the following example.

Example 6.10. Let $g(x) = -x^{-1}$ (a strictly-increasing function). Suppose there are two agents with $A_1 = \{a\}$, $A_2 = \{b\}$, $C_1 = 2$, $C_2 = 1$. Then, the unique decomposable distribution $\delta^* = (2, 1)$ has the same g-welfare $(-2/2 - 1/1 = -2)$ as the non-decomposable distribution $\delta = (1.5, 1.5)$ $(-2/1.5 - 1/1.5 = -2)$.

The Nash product rule is often considered a compromise between maximizing utilitarian welfare $(\sum_{i \in N} C_i \cdot u_i)$ and egalitarian welfare (maximize the utility of the agent with smallest utility; notice that the conditional egalitarian rule is a refinement). This can be seen when considering the family of g-welfare functions $\sum_{i \in N} C_i \cdot \text{sgn}(p) \cdot u^p$ for $p \neq 0$ where the limit $p \to 0$ corresponds to $\sum_{i \in N} C_i \cdot \log(u_i)$ and $p \to -\infty$ approaches egalitarian welfare.

It is interesting to check whether the equivalence between conditional egalitarian welfare and Nash welfare extends to a larger class of g-welfare functions. This is indeed the case, as the following theorem shows; its proof can be found in Appendix A.1.

Theorem 6.11. *Assume $g : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$*

(1) *is strictly increasing on $\mathbb{R}_{\geq 0}$ and differentiable on $\mathbb{R}_{>0}$, and*

(2) *$xg'(x)$ is non-increasing on $\mathbb{R}_{>0}$.*

Then, the equilibrium distribution maximizes g-welfare among all decomposable distributions.

Property (1) ensures that the social welfare is indeed increasing when an individual's utility increases and small changes in individual utilities only cause small changes in the total social welfare. Property (2) implies that increasing utilities are discounted "at least logarithmically" when being translated to welfare.

In particular, Theorem 6.11 holds for all g-welfare functions $\sum_{i\in N} C_i \cdot \text{sgn}(p) \cdot u^p$ with $p < 0$. However, it ceases to hold when $p > 0$, as the following proposition (whose proof is deferred to Appendix A.2) shows.

Proposition 6.12. *For each $p > 0$, maximizing the g-welfare with respect to $g(u) = u^p$ subject to decomposability does not always return the equilibrium distribution.*

Note that Theorem 6.11 cannot be further strengthened by allowing different functions g for different agents.

Example 6.13 (Maximizing different individual welfare functions does not result in the equilibrium distribution). There are two agents with $A_1 = \{a\}$, $A_2 = \{a, b\}$, $C_1 = 2$, $C_2 = 2$. The unique equilibrium distribution is $(2, 2)$. But maximizing the sum of strictly-concave welfare functions

$$-u_1(\delta)^{-1} - u_2(\delta)^{-2}/2$$

finds the distribution $(2.272, 1.728)$, as it gives more weight to agent 1. This distribution is decomposable but not in equilibrium, as agent 2 contributes to a non-critical project a.

This example does not come as a surprise, as different functions g result in different translations of their utilities to welfare, rendering some agents more important than others.

Theorem 6.11 stresses the fact that *EDR* can be motivated not only from a game-theoretic and axiomatic point of view, but also from a welfarist perspective.

6.4 Convergence to equilibrium

In this section, we propose a simple best-response-based spending dynamics for binary weights that converges to the equilibrium distribution δ^*. This enables a decentralized implementation of the mechanism in which agents do not have to reveal their preferences explicitly.

Denote by δ^t the distribution at time step t (along with its associated decomposition), e.g., δ^0 equals the null vector as no agent $i \in N$ has yet distributed her contribution C_i. At each time step t, allow one agent i_t to contribute or redistribute her entire contribution in such a way that her utility is maximized for the new distribution δ^{t+1}, i.e.,

$$\delta_{i_t}^{best} := \arg \max_{\delta_{i_t} \in \Delta(C_{i_t})} u_{i_t}\left(\delta_{i_t} + \sum_{j \neq i_t} \delta_j^t\right);$$

$$\delta^{t+1} = \delta_{i_t}^{best} + \sum_{j \neq i_t} \delta_j^t.$$

Lemma 6.14. *For every time step t and agent i_t, there exists a unique best response $\delta_{i_t}^{best}$.*

Proof. Since a best response corresponds to a solution of a maximization problem over the closed and bounded set of possible distributions $\delta_{i_t} + \sum_{j\neq i_t} \delta_j^t$ with the continuous objective function u_{i_t}, existence is guaranteed.

To show uniqueness, observe that for the distribution at step $t+1$ (which for simplified notation we denote by $\delta := \delta^{t+1}$), we have $\delta_{i_t}(T_{\delta,i_t}) = C_{i_t}$, that is, agent i_t distributes all her contribution on her critical projects in δ. In any other response δ'_{i_t}, agent i_t must contribute less to at least one project of T_{δ,i_t}. Therefore, her utility must be lower than $u_{i_t}(\delta)$, so δ'_{i_t} cannot be a best response. □

Theorem 6.15. *Let $\mathcal{S} = (i_0, i_1, i_2, \dots)$ be an infinite sequence of agents updating their individual distributions by best responses. If each agent $i \in N$ appears infinitely often in $\mathcal{S}$, the dynamics converges to the equilibrium distribution, that is,* $\lim_{t\to\infty} \delta^t = \delta^*$.

The proof of Theorem 6.15 is deferred to Appendix A.3.

Just like the question of whether the equilibrium distribution can be computed by a linear program for general Leontief utilities, it is open whether the best response dynamics converges to the equilibrium distribution in the case of general Leontief utilities. A proof of such a statement would need to be quite different from that of Theorem 6.15, because it cannot use the lexicographically ordered distribution vector as a potential function anymore (see Example 6.8).

7 CONCLUSION AND FURTHER DIRECTIONS

All in all, *EDR* turns out to be an exceptionally attractive rule for donor coordination with Leontief utilities. It satisfies virtually all desirable properties one could hope for and can be computed via convex programming. In the case of binary weights, *EDR* maximizes a wide range of possible welfare functions and can be computed via linear programming or a simple spending dynamics. These results stand in sharp contrast to the previously studied case of sum-based utilities, where a far-reaching impossibility has shown the incompatibility of efficiency, strategyproofness, and a very weak form of fairness [Brandl et al., 2021]. The literature in this stream of research has produced various rules such as the *conditional utilitarian rule*, the *Nash product rule*, the *random priority rule*, or the *sequential utilitarian rule* which trade off these properties against one another [Aziz et al., 2019, Bogomolnaia et al., 2005, Brandl et al., 2022, 2021, Duddy, 2015].

An interesting question for future work is to find more general types of utility functions for which maximizing Nash welfare results in equilibrium distributions. It turns out that the distribution returned by *EDR* remains in equilibrium beyond the case of Leontief utilities. In fact, whenever each agent i has a separably-additive, strictly-concave utility function, i.e., $u_i(\delta) = \sum_{x\in A_i} g_i(\delta(x))$ where $g_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is strictly concave, strictly increasing, and continuously differentiable, the equilibrium distribution coincides with that of the corresponding Leontief utilities with binary weights. A complete classification of distribution rules for arbitrary utility functions is still pending.

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A OMITTED PROOFS

A.1 Proof of Theorem 6.11

THEOREM 6.11. *Assume* $g : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$

(1) *is strictly increasing on* $\mathbb{R}_{\geq 0}$ *and differentiable on* $\mathbb{R}_{>0}$*, and*
(2) $xg'(x)$ *is non-increasing on* $\mathbb{R}_{>0}$.

Then, the equilibrium distribution maximizes g-welfare among all decomposable distributions.

For the proof, we need some additional definitions and lemmas. Recall that $[z] := \{1, 2, \ldots, z\}$ for each positive integer z.

Definition A.1. Given any distribution δ, define $\mathcal{P}(\delta)$ as a partition of the projects into subsets allocated the same amount. That is, $\mathcal{P}(\delta) := (X_1, \ldots, X_p)$ for some integer $p \geq 1$, where $\cup_{k=1}^{p} X_k = A$, and for each $k \in [p]$, all projects in X_k receive the same amount, $\delta(x) = w_k$ for all $x \in X_k$, and the amounts are ordered such that $0 \leq w_1 < \cdots < w_k$.

Note that $w_1 = 0$ if and only if there exist projects that receive no funding.

LEMMA A.2. *Let* δ^* *be the equilibrium distribution, and* $(X_1^*, \ldots, X_p^*) = \mathcal{P}(\delta^*)$ *be its project partition. For each* $k \geq 1$*, let* N_k^* *be the set of agents who approve one or more projects of* X_k^**, but do not approve any project of* $\cup_{\ell<k} X_\ell^*$*. Then in equilibrium, the agents of* N_k^* *contribute only to projects of* X_k^**, that is:*

$$\delta^*(X_k^*) = C_{N_k^*}, \text{ and}$$
$$w_k^* = C_{N_k^*}/|X_k^*| = \delta^*(X_k^*)/|X_k^*|.$$

PROOF. The utility of all agents in N_k^* is w_k^*, so the set of their critical projects is contained in X_k^*. In equilibrium they contribute only to projects in X_k^* by Lemma 4.4.

All projects in X_k^* receive the same amount, so this amount must be $C_{N_k^*}/|X_k^*|$. □

Note that, if there are projects not approved by any agent (or approved only by agents who contribute 0), then all these projects will be in X_1^*, and we will have $w_1^* = C_{N_1^*} = 0$.

Definition A.3. A distribution δ is called *weakly decomposable* if it has a decomposition in which each agent i only contributes to projects x with $\delta^*(x) \geq u_i(\delta^*)$, where δ^* denotes the equilibrium distribution.

With binary weights, $x \in A_i$ implies $\delta^*(x) \geq u_i(\delta^*)$, so every decomposable distribution is weakly decomposable. Therefore, it is sufficient to prove that δ^* maximizes g-welfare among all weakly decomposable distributions.

The set of weakly decomposable distributions is again convex and can be characterized as follows.

LEMMA A.4. *A distribution* δ *is weakly decomposable if and only if, for every* $\ell \in [p]$,

$$\delta\left(\cup_{k=\ell}^{p} X_k^*\right) \geq \delta^*\left(\cup_{k=\ell}^{p} X_k^*\right). \tag{9}$$

PROOF. A distribution δ is weakly decomposable if and only if there exists a decomposition of δ where for every $\ell \in [p]$, agents of N_ℓ^* only contribute to projects of $\cup_{k=\ell}^{p} X_k^*$. This holds if and only if $\delta\left(\cup_{k=\ell}^{p} X_k^*\right) \geq \sum_{k=\ell}^{p} C_{N_\ell^*}$ for every $\ell \in [p]$. By Lemma A.2, this is equivalent to the condition $\delta\left(\cup_{k=\ell}^{p} X_k^*\right) \geq \delta^*\left(\cup_{k=\ell}^{p} X_k^*\right)$ for every $\ell \in [p]$. □

To simplify the proof of Theorem 6.11, we introduce the following class of profiles.

Definition A.5. A profile is called *reduced* if, in its equilibrium distribution δ^*, for every agent i, there exists a $k \in [p]$ such that $A_i \subseteq X_k^*$, that is, all projects approved by an agent belong to the same class in the partition induced by δ^*.

Note that, in a reduced profile, all projects approved by agent i receive in equilibrium the same amount $u_i(\delta^*)$, and therefore are all critical for i, that is, $T_{\delta^*,i} = A_i$ for all $i \in N$.

Lemma A.6. *If Theorem 6.11 is true for reduced profiles, then it is true for all profiles.*

Proof. Let P be any profile, and δ^* its equilibrium distribution. Let P' be its reduced profile where, compared to P, every agent i has removed her approval from every project x with $\delta^*(x) > u_i(\delta^*)$. Then, δ^* is the equilibrium distribution for P', too (by the same decomposition). By assumption, Theorem 6.11 is true for P', so δ^* maximizes g-welfare among all distributions that are weakly decomposable with respect to P'. Since the equilibrium distribution is the same in P and P', the set of weakly decomposable distributions is the same too.

The profile P differs from P' by having additional approvals, which could only decrease the maximal possible g-welfare. But δ^* yields the same welfare in P and P'. Therefore, δ^* necessarily maximizes g-welfare among all distributions that are weakly decomposable with respect to P, too. □

Proof of Theorem 6.11. Based on Lemma A.6, we assume w.l.o.g. that we are given a reduced profile. Let $X_1^*, \ldots, X_p^*$, and $N_1^*, \ldots, N_p^*$ be the partitioning of projects and agents induced by the equilibrium distribution δ^*, and $w_1^* < \cdots < w_p^*$ the corresponding allocations. By Lemma A.2, each project in X_k^* receives $w_k^* = \delta^*(X_k^*)/|X_k^*|$, and every agent $i \in N_k^*$ has utility w_k^*. Since the profile is reduced, $T_{\delta^*,i} = A_i \subseteq X_k^*$ for all $i \in N_k^*$.

Let δ be any weakly decomposable distribution different than δ^*. We prove that δ does not maximize g-welfare among weakly decomposable distributions by showing a modification δ' of δ, which is weakly decomposable but has a higher g-welfare than δ.

Since $\delta \neq \delta^*$ and both distributions sum up to C_N, there must be projects $x^-, x^+ \in A$ with $\delta(x^-) < \delta^*(x^-)$ and $\delta(x^+) > \delta^*(x^+)$, respectively. Consequently, one of the following two cases has to apply:

- If $\delta(X_k^*) = \delta^*(X_k^*)$ for all $k \in [p]$, let $X_r^* = X_s^*$ $(r = s)$ be a class that contains a project x^- with $\delta(x^-) < \delta^*(x^-)$.
- Otherwise, let r be the largest index in $[p]$ for which $\delta(X_r^*) \neq \delta^*(X_r^*)$. Weak decomposability of δ and Lemma A.4 imply that $\delta(X_r^*) > \delta^*(X_r^*)$. As $\delta(X_k^*) = \delta^*(X_k^*)$ for all $k > r$, there must be an $s \leq r$ such that there exists a project x^- in X_s^* with $\delta(x^-) < \delta^*(x^-)$; choose $s \leq r$ to be the largest index with this property.

In both cases, we define $X^- \subseteq X_s^*$ as the set of all projects x in X_s^* with $\delta(x) < \delta^*(x)$, and $X^+ \subseteq X_r^*$ as the set of all projects x in X_r^* with $\delta(x) > \delta^*(x)$; both sets must be non-empty by construction. The case $r > s$ is depicted in Figure 1.

Starting from δ, transfer a sufficiently small amount ε uniformly from X^+ to X^-; call the resulting distribution δ'. We choose ε small enough such that it does not change the order relations between projects inside and outside X^+ and X^-, that is, for all $x^- \in X^-$ and $x^+ \in X^+$: $\delta'(x^+) > \delta'(x)$ for all $x \in A$ with $\delta(x^+) > \delta(x)$, and analogously, $\delta'(x^-) < \delta'(x)$ for all $x \in A$ with $\delta(x^-) < \delta(x)$. In particular, since $\delta(x^+) > \delta^*(x^+) \geq \delta^*(x^-) > \delta(x^-)$, we have $\delta'(x^+) > \delta'(x^-)$.

We claim that δ' is weakly decomposable. By Lemma A.4, it suffices to show that (9) holds for δ', that is, $\delta'\left(\cup_{k=\ell}^{p} X_k^*\right) \geq \delta^*\left(\cup_{k=\ell}^{p} X_k^*\right)$ for every $\ell \in [p]$. Note that $\delta'\left(\cup_{k=\ell}^{p} X_k^*\right) = \delta\left(\cup_{k=\ell}^{p} X_k^*\right)$ for all $\ell \leq s$ and all $\ell \geq r+1$, so for these indices, (9) for δ' follows from the weak-decomposability

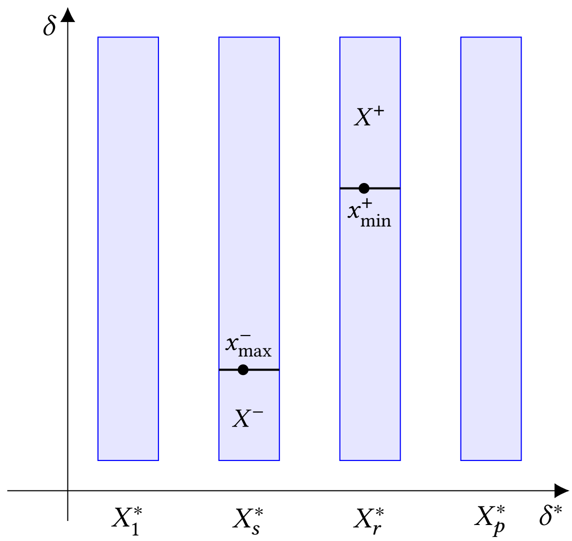

Fig. 1. Project sets in the proof of Theorem 6.11, for the case $r > s$. The horizontal position of a project denotes its allocation in δ^*; the vertical position denotes its allocation in δ.

of δ. It therefore remains to prove (9) for $\ell \in \{s+1, \ldots, r\}$. This set is non-empty only when $s < r$, which is possible only in the second case above.

Our choices of r and s ensure that $\delta\left(\cup_{k=r}^{p} X_k^*\right) > \delta^*\left(\cup_{k=r}^{p} X_k^*\right)$ and $\delta\left(\cup_{k=1}^{s} X_k^*\right) < \delta^*\left(\cup_{k=1}^{s} X_k^*\right)$. For ε sufficiently small, the same inequalities hold between δ' and δ^*. Moreover, for $s < \ell \le r$,

$$\delta'\left(\cup_{k=\ell}^{p} X_k^*\right) = \delta'\left(\cup_{k=r}^{p} X_k^*\right) + \delta'\left(\cup_{k=\ell}^{r-1} X_k^*\right) > \delta^*\left(\cup_{k=r}^{p} X_k^*\right) + \delta\left(\cup_{k=\ell}^{r-1} X_k^*\right) \ge \delta^*\left(\cup_{k=\ell}^{p} X_k^*\right)$$

where the first inequality holds because $\delta'\left(\cup_{k=r}^{p} X_k^*\right) > \delta^*\left(\cup_{k=r}^{p} X_k^*\right)$ and $\delta'(X_k^*) = \delta(X_k^*)$ for all $k \notin \{r, s\}$, and the second inequality holds because, for each $k \in \{s+1, \ldots, r-1\}$, all projects x in X_k^* satisfy $\delta(x) \ge \delta^*(x)$ by definition of s. Therefore, by Lemma A.4, δ' is still weakly decomposable.

We now analyze the effect of this redistribution on the agents' utilities. For that, we prove an auxiliary claim on critical projects of agents under δ. Define $x^+_{\min} \in \arg\min_{x^+ \in X^+} \delta(x^+)$ as a project from X^+ with minimal allocation in δ and $x^-_{\max} \in \arg\max_{x^- \in X^-} \delta(x^-)$ as a project from X^- with maximal contribution in δ.

Claim. For every agent $i \in N$, either $T_{\delta,i} \cap X^- = \emptyset$ or $T_{\delta,i} \subseteq X^-$. Similarly, either $T_{\delta,i} \cap X^+ = \emptyset$ or $T_{\delta,i} \subseteq X^+$.

Proof of claim. We prove the claim for X^-; the proof for X^+ is analogous. By definition of critical projects, $T_{\delta,i} \subseteq A_i$. Since the profile is reduced, A_i is contained in a single partition class. If this partition class is not the one that contains X^-, namely X_s^*, then $T_{\delta,i} \cap X^- = \emptyset$. Otherwise, $T_{\delta,i} \subseteq X_s^*$. Now, if $u_i(\delta) > \delta(x^-_{\max})$, then $\delta(x) > \delta(x^-_{\max})$ for every $x \in T_{\delta,i}$, so $T_{\delta,i} \cap X^- = \emptyset$; and if $u_i(\delta) \le \delta(x^-_{\max})$, then $\delta(x) \le \delta(x^-_{\max})$ for every $x \in T_{\delta,i}$, so $T_{\delta,i} \subseteq X^-$.

Back to proof of theorem. Denote by "losers" the agents who lose utility from the redistribution. The claim implies that all the losers have $T_{\delta,i} \subseteq X^+$; each of them loses $\varepsilon/|X^+|$. Moreover, all losers have $A_i \subseteq X^+$: this is because $A_i \subseteq X_r^*$ (since the profile is reduced), and $\delta(x_A) \ge \delta(x_T) \ge \delta(x^+_{\min})$

for all $x_A \in A_i$ and $x_T \in T_{\delta,i}$. Therefore, in equilibrium, all losers give all their contributions to projects in X^+. This implies that the contributions of all losers sum up to at most $\delta^*(X^+) = w_r^* \cdot |X^+|$. Then, for every loser i,

$$g\left(u_i(\delta)\right) - g\left(u_i(\delta')\right) \leq g\left(\delta(x^+_{\min})\right) - g\left(\delta(x^+_{\min}) - \frac{\varepsilon}{|X^+|}\right) \tag{10}$$

by concavity of g (which follows from the assumption that $xg'(x)$ is non-increasing).

Denote by "gainers" the agents who gain utility from the redistribution. The claim implies that every agent with $T_{\delta,i} \cap X^- \neq \emptyset$ is a gainer; each of them gains $\varepsilon/|X^-|$. Moreover, every agent with $A_i \cap X^- \neq \emptyset$ is a gainer: this is because $A_i \cap X^- \neq \emptyset$ implies $\delta(x_A) \leq \delta(x^-_{\max})$ for at least one project $x_A \in A_i$, and $\delta(x_T) \leq \delta(x_A)$ for all projects $x_T \in T_{\delta,i}$. Therefore, in equilibrium, every agent who contributes a positive amount to at least one project in X^- must be a gainer. So the contributions of all gainers must sum up to at least $\delta^*(X^-) = w_s^* \cdot |X^-|$. Then, for every gainer i,

$$g\left(u_i(\delta')\right) - g\left(u_i(\delta)\right) \geq g\left(\delta(x^-_{\max}) + \frac{\varepsilon}{|X^-|}\right) - g\left(\delta(x^-_{\max})\right) \tag{11}$$

by concavity of g.

Therefore, by (10) and (11), the increase in g-welfare from δ to δ' is at least

$$\begin{aligned} & w_s^* \cdot |X^-| \cdot \left[g\left(\delta(x^-_{\max}) + \frac{\varepsilon}{|X^-|}\right) - g\left(\delta(x^-_{\max})\right)\right] \\ & - w_r^* \cdot |X^+| \cdot \left[g\left(\delta(x^+_{\min})\right) - g\left(\delta(x^+_{\min}) - \frac{\varepsilon}{|X^+|}\right)\right]. \end{aligned} \tag{12}$$

Since g is strictly concave,

$$\begin{aligned} g\left(\delta(x^-_{\max}) + \frac{\varepsilon}{|X^-|}\right) - g(\delta(x^-_{\max})) &> \frac{\varepsilon}{|X^-|} \cdot g'\left(\delta(x^-_{\max}) + \frac{\varepsilon}{|X^-|}\right); \\ g(\delta(x^+_{\min})) - g\left(\delta(x^+_{\min}) - \frac{\varepsilon}{|X^+|}\right) &< \frac{\varepsilon}{|X^+|} \cdot g'\left(\delta(x^+_{\min}) - \frac{\varepsilon}{|X^+|}\right). \end{aligned}$$

Plugging this into (12), we get that the increase in g-welfare is larger than

$$w_s^* \cdot |X^-| \cdot \frac{\varepsilon}{|X^-|} \cdot g'\left(\delta(x^-_{\max}) + \frac{\varepsilon}{|X^-|}\right) - w_r^* \cdot |X^+| \cdot \frac{\varepsilon}{|X^+|} \cdot g'\left(\delta(x^+_{\min}) - \frac{\varepsilon}{|X^+|}\right).$$

By our choice of ε, we have $w_r^* = \delta^*(x^+_{\min}) < \delta(x^+_{\min})$, so $w_r^* < \delta(x^+_{\min}) - \varepsilon/|X^+|$ for sufficiently small ε. Similarly, $w_s^* = \delta^*(x^-_{\max}) > \delta(x^-_{\max}) + \varepsilon/|X^-|$ for sufficiently small ε. Therefore, the increase in g-welfare is larger than

$$\varepsilon \cdot \left(\delta(x^-_{\max}) + \frac{\varepsilon}{|X^-|}\right) \cdot g'\left(\delta(x^-_{\max}) + \frac{\varepsilon}{|X^-|}\right) - \varepsilon \cdot \left(\delta(x^+_{\min}) - \frac{\varepsilon}{|X^+|}\right) \cdot g'\left(\delta(x^+_{\min}) - \frac{\varepsilon}{|X^+|}\right). \tag{13}$$

By our choice of ε, $\delta(x^-_{\max}) + \varepsilon/|X^-| < \delta(x^+_{\min}) - \varepsilon/|X^+|$. By the assumption on g, $xg'(x)$ is non-increasing in x. Therefore, the expression in (13) is at least 0, so the increase in g-welfare from δ to δ' is larger than 0. This means that δ does not maximize g-welfare.

Since δ was any weakly decomposable distribution different than δ^*, we conclude that δ^* maximizes g-welfare subject to weak decomposability in any reduced profile. By Lemma A.6, the same is true in any profile. □

A.2 Proof of Proposition 6.12

Proposition 6.12. *For each $p > 0$, maximizing the g-welfare with respect to $g(u) = u^p$ subject to decomposability does not always return the equilibrium distribution.*

Proof. For a fixed $p > 0$, consider a profile consisting of two agents with binary weights and approval sets $\{a\}$ and $\{a, b\}$, and respective contributions $C_1 = \max\left((2^{p-1} \cdot p)^{-1/p}, 2\right)$ and $C_2 = 1$. Since $C_1 \geq 2$, the equilibrium distribution is $(C_1, 1)$. We claim that the decomposable distribution $(C_1 + 1, 0)$ yields a higher g-welfare, that is,

$$C_1 \cdot g(C_1 + 1) + 1 \cdot g(0) > C_1 \cdot g(C_1) + 1 \cdot g(1)$$
$$\iff C_1 \cdot (g(C_1 + 1) - g(C_1)) > 1.$$

For every $p \geq 1$, g is convex, so

$$g(C_1 + 1) - g(C_1) \geq g'(C_1) \cdot 1 = p \cdot C_1^{p-1}$$
$$\implies C_1 \cdot (g(C_1 + 1) - g(C_1)) \geq p \cdot C_1^p \geq p \cdot 2^p \geq 2 > 1.$$

For every $0 < p < 1$, g is strictly concave, so

$$\begin{aligned} g(C_1 + 1) - g(C_1) > g'(C_1 + 1) \cdot 1 &= p \cdot (C_1 + 1)^{p-1} \\ &> p \cdot (2C_1)^{p-1} \qquad \text{(since } p - 1 < 0 \text{ and } C_1 > 1) \end{aligned}$$

and so

$$\begin{aligned} C_1 \cdot (g(C_1 + 1) - g(C_1)) &> 2^{p-1} \cdot p \cdot C_1^p \\ &\geq 2^{p-1} \cdot p \cdot \left(\left(\frac{1}{2^{p-1}p}\right)^{\frac{1}{p}}\right)^p = 1. \end{aligned}$$

In both cases, the equilibrium distribution does not maximize g-welfare. □

A.3 Proof of Theorem 6.15

Theorem 6.15. *Let $\mathcal{S} = (i_0, i_1, i_2, \dots)$ be an infinite sequence of agents updating their individual distributions by best responses. If each agent $i \in N$ appears infinitely often in $\mathcal{S}$, the dynamics converges to the equilibrium distribution, that is, $\lim_{t\to\infty} \delta^t = \delta^*$.*

By the theorem assumption, there exists a time T by which all agents have already appeared at least once in $\mathcal{S}$. It is sufficient to prove the theorem for the subsequence starting at T. Therefore, from now on, we assume without loss of generality that at step $t = 0$, all agents have already appeared at least once in $\mathcal{S}$, and thus, have contributed the entire amount C_i.

Denote the amount of shifted contributions in step t by

$$c_t := 1/2 ||\delta^t - \delta^{t+1}||_1.$$

Thus, when moving from δ^t to δ^{t+1} in round t, agent i_t redistributes c_t from a set of projects $A^-_{i_t}$ to another set $A^+_{i_t}$ with $A^+_{i_t} \cap A^-_{i_t} = \emptyset$. Since the agent is only allowed to redistribute her individual distribution, $\delta^t_{i_t}(A^-_{i_t}) \geq c_t$. Furthermore, she redistributes according to her best response, so $u_{i_t}(\delta^{t+1}) = \delta^{t+1}(x^+)$ for every $x^+ \in A^+_{i_t}$, and $\delta^{t+1}_{i_t}(x) = 0$ for all $x \in A$ with $\delta^{t+1}(x) > u_{i_t}(\delta^{t+1})$. An illustrative example is given in Figure 2. In particular, $\delta^{t+1}(x^-) \geq u_{i_t}(\delta^{t+1}) = \delta^{t+1}(x^+)$ for all $x^- \in A^-_{i_t}$ and $x^+ \in A^+_{i_t}$.

The proof of Theorem 6.15 proceeds in three steps. First, we prove that the total amount of redistributions $\sum_{t=0}^{\infty} c_t$ is finite. Second, we show that the amount an arbitrary agent wants to

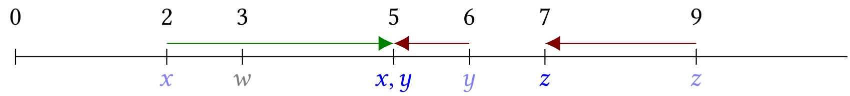

Fig. 2. An instance with four projects (named w, x, y, z), $\delta^t = (3, 2, 6, 9)$, and an agent i_t with $\delta^t_{i_t} = (0, 2, 2, 2)$ and $A_{i_t} = \{x, y, z\}$. Then, $\delta^{best}_{i_t} = (0, 5, 1, 0)$, $\delta^{t+1} = (3, 5, 5, 7)$, $c_t = 3$, $A^-_{i_t} = \{y, z\}$, $A^+_{i_t} = \{x\}$.

redistribute converges to 0. Finally, we conclude that this can only be the case if the dynamics converges to the equilibrium distribution.

Lemma A.7. *For any sequence $\mathcal{S}$, it holds that $\sum_{t=0}^{\infty} c_t < \infty$.*

Proof. First, observe that an agent's best response going from δ^t to δ^{t+1} can be described by the following continuous process: As long as the agent spends a positive amount on a non-critical project, transfer money from such a project to all critical projects equally, until either (i) at least one more project becomes critical, or (ii) the agent no longer spends a positive amount on a non-critical project. This process can be interpreted as a sequence of transfers, where each transfer goes from a project with higher distribution to one with lower distribution, such that after the transfer, the distribution of the former project remains at least as high as that of the latter.

For each t, since the difference between δ^t and δ^{t+1} is caused by transfers, and each amount ε transferred from one project to another project causes a change of ε in distribution for both projects, c_t is upper-bounded by the amount of transfer made going from δ^t to δ^{t+1}. Hence, it suffices to prove that the total amount of transfer made over all time steps t of the dynamics is finite.

To prove the claim, we define a potential function: $\Phi(\delta) := \sum |\delta(x) - \delta(y)|$, where the sum is over all (unordered) pairs of distinct projects $x, y \in A$. Consider what happens to the potential when an amount ε is transferred from a project p to another project q such that $\delta(p) - \varepsilon \geq \delta(q) + \varepsilon$:

- The term $|\delta(p) - \delta(q)|$ decreases by exactly 2ε.
- For every $x \notin \{p, q\}$, if $\delta(x) \leq \delta(q) + \varepsilon$ then $|\delta(p) - \delta(x)|$ decreases by ε and $|\delta(q) - \delta(x)|$ increases by at most ε; if $\delta(x) \geq \delta(p) - \varepsilon$ then $|\delta(p) - \delta(x)|$ increases by at most ε and $|\delta(q) - \delta(x)|$ decreases by ε; if $\delta(q) + \varepsilon \leq \delta(x) \leq \delta(p) - \varepsilon$ then both $|\delta(q) - \delta(x)|$ and $|\delta(p) - \delta(x)|$ decrease by ε.
- Finally, for $x, y \notin \{p, q\}$, the term $|\delta(x) - \delta(y)|$ is unchanged.

Thus, the potential $\Phi(\delta)$ decreases by at least 2ε. Since $\Phi(\delta^t) \geq 0$ for all t, the total amount of transfer in the entire sequence is at most $\Phi(\delta^0)/2$, which is finite. □

Note that each transfer in the proof of Lemma A.7 constitutes a lexicographic improvement with respect to the distribution vector, implying $\delta^{t+1} \geq_{lex} \delta^t$ for all t.

Define $d_i(\delta)$ as the amount of contribution that would be shifted by an agent i if the current distribution (along with its associated decomposition) were δ and it was her turn to respond. Note that we define $d_i(\delta)$ for all agents, not only the one who actually plays her best response; in particular, $d_{i_t}(\delta^t) = c_t$ for all t. Note also that δ is the equilibrium distribution if and only if $d_i(\delta) = 0$ for all $i \in N$.

Lemma A.8. *For any sequence $\mathcal{S}$, time step $t \geq 0$, and agent $i \in N$:*

$$d_i(\delta^t) \leq d_{i_t}(\delta^t) + d_i(\delta^{t+1}).$$

Intuitively, the lemma can be seen as a "triangle inequality": the left-hand side denotes the direct distance from δ^t towards i's optimal redistribution; the right-hand side denotes the distance along an indirect path that first goes to δ^{t+1} and then proceeds towards i's optimal redistribution.

Proof. Let i be an arbitrary agent with possible shift $d_i(\delta^t)$ at step t. If $d_i(\delta^t) \leq d_{i_t}(\delta^t)$, the statement holds trivially. Hence, assume that $d_i(\delta^t) > d_{i_t}(\delta^t)$. In particular, $i \neq i_t$.

Let $\tilde{\delta}_i^{t+1}$ and $\tilde{\delta}^{t+1}$ be the (hypothetical) individual distribution of agent i and the overall distribution had she been able to implement her best response.

Denote the sets of projects that would be affected by agent i's best response at δ^t by $A_i^- := \{x^- \in A_i : \tilde{\delta}_i^{t+1}(x^-) < \delta_i^t(x^-)\}$ and $A_i^+ := \{x^+ \in A_i : \tilde{\delta}_i^{t+1}(x^+) > \delta_i^t(x^+)\}$. Then,

$$\begin{aligned}\tilde{\delta}^{t+1}(x^-) &= \sum_{j\neq i} \delta_j^t(x^-) + \tilde{\delta}_i^{t+1}(x^-)\\ &\geq \sum_{j\neq i} \delta_j^t(x^+) + \tilde{\delta}_i^{t+1}(x^+) = \tilde{\delta}^{t+1}(x^+) \text{ for all } x^- \in A_i^- \text{ and } x^+ \in A_i^+;\text{ and}\end{aligned} \tag{14}$$

$$\tilde{\delta}_i^{t+1} = 0 \text{ for all } x \in A \text{ with } u_i(\tilde{\delta}^{t+1}) < \tilde{\delta}^{t+1}(x) \tag{15}$$

hold by definition of best responses.

Now, a lower bound for $d_i(\delta^{t+1})$ is given by the amount shifted from projects in A_i^- under i's best response in step $t+1$. Again, denote by $\tilde{\delta}_i^{t+2}$ and $\tilde{\delta}^{t+2}$ agent i's best response in step $t+1$ and the corresponding overall distribution.

Consider first the special case in which agent i_t did not change her contribution to projects in $A_i^- \cup A_i^+$, that is, $\delta^t(x) = \delta^{t+1}(x)$ for all $x \in A_i^- \cup A_i^+$. If $d_i(\delta^{t+1}) < d_i(\delta^t)$, then a lower amount is transferred from projects in A_i^- and to projects in A_i^+ in i's best response at δ^{t+1} than in i's best response at δ^t, so by (14), there exist projects $x^- \in A_i^-$ and $x^+ \in A_i^+$ such that $\tilde{\delta}^{t+2}(x^-) > \tilde{\delta}^{t+2}(x^+) \geq u_i(\tilde{\delta}^{t+2})$ and $\tilde{\delta}_i^{t+2}(x^-) > 0$. This contradicts (15) with $t+2$ instead of $t+1$. Thus, $d_i(\delta^{t+1}) \geq d_i(\delta^t)$ and the claim follows.

Consider now the general case, in which agent i_t may have changed her contribution to some projects in $A_i^- \cup A_i^+$. We claim that the total transfer of i_t and then i (i.e., $d_{i_t}(\delta^t) + d_i(\delta^{t+1})$) cannot be less than the transfer if i were to act alone (i.e., $d_i(\delta^t)$). The reason is similar to the previous paragraph: If this total transfer is less than $d_i(\delta^t)$, then there exist projects $x^- \in A_i^-$ and $x^+ \in A_i^+$ such that $\tilde{\delta}^{t+2}(x^-) > \tilde{\delta}^{t+2}(x^+) \geq u_i(\tilde{\delta}^{t+2})$ and $\tilde{\delta}_i^{t+2}(x^-) > 0$, which is a contradiction. Hence, $d_{i_t}(\delta^t) + d_i(\delta^{t+1}) \geq d_i(\delta^t)$, as desired. □

For any agent $j \in N$ and time step t, we know that j will get the chance to redistribute her contribution in the future, since $\mathcal{S}$ contains every agent infinitely often. Denote this next time step by t'. So,

$$\begin{aligned}\sum_{\ell=t}^{t'} c_\ell &= \sum_{\ell=t}^{t'} d_{i_\ell}(\delta^\ell)\\ &\geq \sum_{\ell=t}^{t'} \left(d_j(\delta^\ell) - d_j(\delta^{\ell+1})\right) && \text{(by Lemma A.8)}\\ &= d_j(\delta^t) - d_j(\delta^{t'+1})\\ &= d_j(\delta^t) && \text{(as } d_j(\delta^{t'+1}) = 0 \text{ after agent } j\text{'s best response.)}\end{aligned}$$

Combining this with Lemma A.7 implies the following corollary.

Corollary A.9. *For any sequence $\mathcal{S}$ and agent $j \in N$,* $\lim_{t\to\infty} d_j(\delta^t) = 0$.

We can now finish the proof of Theorem 6.15.

Proof of Theorem 6.15. For any $\mathcal{S}$, since $(\delta^t)_{t\in\mathbb{N}}$ is an infinite sequence in the closed set of decomposable distributions in the (bounded) simplex $\Delta(C_N)$, the Bolzano-Weierstrass theorem states that it has a convergent subsequence $(\delta^{t_k})_{k\in\mathbb{N}}$ with decomposable limit $\delta \in \Delta(C_N)$. By Corollary A.9, $\lim_{k\to\infty} d_i(\delta^{t_k}) = 0$, implying $d_i(\delta) = 0$ for every agent $i \in N$, and so $\delta = \delta^*$. However, as $(\delta^t)_{t\in\mathbb{N}}$ is lexicographically increasing (as shown in the proof of Lemma A.7), every δ^t is decomposable by definition of the dynamics, and δ^* is lexicographically maximal among all decomposable distributions (Theorem 6.5), we conclude that $\lim_{t\to\infty} \delta^t = \delta^*$. □

B WELFARE-MAXIMIZING DISTRIBUTIONS

Let g be a strictly-increasing function. The *g-welfare* of a distribution δ is defined as the following weighted sum:

$$g\text{-welfare}(\delta) := \sum_{i\in N} C_i \cdot g(u_i(\delta)).$$

Quantifying welfare enables us to compare and rank all possible utility vectors, which by Lemma 4.9 induces a social welfare ordering over all distributions $\delta \in \Delta(C_N)$ by g-welfare(δ).

Inversely, every continuous social welfare ordering without any "welfare dependencies" between the agents' utilities can be represented by a g-welfare function; see Chapter 2 in the book by Moulin [1988] for a detailed discussion. Additionally weighting agents by their contributions, we arrive at the very expressive class of g-welfare functions.

A distribution is called *g-welfare-maximizing* if it maximizes the g-welfare, i.e., it always chooses a maximal element of the corresponding social welfare ordering. Clearly, every g-welfare-maximizing distribution is efficient. When g is concave (equivalently: when the induced social welfare ordering satisfies the Pigou-Dalton principle), a g-welfare-maximizing distribution can be found by solving a convex program where the variables are $(u_i)_{i\in N}$ and $(\delta_x)_{x\in A}$:

$$\begin{aligned}
\text{maximize} \quad & \sum_{i\in N} C_i \cdot g(u_i) && (16)\\
\text{s.t.} \quad & \sum_{x\in A} \delta_x \leq C_N \\
& u_i \leq \delta_x / v_{i,x} && \text{for all } i \in N, x \in A_i \\
& u_i \geq 0 && \text{for all } i \in N \\
& \delta_x \geq 0 && \text{for all } x \in A.
\end{aligned}$$

The following technical lemmas prove uniqueness of the welfare-maximizing distribution when g is strictly concave (and strictly increasing).

Lemma B.1. *For every strictly-concave strictly-increasing function g, every constant $t \in (0,1)$, and every two distributions $\delta \neq \delta'$,*

$$g\text{-}\textit{welfare}\,(t\delta + (1-t)\delta') \;>\; \min(g\text{-}\textit{welfare}(\delta'), g\text{-}\textit{welfare}(\delta)).$$

Proof. For every agent $i \in N$, by the concavity of the minimum operator,

$$u_i\,(t\delta + (1-t)\delta') \geq t u_i(\delta) + (1-t) u_i(\delta).$$

Therefore,

$$\begin{aligned}
g\text{-welfare}\,(t\delta + (1-t)\delta') &\geq \sum_{i\in N} C_i \cdot g\,(t \cdot u_i(\delta) + (1-t) \cdot u_i(\delta')) \;\; \text{(by monotonicity)} \\
&> t \sum_{i\in N} C_i \cdot g(u_i(\delta)) + (1-t) \sum_{i\in N} C_i \cdot g(u_i(\delta')) \;\text{(by strict concavity)}
\end{aligned}$$

$$= t \cdot g\text{-welfare}(\delta) + (1-t)\cdot g\text{-welfare}(\delta')$$
$$\geq \min(g\text{-welfare}(\delta'), g\text{-welfare}(\delta)). \qquad \square$$

Lemma B.2. *For every strictly-concave strictly-increasing function g, there is a unique g-welfare-maximizing distribution.*

Proof. Assume for contradiction that there exist two different g-welfare-maximizing distributions δ and δ'. Since both distributions are efficient, by Lemma 4.9 they induce two different utility vectors $(u_i(\delta))_{i\in N}$ and $(u_i(\delta'))_{i\in N}$. By Lemma B.1, for any $t \in (0,1)$,

$$g\text{-welfare}\,(t\delta + (1-t)\delta') > \min(g\text{-welfare}(\delta'), g\text{-welfare}(\delta))$$
$$= g\text{-welfare}(\delta') = g\text{-welfare}(\delta).$$

This contradicts the assumption that δ and δ' are g-welfare-maximizing. $\square$

C CONTRIBUTION INCENTIVE-COMPATIBILITY

For sum-based utilities [Brandl et al., 2022], under a *contribution incentive-compatible* distribution rule, every agent i increases her utility from the rule by at least her additional contribution, i.e., $u_i(f(P')) \geq u_i(f(P)) + C_i$. For Leontief utilities with binary weights, agent i is only able to generate utility $C_i/|A_i|$ from a contribution C_i on her own, leading to an adapted definition of this property in our setting.

Definition C.1 (Contribution incentive-compatibility). A distribution rule f satisfies *contribution incentive-compatibility* if for every two profiles P and P' where in P', agent i contributes an additional amount of C_i, $u_i(f(P')) \geq u_i(f(P)) + C_i/|A_i|$.

Thus, agent i receives at least as much utility by contributing C_i to the mechanism as by saving it and distributing this contribution equally among all projects in A_i.

One could consider a stronger version of contribution incentive-compatibility, where the agent does not have to distribute equally among all projects in A_i, but instead may pick a clever distribution in order to optimize her utility. However, the following example shows that the resulting axiom cannot be satisfied by any distribution rule. We note that this strengthening may be rather unrealistic in any case, since it seems difficult for an agent to know the outcome of the rule based on other agents' contributions and strategize based on that.

Example C.2 (Strengthening of contribution incentive-compatibility cannot be satisfied). Let $n = 2$, $A = \{w, x, y, z\}$, $A_1 = \{w, x\}$, $A_2 = \{x, y, z\}$, and $C_1 = C_2 = 6$. Assume for contradiction that there exists a distribution rule f that satisfies this strengthening.

For $C_2 = 0$, f must return the distribution $(3, 3, 0, 0)$, since otherwise agent 1 can achieve a higher utility by not contributing to f. Similarly, for $C_1 = 0$, f must return the distribution $(0, 2, 2, 2)$, since otherwise agent 2 can achieve a higher utility by not contributing to f.

Now, consider $C_1 = C_2 = 6$. The output of f must yield utility at least 4 to agent 1; otherwise agent 1 can choose not to contribute to f and instead distribute 4 to w and 2 to x. Similarly, the output of f must yield utility at least 3 to agent 2; otherwise agent 2 can choose not to contribute to f and instead distribute 3 to each of y and z. However, a distribution that yields utility at least 4 to agent 1 and at least 3 to agent 2 must have a total contribution of at least $4 + 4 + 3 + 3 > 6 + 6$, a contradiction.

In contrast to this strengthening, contribution incentive-compatibility can be satisfied, e.g., by the trivial distribution rule that distributes each agent's contribution equally among her approved projects, sometimes referred to as the *uncoordinated rule.*

Nevertheless, contribution incentive-compatibility might clash with strategyproofness as well as with efficiency.

Example C.3 (Contribution incentive-compatibility and strategyproofness cannot be satisfied together). Let $A = \{x, y\}$, and assume we are given three agents with contribution 0 if not stated otherwise. Let f be any distribution rule satisfying contribution incentive-compatibility (CIC); we will prove that f is not strategyproof.

- If $C_1 = 1$ and $A_1 = \{x\}$, by CIC, f must return the distribution $(1, 0)$.
- If $C_1 = C_2 = 1$, $A_1 = \{x\}$, and $A_2 = \{y\}$, by CIC on agent 2, f must assure agent 2 a utility of at least 1. By an analogous argument, f must give agent 1 a utility of at least 1, so the returned distribution must be $(1, 1)$.
- If $C_1 = 1$ and $A_1 = \{x, y\}$, by CIC, f must return the distribution $(1/2, 1/2)$.
- If $C_1 = C_2 = 1$, $A_1 = \{x, y\}$, and $A_2 = \{y\}$, by CIC on agent 2, f must give agent 2 a utility of at least $3/2$, so must return a distribution $(a, 2 - a)$ with $a \leq 1/2$.
  Now, if agent 1 reports $A_1' = \{x\}$ instead, then her utility improves to 1. Therefore, f is not strategyproof.

Example C.4 (Contribution incentive-compatibility and efficiency cannot be satisfied together). Let $A = \{x, y, z\}$ and consider four agents with approval sets $A_1 = \{x\}$, $A_2 = \{x, y\}$, $A_3 = \{y, z\}$, $A_4 = \{z\}$ and contributions $C_1 = C_4 = 6$, $C_2 = C_3 = 12$. We will consider several combinations of the agents.

If only agents 1 and 3 contribute, then a contribution incentive-compatible rule must return the distribution $(6, 6, 6)$. Therefore, if agents 1, 2 and 3 are present, the utility of agent 2 must be at least 12, so the distribution must be $(12, 12, 6)$. Applying CIC backwards for agent 1 implies that, when only agents 2 and 3 contribute, project x should receive at most 6.

By analogous arguments, if only agents 2 and 4 contribute, then the distribution is $(6, 6, 6)$; if agents 2, 3 and 4 contribute, the utility of agent 3 must be at least 12, so the distribution must be $(6, 12, 12)$. Applying CIC backwards for agent 4 implies that, when only agents 2 and 3 are present, project z should receive at most 6.

Consequently, when only agents 2 and 3 contribute, the utility of each agent must be at most 6. Such a distribution can never be efficient: it is dominated, e.g., by the distribution $(8, 8, 8)$.

Remark 5. As already mentioned, the uncoordinated rule trivially satisfies CIC. Thus, the above examples imply that it is neither efficient nor strategyproof. The latter fact is somewhat surprising, since this uncoordinated approach is just "no mechanism at all". Here, the manipulability does not arise from a feature of the mechanism but rather from the binary weights. Without coordination, each donor, who wants to maximize her utility, needs to know what other donors do. A strategyproof rule would allow donors to avoid this mental burden.

D STRATEGYPROOFNESS DOES NOT IMPLY PREFERENCE-MONOTONICITY

For binary sum-based utilities, strategyproofness implies preference-monotonicity [Brandl et al., 2021]. This does not hold for binary Leontief utilities, even when demanding that contributions are only allocated to projects that are approved by at least one agent.

Example D.1 (Strategyproofness does not imply preference-monotonicity). Let $n = 2$, $A = \{x, y, z\}$, and $C_1 = C_2 = 1$, both agents have binary Leontief utilities, and consider the following distribution rule:

- If $A_2 \neq \{x, y, z\}$, split the entire contribution equally among all projects in A_2.
- Else ($A_2 = \{x, y, z\}$), if $A_1 = \{x, y\}$ or $\{z\}$, return the distribution $(0, 0, C_1 + C_2)$.
- Else, return the distribution $(0, {}^1/_2(C_1 + C_2), {}^1/_2(C_1 + C_2))$.

It is clear that this distribution rule does not allocate any contribution to projects with zero approvals. Further, contributing less does not change the relative distribution, which is why we only have to consider manipulations of the approval sets when proving strategyproofness.

- Consider agent 2. If $A_2 \neq \{x, y, z\}$, the agent already receives the highest possible utility and has no incentive to deviate. Otherwise, $A_2 = \{x, y, z\}$, and any deviation yields utility 0 to the agent as then, at least one project does not receive any contribution.
- Consider agent 1. If $A_2 \neq \{x, y, z\}$, the distribution rule ignores A_1, so the agent has no incentive to deviate. Otherwise, $A_2 = \{x, y, z\}$.
  - If $x \in A_1$, agent 1's utility is 0 no matter what she reports.
  - Else ($x \notin A_1$), if $y \in A_1$, agent 1 gets utility 1 by being truthful, and cannot get a higher utility by deviating.
  - Else, $A_1 = \{z\}$. In this case, agent 1 already gets the maximum utility of 2 by being truthful.

Finally, the distribution rule is not preference-monotonic: if $A_1 = \{x\}$ and $A_2 = \{x, y, z\}$, then $\delta(y) = 1$, and it decreases to 0 when agent 1 additionally approves y.

E EQUILIBRIUM DISTRIBUTIONS ARE RATIONAL-VALUED

In this section we prove that, if the agents' valuations $v_{i,x}$ and contributions C_i are rational numbers, then the equilibrium distribution δ^* is rational-valued.

For each $i \in N$, let T_i be a non-empty set of projects. Consider the following linear program (LP), with variables u_i (for $i \in N$), δ_x (for $x \in A$), and $\delta_{i,x}$ (for $i \in N$ and $x \in A$):

$$\begin{aligned}
\delta_x &= u_i \cdot v_{i,x} && \text{for all } i \in N, x \in T_i; \\
\delta_x &\geq u_i \cdot v_{i,x} && \text{for all } i \in N, x \in A \setminus T_i; \\
\sum_{x \in T_i} \delta_{i,x} &= C_i && \text{for all } i \in N; \\
\sum_{i \in N} \delta_{i,x} &= \delta_x && \text{for all } x \in A.
\end{aligned}$$

Every solution to this LP (if it has a solution) indicates a distribution $\delta(x) = \delta_x$ for all x, with a decomposition $\delta_i(x) = \delta_{i,x}$ for all i, x, such that each agent i contributes only to projects in T_i, and all the projects in T_i are critical for i. By Lemma 4.4, such a distribution has to coincide with the equilibrium distribution.

The equilibrium distribution δ^* is a solution to the above LP whenever $T_i = T_{\delta^*,i}$ for all $i \in N$. By assumption, the coefficients of this LP are all rational. Therefore, by well-known properties of linear programming, the LP has a rational solution.

Unfortunately, the proof cannot be used directly for computing the equilibrium distribution in polynomial time, since the proof requires us to know $T_{\delta^*,i}$. So it remains open whether δ^* can be computed in polynomial time using linear programming.