Strategy-Proofness and Proportionality in Party-Approval Multi-Winner Elections

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Abstract

In party-approval multi-winner elections, the goal is to allocate the seats of a fixed-size committee to parties based on approval ballots of the voters over the parties. In particular, each voter can approve multiple parties and each party can be assigned multiple seats. Two central requirements in this setting are proportional representation and strategy-proofness. Intuitively, proportional representation requires that every sufficiently large group of voters with similar preferences is represented by some committee members. On the other hand, strategy-proofness demands that no voter can benefit by misreporting her true preferences. We show that these two basic axioms are incompatible for anonymous party-approval multi-winner voting rules, thus proving a far-reaching impossibility theorem. The proof of this result is obtained by formulating the problem in propositional logic and then letting a SAT solver show that the formula is unsatisfiable. Additionally, we demonstrate how to circumvent this impossibility by considering a weakening of strategy-proofness, which requires that only voters who do not approve any elected party cannot manipulate. While most common voting rules fail even this weak notion of strategy-proofness, we characterize Chamberlin-Courant approval voting within the class of Thiele rules based on this strategy-proofness notion, thus giving an escape route to our impossibility.

1 Introduction

One of the central problems in multi-agent systems is collective decision making: given the preferences of multiple agents over a set of alternatives, a joint decision has to be made (see, e.g., Nisan et al., 2007; Shoham and Leyton-Brown, 2009; Brandt et al., 2016). While classic literature for this problem focuses on the case of choosing a single alternative as the winner, there is also a wide range of scenarios where a whole set of winners needs to be elected. For instance, this is the case in parliamentary elections, where the seats of a parliament are assigned to parties based on the voters' preferences. In the literature, parliamentary elections are studied under the name *apportionment* and a crucial assumption for their analysis is that voters are only allowed to vote for a single party (Balinski and Young, 1982; Pukelsheim, 2014). However, this assumption has recently been criticized because of its lack of flexibility and expressiveness (Brill et al., 2018, 2020). Following Brill et al. (2020), we thus study *party-approval (P-APP) elections* in this paper. In this setting, the parliament, or more formally a multiset of fixed size, is elected based on the approval ballots of the voters, i.e., each voter now reports which parties she finds acceptable instead of only her most preferred one. More generally, P-APP elections distribute k copies of an indivisible resource to some alternatives based on the voters' preferences. This setting thus also has a number of applications other than parliamentary elections, such as recommender systems and collaborative filtering. The task might be, for example, to choose a fixed number of movies to present to users of a streaming platform proportionally taking into account the users' preferences over movie genres. However, we use the language of parliamentary elections throughout the paper because it provides the most intuitive terminology.

Two central desiderata for P-APP elections are *proportional representation* and *strategy-proofness*. Roughly, proportional representation requires that the chosen committee should proportionally reflect the voters' intentions. Or, in other words, each sufficiently large group of voters that agrees on a party should be represented by some party in the committee. Indeed, Brill et al. (2020) have shown that in P-APP elections core-stable committees, i.e., committees satsifying one of the highest degrees of proportionality, are guaranteed to exist. On the other hand, strategy-proofness requires that no voter can benefit by lying about her true preferences. This axiom is not well-understood for P-APP elections yet and we thus investigate in this paper the trade-off between strategy-proofness and proportional representation.

In contrast to P-APP elections, in *approval-based committee (ABC) elections* both proportional representation and strategy-proofness have received significant attention. In ABC elections, voters report their approval preferences over individual candidates instead of parties and each candidate can receive at most one seat in the committee, whereas parties can have multiple seats in P-APP elections (see Lackner and Skowron (2021) for an excellent survey on ABC voting). Consequently, ABC elections are closely connected to P-APP elections: every ABC voting rule is also a P-APP voting rule if there are less seats in the committee than parties available and, conversely, we can also model P-APP elections with ABC elections by simply replacing a party with its members, i.e., adding for each party sufficiently many clones. More formally, P-APP elections can be seen as a restriction on the domain of possible preferences in ABC voting: for each party, voters can either approve all or none of its members.

Moreover, for ABC elections, there is a large amount of work on proportional representation (e.g., Aziz et al., 2017; Sánchez-Fernández et al., 2017; Brill et al., 2022; Skowron, 2021; Peters and Skowron, 2020) and strategy-proofness (e.g., Aziz et al., 2015; Peters, 2018; Kluiving et al., 2020; Duddy, 2014; Lackner and Skowron, 2018; Botan, 2021). For instance, Aziz et al. (2017) investigate ABC voting rules with respect to a proportionality axiom called justified representation, and Sánchez-Fernández et al. (2017) analyze the stronger concept of proportional justified representation. The main message from these results is that there are few ABC voting rules that guarantee strong representation axioms. On the other hand, the results for strategy-proofness are more negative: Aziz et al. (2015) and Janson (2016) show for specific voting rules that they fail strategy-proofness and Lackner and Skowron (2018) characterize approval voting, which chooses the candidates that are approved by the most voters, essentially as the only ABC voting rule that satisfies strategy-proofness. These negative results culminated in a strong impossibility by Peters (2018) who shows that even minimal notions for proportional representation are in conflict with strategy-proofness for weakly efficient ABC voting rules (see also (Peters, 2021) for an errandum and (Duddy, 2014; Kluiving et al., 2020) for related results). Thus, our pursuit of strategy-proofness and proportional representation in P-APP elections can also be seen as attempt to circumvent these negative results for ABC voting.

Even more, there are hints that strategy-proofness and proportional representation might be compatible for P-APP elections because these axioms can be simultaneously satisfied in related settings. For instance, in apportionment—where each voter is only allowed to vote for a single party instead of a set of parties—it is known that strategy-proofness and proportional representation are compatible (Balinski and Young, 1982; Pukelsheim, 2014). This positive results also holds in *fair mixing*, where a divisible resource, such as money or time, is allocated to the parties based on the approval ballots of the voters (Bogomolnaia et al., 2005; Duddy, 2015; Aziz et al., 2019a; Brandl et al., 2021). In particular, Aziz et al. (2019a) show that the *conditional utilitarian rule* satisfies strategy-proofness and proportional representation in this setting. Since P-APP elections are a discrete special case of fair mixing in which we distribute k copies of an indivisible resource instead of a divisible resource, it seems reasonable to conjecture that there are also P-APP voting rules that satisfy strategy-proofness and proportional representation simultaneously.

We show that this unfortunately is not the case: strategy-proofness conflicts even with minimal notions of proportional representation in the P-APP setting. Namely, we formalize proportional representation with the concepts of *weak representation* and *weak proportional representation.* The latter axiom requires that a party has at least ℓ seats in a committee of size k if it is uniquely approved by at least an $\frac{\ell}{k}$ fraction of the voters, whereas weak representation only concerns itself with the special case that $\ell = 1$. Based on these two axioms, we show two impossibilities (where k denotes the committee size, m the number of parties, and n the number of voters):

Theorem 1. *No P-APP voting rule simultaneously satisfies anonymity, weak representation, and strategy-proofness if $k \geq 3$, $m \geq k+1$, and $2k$ divides n.*

Theorem 2. *No P-APP voting rule simultaneously satisfies anonymity, weak proportional representation, and strategy-proofness if $k \geq 3$, $m \geq 4$, and $2k$ divides n.*

Our first result is closely connected to the impossibility by Peters (2018) stating that no weakly efficient ABC voting rule satisfies strategy-proofness and weak representation. While we use stronger notions of strategy-proofness and proportional representation than Peters (2018) and additionally require anonymity, our result does not need to employ efficiency and the P-APP setting is more flexible than the ABC setting. This is, for instance, highlighted by the fact that Theorem 1 immediately implies that also no ABC voting rule satisfies the given axioms. The main draw back of this result is the requirement that there must be more parties than seats in the committee. While this assumption is true for many applications inspired from ABC voting, this is not the case in our initial example of parliamentary elections. To also cover such scenarios, we extend in our second theorem the impossibility to the case that $k \geq m$ by requiring weak proportional representation instead of weak representation.

Both impossibility results are proven by a computer-aided approach based on SAT solving, which has recently led to a number of sweeping impossibility results (Brandl et al., 2018, 2021; Brandt et al., 2022, e.g.,). This approach often even allows to extract a human-readable proof by considering miniminal unsatisfiable sets (MUSes). However, the MUSes which we are confronted with are of enormous size: the smallest MUS which we found so far requires 635 profiles and over 20.000 strategy-proofness applications, which is by far the largest computer-generated proof found in social choice theory (currently, the largest proof is due to Brandl et al. (2021) requiring 386 profiles). Unfortunately, the sheer size of the proof makes the extraction and verification of the proof very tedious if not impossible for humans. We thus offer three ways to verify our results: *(i)* we make our program for producing the impossibility publicly available Delemazure et al. (2021); *(ii)* we present a human-readable proof of a slightly weaker statement which additionally requires Pareto-optimality; and *(iii)* we verify the correctness of our impossibilities with the highly trustworthy interactive theorem prover Isabelle (Nipkow et al., 2002). In particular, the verification with Isabelle can be considered a full substitute of a human-readable proof because Isabelle allows us to formulate the setting and our axioms in higher order logic and then shows the impossibilities by only applying verified inference steps.

Finally, we also demonstrate an escape route to these strong impossibilities by investigating a weakening of strategy-proofness which only requires that voters who do not approve any party in the elected committee cannot benefit by manipulating. Perhaps surprisingly, almost all commonly considered committee voting rules fail even this weak strategy-proofness notion. On the other hand, we prove that Chamberlin-Courant approval voting (CCAV) always satisfies this notion of strategy-proofness for unrepresented voters and can even characterize CCAV within all Thiele rules based on this axiom. Since this rule also satisfies weak representation, this result offers an attractive escape route to Theorem 1.

2 Preliminaries

Let $N = \{1, \dots, n\}$ denote a set of n voters and $\mathcal{P} = \{a, b, c, \dots\}$ denote a set of m parties. All voters $i \in N$ are assumed to have a *dichotomous preference relation* over the parties, i.e., voters partition the parties into two sets, one containing the acceptable parties and one containing the unacceptable ones. The *approval ballot* $\emptyset \subsetneq A_i \subseteq A$ of a voter i contains the parties she finds acceptable. When writing down explicit approval ballots, we usually omit the brackets and commas between the parties, e.g., $A_i = xy$ means that voter i approves parties x and y. Next, let $\mathcal{A}$ denote the set of all possible approval ballots. An *approval profile* A is the collection of the approval ballots of all voters, i.e., A is an element of $\mathcal{A}^n$.

Given an approval profile A, the goal in *party-approval multi-winner (P-APP) elections* is to assign a fixed number of seats to the parties. We call such an outcome a *committee*, which is formally a multiset of parties $W : \mathcal{P} \to \mathbb{N}$. In particular, each party can be chosen multiple times and we denote by $W(x)$ the number of seats assigned to party x. We extend this notation also to sets of parties $X \subseteq \mathcal{P}$ by defining $W(X) = \sum_{x \in X} W(x)$. Furthermore, we denote specific committees by square brackets, e.g., $[a, a, b]$ is the committee which contains party a twice and party b once. The size of a committee W is the total number of allocated seats, i.e., $|W| = W(\mathcal{P})$, and we define $\mathcal{W}_k$ as the set of all committees of size k.

For finding the winning committee for an approval profile A, we use *party-approval committee (P-APP) voting rules*. Formally, such a rule is a function f which takes an approval profile $A \in \mathcal{A}^n$ and a target committee size k as input and returns a committee $W \in \mathcal{W}_k$. In particular, P-APP voting rules are resolute, i.e., there is always a single winning committee. Since we will often need to reason about the number of seats assigned to a party by a P-APP voting rule f, we define $f(A, k, x)$ as the number of seats assigned to party x by f for the profile A when choosing a committee of size k. Just as for committees, we extend this notion also to sets by defining $f(A, k, X) = \sum_{x \in X} f(A, k, x)$.

Two well-known properties of voting rules are anonymity and Pareto-optimality. Informally, anonymity requires that all voters are treated equally. On a formal level, we say a P-APP voting rule f is *anonymous* if $f(A, k) = f(A', k)$ for all approval profiles A, A' and all committee sizes $k \in \mathbb{N}$ such that there is a permutation $\pi : N \to N$ with $A'_i = A_{\pi(i)}$. For anonymous P-APP voting rules, we can view approval profiles as multisets of preferences instead of tuples as the order of the voters does not matter. Next, *Pareto-optimality* is a well-known efficiency property based on the notion of Pareto-dominance. A party x *Pareto-dominates* another party y in an approval profile A if $y \in A_i$ implies $x \in A_i$ for all $i \in N$ and there is a voter $i \in N$ with $x \in A_i$, $y \notin A_i$. Then, *Pareto-optimality* requires of a P-APP voting rule f that $f(A, k, y) = 0$ for all approval profiles A, committee sizes k, and parties y that are Pareto-dominated in A.

2.1 Proportional Representation

One of the central desiderata in committee elections is that the chosen committee should proportionally represent the voters' preferences. For formalizing this idea, the quota $\frac{n}{k}$ plays a central role because, intuitively, a group of voters $G \subseteq N$ with $|G| \geq \frac{n}{k}$ that agrees on a party deserves representation in a committee of size k. This simple idea leads to the notion of justified representation introduced by Aziz et al. (2017): a P-APP voting rule f satisfies *justified representation (JR)* if $f(A, k, \bigcup_{i\in G} A_i) \geq 1$ for every profile A, committee size k, and group of voters G such that $|G| \geq \frac{n}{k}$ and $\bigcap_{i\in G} A_i \neq \emptyset$. While justified representation is already a rather weak representation axiom, we will consider in this paper a further weakening which we call weak representation. Intuitively, this axiom additionally requires that all voters in G commonly approve the same single party x.

Definition 1 (Weak Representation)**.** A P-APP voting rule f satisfies *weak representation (WR)* if $f(A, k, x) \geq 1$ for every profile A, committee size k, party x, and group of voters G such that $|G| \geq \frac{n}{k}$ and $A_i = x$ for all $i \in G$.

Observe that weak representation can be easily satisfied if we have more seats in the committee than parties; we can then simply assign at least one seat to every party. This, however, clearly contradicts the idea of proportional representation because even if all voters uniquely approve the same party, the other parties get seats. Sánchez-Fernández et al. (2017) therefore introduced the concept of *proportional justified representation (PJR)* which requires of a P-APP voting rule f that $f(A, k, \bigcup_{i\in G} A_i) \geq \ell$ for all approval profiles A, committee sizes k, and groups of voters G such that $|G| \geq \ell\frac{n}{k}$ and $\bigcap_{i\in G} A_i \neq \emptyset$. Or, in other words, if an $\frac{\ell}{k}$ fraction of the voters agrees on some party, this group should be represented by at least ℓ seats in the committee. Analogous to weak representation, we will consider a weakening of proportional justified representation which requires that all voters in G commonly approve the same single party x.

Definition 2 (Weak Proportional Representation)**.** A P-APP voting rule f satisfies weak proportional representation (WPR) if $f(A, k, x) \geq \ell$ for every profile A, committee size k, party x, and group of voters G such that $|G| \geq \ell\frac{n}{k}$ and $A_i = x$ for all $i \in G$.

2.2 Strategy-proofness

The main focus of this paper is to study P-APP voting rules with respect to strategy-proofness. Intuitively, strategy-proofness requires that a voter cannot benefit by lying about her true preferences. Consequently, if a P-APP voting rule fails strategy-proofness, we cannot expect the voters to submit their true preferences, which may lead to socially undesirable outcomes.

Definition 3 (Strategy-Proofness (SP))**.** A P-APP voting rule f is *strategy-proof* if $f(A, k, A_i) \geq f(A', k, A_i)$ for all approval profiles A, A', committee sizes k, and voters $i \in N$ such that $A_j = A'_j$ for all $j \in N \setminus \{i\}$. We call f *manipulable* if it is not *strategy-proof*.

Less formally, strategy-proofness requires that no voter can increase the number of the seats assigned to her approved parties by lying about her true preferences. The motivation for this notion of strategy-proofness stems from the assumption that voters are indifferent between their approved parties; then, only the number of seats assigned to these parties matters and voters will report the preferences that maximize this number. While rather strong, this cardinality-based strategy-proofness notion is commonly used in ABC voting (see, e.g., Aziz et al., 2015; Lackner and Skowron, 2018; Botan, 2021) and equivalent notions are used in fair mixing and randomized social choice (Gibbard, 1977; Bogomolnaia et al., 2005; Aziz et al., 2019b).

Since we will show that strategy-proofness conflicts with even minimal representation properties, we consider also a weakening of this axiom to find more positive results: *strategy-proofness for unrepresented voters* requires that only voters without representation in the committee cannot manipulate.

Definition 4 (Strategy-Proofness for Unrepresented Voters)**.** A P-APP voting rule f is *strategy-proof for unrepresented voters* if $f(A, k, A_i) \geq f(A', k, A_i)$ for all approval profiles A, A', committee sizes k, and voters $i \in N$ such that $A_j = A'_j$ for all $j \in N \setminus \{i\}$ and $f(A, k, A_i) = 0$.

We believe that strategy-proofness for unrepresented voters is an important weakening of strategy-proofness because voters without any representation in the committee have the strongest incentive to manipulate without any risk of getting worse off. Indeed, voters who have some representation in the committee may already be content with the outcome since the benefit of approving more members of the committee is less straightforward than that of being represented at all. Moreover, voters with representation may also be more cautious to manipulate due to the fear of losing their representation when misstating their preferences. Both arguments do not apply to unrepresented voters, which shows why these voters might be particularly prone to manipulate.

2.3 P-APP voting rules

Finally, we introduce three classes of P-APP voting rules. On the one side, these rules will be used to illustrate our results on strategy-proofness and, on the other side, we investigate these classes with respect to strategy-proofness for unrepresented voters in Section 4. In more detail, we introduce subsequently the classes of Thiele rules, sequential Thiele rules, and divisor methods based on majoritarian portioning. Note that, even though we define P-APP voting rules for a fixed numbers of voters n and parties m, all subsequent rules are independent of such details.

Thiele rules. Thiele rules are maybe the most well-studied class of rules in the ABC setting. Introduced by Thiele (1895), these rules optimize a score of the winning committee w.r.t. a weight vector w: each voter assigns $\sum_{i=1}^{\ell} w_\ell$ points to a committee in which she approves ℓ members and the Thiele rule defined by w then chooses the committee with maximal total score. Formally, a *w-Thiele rule* f is defined by a non-increasing and non-negative weight vector $w = (w_1, w_2, \dots)$ and chooses for each committee size k the committee $W \in \mathcal{W}_k$ that maximizes the score $s_w(W, A) = \sum_{i \in N} \sum_{j=1}^{W(A_i)} w_j$. Throughout the paper, we assume that $w_1 = 1$; this is without loss of generality since Thiele rules are invariant to scaling. There are many well-known Thiele rules such as *approval voting (AV)* (defined by $w = (1, 1, 1, \dots)$), *proportional approval voting (PAV)* (defined by $w = (1, \frac{1}{2}, \frac{1}{3}, \dots)$), and *Chamberlin-Courant approval voting (CCAV)* (defined by $w = (1, 0, 0, \dots)$).

Sequential Thiele rules. Sequential Thiele rules are closely related to Thiele rules: instead of optimizing the score of the committee, these rules proceed in rounds and greedily choose in each iteration the party that increases the score of the committee the most. For formally defining these rules, let W^x denote the committee derived from another committee W (with one seat less) by assigning an additional seat to party x. Then, a *sequential w-Thiele rule* is defined by a non-increasing and non-negative weight vector $w = (w_1, w_2, \dots)$ and assigns the i-th seat to the party x that maximizes $s(A, W^x_{i-1})$ (where W_{i-1} denotes the committee computed by the previous $i-1$ rounds). After k rounds this process stops and the final committee W_k is returned. Two important examples of sequential Thiele rules are *sequential proportional approval voting (seqPAV)*, (defined by $w = (1, \frac{1}{2}, \frac{1}{3}, \dots))$ and *sequential Chamberlin-Courant approval voting (seqCCAV)* (defined by $w = (1, 0, 0, \dots)$). Also, note that AV and its sequential version are equivalent.

Divisor methods based on majoritarian portioning. Brill et al. (2020) introduced the concept of composite P-APP voting rules. These rules first apply a *portioning method*, which take the voters' approval ballots as input and returns a weight for each party. As second step, an *apportionment method* assigns the seats in the committee based on these weights. One important special case of this concept are *divisor methods based on majoritarian portioning* because many of these rules satisfy strong representation axioms (Brill et al., 2020). These methods first apply *majoritarian portioning* to compute the weights w_x. Majoritarian portioning works in rounds and in each round, we determine the party x that is approved by the most voters. Then, we set w_x to the number of voters who approve x and remove all corresponding voters from the profile. Thereafter, the next round repeats this process with the reduced profile until no voters are left. Finally, we set $w_x = 0$ for all parties x that have no weight after all voters were removed. After the majoritarian portioning, we use a divisor method to allocate the seats to the parties based on the weights w_x. *Divisor methods* are defined by a monotone function $g : \mathbb{N}_0 \to \mathbb{R}_{>0}$ and work in rounds: in the i-th round, the next seat is assigned to the party x that maximizes $\frac{w_x}{g(t^x_{i-1})}$, where w_x is the weight computed in the portioning step and t^x_{i-1} is the number of seats allocated to x in the previous $i-1$ rounds. Examples of well-known divisor methods are Jefferson's method (where $g(x) = x + 1$) and Hill's method (where $g(x) = \sqrt{x(x + 1)}$) (Balinski and Young, 2001, p. 99). However, even AV can be seen as a divisor method based on majoritarian portioning by choosing $g(x) = 1$.

Note that all P-APP voting rules we consider are in principle irresolute, i.e., they may declare multiple committees as tied winners of an election. Since we investigate in this paper resolute voting rules, we assume that ties are always broken lexicographically.[1] Formally, this means that for every $k \in \mathcal{N}$, there is a linear tie-breaking order $\succ_k$ on the committees $W \in \mathcal{W}_k$ and, if a P-APP voting rule f declares multiple committees as tied winners, we choose the best one according to $\succ_k$. Similarly, if a sequential Thiele rule, majoritarian portioning, or a divisor methods is tied between multiple parties in a step, the choice is made according to $\succ_1$. The assumption of lexicographic tie-breaking is standard in the literature on strategy-proofness (e.g, Bartholdi, III et al., 1989; Faliszewski et al., 2010; Aziz et al., 2015).

3 Impossibility Results

In this section, we discuss the incompatibility of strategy-proofness and proportional representation for P-APP voting rules by proving two sweeping impossibility theorems. As first result, we show that no anonymous P-APP voting rule satisfies strategy-proofness and weak representation.

[1]More specifically, our results only require to break ties between two committees at any time. Therefore, we can also use any intransitive tie-breaking order, which is more general than lexicographic tie-breaking.

Theorem 1. *No P-APP voting rule simultaneously satisfies anonymity, weak representation, and strategy-proofness if $k \geq 3$, $m \geq k+1$, and $2k$ divides n.*

This result shows that strategy-proofness and weak representation are incompatible for P-APP elections. Hence, this model does not allow to escape the incompatibility of the given axioms for the ABC setting (Peters, 2018). Even more, Theorem 1 emphasizes again the incompatibility of strategy-proofness and representation axioms since it implies that no ABC voting rule satisfies anonymity, weak representation, and strategy-proofness, thus demonstrating the first impossibility without efficiency for the ABC setting.

Next, note that Theorem 1 requires specific committee sizes, numbers of parties, and numbers of voters. In particular, two constraints might seem prohibitive: the assumption that $2k$ divides n and the assumption that $m \geq k+1$. The first assumption is merely a technical one as we—just as other authors (Peters, 2018; Kluiving et al., 2020)—could not find an argument to generalize the impossibility to arbitrary values of n. However, almost all commonly studied P-APP voting rules satisfy independence of indifferent voters, which requires that adding voters who approve all parties does not affect the outcome. For voting rules satisfying this axiom, we can extend the impossibility by simply adding voters who approve all parties, which shows that Theorem 1 applies to almost all common P-APP voting rules if $n \geq 2k$.

On the other side, the assumption that $m \geq k+1$ is crucial for Theorem 1: if $k \geq m$, every rule that always returns the same committee W with $W(x) \geq 1$ for all parties x satisfies the considered axioms. Moreover, while the requirement that there are less seats in the committee than parties available applies for a number of applications inspired from ABC voting (e.g., recommender systems), this is clearly not the case for parliamentary elections. However, our second impossibility shows that there are still no attractive strategy-proof P-APP voting rules if $k \geq m$ because we can generalize Theorem 1 to arbitrary $m \geq 4$ by requiring weak proportional representation instead of weak representation.

Theorem 2. *No P-APP voting rule simultaneously satisfies anonymity, weak proportional representation, and strategy-proofness if $k \geq 3$, $m \geq 4$, and $2k$ divides n.*

In summary, Theorems 1 and 2 show that strategy-proofness is incompatible with minimal notions of proportional representation in the P-APP setting.

We believe that also the proofs of these results are of interest: for showing Theorems 1 and 2, we rely on a computer-aided approach based on SAT solving. In the realm of social choice, this technique was pioneered by Tang and Lin (2009) and has by now been used to prove a wide variety of results (e.g., Brandt and Geist, 2014; Brandl et al., 2018; Peters, 2018; Endriss, 2020; Brandt et al., 2022; Brandl et al., 2021); Geist and Peters (2017) provide an excellent overview of this technique. In essence, all of these computer-generated proofs follow the same steps: first, the problem of finding a voting rule for fixed numbers of voters and parties that satisfies certain axioms is encoded as logical formulas and then a SAT solver is used to show that the formula is unsatisfiable. This means that no voting rule satisfies the given axioms, thus proving an impossibility theorem for fixed values of m, n, and k. As second step, this impossibility is verified, either by means of a human-readable proof or by using an interactive theorem prover. Finally, the impossibility, which is so far only shown for fixed values of k, m, and n, is lifted to larger parameters by inductive arguments.

We follow this approach and thus explain first in Section 3.1 how we encode the problem of finding an anonymous P-APP voting rules that satisfies strategy-proofness and weak representation. This encoding results in an unsatisfiable formula for committees of size $k = 3$, $m = 4$ parties, and

$n = 6$ voters, thus proving that there is no P-APP voting rule that satisfies our axioms for these parameters. Hence, we discuss in Section 3.2 the base case and its verification in more detail and generalize the impossibility from fixed values of k, m, and n to almost arbitrary committee sizes and numbers of parties and voters in Section 3.3. Finally, we discuss several remarks in Section 3.4.

3.1 Computer-Aided Theorem Proving

In this section, we explain our computer-aided theorem proving approach; for a more detailed overview of this technique, we refer the reader to the survey by Geist and Peters (2017). The first point to notice for computer-aided theorem proving is that for a fixed committee size k and fixed numbers of voters n and parties m, there is a finite but very large number of P-APP voting rules. More specifically, there are $\binom{k+m-1}{k}$ possible committees, $2^m - 1$ possible preferences and thus $(2^m - 1)^n$ possible approval profiles.[2] Hence, there are a total of $\binom{k+m-1}{k}^{(2^m-1)^n}$ P-APP voting rules and we could, at least theoretically, check for every function whether it satisfies the given axioms. However, for our choice of parameters $k = 3$, $m = 4$, and $n = 6$, there are roughly 6.2×10^{14819544} such functions, which is far too much for enumeration.

Thus, instead of enumerating all possible P-APP voting rules, we formulate constraints that enforce P-APP rules to satisfy anonymity, weak representation, and strategy-proofness. In more detail, we will construct a logical formula such that any P-APP voting rule satisfying our conditions corresponds to a satisfying assignment of the formula and vice versa. By showing that there is no satisfying assignment for the logical formula, it hence follows that there is also no P-APP voting rule which meets our conditions. Moreover, there are computer programs, so-called SAT solvers that can efficiently decide whether formulas with millions of constraints are satisfiable. Hence, we can prove our impossibility for fixed numbers of k, m, and n by constructing the logical formula and letting a SAT solver show that it is unsatisfiable.

It remains to explain how we construct the logical formula. The central idea for encoding a P-APP voting rule that satisfies certain properties is rather simple: we will use variables of the form $x_{A,W}$ which indicate that the encoded rule f chooses the committee W for the profile A, i.e., $x_{A,W} = 1$ if and only if $f(A, k) = W$. However, when using this straightforward notation for all profiles A, the mere number of profiles becomes prohibitive because there are over 11.000.000 profiles for $k = 3$, $m = 4$, and $n = 6$. We thus apply several optimizations, similar to those suggested by Brandl et al. (2018). Firstly, we can use anonymity to reduce the number of considered profiles drastically. Essentially this axiom states that the order of the voters does not matter as the outcome cannot change by reordering the voters. Hence, anonymity allows us to view approval profiles as multisets of approval ballots instead of tuples. We do encode anonymity therefore not as constraints but by changing the domain of P-APP voting rules from ordered lists of preferences to multisets of preferences. As second optimization, we simply exclude approval profiles that are not needed for the proof. In particular, we impose three conditions on approval profiles: *(i)* no voter is allowed to approve all parties, *(ii)* no party can be approved by more than four voters, and *(iii)* the total number of approvals given by all voters does not exceed eleven. We call the domain of all anonymous profiles that satisfy these conditions $\mathcal{A}_{SAT} \subseteq \mathcal{A}^n$, and note that $\mathcal{A}_{SAT}$ only consists of 7.293 profiles. Moreover, it is straightforward that, if there is no P-APP voting rule satisfying strategy-proofness, weak representation, and anonymity on $\mathcal{A}_{SAT}$, there is also no such function on the full domain of approval profiles.

[2] For simplicity, we ignore anonymity here. If we only count anonymous approval profiles, the order of the voters does not matter and there are $\binom{2^m+n-2}{n}$ profiles.

After specifying our variables and the domain, we turn now to the constraints of our formula. First, we specify that the formula indeed encodes a P-APP function f, i.e., for every profile $A \in \mathcal{A}_{SAT}$, there is exactly one committee $W \in \mathcal{W}_k$ such that $x_{A,W} = 1$. For this, we add two types of clauses for every profile A: the first one specifies that at least one committee W is chosen for A and the second one that no more than one committee can be chosen.

$$\bigvee_{W \in \mathcal{W}_k} x_{A,W} \qquad \forall A \in \mathcal{A}_{SAT} \tag{1}$$

$$\bigwedge_{W,W' \in \mathcal{W}_k : W \neq W'} \neg x_{A,W} \vee \neg x_{A,W'} \qquad \forall A \in \mathcal{A}_{SAT} \tag{2}$$

Next, we encode that f satisfies weak representation. This axiom just forbids some outcomes: if at least $\frac{n}{k}$ voters uniquely approve a party x in a profile A, this party must have a seat in the committee. Hence, any committee W with $W(x) = 0$ is infeasible for A. This condition corresponds to the following formula, where $uAV(A, x)$ denotes how many voters uniquely approve party x in profile A.

$$\neg x_{A,W} \qquad \forall A \in \mathcal{A}_{SAT}, W \in \mathcal{W}_k : \exists x \in \mathcal{P} : uAV(A, x) \geq \frac{n}{k} \wedge W(x) = 0 \tag{3}$$

Note that in our code (Delemazure et al., 2021), we do not add these singleton clauses to the formula, but directly remove them from the formula. In more detail, instead of writing the constraint $\neg x_{A,W}$ for some profile A and committee W, we remove all constraints that contain $\neg x_{A,W}$ and we remove $x_{A,W}$ from constraint (1) for the profile A. This corresponds to the technique of unit propagation (see, e.g., Biere et al., 2009) and helps to reduce the size of the logical formula.

As last axiom, we need to encode strategy-proofness. We can express this axiom again by excluding some outcomes: if A and A' only differ in the preference of a single voter i, then $f(A, k, A_i) \geq f(A', k, A_i)$. Or, in other words, if f chooses committee W for A, it cannot choose a committee W' for A' with $W'(A_i) > W(A_i)$. Hence, we can encode strategy-proofness as follows.

$$\neg x_{A,W} \vee \neg x_{A',W'} \qquad \forall A, A' \in \mathcal{A}_{SAT} \text{ such that } W'(A_i) > W(A_i) \text{ and } A' \text{ differs in a single preference } A_i \text{ from } A \tag{4}$$

While constraints (1) to (4) are already unsatisfiable, we add two more types of constraints to speed up the SAT solving. Firstly, we show that the combination of strategy-proofness and weak representation restricts the set of feasible outcomes for many profiles even further.

Lemma 1. *Let f denote an anonymous P-APP voting rule that satisfies strategy-proofness and weak representation, and let k denote the target committee size. Moreover, we consider a profile A and define $X = \{x \in \mathcal{P} : \exists G \subseteq N : |G| \geq \frac{n}{k} \wedge \forall i \in G : A_i = x\}$ as the set of parties that are each uniquely approved by at least $\frac{n}{k}$ voters in A. If there is a set of voters $G = \{i_1, \dots i_\ell\} \subseteq N$ such that $|G| \geq \frac{n}{k}$, $A_{i_1} \subseteq A_{i_2} \subseteq \dots \subseteq A_{i_\ell}$, and $A_{i_1} \not\subseteq X$, then $f(A, k, A_{i_\ell}) \geq |A_{i_\ell} \cap X| + 1$.*

The proof of this lemma works by contradiction: if there was such a group G and a profile A such that $f(A, A_{i_\ell}) \leq |X \cap A_{i_\ell}|$, then the voters in G can manipulate by one after another deviating to approve only a single party $y \in A_{i_1} \setminus X$. The details can be found in the appendix.

Just as weak representation, this lemma shows that some committees are infeasible for some profiles, and we can therefore encode it in the same way. For instance, this lemma requires for the profile A shown below that at least one seat has to be given to a or b when choosing a committee of size $k = 3$.

$$A: \quad a \quad ab \quad b \quad c \quad cd \quad d$$

As a last point, we observe that some profiles are very symmetric with respect to the parties, which allows us to treat many outcomes analogously. For instance, in the profile A the committee $[a, b, c]$ is symmetric to the outcome $[a, b, d]$. Hence, if there was a P-APP voting rule satisfying anonymity, weak representation, strategy-proofness, and $f(A, k) = [a, b, c]$, we can also construct another rule f' satisfying the same constraints for which $f'(A, k) = [a, b, d]$. The next lemma formalizes this observation.

Lemma 2. *Let f denote an anonymous P-APP voting rule that satisfies strategy-proofness and weak representation and let $\tau : \mathcal{P} \mapsto \mathcal{P}$ denote a permutation of the parties. Then, the P-APP voting rule $f^\tau(A, k, x) = f(\tau(A), k, \tau(x))$, where $\tau(A)$ denotes the profile such that $\tau(x) \in \tau(A)_i$ if and only if $x \in A_i$ for all $i \in N$ and $x \in \mathcal{P}$, satisfies the same axioms as f.*

Note that this lemma is particularly helpful for the profile A shown above as there are only two canonical committees possible: Lemma 1 requires that at least one seat is assigned to a or b and that at least one seat is assigned to c or d, and all remaining committees are symmetric to either $[a, a, c]$ or $[a, b, c]$. Hence, if we show that no P-APP voting rule satisfies anonymity, weak representation, strategy-proofness, and $f(A, k) \in \{[a, a, c], [a, b, c]\}$, then Lemma 2 implies that no such rule exists in general. Thus, we add a last constraint to our formula stating that only these two outcomes are possible for the approval profile A shown above.

3.2 Induction Basis

When building the logical formula as described in the last section for the parameters $k = 3$, $m = 4$, and $n = 6$, the corresponding formula contains 7.044.614 constraints on 70.072 variables. A state-of-the-art SAT solvers, such as Glucose (Audemard and Simon, 2018), then shows that the formula is unsatisfiable in a few seconds. This means that no P-APP voting rule satisfies anonymity, strategy-proofness, and weak representation simultaneously for the considered parameters, thus resulting in the subsequent proposition. Note that this proposition also serves as base case for Theorem 2 since weak proportional representation implies weak representation.

Proposition 1. *There is no P-APP voting rule that satisfies anonymity, weak representation, and strategy-proofness if $k = 3$, $m = 4$, and $n = 6$.*

After proving such a statement by SAT solving, it is common to investigate the impossibility in more detail by considering minimal unsatisfiable sets (MUSes) of the logical formula. A MUS is a subset of the original constraints that is still unsatisfiable, but removing a single constraint of the MUS makes the problem satisfiable. The reason for considering MUSes is that they usually are significantly smaller than the original formula: even if the original problem contains millions of constraints, a MUS usually consists of a few hundred. Hence, it is often even possible to extract a human-readable proof from a MUS. Moreover, there are software tools for the extraction of a MUS from a logical formula, which often gives a convenient way for verifying the correctness of an impossibility.

Unfortunately, this approach does not work for Proposition 1: two state-of-the-art MUS extractors, namely haifamuc and muser2 (Belov and Marques-Silva, 2012; Nadel et al., 2014), require

```
weak_representation n 𝒫 k f =
 (is_anonymous_PAPP_function n 𝒫 k f ∧
 (∀A x. is_profile n 𝒫 A ∧ k * count A {x} ≥ n → count f(A) x ≥ 1))
```

Figure 1: The Isabelle/HOL code for weak representation. Note that we assume in our Isabelle implementation that P-APP voting rules are defined for fixed committee sizes k as this is sufficient to prove Proposition 1. The function `weak_representation n m k f` states the following: if f is an anonymous P-APP voting rule for n voters, the set of parties $\mathcal{P}$, and committees of size k, then it must hold for all approval profiles A (that are valid profiles for n voters and the parties $\mathcal{P}$) and all parties x that $f(A)$ assigns at least one seat to x if at least $\frac{n}{k}$ voters report the ballot x.

hours for finding a MUS at all and the corresponding MUSes are enormous. The smallest MUS we found so far uses over 20.000 constraints and 635 profiles, which is by far the largest MUS found in social choice theory.[3] Moreover, these numbers indicate that it would be very difficult or even impossible for humans to prove the result by hand. On the other hand, the size and the structure of the MUS make even the verification of the result challenging: any human-readable proof derived from the MUS would likely require several hundred cases and thousands of inference steps. Because of that, we do not include a human-readable proof of Proposition 1 and use a threefold approach for verifying our result instead: firstly, we made all our code publicly available, thus enabling other researchers to verify the correctness of our program and reproduce the impossibility (Delemazure et al., 2021).

Secondly, we have verified the correctness of Proposition 1 by formulating this impossibility with the interactive theorem prover Isabelle/HOL (Nipkow et al., 2002). Note that this approach has already been used for successfully verifying involved computer-aided theorems in social choice theory (see, e.g., Brandl et al., 2018; Brandt et al., 2022). The advantage of interactive theorem provers like Isabelle over, e.g., SAT solvers is that they support much more expressive logics (in our case Higher-Order Logic). Hence, we can formalize not just a SAT encoding of the result (and have to trust the correctness of the encoding), but the entire theorem with all the mathematical notions expressed in a natural way that is very close to how they are defined in Section 2. For instance, weak representation can be formalized in Isabelle as shown in Figure 1. As a consequence of this, Isabelle/HOL verifies the theorem directly based on the axioms: our Isabelle implementation shows that if there is an anonymous party-approval rule for 6 agents, 4 alternatives, committee size 3 that satisfies strategy-proofness and weak representation, 'False' follows. In particular, this releases us from the need to check all intermediate steps encoded in Isabelle, such as proving Lemma 1 and Lemma 2 or the translation of the axioms to a logical formula, because Isabelle checks the correctness of these steps for us. Here, it is important to note that Isabelle/HOL is highly trustworthy because all proofs have to pass through the *inference kernel*, a relatively small piece of code that only knows the most basic logical inference steps. Thus, to trust the correctness of our result, one need only trust that the inference kernel works correctly and that our formal definitions are faithful to their versions in this paper.

As a third point for verifying Proposition 1, we provide a human-readable proof for a weakening of the result that additionally requires Pareto-optimality. The proof of this corollary is derived from our computer-aided approach described in Section 3.1 with two small modifications: firstly, we of

[3]So far, the largest computer-generated proof in social choice theory required 386 profiles (Brandl et al., 2021).

A^1	a	a	b	abc	d	ad
A^2	a	a	b	ac	d	bd
A^3	a	a	b	ac	abc	d
A^4	a	a	ac	abc	d	bd
A^5	a	a	b	ac	d	ad
A^6	a	a	b	ac	ad	bd

Table 1: Profiles used in the proof sketch for Corollary 1

course need to encode Pareto-optimality and secondly, we encode a manual case distinction because this approach leads to a shorter proof. Note that we can encode Pareto-optimality similarly to weak proportionality by excluding committees that contain Pareto-dominated parties. For the case distinction, we add an additional constraint to the formula, which specifies the outcome for a single profile. For each case, we let the SAT solver then show that the formula is unsatisfiable, extract a MUS, and translate these MUSes to a human-readable proof. In contrast to Proposition 1, this approach results in significantly smaller MUSes (one containing 31 profiles and the other containing 76), thus enabling us to extract a human-readable proof. Nevertheless, this proof is still quite tedious and we thus only consider one of the cases in the main body. We refer to the appendix for a full proof.

Corollary 1. *There is no anonymous P-APP voting rule that satisfies weak representation, Pareto-optimality, and strategy-proofness if $k = 3$, $m = 4$, and $n = 6$.*

Proof sketch. Assume for contradiction that there is an anonymous P-APP voting rule f that satisfies strategy-proofness, Pareto-optimality, and weak representation for $k = 3$, $m = 4$, and $n = 6$. Note that the committee size is fixed during the proof and we thus write $f(A)$ instead of $f(A, 3)$. For deriving a contradiction, we need to distinguish several cases and we focus in this proof sketch on a particularly simple one. Thus, consider the profiles in Table 1 and assume that $f(A^1) = [a, a, a]$ and $f(A^2) \in \{[a, a, b], [a, a, d]\}$.

As first step, we determine the committee for the profile A^3. Note for this that strategy-proofness requires that $f(A^3, ac) \geq 3$ as voter 4 can otherwise manipulate by deviating back to A^1. Moreover, Pareto-optimality implies that c is not chosen for A^3 and we thus infer that $f(A^3) = [a, a, a]$.

As second step, observe that strategy-proofness from A^3 to A^4 requires that b is not in $f(A^4)$. Furthermore, Pareto-optimality entails that c is also not in $f(A^4)$. Finally, Lemma 1 shows that $f(A^4, abc) \geq 2$ because of voters 3 and 4, and that $f(A^4, bd) \geq 1$ because of voters 5 and 6. Combining these facts shows that $f(A^4) = [a, a, d]$.

With $f(A^4) = [a, a, d]$, it follows that $f(A^2) \neq [a, a, b]$; otherwise voter 4 can manipulate by deviating from A^4 to A^2. Since we assume that $f(A^2) \in \{[a, a, b], [a, a, d]\}$, we thus derive that $f(A^2) = [a, a, d]$.

Subsequently, consider the profile A^5 and note that $f(A^5, c) = 0$ because of Pareto-optimality. Since strategy-proofness requires that $f(A^5, ac) \geq f(A^1, ac)$, we infer thus that $f(A^5) = [a, a, a]$.

Finally, we investigate the profile A^6 and note that Lemma 1 implies that $f(A^6, bd) \geq 1$. Moreover, strategy-proofness from A^5 to A^6 requires that $f(A^6, d) = 0$ because voter 5 can manipulate otherwise. Hence, it follows that $f(A^6, b) \geq 1$. However, this means that voter 5 can manipulate by deviating from A^6 to A^2 because $f(A^6, ad) \leq 2$ and $f(A^2, ad) = 3$. Hence, our assumptions contradict each other. □

3.3 Induction Step

Finally, for proving Theorems 1 and 2, we need to generalize Proposition 1 from committees of size $k = 3$, $m = 4$ parties, and $n = 6$ voters to larger parameters. For this, we use inductive arguments to generalize Proposition 1. We focus in this section only on the inductive arguments based on weak representation because two of the three inductive arguments also hold when using weak proportional representation and the argument for the remaining one is rather similar.

Before proving these inductive arguments, we first introduce two auxiliary facts.

Proposition 2. *Let f denote a strategy-proof P-APP voting rule and $k \in \mathbb{N}$. It holds that $f(A, k, A_i) \geq f(A', k, A_i)$ for all sets of voters $I \subseteq N$, voters $i \in I$, and profiles A, A' such that $A_j = A'_j$ for all $j \in N \setminus I$ and $A_{i_1} = A_{i_2}$ for all $i_1, i_2 \in I$.*

Proof. Consider a strategy-proof P-APP voting rule f, a set of voters I, and two profiles A and A' as specified above. Define $A^0, \dots, A^{|I|}$ as the sequence of approval profiles such that $A^0 = A$, $A^{|I|} = A'$, and A^{l+1} differs from A^l by letting a single voter $i \in I$ deviate from A_i to A'_i. Since $A_{i_1} = A_{i_2}$ for all $i_1, i_2 \in I$, strategy-proofness implies for every $l \in \{0, \dots, |I|-1\}$ that $f(A^l, k, A_i) \geq f(A^{l+1}, k, A_i)$. Finally, combining these inequalities yields the lemma. □

Proposition 3. *Assume that f is a P-APP voting rule that satisfies strategy-proofness and weak representation. For every subset of parties $X \subseteq \mathcal{P}$, profile A, committee size $1 \leq k \leq n$ and integer $\ell \leq k$, it holds that $f(A, X) \geq \ell$ if $|X| \geq \ell$ and at least $\ell \cdot \lceil \frac{n}{k} \rceil$ voters report X as their approval ballot.*

Proof. Consider arbitrary values for k, m, and n and assume that f is a P-APP voting rule satisfying strategy-proofness and weak representation. We prove the statement by induction on ℓ and first consider the case that $\ell = 1$. Thus, let A denote a profile and X a non-empty set of parties such that $A_i = X$ for at least $\lceil \frac{n}{k} \rceil$ voters. Moreover, consider the profile A' derived from A by letting all voters who initially approved X only approve a single party $x \in X$. It follows from weak representation that $f(A', k, x) \geq 1$ and thus $f(A', k, X) \geq 1$. Finally, Proposition 2 implies that $f(A, k, X) \geq f(A', k, X) \geq 1$, which proves the induction basis.

For the induction step, we assume that the statement holds for all profiles A, sets of parties X, and integers $1, \dots, \ell$ and we prove the statement for $\ell + 1$. Hence, consider a profile A and set of parties X such that $|X| \geq \ell + 1$ and at least $(\ell + 1) \cdot \lceil \frac{n}{k} \rceil$ voters report X. Next, consider the profile A' derived from A by letting $\lceil \frac{n}{k} \rceil$ voters who approve X in A report a single party $x \in X$ and the remaining voters who approve X in A report $A \setminus \{x\}$. Weak representation requires for A' that x needs to be chosen at least once. Moreover, the induction hypothesis requires that $f(A', k, X \setminus \{x\}) \geq \ell$ because $|X \setminus \{x\}| \geq \ell$ and at least $\ell \cdot \lceil \frac{n}{k} \rceil$ voters report $X \setminus \{x\}$. Hence, $f(A', k, X) \geq \ell + 1$ and Proposition 2 implies in turn that $f(A, k, X) \geq f(A', k, X) \geq \ell + 1$. □

We are now ready to discuss the inductive arguments and start by showing that we can multiply the number of voters. An analogous argument has been proposed by Peters (2018) for ABC voting rules. Also, note that Lemma 3 also works when using weak proportional representation instead of weak representation.

Lemma 3. *Let $\ell \in \mathbb{N}$ denote an integer. If there is no P-APP voting rule that satisfies anonymity, weak representation, and strategy-proofness for committees of size k, m parties, n voters, there is no such rule for committees of size k, m parties, $\ell \cdot n$ voters.*

Proof. We prove the contrapositive of the lemma: if there is a committee voting rule that meets all conditions of the lemma for committees of size k, m parties, and $\ell \cdot n$ voters, then there is also such a rule if there are only n voters. Hence, assume that f denotes a P-APP voting rule satisfying anonymity, strategy-proofness and weak representation for $\ell \cdot n$ voters. The following P-APP rule g for n voters satisfies the same axioms as f: given a profile A for n voters, g copies every voter ℓ times to derive the profile A'. Then $g(A) = f(A')$.

First, note that g inherits anonymity from f as permuting the voters in A only leads to a permutation of the voters in A'. Moreover, g satisfies weak representation: if $j \geq \frac{n}{k}$ voters uniquely approve a party x in A, then $\ell \cdot j \geq \ell \frac{n}{k}$ voters approve x uniquely in A'. Since f satisfies weak representation, this means that $g(A, k, x) = f(A', k, x) \geq 1$. Finally, it follows from Proposition 2 that g is strategy-proof. For this consider two approval profiles A and $\bar{A}$ that only differ in the preference of a single voter i. The profiles A' and $\bar{A}'$ derived by cloning every voter ℓ times thus differ only in the clones of voter i. Hence, Proposition 2 implies that $f(A', k, A'_i) \geq f(\bar{A}', k, A'_i)$, which proves that $g(A, k, A_i) \geq g(\bar{A}, k, A_i)$. □

As second inductive argument, we show that we can increase the number of parties arbitrarily. The lemma holds again when using weak proportional representation instead of weak representation.

Lemma 4. *If there is no P-APP voting rule that satisfies anonymity, strategy-proofness, and weak representation for committees of size k, m parties, and n voters, there is also no such a rule for committees of size k, $m+1$ parties, n voters.*

Proof. We prove the contrapositive of the lemma: if there is a P-APP voting rule f that satisfies all axioms for $m+1$ parties, there is also such a voting rule for m parties. For proving this, we define a P-APP voting rule g for m parties based on f. To this end, let A^{xy} be the profile derived from A by cloning a party x (and using y as the name for the new party). More formally, this means $A_i^{xy} = A_i \cup \{y\}$ if $x \in A_i$ and $A_i^{xy} = A_i$ otherwise, where y denotes a new party. Then, g is defined as follows: $g(A, k, z) = f(A^{xy}, k, z)$ for all $z \in \mathcal{P} \setminus \{x\}$ and $g(A, k, x) = f(A^{xy}, k, xy)$.

First, note that g is a well-defined P-APP voting rule for m parties as its output committee never contains y. Moreover, g clearly inherits anonymity from f. It is also not hard to see that g is strategy-proof: if not, there are profiles A^1 and A^2 that differ only in the preference of a single voter i such that $g(A^2, k, A_i^1) > g(A^1, k, A_i^1)$. However, this implies that f is manipulable because for all profiles A and sets of parties X, it holds that $g(A, k, X) = f(A^{xy}, k, X)$ if $x \notin X$ and $g(A, k, X) = f(A^{xy}, k, X \cup \{y\})$ otherwise. Since every voter in A^{xy} approves y if and only if she approves x in A, it follows therefore for the profiles $\bar{A}^1 = (A^1)^{xy}$ and $\bar{A}^2 = (A^2)^{xy}$ that $f(\bar{A}^2, k, \bar{A}_i^1) > f(\bar{A}^1, k, \bar{A}_i^1)$, thus contradicting that f is strategy-proof. Finally, g satisfies also weak representation, which follows by a case distinction on the parties. If a party $z \neq x$ is uniquely approved by $\lceil \frac{n}{k} \rceil$ voters in the original profile A, then it is also uniquely approved by these in A^{xy}. Hence, $g(A, z) = f(A^{xy}, z) \geq 1$ due to the weak representation of f. On the other hand, if x is uniquely approved by at least $\lceil \frac{n}{k} \rceil$ voters, then these voters approve xy in A^{xy}. Hence, Proposition 3 shows that $g(A, k, x) = f(A^{xy}, k, xy) = 1$. This shows that g satisfies all required axioms. □

Finally, we prove that we can also increase the size of the committee if we increase the number of voters and parties. Note that based on weak proportional representation, we can prove a stronger inductive argument leading to the improved bound on the number of parties in Theorem 2. This argument relies on a similar idea and is thus deferred to the appendix.

Lemma 5. *Assume that n, k are integers such that n is a multiple of k. If there is no P-APP voting rule that satisfies anonymity, strategy-proofness, and weak representation for committees of size k, m parties, and n voters, there is also no such rule for committees of size $k+1$, $m+1$ parties, and $\frac{n(k+1)}{k}$ voters.*

Proof. Once again, we prove the contrapositive of the lemma and thus assume that there is a voting rule f that satisfies anonymity, strategy-proofness, and weak representation for committees of size $k+1$, $m+1$ parties, and $\frac{n(k+1)}{k}$ voters. Given this rule f, we construct a P-APP voting rule g that satisfies the same axioms for committees of size k, m parties and n voters. To this end, let A^{xy} denote the profile derived from A by adding y as clone of x and adding $\frac{n}{k}$ voters who approve xy. More formally, A^{xy} is defined as follows, where y is a new party: $A_i^{xy} = A_i \cup \{y\}$ for all $i \in N$ with $x \in A_i$, $A_i^{xy} = A^i$ for all $i \in N$ with $x \notin A_i$, and there are $\frac{n}{k}$ voters $i \notin N$ with $A_i^{xy} = xy$. Finally, we define g by $g(A, k, z) = f(A^{xy}, k+1, z)$ for all $z \neq x$ and $g(A, k, x) = f(A^{xy}, k+1, xy) - 1$.

First, note that g is a well-defined P-APP voting rule for committees of size k as Proposition 3 ensures that $f(A^{xy}, k+1, xy) \geq 1$ for all profiles A. Thus $g(A, k, x) \geq 0$ and, since f returns a committee of size $k+1$, g returns a committee of size k. Next, it is apparent that g inherits anonymity of f. Moreover, a similar argument as in Lemma 4 shows that g is strategy-proof. Finally, note that g satisfies weak representation. For every party $z \neq x$: if z is uniquely approved by at least $\frac{n}{k}$ voters in a profile A, it is also uniquely approved by these voters in A^{xy}. Since $\frac{n(k+1)}{k(k+1)} = \frac{n}{k}$, it thus follows that $f(A^{xy}, k+1, z) \geq 1$ due to the weak representation of f and, by definition, we thus have $g(A, k, z) \geq 1$. On the other hand, if x is uniquely approved by at least $\frac{n}{k}$ voters in A, there are at least $2\frac{n}{k}$ voters in A^{xy} that exactly approve xy in A^{xy}. Since $\frac{n(k+1)}{k(k+1)} = \frac{n}{k}$, we infer from Proposition 3 that $f(A^{xy}, k+1, xy) \geq 2$, which implies that $g(A, k, x) \geq 1$. Hence, g satisfies all required axioms, which proves the lemma. □

3.4 Remarks

We conclude this section by discussing remarks about Theorems 1 and 2.

Remark 1. Approval voting satisfies all axioms of Theorem 1 but weak representation, and Chamberlin-Courant approval voting satisfies all axioms but strategy-proofness. These examples show that these axioms are required for the impossibility. On the other hand, we cannot show that anonymity is necessary for the impossibility and we conjecture that this axiom can be omitted.

Remark 2. If we consider electorates where the committee size k is a multiple of the number of voters n, there are voting rules that satisfy PJR, anonymity, and strategy-proofness: we can just let every voter choose $\frac{k}{n}$ parties of the committee independently of the approval ballots of the other voters. This is an important difference to the impossibility by Peters (2018), which holds also in the case that $n = k$.

Remark 3. If $k = 2$, we can define a P-APP voting rule f that satisfies anonymity, strategy-proofness, and even PJR. For defining this rule, let $s(A, x) = |\{i \in V : x \in A_i\}|$ denote the approval score of party x in profile A, let $maxs(A) = \max_{x \in \mathcal{P}} s(A, x)$ denote the maximal approval score in profile A, and let $AV(A) = \{x \in \mathcal{P} : s(A, x) = maxs(A)\}$ denote the set of parties with maximal approval scores. Moreover, let $\succ = x_1 \succ x_2 \succ \cdots \succ x_m$ denote an external tie-breaking order. As first step, f removes all clones from the set $AV(A)$ according to the tie-breaking order $\succ$, i.e., if $x, y \in AV(A)$ and for all voters $i \in N$, $x \in A_i$ if and only if $y \in A_i$, we remove y from $AV(A)$

if $x \succ y$ and otherwise x. Based on this reduced set $AV(A)'$, we define f as follows: if $|AV(A)'| = 1$ or $masxs(A) \neq \frac{n}{2}$, choose the maximal element in $AV(A)'$ according to $\succ$ twice. Else, we choose the maximal and next best party in $AV(A')$.

Also, note that the bound on m is tight: if $k = 2$, the rule sketched above satisfies PJR and strategy-proofness for every m, and if $k = 3$ and $m \leq 3$, we can constantly choose a committee containing each party at least once to satisfy WR and strategy-proofness. Finally, the SAT solver shows that if $m = 4$ and $k = 3$, we have a possibility if $n \in \{4, 5\}$.

Remark 4. Theorem 2 also holds when replacing weak proportional representation with weak representation and Pareto-optimality. The reason for this is that for almost all profiles, the combination of weak representation, strategy-proofness and Pareto-optimality implies weak proportional representation. In more detail, this claim holds for all profiles but those where we can split the voters into two sets N_1 and N_2 such that $A_i = x$ for all $i \in N_1$ and $A_i \in \{\mathcal{P}, \mathcal{P} \setminus \{x\}\}$ for all other voters. As neither our proof nor the inductive arguments requires such profiles, it follows thus that Theorem 2 also holds with the modified set of assumptions.

Remark 5. We use a significantly stronger strategy-proofness notion than Peters (2018) for our impossibility theorem. Peters (2018) only considers a rule manipulable if there are profiles A, A' such that $A_{-i} = A'_{-i}$, $A_i \cap f(A) \subsetneq A_i \cap f(A')$ and $A'_i \subseteq A_i$. Perhaps surprisingly, our computer program finds a satisfying assignment when using this weaker strategy-proofness notion and even when dropping the condition on A'_i. However, the corresponding P-APP voting rule seems merely technical and we could not find a closed form representation.

4 Strategy-proofness for unrepresented voters

In order to circumvent the impossibilities of Theorems 1 and 2, we consider in this section a restricted form of strategy-proofness: instead of prohibiting all voters from manipulating, we only require that voters who do not approve any member of the elected committee are unable to manipulate. This idea is formalized by strategy-proofness for unrepresented voters (Definition 4), which is severely weaker than the strategy-proofness notion we considered so far and thus may lead to more positive results. Indeed, we demonstrate that we can circumvent Theorem 1 based on this strategy-proofness notion because Chamberlin-Courant approval voting (CCAV) satisfies strategy-proofness for unrepresented voters and all other axioms of Theorem 1 for all target committee sizes, voters and parties. This result is all the more surprising given that all other Thiele rules, sequential Thiele rules, and divisor methods based on majoritarian portioning (except approval voting) fail strategy-proofness for unrepresented voters. As a consequence of these two observations we can even characterize CCAV within Thiele rules based on this strategy-proofness notion.

As a first result, we prove that, even though strategy-proofness for unrepresented voters may seem weak, it is a challenging property that most P-APP voting rules fail for some election instance.

Theorem 3. *For every Thiele rule (but AV and CCAV), every sequential Thiele rule (but AV), and every divisor method based on majoritarian portioning (but AV), there are a committee size k, a number of parties m, and a number of voters n such that the considered method fails strategy-proofness for unrepresented voters.*

Proof. For each of the three classes we discuss a separate counter example below.

Thiele rules. Let f denote a w-Thiele rule other than AV and CCAV. Since f is not approval voting, there is an index j with $w_j < w_1$ and we denote with j^* the smallest such index. In particular, this means that $\forall j < j^*, w_j = w_1 = 1$. Moreover, if $w_{j^*} = 0$, then $j^* \geq 3$ because f is not CCAV.

We build the profile P with $m = 2j^*$ parties $\mathcal{P} = \{a_1, \dots, a_{j^*}, b_1, \dots, b_{j^*}\}$ and $n = 2 \cdot \binom{2j^*}{j^*} - 2$ voters and aim to elect a committee of size $k = j^*$. The approval ballots of the voters are defined as follows:

- A single voter v_1 reports $\{a_1, \dots, a_j\}$.
- A single voter v_2 reports $\{b_1\}$.
- For every subset of $\mathcal{P}$ of size $k = j^*$ which is not $\{a_1, \dots, a_{j^*}\}$ or $\{b_1, \dots, b_{j^*}\}$ (i.e., it contains elements from the a's family and the b's family), add two voters with a ballot corresponding to this subset. These are voters of category 1.

Note that every party appears exactly in $n_c = 2\binom{2j^*-1}{j^*-1} - 2$ ballots of the voters of category 1. This means that every committee W of size $k = j^*$ gets a total of $\sum_{x \in \mathcal{P}} W(x) |\{i \in N_{C1} : x \in A_i\}| = j^* n_c$ approvals from voters of category 1. We use this fact to compute the scores of a committee W derived from these voters. Thus, observe first that the committees $[a_1, \dots, a_{j^*}]$ and $[b_1, \dots, b_{j^*}]$ receive a score of $j^* n_c$ from the voters of category 1 as none of these voters approve all parties in the committee. Since $w_1 = \dots = w_{j^*-1} = 1$, every approval brings therefore 1 point, resulting in the score above. On the other hand, for every other committee W, there are at least two voters who approve all parties in W. Hence, these voters assign the committee a score of $j^* - 1 + w_{j^*}$. Note that the total sum of approvals is constant, and we thus derive that the remaining voters of category 1 assign at most $j^*(n_c - 2)$ points to W. Hence, the score of W among voters of category 1 is upper bounded by $j^* n_c - 2(1 - w_{j^*})$.

Finally, if we add the last two voters, the score of $[a_1, \dots, a_j]$ is $j^* n_c + j^* - 1 + w_{j^*}$, the score of $[b_1, \dots, b_j]$ is $j^* n_c + 1 < j^* n_c + j^* - 1 + w_{j^*}$ (because either $j^* \geq 3$ or $w_{j^*} > 0$), and the scores of other committees is at most $j^* n_c - 2(1 - w_{j^*}) + j^* < j^* n_c + j^* - 1 + w_{j^*}$ (since $w_{j^*} < 1$). Hence, f elects the committee $[a_1, \dots, a_{j^*}]$ for A.

Now, consider the profile A' derived from A by changing the approval ballot of voter v_2 to $\{b_1, \dots, b_{j^*}\}$. Then, the score of the committee $[a_1, \dots, a_{j^*}]$ does not change and the score of $[b_1, \dots, b_j]$ is now the same than the score of $\{a_1, \dots, a_{j^*}\}$. Moreover, the same argument as in the last paragraph shows that the score of all other committees W is still strictly lower than the score of $[a_1, \dots, a_{j^*}]$. This means that the committees $[a_1, \dots, a_{j^*}]$ and $[b_1, \dots, b_{j^*}]$ are now tied w.r.t. these scores. Finally, if we assume that the lexicographic tie-breaking prefers the latter committee, $f(A', j^*) = [b_1, \dots, b_{j^*}]$ and voter v_2 can thus manipulate even though she approves none of the parties in $f(A, j^*)$.

Sequential Thiele rules. Consider any sequential w-Thiele rule f other than approval voting. Since, the vector w is decreasing and f is not approval voting, there is an index j such that $w_j < 1$. Let j^* denote the first such index. Moreover we define $\ell \in \mathbb{N}$, $\ell \geq 4$, as the smallest integer such that $w_{j^*} < \frac{\ell-2}{\ell}$. Finally, consider the following two profiles A and A' with $m = 4$ parties and $n = 4 \cdot \ell + 1$ voters (the numbers before the preference relation indicate how often a preference relation is reported, e.g., ℓ voters approve the set ab in A),

$$
\begin{array}{lllllll}
A: & 1\colon b & \ell\colon ab & \ell\colon bd & \ell\colon ac & \ell-1\colon cd & 1\colon d \\
A': & 1\colon b & \ell\colon ab & \ell\colon bd & \ell\colon ac & \ell-1\colon cd & 1\colon ad
\end{array}
$$

We will now show that an unrepresented voter can manipulate f in A if $k = j^*$. Note for this that $w_1 = \dots = w_{j^*-1} = 1$, which implies that f assigns the first j^*-1 seats to the approval winner. For the profile A, this means that these seats go to party b as it is approved by $2\ell+1$ voters. Finally, the last seat goes to party c. For proving this claim, let W^x denote the committee which assigns j^*-1 seats to party b and the last seat to party x. Then, $s(W^c, A) = (j^*-1)(2\ell+1)+2\ell-1$, $s(W^a, A) = s(W^d, A) = (j^*-1)(2\ell+1)+\ell\cdot(1+w_{j^*}) < (j^*-1)(2\ell+1)+\ell(1+\frac{\ell-2}{\ell}) = (j^*-1)(2\ell+1)+2\ell-2$, and $s(W^b, A) = (j^*-1)(2\ell+1)+(2\ell+1)w_{j^*} < (j^*-1)(2\ell+1)+(2\ell+1)\frac{\ell-2}{\ell} < (j^*-1)(2\ell+1)+2\ell-3$, which proves our claim.

Next, consider the profile A'. Just as for A, f assigns the first j^*-1 seats to the approval winner, which is in this case a as both a and b are approved by $2\ell+1$ voters and the lexicographic tie-breaking chooses a. Finally, analogous computations as for A show that the last seat then goes to d, i.e., $f(A', j^*)$ chooses the committee that assigns j^*-1 seats to a and one seat to d. Since A and A' differ only in the preference of the last voter (who approves only d) and $f(A, j^*, d) = 0 < 1 = f(A', j^*, d)$, this proves that an unrepresented voter can manipulate f.

Divisor methods based on majoritarian portioning. Let f denote a divisor method based on majoritarian portioning other than approval voting. This means that there is a monotone function $g : \mathbb{N}_0 \to \mathbb{R}_{>0}$ such that f can be computed as follows: in the i-th round, f assigns the next seat to the party x that maximizes $\frac{w_x}{g(t_x^i)}$ (w_x denotes the weight of party x computed by majoritarian portioning, and t_x^i denotes the number of seats assigned to x in all previous iterations). Now, since f is not approval voting, there are integers $\ell, j \in \mathbb{N}$ such that $\frac{\ell+1}{g(j)} \leq \frac{\ell}{g(0)}$. Let ℓ^*, j^* denote a pair of such indices that minimize j^*, i.e., for all $j' < j^*$ and $\ell \in \mathbb{N}_0$, it holds that $\frac{\ell+1}{g(j')} > \frac{\ell}{g(0)}$.

As next step, we show that we may assume that $\ell^* \geq 2$. If this is not the case then, $\ell^* = 1$. Note that $\ell^* = 0$ is impossible as $\frac{1}{g(j)} > \frac{0}{g(0)}$. Now, if $\ell^* = 1$, our conditions require that $\frac{2}{g(j^*)} \leq \frac{1}{g(0)}$ and $\frac{2}{g(j)} > \frac{1}{g(0)}$ for every $j \in \mathbb{N}_0$ with $j < j^*$. We will show subsequently that we can set ℓ^* to 3. In particular, we have $\frac{4}{g(j^*)} \leq \frac{2}{g(j^*)} < \frac{3}{g(0)}$, showing that our first condition is met. Next, consider the condition that $\frac{4}{g(j)} > \frac{3}{g(0)}$ for all $j < j^*$ and assume for contradiction that there is $j' \in \mathbb{N}$ such that $j < j^*$ and $\frac{4}{g(j')} \leq \frac{3}{g(0)}$. Since $\frac{4}{g(0)} > \frac{3}{g(0)}$, we can thus find an index $j'' \in \mathbb{N}$ such that $j'' < j^*$, $\frac{4}{g(j)} > \frac{3}{g(0)}$ for all $j < j''$ and $\frac{4}{g(j'')} \leq \frac{3}{g(0)}$. However, this contradicts the definition of j^*, as j^* is the minimal integer that meets our requirements. Hence, we can set ℓ^* to 3.

We use the insights of the last two paragraphs to define two profiles A and A' on $n = 4\ell^* + 4$ voters and $m = 4$ parties on which f fails strategy-proofness for unrepresented voters if $k = j^* + 1$. In the profiles, the numbers before the preference relation indicate how often a preference relation is reported, e.g., ℓ^* voters approve the set ab in A.

A:	2: c	2: d	ℓ^*: ac	ℓ^*: cd	ℓ^*: ab	ℓ^*: bd
A':	2: c	2: ad	ℓ^*: ac	ℓ^*: cd	ℓ^*: ab	ℓ^*: bd

In profile A we see that in total, a and b are each approved by $2\ell^*$ voters and c and d are each approved by $2\ell^*$ voters, while in A' two voters additionally approve a. When applying majoritarian portioning (with lexicographic tie-breaking), we derive for A that $w_a = 0$, $w_b = 2\ell^*$, $w_c = 2\ell^* + 2$, and $w_d = 2$. In more detail, we first allocate $2\ell^* + 2$ votes to c because of the lexicographic tie-breaking. After removing all voters who approve c, the approval score of d is $\ell^* + 2$ and the approval score of b is $2\ell^*$. Since $\ell^* \geq 2$, we next allocate $2\ell^*$ votes to b. Finally, only the two voters who approve d remain, giving d a weight of 2. Now, using the definition of ℓ^* and j^*, it follows that f

assigns the first j^* seats to c because $\frac{\ell^*+1}{g(j')} > \frac{\ell^*}{g(0)}$ for all $j' < j^*$. In contrast, $\frac{\ell^*+1}{g(j^*)} \leq \frac{\ell^*}{g(0)}$ implies that the last seat goes to b. Hence, f outputs a committee W with $W(c) = j^*$ and $W(b) = 1$

Analogous computations as for A show that majoritarian portioning results in the following weights for A': $w_a = 2\ell^* + 2$, $w_b = 0$, $w_c = 2$, $w_d = 2\ell^*$. Then, a symmetric analysis as for A shows that f elects the committee W' with $W'(a) = j^*$ and $W'(d) = 1$. Finally, note that A and A' only differ in the ballot of the second type of voters. Since these voters approve only d in A but $W(d) = 0$, strategy-proofness for unrepresented voters requires that d cannot obtain a seat if these voters manipulate. However, by letting them deviate one after another, the outcome changes eventually in their favor, which means that f can be manipulated by unrepresented voters.

□

While the theorem only shows that all of the considered rules fail strategy-proofness for unrepresented voters for some specific values of k, m, and n, it is often easy to generalize the constructions. In particular, for many rules, the counterexamples can be extended to larger committee sizes k by adding few parties and adding voters who uniquely approve these parties. By doing so, one can enforce that a specific number of seats is allocated to the new parties, and the remaining seats will be assigned as in the initial counter example.

Next, observe that Theorem 3 contains exceptions: approval voting and Chamberlin-Courant approval voting are not shown to fail strategy-proofness for unrepresented voters. Even more, we can show that both rules indeed satisfy this axiom. While this is not surprising for approval voting as it even satisfies full strategy-proofness, this observation offers an escape route to Theorem 1 as CCAV also satisfies weak representation. Even more, we can characterize CCAV within Thiele rules based on strategy-proofness for unrepresented voters and weak representation as all other Thiele rules fail one of these conditions.

Theorem 4. *CCAV is the only Thiele rule that satisfies weak representation and strategy-proofness for unrepresented voters for all committee sizes k, numbers of parties m, and numbers of voters n.*

Proof. First, note that Theorem 3 shows that all Thiele rules but CCAV and AV fail strategy-proofness for unrepresented voters for some values of k, m, and n. Since it is straightforward to see that AV fails weak representation, it follows that no Thiele rule but CCAV can satisfy the given axioms. Hence, it only remains to show that CCAV indeed satisfies strategy-proofness for unrepresented voters and weak representation.

We first consider strategy-proofness and assume for contradiction that CCAV fails this axiom. To this end we assume that there are values of m, n, and k such that there is a voter $i \in N$ and profiles A^1 and A^2 on m parties and n voters such that $CCAV(A^2, k, A^1_i) > CCAV(A^1, k, A^1_i) = 0$ and $A^1_j = A^2_j$ for all $j \in N \setminus \{i\}$. For a simpler notation, let $W^1 = CCAV(A^1, k)$ and $W^2 = CCAV(A^2, k)$, and define $s(W, A) = |\{i \in N : A_i \cap X \neq \emptyset\}|$ as the CCAV-score of a committee W in a profile A. Now, by the definition of W^1 and W^2, it follows that $s(W^1, A^1) \geq s(W^2, A^1)$ and $s(W^2, A^2) \geq s(W^1, A^2)$. Moreover, since $W^1(A^1_1) = 0$ and $A^1_j = A^2_j$ for all other voters $j \in N \setminus \{i\}$, we can infer that $s(W^1, A^2) \geq s(W^1, A^1)$. Finally, we have by assumption that $W^2(A^1_i) > 0$, which implies that $s(W^2, A^1) \geq s(W^2, A^2)$ since $A^1_j = A^2_j$ for all $j \in N \setminus \{i\}$. Combining these inequalities results in

$$s(W^2, A^2) \geq s(W^1, A^2) \geq s(W^1, A^1) \geq s(W^2, A^1) \geq s(W^2, A^2).$$

Clearly this chain of inequalities can only be true if all scores are equal. However, our lexicographic tie-breaking implies then that we choose either W^1 or W^2 for both A^1 and A^2, which conflicts with the assumption that $W^1 = CCAV(A^1, k)$ and $W^2 = CCAV(A^2, k)$. This contradiction shows that the initial assumption is wrong and CCAV is thus strategy-proof for unrepresented voters.

Finally, we show that CCAV satisfies weak representation. We assume again that this is not the case, which means that there is a party x, a group of voters $G \subseteq N$, a profile A, and a committee size k such that $CCAV(A, k, x) = 0$, $A_i = x$ for all $i \in G$, and $|G| \geq \frac{n}{k}$. We define again $W = CCAV(A, k)$ and let $s(W, A)$ denote the CCAV score of W in A. Since no voter in G approves a party in W, we infer that $s(W, A) \leq n - \frac{n}{k} = \frac{n(k-1)}{k}$. Moreover, as the committee W consists of k seats, it follows that, on average, a single seat contributes $\frac{n(k-1)}{k^2}$ to this score. Hence, there is a party y with $W(y) \geq 1$ such that the committee W' of size $k-1$ defined by $W'(y) = W(y) - 1$ and $W'(z) = W(z)$ for all $z \in \mathcal{P} \setminus \{y\}$ has a score of $s(W', A) \geq s(W, A) - \frac{n(k-1)}{k^2}$. Finally, consider the committee W'' derived from W' by adding x. Since the voters in G approve a party in this committee, we can compute that

$$s(W'', A) \geq S(W, A) - \frac{n(k-1)}{k^2} + \frac{n}{k} = s(W, A) + \frac{n}{k^2} > s(W, A).$$

This proves that the CCAV score of W'' is larger than the one of W, which contradicts that $W = CCAV(A, k)$. Hence, the initial assumption that CCAV fails weak representation is wrong. □

Remark 6. Note that CCAV becomes highly indecisive if $k \geq m$ because it is then always possible to elect a committee that chooses one of the approved parties of each voter. In this case, many seats of the committee will be assigned by the tie-breaking. Note that similar arguments apply for all w-Thiele rules that have an index i with $w_i = 0$: if there are more than im seats in the committee, these rules can choose a score maximizing committee, which guarantees every voter i representatives. In particular, this shows that such rules are strategy-proof for unrepresented voters if $k \geq im$ because there are simply no unrepresented voters. Consequently, it is necessary for Theorem 4 to quantify over the numbers of seats in the committee k, parties m, and voters n.

Remark 7. All results of this section carry directly over into the ABC setting. For Theorem 3, this follows from the fact that we can simulate P-APP voting rules with ABC voting rules by cloning every party k times. After this transformation, all considered classes of rules behave exactly as their ABC equivalent. For Theorem 4, our claim follows by noting that the proof does not rely on the fact that we are in the P-APP setting.

Remark 8. In much of the literature, committee voting rules are modelled as functions that return sets of committees to avoid the tie-breaking (see, e.g., Faliszewski et al., 2017). It is also possible to show that Theorems 3 and 4 hold for irresolute P-APP voting rules when using suitable set extensions (such as Kelly's (Kelly, 1977)) to compare sets of committees.

Remark 9. We focus in this paper on divisor methods based on majoritarian portioning as examples of composite rules. However, Brill et al. (2020) introduce also composite P-APP voting rules based on different portioning methods. For many of these rules, it is not difficult to show that they also fail strategy-proofness for unrepresented voters, i.e., the whole class of composite P-APP voting rules seems rather prone to manipulation. We focus on the class of rules based on majoritarian portioning here since these were shown to often provide high degrees of proportionality (Brill et al., 2020).

5 Conclusion

In this paper, we study the compatibility of strategy-proofness and proportional representation for party-approval multi-winner (P-APP) elections. In P-APP elections, a multiset of the parties is chosen based on the voters' approval ballots and this setting thus constitutes an important generalization of approval-based committee elections (Brill et al., 2020). Unfortunately, it turns out that strategy-proofness and minimal notions of proportional representation are incompatible even in this more flexible setting. In more detail, we prove two impossibilities: First, we show that no anonymous P-APP voting rule satisfies both strategy-proofness and weak representation. This result is related to an impossibility by Peters (2018) in the more restricted ABC setting and shows that, in general, not even minimal notions of proportional representation are compatible with strategy-proofness. However, the result requires that there are more parties than seats in the committee, a restriction which many applications of P-APP elections do not meet. We thus prove a second impossibility theorem for this case by strengthening weak representation to weak proportional representation. In the final chapter we consider a weakening of strategy-proofness to escape these sweeping impossibility results. In particular, we investigate strategy-proofness for unrepresented voters, which only requires that voters who do not approve any member of the committee cannot manipulate. Perhaps surprisingly, most commonly studied P-APP voting rules fail even this very weak strategy-proofness notion. This observation allows us to characterize Chamberlin-Courant approval voting as the only Thiele rule that satisfies strategy-proofness for unrepresented voters and weak representation for all combinations of numbers of parties, voters, and seats in the committee. Thus, this observation offers an escape route to our first impossibility theorem and highlights the appeal of Chamberlin-Courant approval voting.

While we show the characterization of Chamberlin-Courant approval voting by hand, our impossibility results are derived by computer-aided techniques. In particular, we formalize the problem of finding an anonymous P-APP voting rule that satisfy strategy-proofness and weak representation as a logical formula for fixed numbers of parties, voters, and seats in the committee and let a SAT solver show that the constructed formula is unsatisfiable. This shows that no such function exists for the given parameters and we use hand-crafted inductive arguments to generalize the impossibility to larger values of m, n, and k. Notably, the smallest proof found by this computer-aided approach requires 635 profiles and more than 20.000 strategy-proofness applications, making it very tedious or even impossible to check the proof by hand. As solution, we verify the result by Isabelle, a highly trustworthy interactive theorem prover, and include the full proof of a slightly weaker claim in the appendix.

Our work also offers several directions for future work. Firstly, we use a cardinality-based strategy-proofness notion for our impossibility results, and the SAT solver suggests that the impossibilities ceases to hold for weaker strategy-proofness notions. Thus, it might be worthwhile to analyze slightly weaker strategy-proofness notions in this setting. Secondly, we feel that the notion of strategy-proofness for unrepresented voters deserves more attention; for instance, while CCAV satisfies it together with weak representation, we have to leave it open whether weak proportional representation is compatible with this strategy-proofness notion. Finally, one can see strategy-proofness and strategy-proofness for unrepresented voters as two extreme cases of a parameterization of strategy-proofness. Since we derive different results based on the parametrization, it might also be interesting to consider quantified strategy-proofness notions for P-APP elections.

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A Omitted proofs of Section 3.1 and Section 3.3

In this section, we provide the proofs omitted from Section 3. In more detail, we prove Lemma 1, Lemma 2, and the missing inductive argument for showing Theorem 2. We start by showing Lemma 1.

Lemma 1. *Let f denote an anonymous P-APP voting rule that satisfies strategy-proofness and weak representation, and let k denote the target committee size. Moreover, we consider a profile A and define $X = \{x \in \mathcal{P} : \exists G \subseteq N : |G| \geq \frac{n}{k} \land \forall i \in G : A_i = x\}$ as the set of parties that are each uniquely approved by at least $\frac{n}{k}$ voters in A. If there is a set of voters $G = \{i_1, \dots i_\ell\} \subseteq N$ such that $|G| \geq \frac{n}{k}$, $A_{i_1} \subseteq A_{i_2} \subseteq \dots \subseteq A_{i_\ell}$, and $A_{i_1} \not\subseteq X$, then $f(A, k, A_{i_\ell}) \geq |A_{i_\ell} \cap X| + 1$.*

Proof. Let f denote an anonymous P-APP voting rule that satisfies strategy-proofness and weak representation and consider an arbitrary profile A and a set of voters $G = \{i_1, \dots, i_\ell\}$ as defined in the lemma. Moreover, let X denote the set of parties that are uniquely approved by at least $\frac{n}{k}$ voters each. Note that weak representation forces f to assign at least one seat to every party in X. For the sake of contradiction, we suppose that $f(A, A_{i_\ell}) \leq |A_{i_\ell} \cap X|$. Since every party in X is chosen at least once, it thus follows that this inequality is tight. Moreover, by assumption, there is a party $y \in A_{i_1} \subseteq A_{i_2} \subseteq \dots \subseteq A_{i_\ell}$ such that $y \notin X$. In particular, this means that no voter in G approves one of the parties in X uniquely, i.e., the parties in $A_{i_\ell} \cap X$ will be assigned at least one seat independent of the ballots of the voters in G.

As next step, we consider the sequence of approval profiles $A^{\ell+1}, \cdots, A^1$ such that $A^{\ell+1} = A$ and A^j is derived from A_{j+1} by letting voter $i_j \in G$ change her approval ballot to y. Now consider a single step from A^{j+1} to A^j and suppose that $f(A^{j+1}, A^{j+1}_{i_j}) \leq |A^{j+1}_{i_j} \cap X|$.By our previous observation, it follows that every party in $A_{i_j} \cap X$ is chosen at least once and thus the above inequality is tight. Moreover, strategy-proofness requires that $|A^{j+1}_{i_j} \cap X| = f(A^{j+1}, A^{j+1}_{i_j}) \geq f(A^j, A^{j+1}_{i_j})$. Since every party in $A^{j+1}_{i_j} \cap X$ must also be chosen at least once for A^j, it follows that each of these parties are exactly chosen once for this profile. In particular, we can derive that no party in $A^{j+1}_{i_j} \setminus X$ can be elected for A^j. Since $A^j_{j-1} = A_{j-1} \subseteq A_j$, it thus follows that $f(A^j, A^j_{j-1}) \leq |A^j_{j-1} \cap X|$, too. Finally, note that we can now repeat the same argument for A^{j-1}. Since we assume that $f(A, A_{i_\ell}) = f(A^{i_\ell+1}, A_{i_\ell}) \leq |A_{i_\ell} \cap X|$, we infer thus inductively that $f(A^1, A_{i_1}) \leq |A_{i_1} \cap X|$. In particular, this still means that no party in $A_{i_1} \setminus X$ can be chosen. However, all voters in G now approve only y and weak representation thus requires that this party is chosen. This is a contradiction and thus shows that $f(A, A_{i_\ell}) \geq |A_{i_\ell} \cap X|$. □

Next, we show our second auxiliary lemma for the SAT encoding.

Lemma 2. *Let f denote an anonymous P-APP voting rule that satisfies strategy-proofness and weak representation and let $\tau : \mathcal{P} \mapsto \mathcal{P}$ denote a permutation of the parties. Then, the P-APP voting rule $f^\tau(A, k, x) = f(\tau(A), k, \tau(x))$, where $\tau(A)$ denotes the profile such that $\tau(x) \in \tau(A)_i$ if and only if $x \in A_i$ for all $i \in N$ and $x \in \mathcal{P}$, satisfies the same axioms as f.*

Proof. Let f denote an anonymous P-APP voting rule that satisfies strategy-proofness and weak presentation for some values of k, m, and n, and let $\tau : P \mapsto P$ denote an arbitrary permutation on the parties. Moreover, consider the P-APP voting rule f^τ as defined in the lemma. First, it is apparent that f^τ is anonymous as permuting the voters in A also permutes the voters in $\tau(A)$. Thus, f^τ inherits this property from f. Next, f^τ satisfies weak representation: if a party x is uniquely approved by at least $\frac{n}{k}$ voters in A, then $\tau(x)$ is uniquely approved by at least $\frac{n}{k}$ voters in $\tau(A)$. Since f satisfies weak representation, it thus follows that $f^\tau(A,k,x) = f(\tau(A),k,\tau(x)) \geq 1$, proving that f^τ also satisfies weak representation. Finally, f^τ also inherits strategy-proofness from f; otherwise there are two profiles A and A' and a voter $i \in N$ such that $A_j = A'_j$ for all $j \in N \setminus \{i\}$ and $f^\tau(A',k,A_i) > f^\tau(A,k,A_i)$. However, it is easily seen that this implies that $f(\tau(A'),k,\tau(A)_i) > f(\tau(A),k,\tau(A)_i)$, proving that f fails strategy-proofness, too. As this contradicts our initial assumption f^τ must be strategy-proof. □

As the final point of this section, we show that weak proportional representation allows us to increase the number of seats in the committee without changing the number of parties. Together with Proposition 1, Lemma 3, and Lemma 4, this then proves Theorem 2.

Lemma 6. *Assume that n, k are integers such that n is a multiple of k. If there is no P-APP voting rule that satisfies anonymity, strategy-proofness, and weak proporitonal representation for committees of size k, m parties, and n voters, there is also no such rule for committees of size $k+1$, m parties, and $\frac{n(k+1)}{k}$ voters.*

Proof. We prove the contraposition of the lemma and thus assume that there is a P-APP voting rule f that satisfies anonymity, strategy-proofness, and weak proportional representation for committees of the size $k+1$, m parties, and $\frac{n(k+1)}{k}$ voters. Based on f, we construct next a P-APP voting rule g for committees of size k, m parties, and $\frac{n(k+1)}{k}$ voters that satisfies the same axioms. To this end, let $x \in \mathcal{P}$ denote an arbitrary party. Then g is defined as follows: given a profile A on n voters, g appends $\frac{n}{k}$ voters who only approve x to derive a new profile A'. Finally, $g(A,k,z) = f(A',k+1,z)$ for all $z \in \mathcal{P} \setminus \{x\}$ and $g(A,k,x) = f(A',k+1,x)$.

We next show that g indeed satisfies all requirements. First, note that g is a well-defined P-APP voting rule for m, n, and k as $g(A,k,z) \geq 0$ for all $z \in \mathcal{P}$. In particular, observe here that $g(A,k,x) = f(A',k+1,x) - 1 \geq 0$ because weak proportional representation requires that $f(A',k+1,x) \geq 1$: at least $\frac{n}{k} \geq \frac{n(k+1)}{k(k+1)}$ voters approve x uniquely guaranteeing it one seat in the committee. Next, g is of course anonymous if f satisfies this property. As third point, it is easy to see that g satisfies strategy-proofness. The key observation is that for all profiles A, $g(A,k)$ differs from $f(A',k+1)$ only in the fact that f chooses x one time more. Hence, if there were profiles A, $\bar{A}$ and a voter i such that $A_j = A'_j$ for all $j \in N \setminus \{i\}$ and $g(\bar{A},k,A_i) > g(A,k,A_i)$, then also $f(\bar{A},k+1,A_i) > g(A,k+1,A_i)$. Or in other words, if g is manipulable, f is so too, which contradicts our assumptions. Finally, g satisfies weak proportional representation. For showing this, consider a profile A and a party z such that $j \geq \frac{\ell n}{k}$ uniquely approve z. Now, if $z \neq x$, then the same j candidates approve z uniquely and weak proportional representation requires that $f(A',k+1,z) \geq \ell$ because $j \geq \frac{\ell n}{k} = \frac{\ell n(k+1)}{k(k+1)}$. Since $g(A,k,z) = f(A,k+1,z)$, weak proportional representation is met in these cases. On the other hand, if $z = x$, there are $j + \frac{n}{k} \geq \frac{(\ell+1)n}{k}$ voters who approve x uniquely in A'. Similar computations as before show that weak proportional representation requires thus that $f(A',k+1,x) \geq \ell+1$, which proves that $g(A,k,x) \geq \ell$. Hence, g satisfies all required axioms, proving the lemma. □

B Proof of Corollary 1

In this appendix, we discuss the proof of Corollary 1. We derive the proof of this result by only slightly modifying the computer program used to show Proposition 1. In more detail, there are two differences: firstly, we additionally encode that the P-APP voting rule should satisfy Pareto-optimality and secondly, we hard-code the case distinction over the profile A^+ (see below) because this lead to a shorter proof. In particular, we did not change the encoding of strategy-proofness or weak representation. Consequently, the proof of Corollary 1 showcases that the reasoning of our computer program is correct.

Corollary 1. *There is no anonymous P-APP voting rule that satisfies weak representation, Pareto-optimality, and strategy-proofness if $k = 3$, $m = 4$, and $n = 6$.*

Proof. Assume for contradiction that there is an anonymous P-APP voting rule g that satisfies strategy-proofness, Pareto-optimality, and weak representation for $k = 3$, $m = 4$, and $n = 6$. Using Lemma 2, it follows that there is also a P-APP voting rule f that satisfies these axioms and that $f(A^*, 3) = [a, a, b]$ or $(A^*) = [a, b, c]$ for the profile A^* shown below.

$$A^* \quad a \quad b \quad ab \quad c \quad d \quad cd$$

As next step, we consider the profile A^+. First, note that a Pareto-dominates b in A^+, thus entailing that $f(A^+, 3, b) = 0$. Moreover, strategy-proofness requires that d is not in $f(A^+, k)$: otherwise, the voter with preference abc can manipulate by deviating to A^*. Finally, Lemma 1 implies thus that at least one seat is allocated to a and c each. Hence, it follows that $f(A^+, 3) = [a, a, c]$ or $f(A^+) = [a, c, c]$. We proceed with a case distinction for these two options.

$$A^+ \quad a \quad ab \quad c \quad abc \quad d \quad cd$$

Case 1: $f(A^+) = [a, c, c]$

As first case, we suppose that $f(A^+, 3) = [a, c, c]$. We consider the following 31 profiles shown in Table 2 to derive a contradiction, where $A^1 = A^+$ and $A^2 = A^*$.

For deriving a contradiction in this case, we use another case distinction with respect to the profile A^{18}. Note for this that Pareto-optimality requires that $f(A^{18}, 3, b) = 0$. Moreover, weak representation implies that c is allocated at least one seat in $f(A^{18}, 3, c)$. Finally, Lemma 1 for voter 4 shows that $f(A^{18}, 3, ac) \geq 2$ and for voter 6 that $f(A^{18}, 3, cd) \geq 2$. These conditions are only met by four committees: $f(A^{18}, 3) = [a, c, c]$, $f(A^{18}, 3) = [a, c, d]$, $f(A^{18}, 3) = [c, c, c]$, and $f(A^{18}, 3) = [c, c, d]$. We consider another case distinction with respect to these four cases.

For these four cases and all subsequent proofs, we introduce some short-hand notation. First, for every set X and profile A^k, we define $x^k_X = f(A^k, 3, X)$, i.e., the number of seats in x^k_X denotes how many seats are assigned to parties in X by $f(A^k, 3)$. Second, we write down our derivations in table form to keep the proof as short as possible. Each row of these tables contains a profile and the possible committees for this profile, and gives an explanation of why no other committee is feasible in form of constraints. Since we derive these proofs from our computer program, there are exactly four different reasons to why a committee is not possible: *(i)* Pareto-optimality (PO) requires that a party is not in the committee; *(ii)* weak representation (WR) requires that a party is chosen at least once; *(iii)* Lemma 1 requires that a sufficient number of seats are allocated to a set of parties, and *(iv)* strategy-proofness (SP) relates the outcome of two profiles. We explain this notation exemplarily in Case 1.1.

Case 1.1: $f(A^{18}, 3) \notin \{[a, c, d], [c, c, d]\}$

We can infer directly from $f(A^1, 3) = [a, c, c]$ that $f(A^{18}, 3) \notin \{[a, c, d], [c, c, d]\}$.

A^1	a	ab	c	abc	d	cd
A^2	a	b	ab	c	d	cd
A^3	a	b	c	bc	abc	d
A^4	a	b	c	abc	d	cd
A^5	a	ab	c	c	c	ad
A^6	a	ab	c	c	abc	d
A^7	a	ab	c	c	d	ad
A^8	a	ab	c	c	d	cd
A^9	a	ab	c	c	ad	cd
A^{10}	a	c	c	c	abc	d
A^{11}	a	c	c	c	d	ad
A^{12}	a	c	c	c	d	cd
A^{13}	a	c	c	c	ad	ad
A^{14}	a	c	c	c	ad	cd
A^{15}	a	c	c	bc	abc	d
A^{16}	a	c	c	abc	d	d
A^{17}	a	c	c	abc	d	ad
A^{18}	a	c	c	abc	d	cd
A^{19}	a	c	bc	bc	abc	d
A^{20}	b	c	c	c	abc	d
A^{21}	b	c	c	c	d	cd
A^{22}	b	c	c	c	ad	bd
A^{23}	b	c	c	c	bd	cd
A^{24}	b	c	c	ac	abc	d
A^{25}	b	c	c	ac	d	cd
A^{26}	b	c	ac	abc	d	cd
A^{27}	ab	c	c	c	ad	bd
A^{28}	ab	c	c	c	ad	abd
A^{29}	c	c	c	abc	d	ad
A^{30}	c	c	c	abc	d	bd
A^{31}	c	c	c	abc	ad	bd

Table 2: Profiles used for Case 1 of the proof of Corollary 1

	V_1	V_2	V_3	V_4	V_5	V_6	Possible outcomes	Reason
A^1	a	ab	c	abc	d	cd	$[a, c, c]$	Assumption
A^8	a	ab	c	c	d	cd	$[a, c, c]$	$x_b^6 = 0$ (PO) $x_{ab}^6 \geq 1$ (Lemma 1) $x_c^6 \geq x_c^1 = 2$ (SP from A^6 to A^1)
A^{18}	a	c	c	abc	d	cd	$[a, c, c]$, $[c, c, c]$	$x_{abc}^{18} \geq x_{abc}^8 = 3$ (SP from A^{18} to A^8)

The table is to be read as follows: first, we know by assumption that $f(A^{18}) = [a, c, c]$. Next, Pareto-optimality implies for A^8 that b cannot be in the committee. In turn, Lemma 2 implies that a needs to be chosen at least once for this profile. Moreover, strategy-proofness implies that $x_c^8 \geq 2$ because otherwise, voter 4 could manipulate by deviating to A^1. Finally, we derive A^{18} from A^8 by changing the preference of voter 2 from ab to abc. Hence, strategy-proofness implies that all three seats in the committee are distributed to parties in abc; otherwise, a voter can manipulate by redoing this step. In particular, this means that $x_d^{18} = 0$, which proves that $f(A^{18}) \notin \{[a, c, d], [c, c, d]\}$.

Case 1.2: $f(A^{18}, 3) \neq [c, c, c]$

Assume for contradiction that $f(A^{18}, 3) = [c, c, c]$. We derive a conflict at A^{13} as no valid committee exists for this profile.

	V_1	V_2	V_3	V_4	V_5	V_6	Possible outcomes	Reason
A^1	a	ab	c	abc	d	cd	$[a, c, c]$	Assumption
A^{18}	a	c	c	abc	d	cd	$[c, c, c]$	Assumption

A^{10}	a	c	c	c	abc	d	$[c,c,c]$	$x_c^{10} \geq x_c^{18} = 3$ (SP from A^{10} to A^{18})
A^{29}	c	c	c	abc	d	ad	$[c,c,d]$, $[c,d,d]$	$x_b^{29} = 0$ (PO) $x_c^{29} \geq 1$ (WR) $x_{ad}^{29} \geq 1$ (Lemma 1) $x_a^{29} \leq x_a^{10} = 0$ (SP from A^{10} to A^{29})
A^{12}	a	c	c	c	d	cd	$[c,c,c]$	$x_c^{12} \geq x_c^{29} = 3$ (SP from A^{12} to A^{29})
A^{14}	a	c	c	c	ad	cd	$[a,c,c]$, $[a,a,c]$	$x_b^{14} = 0$ (PO) $x_c^{14} \geq 1$ (WR) $x_{ad}^{14} \geq 1$ (Lemma 1) $x_d^{14} \leq x_d^{12} = 0$ (SP from A^{12} to A^{14})
A^{11}	a	c	c	c	d	ad	$[a,c,d]$	$x_b^{11} = 0$ (PO) $x_c^{11} \geq 1$ (WR) $x_{abc}^{11} \leq x_{abc}^{29} = 2$ (SP from A^{29} to A^{11}) $x_{ad}^{11} \leq x_{ad}^{14} = 2$ (SP from A^{14} to A^{11})
A^{6}	a	ab	c	c	abc	d	$[a,c,c]$	$x_b^{6} = 0$ (PO) $x_{ab}^{6} \geq 1$ (Lemma 1) $x_c^{6} \geq x_c^{1} = 2$ (SP from A^{6} to A^{1})
A^{17}	a	c	c	abc	d	ad	$[a,c,d]$	$x_b^{17} = 0$ (PO) $x_c^{17} \leq x_c^{11} = 1$ (SP from A^{11} to A^{17}) $x_{ab}^{17} \leq x_{ab}^{6} = 1$ (SP from A^{6} to A^{17}) $x_{abc}^{17} \geq x_{abc}^{11} = 2$ (SP from A^{17} to A^{11})
A^{7}	a	ab	c	c	d	ad	$[a,c,d]$	$x_b^{17} = 0$ (PO) $x_c^{7} \geq 1$ (WR) $x_{ab}^{7} \geq 1$ (Lemma 1) $x_{abc}^{7} \leq x_{abc}^{17} = 2$ (SP from A^{17} to A^{7})
A^{8}	a	ab	c	c	d	cd	$[a,c,c]$	$x_b^{8} = 0$ (PO) $x_{ab}^{8} \geq 1$ (Lemma 1) $x_c^{8} \geq x_c^{1} = 2$ (SP from A^{8} to A^{1})
A^{9}	a	ab	c	c	ad	cd	$[a,c,c]$	$x_b^{9} = 0$ (PO) $x_{ab}^{9} \geq 1$ (Lemma 1) $x_d^{9} \leq x_d^{8} = 0$ (SP from A^{8} to A^{9}) $x_{cd}^{9} \geq x_{cd}^{7} = 2$ (SP from A^{9} to A^{7})
A^{5}	a	ab	c	c	c	ad	$[a,c,c]$	$x_b^{5} = 0$ (PO) $x_{ab}^{5} \geq 1$ (Lemma 1) $x_c^{5} \geq x_c^{9} = 2$ (SP from A^{5} to A^{9})
A^{13}	a	c	c	c	ad	ad	$\lightning$	$x_b^{13} = 0$ (PO) $x_d^{13} = 0$ (PO) $x_{ab}^{13} \leq x_{ab}^{5} = 1$ (SP from A^{5} to A^{13}) $x_{ad}^{13} \geq x_{ad}^{11} = 2$ (SP from A^{13} to A^{11})

Case 1.3: $f(A^{18}, 3) \neq [a,c,c]$

Assume for contradiction that $f(A^{18}, 3) = [a,c,c]$. We show that this is not possible as no valid committee for A^{31} remains.

	V_1	V_2	V_3	V_4	V_5	V_6	Possible outcomes	Reason
A^1	a	ab	c	abc	d	cd	$[a,c,c]$	Assumption
A^2	a	b	ab	c	d	cd	$[a,a,c]$, $[a,b,c]$	Assumption
A^{18}	a	c	c	abc	d	cd	$[c,c,c]$	Assumption
A^{16}	a	c	c	abc	d	d	$[a,c,d]$	$x^{16}_b = 0$ (PO) $x^{16}_c \geq 1$ (WR) $x^{16}_d \geq 1$ (WR) $x^{16}_{cd} \leq x^{18}_{cd} = 2$ (SP from A^{18} to A^{16})
A^6	a	ab	c	c	abc	d	$[a,c,c]$	$x^6_b = 0$ (PO) $x^6_{ab} \geq 1$ (Lemma 1) $x^6_c \geq x^1_c = 2$ (SP from A^6 to A^1)
A^{17}	a	c	c	abc	d	ad	$[a,c,d]$	$x^{17}_b = 0$ (PO) $x^{17}_{abc} \geq 2$ (Lemma 1) $x^{17}_{ad} \geq x^{16}_{ad} = 2$ (SP from A^{17} to A^{16}) $x^{17}_{ab} \leq x^6_{ab} = 1$ (SP from A^6 to A^{17})
A^7	a	ab	c	c	d	ad	$[a,c,d]$	$x^{17}_b = 0$ (PO) $x^7_c \geq 1$ (WR) $x^7_{ab} \geq 1$ (Lemma 1) $x^7_{abc} \leq x^{17}_{abc} = 2$ (SP from A^{17} to A^7)
A^8	a	ab	c	c	d	cd	$[a,c,c]$	$x^8_b = 0$ (PO) $x^8_{ab} \geq 1$ (Lemma 1) $x^8_c \geq x^1_c = 2$ (SP from A^8 to A^1)
A^9	a	ab	c	c	ad	cd	$[a,c,c]$	$x^9_b = 0$ (PO) $x^9_{ab} \geq 1$ (Lemma 1) $x^9_d \leq x^8_d = 0$ (SP from A^8 to A^9) $x^9_{cd} \geq x^7_{cd} = 2$ (SP from A^9 to A^7)
A^5	a	ab	c	c	c	ad	$[a,c,c]$	$x^5_b = 0$ (PO) $x^5_{ab} \geq 1$ (Lemma 1) $x^5_c \geq x^9_c = 2$ (SP from A^5 to A^9)
A^{13}	a	c	c	c	ad	ad	$[a,c,c]$	$x^{13}_b = 0$ (PO) $x^{13}_d = 0$ (PO) $x^{13}_{ab} \leq x^5_{ab} = 1$ (SP from A^5 to A^{13}) $x^{13}_{ad} \geq x^5_{ad} = 1$ (SP from A^{13} to A^5)
A^{11}	a	c	c	c	d	ad	$[c,c,d]$	$x^{11}_b = 0$ (PO) $x^{11}_{ad} \leq x^{13}_{ad} = 1$ (SP from A^{13} to A^{11}) $x^{11}_{abc} \leq x^{17}_{abc} = 2$ (SP from A^{17} to A^{11})
A^{29}	c	c	c	abc	d	ad	$[c,c,d]$	$x^{29}_b = 0$ (PO) $x^{29}_{ad} \geq 1$ (Lemma 1) $x^{29}_a \leq x^{11}_a = 0$ (SP from A^{11} to A^{29}) $x^{29}_{abc} \geq x^{11}_{abc} = 2$ (SP from A^{29} to A^{11})
A^{15}	a	c	c	bc	abc	d	$[a,c,c]$	$x^{15}_b = 0$ (PO) $x^{15}_{bd} \geq x^6_{bd} = 2$ (SP from A^{15} to A^6) $x^{15}_{cd} \leq x^{18}_{cd} = 2$ (SP from A^{18} to A^{15})

A^{19}	a	c	bc	bc	abc	d	$[a,c,c]$, $[c,c,d]$	$x_b^{19} = 0$ (PO) $x_c^{19} \leq x_c^{15} = 2$ (SP from A^{15} to A^{19}) $x_{bc}^{19} \geq x_{bc}^{15} = 2$ (SP from A^{19} to A^{15})
A^{3}	a	b	c	bc	abc	d	$\mathbb{C}_3 \setminus$ $\{[b,b,b],[b,b,c],$ $[b,c,c],[c,c,c]\}$	$x_{bc}^{3} \leq x_{bc}^{19} = 2$ (SP from A^{19} to A^{3})
A^{4}	a	b	c	abc	d	cd	$[a,c,c]$	$x_{ab}^{4} \leq x_{ab}^{1} = 1$ (SP from A^{1} to A^{4}) $x_{d}^{4} \leq x_{d}^{2} = 0$ (SP from A^{2} to A^{4}) $x_{bc}^{4} \leq x_{bc}^{3} = 2$ (SP from A^{3} to A^{4})
A^{26}	b	c	ac	abc	d	cd	$[c,c,c]$	$x_a^{26} = 0$ (PO) $x_{ac}^{26} \geq x_{ac}^{4} = 3$ (SP from A^{26} to A^{4})
A^{24}	b	c	c	ac	abc	d	$[c,c,c]$	$x_c^{24} \geq x_c^{26} = 3$ (SP from A^{24} to A^{26})
A^{20}	b	c	c	c	abc	d	$[c,c,c]$	$x_c^{20} \geq x_c^{24} = 3$ (SP from A^{20} to A^{24})
A^{30}	c	c	c	abc	d	bd	$[c,c,d]$	$x_a^{30} = 0$ (PO) $x_{bd}^{30} \geq 1$ (Lemma 1) $x_b^{30} \leq x_b^{20} = 0$ (SP from A^{20} to A^{30}) $x_d^{30} \leq x_d^{29} = 1$ (SP from A^{29} to A^{30})
A^{25}	b	c	c	ac	d	cd	$[c,c,c]$	$x_c^{25} \geq x_c^{26} = 3$ (SP from A^{25} to A^{26})
A^{21}	b	c	c	c	d	cd	$[c,c,c]$	$x_c^{21} \geq x_c^{25} = 3$ (SP from A^{21} to A^{25})
A^{23}	b	c	c	c	bd	cd	$[b,b,c]$, $[b,c,c]$	$x_a^{23} = 0$ (PO) $x_c^{23} \geq 1$ (WR) $x_{bd}^{23} \geq 1$ (Lemma 1) $x_d^{23} \leq x_d^{21} = 0$ (SP from A^{21} to A^{23})
A^{22}	b	c	c	c	ad	bd	$[b,b,c]$, $[b,c,c]$, $[b,c,d]$	$x_a^{22} = 0$ (PO) $x_c^{22} \geq 1$ (WR) $x_{cd}^{22} \leq x_{cd}^{23} = 2$ (SP from A^{23} to A^{22})
A^{28}	ab	c	c	c	ad	abd	$[a,c,c]$	$x_b^{28} = 0$ (PO) $x_d^{28} = 0$ (PO) $x_{abd}^{28} \geq 1$ (Lemma 1) $x_a^{28} \leq x_a^{5} = 1$ (SP from A^{5} to A^{28})
A^{27}	ab	c	c	c	ad	bd	$[a,c,c]$, $[b,c,c]$	$x_{acd}^{27} \leq x_{acd}^{28} = 1$ (SP from A^{28} to A^{27}) $x_{ab}^{27} \geq x_{ab}^{22} = 1$ (SP from A^{27} to A^{22})
A^{31}	c	c	c	abc	ad	bd	$\lightning$	$x_{bd}^{31} \geq x_{bd}^{29} = 1$ (SP from A^{31} to A^{29}) $x_{ad}^{31} \geq x_{ad}^{30} = 1$ (SP from A^{31} to A^{30}) $x_{abc}^{31} \geq x_{abc}^{27} = 3$ (SP from A^{31} to A^{27}) $x_{ab}^{31} \leq x_{ab}^{27} = 1$ (SP from A^{27} to A^{31})

Case 2: $f(A^+, 3) = [a, a, c]$

For the second case, we assume that $f(A^+, 3) = [a, a, c]$. Unfortunately, this case is a fair bit more involved and we require the 76 profiles shown in Table 6 to derive a contradiction. Just as in the last case, we define $A^1 = A^+$. Subsequently, we infer the committees for multiple auxiliary profiles before we can derive a contradiction.

A^1	a	ab	c	abc	d	cd	A^{39}	a	a	c	d	d	bcd
A^2	a	a	a	c	abc	d	A^{40}	a	a	c	d	ad	cd
A^3	a	a	a	c	d	ad	A^{41}	a	a	c	d	abd	bcd
A^4	a	a	a	c	d	cd	A^{42}	a	a	ac	abc	d	bd
A^5	a	a	a	c	ad	cd	A^{43}	a	a	ac	d	bd	bd
A^6	a	a	a	abc	d	cd	A^{44}	a	a	abc	d	d	cd
A^7	a	a	a	d	d	bd	A^{45}	a	b	b	d	d	ad
A^8	a	a	a	d	d	cd	A^{46}	a	b	b	d	ad	cd
A^9	a	a	a	d	bd	bd	A^{47}	a	b	ab	d	d	d
A^{10}	a	a	b	b	d	ad	A^{48}	a	b	ab	d	d	ad
A^{11}	a	a	b	ab	d	d	A^{49}	a	b	ab	d	d	cd
A^{12}	a	a	b	c	abc	d	A^{50}	a	b	d	d	d	ad
A^{13}	a	a	b	c	d	bcd	A^{51}	a	b	d	d	d	bd
A^{14}	a	a	b	ac	abc	d	A^{52}	a	b	d	d	ad	bd
A^{15}	a	a	b	ac	d	ad	A^{53}	a	b	d	d	ad	cd
A^{16}	a	a	b	ac	d	bd	A^{54}	a	b	d	d	cd	bcd
A^{17}	a	a	b	ac	ad	bd	A^{55}	a	b	d	ad	cd	bcd
A^{18}	a	a	b	abc	d	ad	A^{56}	a	ab	d	d	d	bd
A^{19}	a	a	b	d	d	ad	A^{57}	a	ab	d	d	cd	bcd
A^{20}	a	a	b	d	d	bd	A^{58}	a	c	c	d	d	ad
A^{21}	a	a	b	d	d	bcd	A^{59}	a	c	c	d	d	abd
A^{22}	a	a	b	d	ad	bd	A^{60}	a	c	ac	d	d	d
A^{23}	a	a	b	d	ad	bcd	A^{61}	a	c	ac	d	d	ad
A^{24}	a	a	b	d	cd	bcd	A^{62}	a	c	ac	d	d	abd
A^{25}	a	a	ab	c	abc	d	A^{63}	a	c	abc	d	d	ad
A^{26}	a	a	ab	d	d	bd	A^{64}	a	c	abc	d	d	abd
A^{27}	a	a	ab	d	d	cd	A^{65}	a	c	d	d	d	ad
A^{28}	a	a	ab	d	d	bcd	A^{66}	a	c	d	d	d	abd
A^{29}	a	a	c	c	d	ad	A^{67}	a	c	d	d	d	cd
A^{30}	a	a	c	c	d	abd	A^{68}	a	c	d	d	d	bcd
A^{31}	a	a	c	abc	d	d	A^{69}	a	c	d	d	ad	cd
A^{32}	a	a	c	abc	d	ad	A^{70}	a	c	d	d	abd	bcd
A^{33}	a	a	c	abc	d	abd	A^{71}	a	ac	d	d	d	cd
A^{34}	a	a	c	abc	d	cd	A^{72}	a	ac	d	d	d	bcd
A^{35}	a	a	c	abc	d	bcd	A^{73}	b	ab	d	d	d	ad
A^{36}	a	a	c	d	d	ad	A^{74}	b	ab	d	d	ad	cd
A^{37}	a	a	c	d	d	abd	A^{75}	c	ac	d	d	d	ad
A^{38}	a	a	c	d	d	cd	A^{76}	c	ac	d	d	d	abd

Table 6: Profiles used for Case 2 of the proof of Corollary 1

Step 2.1: $f(A^{32}, 3) = [a, a, d]$

As first step, we show for the profile A^{32} that $f(A^{32}, 3) = [a, a, d]$. Assume for contradiction that this is not the case and consider the following derivation.

	V_1	V_2	V_3	V_4	V_5	V_6	Possible outcomes	Reason
A^1	a	ab	c	abc	d	cd	$[a, a, c]$	Assumption
A^{34}	a	a	c	abc	d	cd	$[a, a, c]$, $[a, a, d]$	$x^{34}_{cd} \geq 1$ (Lemma 1) $x^{34}_a \geq x^1_a = 2$ (SP from A^{34} to A^1)
A^{31}	a	a	c	abc	d	d	$[a, a, d]$	$x^{31}_b = 0$ (PO) $x^{31}_d \geq 1$ (WR) $x^{31}_{cd} \leq x^{34}_{cd} = 1$ (SP from A^{34} to A^{31})
A^{32}	a	a	c	abc	d	ad	$[a, a, a]$	$x^{32}_{abc} \geq 2$ (Lemma 1) $x^{32}_{ad} \geq x^{31}_{ad} = 3$ (SP from A^{32} to A^{31}) $f(A^{32}, 3) \neq [a, a, d]$ (Assumption)
A^2	a	a	a	c	abc	d	$[a, a, a]$	$x^2_a \geq x^{32}_a = 3$ (SP from A^2 to A^{32})
A^6	a	a	a	abc	d	cd	$[a, a, d]$	$x^6_{cd} \geq 1$ (Lemma 1) $x^6_c \leq x^2_c = 0$ (SP from A^2 to A^6) $x^6_a \geq x^{34}_a = 2$ (SP from A^6 to A^{34})
A^4	a	a	a	c	d	cd	$[a, a, d]$	$x^4_{abc} \leq x^6_{abc} = 2$ (SP from A^6 to A^4) $x^4_a \geq x^{34}_a = 2$ (SP from A^4 to A^{34})
A^3	a	a	a	c	d	ad	$[a, a, a]$	$x^3_a \geq x^{32}_a = 3$ (SP from A^3 to A^{32})
A^5	a	a	a	c	ad	cd	↯	$x^5_{cd} \geq 1$ (Lemma 1) $x^5_d \leq x^3_d = 0$ (SP from A^3 to A^5) $x^5_{ad} \geq x^4_{ad} = 3$ (SP from A^5 to A^4)

Step 2.2: $f(A^{38}, 3) = [a, c, d]$

Next, we investigate the outcome for the profile A^{38} shown below. In particular, we show that $f(A^{38}, 3) = [a, c, d]$. Assume for contradiction that this is not the case and consider the following derivation.

	V_1	V_2	V_3	V_4	V_5	V_6	Possible outcomes	Reason
A^{38}	a	a	c	d	d	cd	$[a, d, d]$	$x^{38}_a \geq 1$ (WR) $x^{38}_d \geq 1$ (WR) $x^{38}_{cd} \geq 2$ (Lemma 1) $f(A^{38}, 3) \neq [a, c, d]$ (Assumption)
A^1	a	ab	c	abc	d	cd	$[a, a, c]$	Assumption
A^{32}	a	a	c	abc	d	ad	$[a, a, d]$	Step 2.1
A^{29}	a	a	c	c	d	ad	$[a, c, d]$	$x^{29}_a \geq 1$ (WR) $x^{29}_c \geq 1$ (WR) $x^{29}_{ad} \geq 2$ (Lemma 1) $x^{29}_{abc} \leq x^{32}_{abc} = 2$ (SP from A^{32} to A^{29})

A^{40}	a	a	c	d	ad	cd	$[a, d, d]$	$x_a^{40} \geq 1$ (WR) $x_{cd}^{40} \geq x_{cd}^{29} = 2$ (SP from A^{40} to A^{29}) $x_{ad}^{40} \geq x_{ad}^{38} = 3$ (SP from A^{40} to A^{38})
A^{36}	a	a	c	d	d	ad	$[a, d, d]$	$x_a^{36} \geq 1$ (WR) $x_d^{36} \geq x_d^{40} = 2$ (SP from A^{36} to A^{40})
A^{34}	a	a	c	abc	d	cd	$[a, a, c]$, $[a, a, d]$	$x_{cd}^{34} \geq 1$ (Lemma 1) $x_a^{34} \geq x_a^1 = 2$ (SP from A^{34} to A^1)
A^{31}	a	a	c	abc	d	d	$[a, a, d]$	$x_b^{31} = 0$ (PO) $x_d^{31} \geq 1$ (WR) $x_{cd}^{31} \leq x_{cd}^{34} = 1$ (SP from A^{34} to A^{31})
A^{63}	a	c	abc	d	d	ad	$[a, d, d]$	$x_{abc}^{63} \geq 1$ (Lemma 1) $x_a^{63} \leq x_a^{36} = 1$ (SP from A^{36} to A^{63}) $x_{ad}^{63} \geq x_{ad}^{31} = 3$ (SP from A^{63} to A^{31})
A^{58}	a	c	c	d	d	ad	$[c, d, d]$	$x_c^{58} \geq 1$ (WR) $x_{abc}^{58} \leq x_{abc}^{63} = 1$ (SP from A^{63} to A^{58})
A^{69}	a	c	d	d	ad	cd	$[d, d, d]$	$x_{cd}^{69} \geq x_{cd}^{58} = 3$ (SP from A^{69} to A^{58}) $x_{ad}^{69} \geq x_{ad}^{38} = 3$ (SP from A^{69} to A^{38})
A^{65}	a	c	d	d	d	ad	$[d, d, d]$	$x_d^{65} \geq x_d^{69} = 3$ (SP from A^{65} to A^{69})
A^{67}	a	c	d	d	d	cd	$[d, d, d]$	$x_d^{67} \geq x_d^{69} = 3$ (SP from A^{67} to A^{69})
A^{71}	a	ac	d	d	d	cd	$[a, d, d]$, $[a, a, d]$	$x_b^{71} = 0$ (PO) $x_d^{71} \geq 1$ (WR) $x_{ac}^{71} \geq 1$ (Lemma 1) $x_c^{71} \leq x_c^{67} = 0$ (SP from A^{67} to A^{71})
A^{61}	a	c	ac	d	d	ad	$[a, d, d]$, $[c, d, d]$	$x_{ac}^{61} \geq 1$ (Lemma 1) $x_{abc}^{61} \leq x_{abc}^{63} = 1$ (SP from A^{63} to A^{61})
A^{60}	a	c	ac	d	d	d	$[a, d, d]$	$x_{ac}^{60} \geq 1$ (Lemma 1) $x_{cd}^{60} \leq x_{cd}^{71} = 2$ (SP from A^{71} to A^{60}) $x_c^{60} \geq x_c^{61} = 2$ (SP from A^{60} to A^{61})
A^{75}	c	ac	d	d	d	ad	↯	$x_{ac}^{75} \geq 1$ (Lemma 1) $x_a^{75} \leq x_a^{65} = 0$ (SP from A^{65} to A^{75}) $x_{ad}^{75} \geq x_{ad}^{60} = 3$ (SP from A^{75} to A^{60})

Step 2.3: Deriving auxiliary profiles

As next step, we infer the outcomes for several profiles based on the knowledge that $f(A^1, 3) = [a, a, c]$, $f(A^{32}, 3) = [a, a, d]$, and $f(A^{38}, 3) = [a, c, d]$. Many of the outcomes derived in this step will be used in the subsequent deductions.

	V_1	V_2	V_3	V_4	V_5	V_6	Possible outcomes	Reason
A^1	a	ab	c	abc	d	cd	$[a, a, c]$	Assumption
A^{32}	a	a	c	abc	d	ad	$[a, a, d]$	Step 2.1
A^{38}	a	a	c	d	d	cd	$[a, c, d]$	Step 2.2

A^{29}	a	a	c	c	d	ad	$[a, c, d]$	$x^{29}_{a} \geq 1$ (WR) $x^{29}_{c} \geq 1$ (WR) $x^{29}_{ad} \geq 2$ (Lemma 1) $x^{29}_{abc} \leq x^{32}_{abc} = 2$ (SP from A^{32} to A^{29})
A^{40}	a	a	c	d	ad	cd	$[a, c, d]$	$x^{40}_{a} \geq 1$ (WR) $x^{40}_{abc} \leq x^{31}_{abc} = 2$ (SP from A^{31} to A^{40}) $x^{40}_{d} \leq x^{38}_{d} = 1$ (SP from A^{38} to A^{40}) $x^{40}_{cd} \geq x^{29}_{cd} = 2$ (SP from A^{40} to A^{29})
A^{34}	a	a	c	abc	d	cd	$[a, a, c]$	$x^{34}_{b} = 0$ (PO) $x^{34}_{ad} \leq x^{40}_{ad} = 2$ (SP from A^{40} to A^{34}) $x^{34}_{a} \geq x^{1}_{a} = 2$ (SP from A^{34} to A^{1})
A^{31}	a	a	c	abc	d	d	$[a, a, d]$	$x^{31}_{b} = 0$ (PO) $x^{31}_{d} \geq 1$ (WR) $x^{31}_{cd} \leq x^{34}_{cd} = 1$ (SP from A^{34} to A^{31})
A^{44}	a	a	abc	d	d	cd	$[a, a, d]$	$x^{44}_{b} = 0$ (PO) $x^{44}_{d} \geq 1$ (WR) $x^{44}_{c} \leq x^{31}_{c} = 0$ (SP from A^{31} to A^{44}) $x^{44}_{abc} \geq x^{38}_{abc} = 2$ (SP from A^{44} to A^{38})
A^{27}	a	a	ab	d	d	cd	$[a, a, d]$	$x^{27}_{b} = 0$ (PO) $x^{27}_{c} = 0$ (PO) $x^{27}_{d} \geq 1$ (WR) $x^{27}_{ab} \geq x^{44}_{ab} = 2$ (SP from A^{27} to A^{44})
A^{8}	a	a	a	d	d	cd	$[a, a, d]$	$x^{8}_{b} = 0$ (PO) $x^{8}_{c} = 0$ (PO) $x^{8}_{abc} \leq x^{44}_{abc} = 2$ (SP from A^{44} to A^{8}) $x^{8}_{a} \geq x^{44}_{a} = 2$ (SP from A^{8} to A^{44})
A^{7}	a	a	a	d	d	bd	$[a, a, d]$	$x^{7}_{b} = 0$ (PO) $x^{7}_{c} = 0$ (PO) $x^{7}_{cd} \leq x^{8}_{cd} = 1$ (SP from A^{8} to A^{7}) $x^{7}_{bd} \geq x^{8}_{bd} = 1$ (SP from A^{7} to A^{8})
A^{9}	a	a	a	d	bd	bd	$[a, a, d]$	$x^{9}_{b} = 0$ (PO) $x^{9}_{c} = 0$ (PO) $x^{9}_{c} \leq x^{7}_{c} = 1$ (SP from A^{7} to A^{9}) $x^{9}_{bd} \geq x^{7}_{bd} = 1$ (SP from A^{9} to A^{7})
A^{43}	a	a	ac	d	bd	bd	$[a, a, d]$	$x^{43}_{b} = 0$ (PO) $x^{43}_{c} = 0$ (PO) $x^{43}_{a} \leq x^{9}_{a} = 2$ (SP from A^{9} to A^{43}) $x^{43}_{ac} \geq x^{9}_{ac} = 2$ (SP from A^{43} to A^{9})
A^{16}	a	a	b	ac	d	bd	$[a, a, b]$, $[a, a, d]$	$x^{16}_{c} = 0$ (PO) $x^{16}_{bd} \geq 1$ (Lemma 1) $x^{43}_{bd} \leq x^{16}_{bd} = 1$ (SP from A^{16} to A^{43})

A^4	a	a	a	c	d	cd	$[a,a,c]$	$x_b^4 = 0$ (PO) $x_{cd}^4 \geq 1$ (Lemma 1) $x_{ad}^4 \leq x_{ad}^{40} = 2$ (SP from A^{40} to A^4) $x_a^4 \geq x_a^{34} = 2$ (SP from A^4 to A^{34})
A^6	a	a	a	abc	d	cd	$[a,a,c]$	$x_b^6 = 0$ (PO) $x_{cd}^6 \geq 1$ (Lemma 1) $x_c^6 \leq x_c^4 = 1$ (SP from A^4 to A^6) $x_{abc}^6 \geq x_{abc}^4 = 3$ (SP from A^6 to A^4)
A^2	a	a	a	c	abc	d	$[a,a,c]$	$x_b^2 = 0$ (PO) $x_c^2 \geq x_c^6 = 1$ (SP from A^2 to A^6) $x_a^2 \geq x_a^{34} = 2$ (SP from A^2 to A^{34})
A^{25}	a	a	ab	c	abc	d	$[a,a,c]$, $[a,a,d]$	$x_b^{25} = 0$ (PO) $x_a^{25} \leq x_a^2 = 2$ (SP from A^2 to A^{25}) $x_{ab}^{25} \geq x_{ab}^2 = 2$ (SP from A^{25} to A^2)
A^{33}	a	a	c	abc	d	abd	$[a,a,d]$	$x_b^{33} = 0$ (PO) $x_a^{33} \leq x_a^2 = 2$ (SP from A^2 to A^{33}) $x_d^{33} \leq x_d^{31} = 1$ (SP from A^{31} to A^{33}) $x_{abd}^{33} \geq x_{abd}^{31} = 3$ (SP from A^{33} to A^{31})
A^{30}	a	a	c	c	d	abd	$[a,c,d]$	$x_b^{30} = 0$ (PO) $x_a^{30} \geq 1$ (WR) $x_c^{30} \geq 1$ (WR) $x_{abc}^{30} \leq x_{abc}^{33} = 2$ (SP from A^{33} to A^{30})
A^{41}	a	a	c	d	abd	bcd	$[a,c,d]$, $[a,d,d]$	$x_b^{41} = 0$ (PO) $x_a^{41} \geq 1$ (WR) $x_{abc}^{41} \leq x_{abc}^{33} = 2$ (SP from A^{33} to A^{41}) $x_{bcd}^{41} \geq x_{bcd}^{30} = 2$ (SP from A^{41} to A^{30})
A^{35}	a	a	c	abc	d	bcd	$[a,a,c]$, $[a,a,d]$	$x_b^{35} = 0$ (PO) $x_{bcd}^{35} \geq 1$ (Lemma 1) $x_{cd}^{35} \leq x_{cd}^{34} = 1$ (SP from A^{34} to A^{35})
A^{12}	a	a	b	c	abc	d	$[a,a,c]$, $[a,a,d]$	$x_{abc}^{12} \geq 1$ (Lemma 1) $x_{bcd}^{12} \leq x_{bcd}^{35} = 1$ (SP from A^{35} to A^{12}) $x_{ab}^{12} \leq x_{ab}^{25} = 2$ (SP from A^{25} to A^{12})

Step 2.4: $f(A^{18}, 3) = [a,a,d]$

In the fourth step, we prove that $f(A^{18}, 3) = [a,a,d]$. For this, we show first by contradiction that $f(R^{18}) \neq [a,a,a]$.

	V_1	V_2	V_3	V_4	V_5	V_6	Possible outcomes	Reason
A^{18}	a	a	b	abc	d	ad	$[a,a,a]$	Assumption
A^{14}	a	a	b	ac	abc	d	$[a,a,a]$	$x_c^{14} = 0$ (PO) $x_{ac}^{14} \geq x_{ac}^{18} = 3$ (SP from A^{14} to A^{18})

A^{42}	a	a	ac	abc	d	bd	$[a,a,d]$	$x_c^{42}=0$ (PO) $x_{abc}^{42}\geq 2$ (Lemma 1) $x_{bd}^{42}\geq 1$ (Lemma 1) $x_b^{42}\leq x_b^{14}=0$ (SP from A^{14} to A^{42})
A^{16}	a	a	b	ac	d	bd	$[a,a,d]$	$x_{abc}^{16}\leq x_{abc}^{42}=2$ (SP from A^{42} to A^{16}) $f(A^{16},3)\in\{[a,a,b],[a,a,d]\}$ (Step 2.3)
A^{15}	a	a	b	ac	d	ad	$[a,a,a]$	$x_c^{15}=0$ (PO) $x_{ac}^{15}\geq x_{ac}^{18}=3$ (SP from A^{15} to A^{18})
A^{17}	a	a	b	ac	ad	bd	↯	$x_{bd}^{17}\geq 1$ (Lemma 1) $x_d^{15}\leq x_d^{17}=0$ (SP from A^{17} to A^{15}) $x_{ad}^{15}\geq x_{ad}^{16}=3$ (SP from A^{15} to A^{16})

Next, we complete the proof of this step by showing that the assumption $f(A^{18},3)\notin\{[a,a,a],[a,a,d]\}$ results in a contradiction.

	V_1	V_2	V_3	V_4	V_5	V_6	Possible outcomes	Reason
A^{18}	a	a	b	abc	d	ad	$[a,a,b]$, $[a,b,d]$	$x_c^{18}=0$ (PO) $x_a^{18}\geq 1$ (WR) $x_{ad}^{18}\geq 2$ (Lemma 1) $x_{abc}^{18}\geq 2$ (Lemma 1) $f(A^{18},3)\neq[a,a,a]$ (Assumption) $f(A^{18},3)\neq[a,a,d]$ (Assumption)
A^{14}	a	a	b	ac	abc	d	$[a,a,b]$, $[a,b,b]$, $[a,b,d]$	$x_c^{14}=0$ (PO) $x_a^{14}\geq 1$ (WR) $x_{ad}^{14}\leq x_{ad}^{18}=2$ (SP from A^{18} to A^{14})
A^{12}	a	a	b	c	abc	d		$x_{ac}^{12}\leq x_{ac}^{14}=2$ (SP from A^{14} to A^{12}) $x_{ad}^{18}\leq x_{ad}^{14}=2$ (SP from A^{14} to A^{18}) $f(A^{12},3)\in\{[a,a,c],[a,a,d]\}$ (Step 2.3)

Step 2.5: $f(A^{22},3)=[a,b,d]$

We continue by showing that $f(A^{22},3)=[a,b,d]$. Once again, we assume that this is not the case and consider the following derivation.

	V_1	V_2	V_3	V_4	V_5	V_6	Possible outcomes	Reason
A^{18}	a	a	b	abc	d	ad	$[a,a,d]$	Step 2.4
A^{10}	a	a	b	b	d	ad	$[a,b,d]$	$x_a^{10}\geq 1$ (WR) $x_b^{10}\geq 1$ (WR) $x_{abc}^{10}\leq x_{abc}^{18}=2$ (SP from A^{18} to A^{10})

A^{22}	a	a	b	d	ad	bd	$[a,d,d]$	$x_a^{22} \geq 1$ (WR) $x_b^{22} \leq x_b^{10} = 1$ (SP from A^{10} to A^{22}) $x_{bd}^{22} \geq x_{bd}^{10} = 2$ (SP from A^{22} to A^{10}) $f(A^{22},3) \neq [a,b,d]$
A^{20}	a	a	b	d	d	bd	$[a,d,d]$	$x_a^{20} \geq 1$ (WR) $x_d^{20} \geq x_d^{22} = 2$ (SP from A^{20} to A^{22})
A^{27}	a	a	ab	d	d	cd	$[a,a,d]$	Step 2.3
A^{26}	a	a	ab	d	d	bd	$[a,a,d]$	$x_c^{26} = 0$ (PO) $x_d^{26} \geq 1$ (WR) $x_{cd}^{26} \leq x_{cd}^{27} = 1$ (SP from A^{27} to A^{26}) $x_b^{26} \leq x_b^{20} = 0$ (SP from A^{20} to A^{26})
A^{11}	a	a	b	ab	d	d	$[a,a,d]$	$x_c^{11} = 0$ (PO) $x_d^{11} \geq 1$ (WR) $x_{bd}^{11} \leq x_{bd}^{26} = 1$ (SP from A^{26} to A^{11})
A^{19}	a	a	b	d	d	ad	$[a,d,d]$	$x_a^{19} \geq 1$ (WR) $x_d^{19} \geq x_d^{22} = 2$ (SP from A^{19} to A^{22})
A^{48}	a	b	ab	d	d	ad	$[a,d,d]$	$x_{ab}^{48} \geq 1$ (Lemma 1) $x_a^{48} \leq x_a^{19} = 1$ (SP from A^{19} to A^{48}) $x_{ad}^{48} \geq x_{ad}^{11} = 3$ (SP from A^{48} to A^{11})
A^{45}	a	b	b	d	d	ad	$[b,d,d]$	$x_c^{45} = 0$ (PO) $x_b^{45} \geq 1$ (WR) $x_{ab}^{45} \leq x_{ab}^{48} = 1$ (SP from A^{48} to A^{45})
A^{52}	a	b	d	d	ad	bd	$[d,d,d]$	$x_{bd}^{52} \geq x_{bd}^{45} = 3$ (SP from A^{52} to A^{45}) $x_{ad}^{52} \geq x_{ad}^{20} = 3$ (SP from A^{52} to A^{20})
A^{51}	a	b	d	d	d	bd	$[d,d,d]$	$x_d^{51} \geq x_d^{52} = 3$ (SP from A^{51} to A^{52})
A^{56}	a	ab	d	d	d	bd	$[a,d,d]$, $[a,a,d]$	$x_c^{56} = 0$ (PO) $x_d^{56} \geq 1$ (WR) $x_{ab}^{56} \geq 1$ (Lemma 1) $x_b^{56} \leq x_b^{51} = 0$ (SP from A^{51} to A^{56})
A^{47}	a	b	ab	d	d	d	$[a,d,d]$	$x_c^{47} = 0$ (PO) $x_{bd}^{47} \leq x_{bd}^{56} = 2$ (SP from A^{56} to A^{47}) $x_d^{47} \geq x_d^{48} = 2$ (SP from A^{47} to A^{48})
A^{50}	a	b	d	d	d	ad	$[d,d,d]$	$x_d^{50} \geq x_d^{52} = 3$ (SP from A^{50} to A^{52})
A^{73}	b	ab	d	d	d	ad	$\lightning$	$x_{ab}^{73} \geq 1$ (Lemma 1) $x_a^{73} \leq x_a^{50} = 0$ (SP from A^{50} to A^{73}) $x_{ad}^{73} \geq x_{ad}^{47} = 3$ (SP from A^{73} to A^{47})

Step 2.6: $f(A^{41},3) = [a,d,d]$

As next step, we show that $f(A^{41},3) = [a,d,d]$. Once again, assume that this is not the case and consider the following derivation.

	V_1	V_2	V_3	V_4	V_5	V_6	Possible outcomes	Reason
A^{41}	a	a	c	d	abd	bcd	$[a,b,d]$	$f(A^{41},3) \in \{[a,b,d],[a,d,d]\}$ (Step 2.3) $f(A^{41},3) \neq [a,d,d]$ (Assumption)
A^{22}	a	a	b	d	ad	bd	$[a,b,d]$	Step 2.5
A^{16}	a	a	b	ac	d	bd	$[a,a,b]$	$x^{16}_{ad} \leq x^{22}_{ad} = 2$ (SP from A^{22} to A^{16}) $f(A^{16},3) \in \{[a,a,b],[a,a,d]\}$ (Step 2.3)
A^{42}	a	a	ac	abc	d	bd	$[a,a,b]$	$x^{42}_{c} = 0$ (PO) $x^{42}_{bd} \geq 1$ (Lemma 1) $x^{42}_{b} \leq x^{16}_{b} = 1$ (SP from A^{16} to A^{42}) $x^{42}_{abc} \geq x^{16}_{abc} = 3$ (SP from A^{42} to A^{16})
A^{14}	a	a	b	ac	abc	d	$[a,a,b]$	$x^{14}_{c} = 0$ (PO) $x^{14}_{bd} \leq x^{42}_{bd} = 1$ (SP from A^{42} to A^{14}) $x^{14}_{b} \geq x^{42}_{b} = 1$ (SP from A^{14} to A^{42})
A^{12}	a	a	b	c	abc	d	$[a,a,d]$	$x^{12}_{ac} \leq x^{14}_{ac} = 2$ (SP from A^{14} to A^{12}) $f(A^{12},3) \in \{[a,a,c],[a,a,d]\}$ (Step 2.3)
A^{13}	a	a	b	c	d	bcd	$[a,c,d]$	$x^{41}_{a} \geq 1$ (WR) $x^{13}_{abc} \leq x^{12}_{abc} = 2$ (SP from A^{12} to A^{13}) $x^{13}_{abd} \leq x^{41}_{abd} = 2$ (SP from A^{41} to A^{13})
A^{24}	a	a	b	d	cd	bcd	$[a,d,d]$	$x^{24}_{c} = 0$ (PO) $x^{24}_{a} \geq 1$ (WR) $x^{24}_{cd} \geq x^{13}_{cd} = 2$ (SP from A^{24} to A^{13})
A^{18}	a	a	b	abc	d	ad	$[a,a,d]$	Step2.4
A^{10}	a	a	b	b	d	ad	$[a,b,d]$	$x^{10}_{c} = 0$ (PO) $x^{10}_{a} \geq 1$ (WR) $x^{10}_{b} \geq 1$ (WR) $x^{10}_{abc} \leq x^{18}_{abc} = 2$ (SP from A^{18} to A^{10})
A^{23}	a	a	b	d	ad	bcd	$[a,d,d]$	$x^{23}_{a} \geq 1$ (WR) $x^{23}_{bcd} \geq x^{10}_{bcd} = 2$ (SP from A^{23} to A^{10}) $x^{23}_{ad} \geq x^{24}_{ad} = 3$ (SP from A^{23} to A^{24})
A^{19}	a	a	b	d	d	ad	$[a,d,d]$	$x^{19}_{a} \geq 1$ (WR) $x^{19}_{d} \geq x^{23}_{d} = 2$ (SP from A^{19} to A^{23})
A^{21}	a	a	b	d	d	bcd	$[a,d,d]$	$x^{21}_{a} \geq 1$ (WR) $x^{21}_{d} \geq x^{23}_{d} = 2$ (SP from A^{21} to A^{23})
A^{27}	a	a	ab	d	d	cd	$[a,a,d]$	Step 2.3
A^{28}	a	a	ab	d	d	bcd	$[a,a,d]$	$x^{28}_{c} = 0$ (PO) $x^{28}_{d} \geq 1$ (WR) $x^{28}_{b} \leq x^{21}_{b} = 0$ (SP from A^{21} to A^{28}) $x^{27}_{cd} \leq x^{21}_{cd} = 1$ (SP from A^{21} to A^{27})
A^{11}	a	a	b	ab	d	d	$[a,a,d]$	$x^{11}_{d} \geq 1$ (WR) $x^{11}_{bcd} \leq x^{28}_{bcd} = 1$ (SP from A^{28} to A^{11})

A^{48}	a	b	ab	d	d	ad	$[a,d,d]$	$x^{48}_{ab} \geq 1$ (Lemma 1) $x^{48}_{a} \leq x^{19}_{a} = 1$ (SP from A^{19} to A^{48}) $x^{48}_{ad} \geq x^{11}_{ad} = 3$ (SP from A^{48} to A^{11})
A^{45}	a	b	b	d	d	ad	$[b,d,d]$	$x^{45}_{c} = 0$ (PO) $x^{45}_{b} \geq 1$ (WR) $x^{45}_{ab} \leq x^{48}_{ab} = 1$ (SP from A^{48} to A^{45})
A^{46}	a	b	b	d	ad	cd	$[b,d,d]$	$x^{46}_{c} = 0$ (PO) $x^{46}_{b} \geq 1$ (WR) $x^{46}_{cd} \geq x^{45}_{cd} = 2$ (SP from A^{46} to A^{45})
A^{55}	a	b	d	ad	cd	bcd	$[d,d,d]$	$x^{55}_{bcd} \geq x^{46}_{bcd} = 3$ (SP from A^{55} to A^{46}) $x^{55}_{ad} \geq x^{24}_{ad} = 3$ (SP from A^{55} to A^{24})
A^{53}	a	b	d	d	ad	cd	$[d,d,d]$	$x^{53}_{d} \geq x^{55}_{d} = 3$ (SP from A^{53} to A^{55})
A^{74}	b	ab	d	d	ad	cd	$[b,d,d]$	$x^{74}_{c} = 0$ (PO) $x^{74}_{ab} \geq 1$ (Lemma 1) $x^{74}_{a} \leq x^{53}_{a} = 0$ (SP from A^{53} to A^{74}) $x^{74}_{cd} \geq x^{48}_{cd} = 2$ (SP from A^{74} to A^{48})
A^{49}	a	b	ab	d	d	cd	$[b,d,d]$	$x^{49}_{c} = 0$ (PO) $x^{49}_{ad} \leq x^{74}_{ad} = 2$ (SP from A^{74} to A^{49}) $x^{49}_{cd} \geq x^{48}_{cd} = 2$ (SP from A^{49} to A^{48})
A^{57}	a	ab	d	d	cd	bcd	$[b,d,d]$	$x^{57}_{c} = 0$ (PO) $x^{57}_{ab} \geq 1$ (Lemma 1) $x^{57}_{b} \leq x^{74}_{b} = 1$ (SP from A^{74} to A^{57}) $x^{57}_{bcd} \geq x^{74}_{bcd} = 3$ (SP from A^{57} to A^{74})
A^{54}	a	b	d	d	cd	bcd	$[b,d,d]$	$x^{54}_{c} = 0$ (PO) $x^{54}_{ab} \leq x^{57}_{ab} = 1$ (SP from A^{57} to A^{54}) $x^{54}_{b} \geq x^{57}_{b} = 1$ (SP from A^{54} to A^{57})
A^{55}	a	b	d	ad	cd	bcd	$\lightning$	$x^{55}_{d} \leq x^{54}_{d} = 2$ (SP from A^{54} to A^{55}) $x^{55}_{bcd} \geq x^{46}_{bcd} = 3$ (SP from A^{55} to A^{46}) $x^{55}_{ad} \geq x^{24}_{ad} = 3$ (SP from A^{55} to A^{24})

Step 2.7: The final contradiction

Using all the gathered knowledge so far, we finally prove that $f(A^1, 3) = [a,a,c]$ is not possible by considering the following derivation.

	V_1	V_2	V_3	V_4	V_5	V_6	Possible outcomes	Reason
A^{41}	a	a	c	d	abd	bcd	$[a,d,d]$	Step 2.6
A^{37}	a	a	c	d	d	abd	$[a,d,d]$	$x^{37}_{a} \geq 1$ (WR) $x^{37}_{d} \geq x^{41}_{d} = 2$ (SP from A^{37} to A^{41})
A^{31}	a	a	c	abc	d	d	$[a,a,d]$	Step 2.3

A^{64}	a	c	abc	d	d	abd	a, d, d	$x_b^{64} = 0$ (PO) $x_{abc}^{64} \geq 1$ (Lemma 1) $x_a^{64} \leq x_a^{37} = 1$ (SP from A^{37} to A^{64}) $x_{abd}^{64} \geq x_{abd}^{31} = 3$ (SP from A^{64} to A^{31})
A^{59}	a	c	c	d	d	abd	$[c, d, d]$	$x_c^{59} \geq 1$ (WR) $x_{abc}^{59} \leq x_{abc}^{64} = 1$ (SP from A^{64} to A^{59})
A^{39}	a	a	c	d	d	bcd	$[a, d, d]$	$x_a^{39} \geq 1$ (WR) $x_c^{39} \geq x_c^{41} = 2$ (SP from A^{39} to A^{41})
A^{70}	a	c	d	d	abd	bcd	$[d, d, d]$	$x_b^{70} = 0$ (PO) $x_{abd}^{70} \geq x_{abd}^{39} = 3$ (SP from A^{70} to A^{39}) $x_{bcd}^{70} \geq x_{bcd}^{59} = 3$ (SP from A^{70} to A^{59})
A^{68}	a	c	d	d	d	bcd	$[d, d, d]$	$x_d^{68} \geq x_d^{70} = 3$ (SP from A^{68} to A^{70})
A^{72}	a	ac	d	d	d	bcd	$[a, a, d]$, $[a, d, d]$	$x_b^{72} = 0$ (PO) $x_d^{72} \geq 1$ (WR) $x_{ac}^{72} \geq 1$ (Lemma 1) $x_c^{72} \leq x_c^{68} = 0$ (SP from A^{68} to A^{72})
A^{62}	a	c	ac	d	d	abd	$[a, d, d]$, $[c, d, d]$	$x_{ac}^{62} \geq 1$ (Lemma 1) $x_{abc}^{64} \leq x_{abc}^{62} = 1$ (SP from A^{62} to A^{64})
A^{60}	a	c	ac	d	d	d	$[a, d, d]$	$x_{bcd}^{60} \leq x_{bcd}^{72} = 2$ (SP from A^{72} to A^{60}) $x_d^{60} \geq x_d^{62} = 2$ (SP from A^{60} to A^{62})
A^{66}	a	c	d	d	d	abd	$[d, d, d]$	$x_d^{66} \geq x_d^{70} = 3$ (SP from A^{66} to A^{70})
A^{76}	c	ac	d	d	d	abd	$\lightning$	$x_{ac}^{76} \geq 1$ (Lemma 1) $x_c^{76} \leq x_c^{60} = 0$ (SP from A^{60} to A^{76}) $x_a^{76} \leq x_a^{66} = 0$ (SP from A^{66} to A^{76})

□