The Banks Set and the Bipartisan Set May Be Disjoint

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Tournament solutions play an important role within social choice theory and the mathematical social sciences at large. We construct a tournament of order 36 that can be partitioned into the Banks set and the bipartisan set. As a consequence, the Banks set, as well as its refinements, such as the minimal extending set and the tournament equilibrium set, can be disjoint from the bipartisan set.

1 Introduction

Many problems in the mathematical social sciences can be addressed using tournament solutions, i.e., functions that associate with each connex and asymmetric relation on a set of alternatives a non-empty subset of the alternatives (e.g., Moulin, 1986; Laslier, 1997; Hudry, 2009; Brandt et al., 2016; Suksompong, 2021). Tournament solutions are most prevalent in social choice theory, where the binary relation is typically assumed to be given by the simple majority rule.

Prominent and well-studied tournament solutions include the top cycle, the uncovered set, the Banks set, the bipartisan set, the Copeland set, and the Slater set. Several studies have addressed whether there are tournaments for which two given tournament solutions return disjoint choice sets. For example, the first tournament proposed in the literature for which the Banks set and the Slater set are disjoint is of order 75 (Laffond and Laslier, 1991).[1] Later, the order was improved to 16 by Charon et al. (1997) and then to 14 by Östergård and Vaskelainen (2010). Östergård and Vaskelainen have also provided a lower bound of 11 on the order by means of an exhaustive computer analysis. For another example, Moulin (1986) presented a tournament of order 13 in which the Banks set and the Copeland set are disjoint and Hudry (1999) proved that this tournament is minimal. Brandt et al. (2015) addressed these questions systematically by exhaustive computer search and defined the *separation index* of two tournament solutions as the order of the

[1] Laffond and Laslier (1991) presented a similar tournament on 139 alternatives in which the Banks set, the Slater, and the Copeland set are all disjoint from each other.

smallest tournament for which the choice sets returned by the two tournament solutions are disjoint.

Whether the Banks set and the bipartisan set always intersect has been identified as a challenging problem by Laffond et al. (1995) who answered the same question for many pairs of tournament solutions. They also note the weaker open problem concerning the relationship between the tournament equilibrium set and the bipartisan set. Brandt et al. (2015, p. 42) write that "perhaps the most interesting open problem regarding the relationships between tournament solutions concerns the bipartisan set and the Banks set. They provide the first tournament in which the Banks set and the bipartisan set are not contained in each other and prove that their separation index is at least 11.

In this paper, we show that the separation index is at most 36 by providing the first tournament in which the Banks set and the bipartisan set are disjoint. It follows from inclusions proven by Schwartz (1990) and Brandt et al. (2017), respectively, that both the tournament equilibrium set and the minimal extending set can also be disjoint from the bipartisan set. In his monograph on tournament solutions, Laslier (1997, p. 235) concludes his summary of the set-theoretic relationships between tournaments solutions by asserting that "it is not known whether the intersection of [the Banks set] and [the bipartisan set] can be empty." Similarly, Brandt et al. (2016, p. 76) laments that "the exact location of [the bipartisan set] in this diagram [their Fig. 3.7] is unknown." Thanks to the tournament described in this paper and the stated inclusions, this is no longer the case.

2 Preliminaries

A *tournament* T is a pair $(A, \succ)$, where A is a finite set of alternatives and $\succ$ a binary relation on A that is both *asymmetric* ($x \succ y$ implies not $y \succ x$ for all $x, y \in A$) and *connex* ($x \neq y$ implies $x \succ y$ or $y \succ x$ for all $x, y \in A$). The relation $\succ$ is usually referred to as *dominance relation* and we say that x dominates y when $x \succ y$. We write $x \succ B$ if $x \succ y$ for all $y \in B$ and refer to the largest set $B \subseteq A$ for which $x \succ B$ as the *dominion* of x. The *order* of a tournament $T = (A, \succ)$ refers to the cardinality of A.

For a given tournament $T = (A, \succ)$, a subset of alternatives $B \subseteq A$ is called *transitive* if $x \succ y$ and $y \succ z$ imply $x \succ z$ for all $x, y, z \in B$. The *Banks set* $BA(T)$ of a tournament, named after Banks (1985), is then defined as the set of all alternatives that are maximal in some inclusion-maximal transitive subset, i.e.,

$$BA(T) = \{x \in A \colon \exists B \subseteq A,\ B \text{ is transitive},\ x \succ B \setminus \{x\}, \text{ and } \nexists y \in A \text{ with } y \succ B\}.$$

Let $M_T \in \{-1, 0, 1\}^{A \times A}$ be the *skew-adjacency matrix* of T, where

$$M_T(x, y) = \begin{cases} 1 & \text{if } x \succ y, \\ -1 & \text{if } y \succ x, \text{ and} \\ 0 & \text{if } x = y. \end{cases}$$

Laffond et al. (1993) and Fisher and Ryan (1995) have shown independently that every tournament T admits a unique probability distribution $p_T \in [0, 1]^A$ such that

$\sum_{x \in A} p_T(x) = 1$ and $\sum_{x \in A} p_T(x) M_T(x,y) \geq 0$ for all $y \in A$. The distribution p_T corresponds to the unique mixed Nash equilibrium of the symmetric zero-sum game M_T (whose value is 0). The *bipartisan set* $BP(T)$ of a tournament T is defined as the support of this equilibrium, i.e.,

$$BP(T) = \{x \in A \colon p_T(x) > 0\}.$$

3 The Result

We are now ready to state our main theorem.

Theorem 1. *There exists a tournament $T = (A, \succ)$ of order 36 such that $BA(T) \cap BP(T) = \emptyset$ and $BA(T) \cup BP(T) = A$.*

Proof. We begin with the formal description of the tournament $T = (A, \succ)$. We define $A := \{v^i_{j,k} \colon i \in \{0,1,2,3\}, j,k \in \{1,2,3\}\}$ and let $\succ$ be given as follows. To simplify notation, we set $3 + 1 := 1$ and $1 - 1 := 3$ and write $*$ when universally quantifying over all possible indices.

$$\begin{aligned}
&\forall i \in \{0,1,2,3\}, j,k \in \{1,2,3\} \colon && v^i_{j,k} \succ v^i_{j,k+1}\\
&\forall i \in \{0,1,2,3\}, j \in \{1,2,3\} \colon && v^i_{j,*} \succ v^i_{j+1,*}\\
&\forall i \in \{1,2,3\} \colon && v^0_{i,*} \succ v^i_{*,*}\\
&\forall i,j \in \{1,2,3\}, k \in \{j-1, j+1\} \colon && v^0_{i+1,j} \succ v^i_{j,*} \succ v^0_{i+1,k}\\
&\forall i,j \in \{1,2,3\}, k \in \{j-1, j+1\} \colon && v^0_{i-1,j} \succ v^i_{*,j} \succ v^0_{i-1,k}\\
&\forall i,j \in \{1,2,3\}, k \in \{j, j+1\} \colon && v^{i+1}_{j-1,*} \succ v^i_{*,j} \succ v^{i+1}_{k,*}
\end{aligned}$$

The following subsets of alternatives will be of particular interest. For all $i \in \{0,1,2,3\}$ and $j \in \{1,2,3\}$, we write

$$\Delta^i := \{v^i_{j,k} : j,k \in \{1,2,3\}\} \quad \text{and} \quad \Delta^i_j := \{v^i_{j,k} \colon k \in \{1,2,3\}\}.$$

The structure of T is visualized in Figure 1.

Next, we compute $BP(T)$. It is easy to see that any alternative in Δ^0 dominates and is dominated by four alternatives in Δ^0, respectively. Moreover, any $v \in A \setminus \Delta^0$ dominates precisely four alternatives in Δ^0, and is dominated by five alternatives in Δ^0. For example, $v^1_{1,3}$ is dominated by the three alternatives in Δ^0_1, by $v^0_{2,1}$, and by $v^0_{3,3}$. Therefore, the optimal distribution p_T assigns probability 1/9 to the alternatives in Δ^0 and probability 0 to all other alternatives. Hence, $BP(T) = \Delta^0$.

To simplify the computation of $BA(T)$, we first observe that T admits the following four graph automorphisms: the map $\varphi \colon A \to A$, defined for all $i,j,k \in \{1,2,3\}$ by

$$\varphi(v^0_{i,j}) := v^0_{i+1,j} \quad \text{and} \quad \varphi(v^i_{j,k}) := v^{i+1}_{j,k},$$

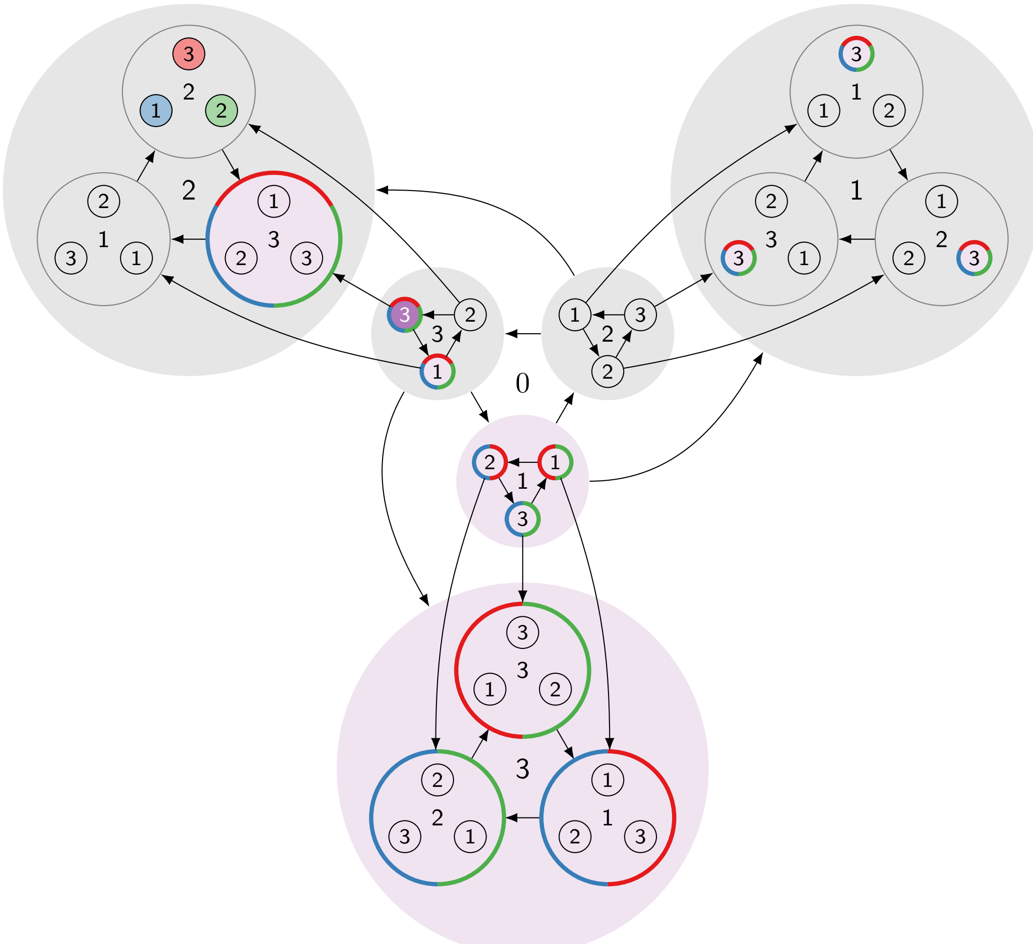

Figure 1: Incomplete depiction of tournament $T = (A, \succ)$ with $BA(T) \cap BP(T) = \emptyset$. The 36 alternatives $\{v^i_{j,k} \colon i \in \{0,1,2,3\}, j,k \in \{1,2,3\}\}$ are labeled such that i denotes the outmost triangle Δ^i, j the triangle Δ^i_j within i, and k the alternative within triangle Δ^i_j. The automorphism group of the tournament admits only two orbits: $\Delta^0 = \{v^0_{j,k} \colon j,k \in \{1,2,3\}\}$ and $A \setminus \Delta^0$. $BP(T) = \Delta^0$ and $BA(T) = A \setminus \Delta^0$. The dominion of the purple alternative $v^0_{3,3}$ is highlighted in light purple. Every transitive subtournament of this dominion is dominated by one of the three alternatives in Δ^2_2, highlighted in blue, green, and red. The dominions of these alternatives are indicated by respectively colored boundaries.

and for any $h \in \{1,2,3\}$ the map $\psi_h\colon A \to A$, defined for all $i,j,k \in \{1,2,3\}$ by

$$\psi_h(v^0_{i,j}) \coloneqq \begin{cases} v^0_{i,j+1}, & \text{if } i = h,\\ v^0_{i,j}, & \text{if } i \neq h, \end{cases} \quad \text{and} \quad \psi_h(v^i_{j,k}) \coloneqq \begin{cases} v^i_{j,k}, & \text{if } i = h,\\ v^i_{j+1,k}, & \text{if } i = h-1, \text{ and}\\ v^i_{j,k+1}, & \text{if } i = h+1. \end{cases}$$

Intuitively, φ rotates the entire tournament counterclockwise by 120°. ψ_h rotates the alternatives in the small triangles Δ^0_h and Δ^{h+1}_* counterclockwise, and the entire small triangles in Δ^{h-1} clockwise. We skip the details of checking that these four maps are indeed graph automorphisms.

It is straightforward to see that for any pair $x, y \in \Delta^0$, an appropriate concatenation of these four automorphisms maps x to y. In fact, the automorphism group induced by these maps admits only two equivalence classes (orbits): Δ^0 and $A \setminus \Delta^0$. In order to prove that $BA(T) \cap BP(T) = \emptyset$, it thus suffices to show that an arbitrary element of $BP(T) = \Delta^0$ fails to be contained in $BA(T)$. To this end, consider alternative $v^0_{3,3}$, highlighted in dark purple in Figure 1. The dominion of this alternative is highlighted in light purple and consists of $v^0_{3,1}$, $v^1_{1,3}$, $v^1_{2,3}$, $v^1_{3,3}$, Δ^2_3, Δ^0_1, and Δ^3. We claim that every transitive subset B of this dominion is dominated by one of the three alternatives in Δ^2_2, highlighted in blue, green, and red in Figure 1. The dominions of these alternatives are indicated by respectively colored boundaries. First, observe that each of the alternatives in Δ^2_2 dominates Δ^2_3, $v^0_{3,1}$, $v^0_{3,3}$, $v^1_{1,3}$, $v^1_{2,3}$, and $v^1_{3,3}$. We can thus focus on the alternatives in Δ^0_1 and Δ^3. Since both of these sets represent triangles (of subtriangles), B can only contain at most two elements of Δ^0_1 and elements of at most two of the subtriangles Δ^3_1, Δ^3_2, and Δ^3_3 of Δ^3. Additionally note that all edges between Δ^0_1 and Δ^3 that are not depicted in Figure 1 go from Δ^3 to Δ^0_1. Thus, for any $j \in \{1,2,3\}$, the triples $v^0_{1,j}$, $v^0_{1,j+1}$, Δ^3_{j+1} and $v^0_{1,j}$, Δ^3_j, Δ^3_{j+1} represent further triangles. From this, it is straightforward to verify that all transitive subsets of $\Delta^0_1 \cup \Delta^3$ are subsets of $\{v^0_{1,2}, v^0_{1,3}\} \cup \Delta^3_1 \cup \Delta^3_2$, or $\{v^0_{1,1}, v^0_{1,3}\} \cup \Delta^3_2 \cup \Delta^3_3$, or $\{v^0_{1,1}, v^0_{1,2}\} \cup \Delta^3_1 \cup \Delta^3_3$. The first set is dominated by $v^2_{2,1}$, the second by $v^2_{2,2}$, and the third by $v^2_{2,3}$, as can be seen by the boundary colors in Figure 1. We conclude that $v^0_{3,3} \notin BA(T)$ and, by the automorphism argument from above, $\Delta^0 \cap BA(T) = \emptyset$.

Since $BA(T) \neq \emptyset$, some alternative in $A \setminus \Delta^0$ has to be contained in $BA(T)$. Moreover, since for any pair $x, y \in A \setminus \Delta^0$, an appropriate concatenation of the four automorphisms described above maps x to y, we obtain that $BA(T) = A \setminus \Delta^0$. □

It can be shown that the theorem holds independently of how the edges in each of the small outer triangles Δ^*_* are chosen. In particular, they need not form cycles. While the tournament described in Theorem 1 was found manually, we have additionally verified the correctness of the statement using a computer. Remarkably, the degree (or Copeland score) of each alternative in the bipartisan set is 19, while that of each alternative in the Banks set is 17. Hence, in this tournament, the alternatives in the Banks set are precisely those with below-average degrees.

The two remaining open questions concerning the inclusion relationships between commonly studied tournament solutions are whether the tournament equilibrium set is contained in the minimal extending set and whether the minimal extending set is contained in the minimal covering set (see Brandt et al., 2017). Another open problem is whether the Banks set satisfies Kelly-strategyproofness (see Brandt et al., 2016).

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