Combinatorial Auctions: Complexity and Algorithms

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Abstract

A combinatorial auction allows bidders to submit bids on bundles of objects and can be considered the pivotal example of a multiple object auctions. They also constitute a paradigmatic problem in algorithmic mechanism design. We provide an overview of both the computational complexity and strategic complexity inherent in the design of such auctions, and discuss how these challenges are addressed in various combinatorial auction formats.

An auction can be defined as "a market institution with an explicit set of rules determining resource allocation and prices on the basis of bids from the market participants" [1]. The competitive process serves to aggregate the scattered information about bidders' valuations and to dynamically set a price. The auction format determines the rules governing when and how a deal is closed [2]. Auctions are typically evaluated using two main criteria, *(allocative) efficiency* and *revenue*. The first one measures whether the objects end up with those bidders who value them most, while the latter focuses on the expected selling price.

Multiple object auctions can be divided in those, where *multiple units* of a single item are sold or where *multiple items* are sold. Of course also combinations are possible, where large quantities (multiple units) of different items get sold or bought, such as large quantities of different types of hard disk drives. Combinatorial auctions are a means to buy or sell multiple items. They have found application in a variety of domains such as the auctioning of spectrum licenses [3], truck load transportation [4], bus routes [5], or industrial procurement [6]. Original designs have been proposed by [7] for the allocation of airport time slots.

Combinatorial auctions address fundamental questions regarding efficiency and prices in markets [8, 9]. These questions have been at the core of algorithmic mechanism design, a discipline at the intersection of Computer Science, Economics, and Operations Research. In this article, we will focus on the design of combinatorial auctions, but we will also address some related types of auctions such as volume discount and multi-attribute auctions. We will mainly look at the efficiency of auction formats as a main goal, as most of the literature in this area does. The article provides a concise introduction and is in parts based on publications such as [10], which we refer to for a more detailed discussion. We will assume that the reader has a basic knowledge about single-object auctions and the respective theory.

1 Complexity in Combinatorial Auctions

Combinatorial auctions have been discussed in the literature, as they allow selling or buying a set of heterogeneous items to or from multiple bidders. Bidders can specify bundle bids, i.e., a price is defined for a subset of the items for auction [10]. The price is only valid for the entire set and the set is indivisible. For example, in a combinatorial auction a bidder might want to buy 10 units of item x and 20 units of item y for a bundle price of $ 100, which might be more than the total of the prices for the items x and y sold individually. We will refer to a bidding language as a set of allowable bid types (e.g., bundle bids or bids on price and quantity) in an auction. A bidding language allowing for bundle bids is also useful in procurement markets with economies of scope, where suppliers have cost complementarities due to reduced production or transportation costs for a set of items. In this case, we will talk about either a combinatorial procurement auction or a combinatorial reverse auction.

Combinatorial auctions have been intensively discussed for the sale of spectrum licenses by the US Federal Communications Commission (FCC) [11]. The FCC divides licenses into different regions. Bidders - usually large telecom companies - often have super-additive preferences for licenses that are adjacent to each other. This can have advantages in advertising a service to the end customer, but also in the infrastructure that needs to be set up. In simultaneous auctions where no bundle bids are allowed, bidders incur the risk that they only win a subset of items from a set of items that they are interested in, and that they end up paying too much for the subset. This is also called the *exposure problem*. These types of preferences can easily be considered in combinatorial auctions. However, the design of combinatorial auctions is such that several types of complexity can arise:

- The auctioneer faces *computational complexity* when determining an optimal allocation. The winner determination problem in combinatorial auctions is an NP-hard problem [12]. In addition, the auctioneer needs to derive ask prices in iterative auctions, which is typically a hard computational problem as well.
- A bidder needs to determine his valuations for $2^m - 1$ bundles, where m is the number of items. We will refer to this as *valuation complexity*. Without restrictions, this would require to elicit 1023 valuations for an auction with only 10 items of interest.
- Even if the bidders knew their valuations perfectly, they would still need to decide how to respond during the auction. The issues relate to when and how they reveal their preferences. We will describe this as *strategic complexity*. Researchers have proposed different auction formats which exhibit various degrees of strategic complexity for bidders [13].
- Finally, *communication complexity* describes the number of messages that need to be exchanged between the auctioneer and the bidders in order to determine the optimal allocation. It has been shown, that the communication complexity in combinatorial auctions is exponential [14].

We will only focus on computational (Section 2) and strategic complexity (Section 3) in the following sections of this article. In Section 4, we will discuss different auction formats that have been suggested in the literature, and how they address these complexities.

2 Computational Complexity

First, we will concentrate on the winner determination problem in combinatorial auctions [15, 16, 17, 18]. It is a good example of the types of optimization

		Bids			
Line	Bids	B1	B2	B3	B4
1	1000t grain in Berlin	1	0	1	1
2	800t grain in Munich	0	1	1	1
3	800t grain in Vienna	1	1	1	0
4	Bid price (in thousands)	\$ 150	\$ 125	\$ 300	\$ 125

Table 1: Example with bundle bids

problems that one encounters in various multi-object auctions. The following example with 4 bids and 3 items illustrates a simple procurement application (see Table 1). The buying organization needs different quantities of grain in different production sites. In this case, the buyer aggregates demand for multiple production sites, as suppliers might be able to provide better prices due to reduced production and transportation costs. Suppliers bid on subsets of the locations and each subset has a bundle price. In this article, we assume suppliers to provide the entire quantity for an item or location. In case they can provide subsets of the quantity, e.g., only 500t of grain for Berlin, this is referred to as a multi-unit combinatorial auction.

Given the bidder valuations for all possible bundles, the efficient allocation can be found by solving the ***Winner Determination Problem (WDP)***. Let $\mathcal{K} = \{1, \ldots, m\}$ denote the set of items indexed by k and $\mathcal{I} = \{1, \ldots, n\}$ denote the set of bidders indexed by i with private valuations $v_i(S) \geq 0$ for bundles $S \subseteq \mathcal{K}$, and p as the price. This means, each bidder i has a valuation function $v_i : 2^{\mathcal{K}} \rightarrow \mathbb{R}_0^+$ that attaches a value $v_i(S)$ to any bundle $S \subseteq \mathcal{K}$. In addition, we assume bidder values $v_i(S)$ to be independent and private (i.e., only known to the bidder), the bidders' utility function to be quasi-linear (i.e., the payoff of a bidder $\pi_i(S) = v_i(S) - p$) with free disposal (i.e., if $S \subset T$ then $v_i(S) \leq v_i(T)$). There are other situations where valuations are interdependent, such as the sale of a tract of land with an unknown amount of oil underground, where the bidders may have different estimates of the amount of oil based on privately conducted tests, but the final value is the same for all bidders. We will focus on independent and private valuations in this article.

The WDP in a forward auction can be formulated as a binary program using the decision variables $x_i(S)$ which indicate whether the bid of the bidder i for the bundle S belongs to the allocation:

$$
\begin{array}{llll}
\max\limits_{x_i(S)} \sum\limits_{S\subseteq\mathcal{K}} \sum\limits_{i\in\mathcal{I}} x_i(S)v_i(S) & & & \\
\text{s.t.} & & & \\
\sum\limits_{S\subseteq\mathcal{K}} x_i(S) & \leq & 1 & \forall i \in \mathcal{I} \\
\sum\limits_{S:k\in S} \sum\limits_{i\in\mathcal{I}} x_i(S) & \leq & 1 & \forall k \in \mathcal{K} \\
x_i(S) & \in & \{0,1\} & \forall i, S
\end{array}
\qquad \textbf{(WDP)}
$$

The first set of constraints guarantees that any bidder can win at most one bundle, which is only relevant for the XOR bidding language. Without this constraint, bidders can win multiple bundles, which is referred to as an OR bidding language.

The XOR language is used because it is fully expressive compared to the OR language, i.e., a bidder can express values for all possible subsets of items. Subadditive valuations, where a bundle is worth less than the sum of individual items, cannot be described appropriately without an exposure risk using the OR bidding language. The second set of constraints ensures that each item is only allocated once. Very early, it has been shown that the WDP (with an OR bidding language) is NP-hard by reducing it to the weighted set packing problem [15].

Theorem 1. *[15] The decision version of the WDP with an OR bidding language is NP-complete, even if restricted to instances where every bid has a value equal to 1, and every bidder bids only on subsets of size of at most 3.*

The same holds for an XOR bidding language, where bidders only bid on subsets of size of at most 2 [19]. The decision version of the WDP refers to the WDP, in which the auctioneer only wants to know, if there is an allocation with a revenue larger than a particular amount. Given those hardness results, one could try to approximate the WDP. Approximation algorithms are polynomial time algorithms with a provable performance guarantee on the deviation from the optimal solution. Unfortunately, it has been shown that for OR and XOR bidding languages, there are no polynomial algorithms that approximate the WDP within certain bounds. A comprehensive overview of complexity results in this area is given in [18].

There are, however, tractable cases if we restrict bids or valuations in a way that gives the bids a structure to allow for efficient solution methods. For example, the *goods are substitutes property* (aka. *substitutes condition*) leads to integral solutions of the LP-relaxation of the WDP.

Definition 1 (Substitutes condition [13]). *Bidder i considers the objects in $\mathcal{K}$ to be substitutes if for all $A \in \mathcal{K}$ and packages S and T not containing A, such that $S \subset T$, $v_i(S \cup \{A\}) - v_i(S) \geq v_i(T \cup \{A\}) - v_i(T)$*

In other words, if items being sold are substitutes, the marginal value of obtaining a particular object A is smaller if the set of objects T already owned is larger than another set S. The substitutes condition would allow for additive valuations but not for complements or super-additive valuations, and will also play a role when determining ask prices in combinatorial auctions in Section 4.

Theorem 2. *[20] Let for all $i \in \mathcal{I}$ the bid values $v_i(S)$, $S \subseteq \mathcal{K}$ of the WDP with an XOR bidding language satisfy the substitutes condition, then the LP-relaxation of the WDP has an integral optimal solution.*

A good overview of tractable cases of the WDP is provided in [21]. Unfortunately, the restrictions on tractable cases are so severe that auctioneers cannot rely on them in most applications of combinatorial auctions. Independent of this, extensive analyses of the empirical hardness of the WDP [22] illustrate that satisfactory performance can be obtained for problem sizes and structures occurring in practice. The problem sizes in many real-world applications have shown to be tractable within acceptable time limits [6].

Apart from bundle bids, other types of advanced bidding languages and respective auction formats have shown to be useful. Apart from traditional multi-unit auctions which allow for the specification of a price for a particular quantity, *volume discount bids* allow to specify supply curves, i.e., unit prices for different quantities of an item sold. Supplier can express economies of scale when bidding on very large quantities (e.g., \$ 500/unit until 1000 units and \$ 450/unit for more than 1000 units). Also here, buyers need to consider various business constraints when selecting such bids. For example, there might be limits on the spend per bidder or group of bidders, and upper and lower bounds on the number of winners. These side constraints as well as limited capacity of suppliers turn the winner determination problem into a hard computational problem [23, 24].

Multi-attribute auctions allow bids on price and qualitative attributes such as delivery time or warranty. In contrast to request for quotes or tenders as they are regularly used in procurement, the purchasing manager specifies a scoring function that is used to evaluate bids. This enables competitive bidding with heterogeneous, but substitutable offers. Multi-attribute auctions differ in the types of scoring rules or functions used, and in the type of feedback that is provided to bidders. Depending on the type of bids submitted, and on the type of scoring function, the auctioneer faces different optimization problems [25].

3 Strategic Complexity

In this section, we will talk about auction formats, which elicit bidders' preferences to an extent that the optimal solution to the WDP, i.e., the economic efficient outcome can be selected by the auctioneer. As outlined in the first paragraph of this article, a central auction design goal is to obtain an *efficient allocation* $X^* = (S_1^*, \ldots, S_n^*)$, where S_i^* is bidder i's optimal bundle.

Definition 2 (Allocative efficiency)**.** *Allocative efficiency is measured as the ratio of the total valuation of the auction outcome X to the maximum possible valuation of an allocation (i.e., the efficient allocation) X^*:*

$$E(X) = \frac{\sum\limits_{i \in \mathcal{I}} v_i(\bigcup_{S \subseteq \mathcal{K}: x_i(S)=1} S)}{\sum\limits_{i \in \mathcal{I}} v_i(\bigcup_{S \subseteq \mathcal{K}: x_i^*(S)=1} S)}$$

Since typically in auctions the bidder valuations are not given and bidders have incentives to lie about their true preferences, strategic complexity is a concern in the design of combinatorial auctions. Strategic complexity is concerned with the effort it takes for a bidder to determine his optimal bidding strategies. In some auction formats bidders might not be willing to reveal their true preferences and rather speculate, which is one of the main sources of inefficiency in auctions.

Incentive compatibility and *strategy proofness* are properties that should lead bidders to reveal their true private valuations to an extent that the auctioneer can determine the efficient allocation, without the need for further speculation about other bidders' preferences. An auction is incentive compatible, if truthful revelation is a Bayes Nash equilibrium. In other words, truth revelation is optimal for a bidder, if and only if, all other bidders in a game with uncertainty about the types of other bidders reveal their valuations truthfully. An auction is strategy proof, if truth revelation is a dominant strategy for bidders, i.e., it is the bidder's best strategy independent of other bidders' types and strategies. In these cases, the strategic complexity of an auction is reduced to a minimum and speculation is not necessary.

Traditional single-object auction theory distinguishes at least four different types of auction formats: first-price sealed bid, Dutch, English, and second-price sealed bid auctions [13]. The first-price sealed bid and the Dutch auction are strategically equivalent, as are the English and the second-price sealed bid. The second-price sealed bid auction or Vickrey auction has a dominant strategy, and the same holds for a simple implementation of the English auction, in which the auctioneer is replaced by an upward ticking clock, and bidders cannot place jump bids, but only drop out at a certain price level. This is often referred to as a Japanese or clock auction. The clock auction can be described as iterative or ascending auction format, where a bidder learns about the willingness-to-pay of other bidders during the course of the auction. Efficiency in dominant strategies is a desirable property of auction mechanisms. There is a generalization of the second-price sealed bid auctions (aka. Vickrey-Clarke-Groves auction) to multiple-item auctions, which maintains its dominant strategy property (see Section 4). It is not obvious that a generalization of the clock auction or any other iterative auction format has similar properties.

General equilibrium models have been developed in Economics to show that in markets with multiple items, the Walrasian price mechanism also known as tâtonnement, which uses item-level or linear prices actually yields the efficient allocation [26] while communicating as few real variables as possible (see [27] and [28]). As a consequence, the First Welfare Theorem shows Pareto-efficiency of allocations obtained at those equilibrium prices. The tâtonnement works as follows: Prices are cried, and agents register how much of each good they would like to offer or purchase. No transactions and no production take place at disequilibrium prices. Then, prices are lowered for goods with positive prices and excess supply, and prices are raised for goods with excess demand until no agent wants to deviate from his allocation.

However, these results assume that all production sets and preferences are convex. The results do not carry over to non-convex economies with indivisible items, such as they often occur in combinatorial auctions. The question is, whether a combinatorial auction mechanism can be fully efficient, and, if so, what types of equilibrium prices are necessary. We will introduce the notion of "Competitive Equilibrium" for the following discussion.

Definition 3 (Competitive Equilibrium, CE [29])**.** *Prices $\mathcal{P}$, and allocation*

X^* are in competitive equilibrium if allocation X^* maximizes the payoff of every bidder and the auctioneer revenue given prices $\mathcal{P}$. The allocation X^* is said to be supported by prices $\mathcal{P}$ in CE.

The first approach would be to use the same Walrasian price mechanism and see, if it produced efficient outcomes in combinatorial auctions, where indivisibilities are present. Unfortunately, without convexity assumptions full efficiency cannot be achieved with simple linear competitive equilibrium prices in a combinatorial auction with unrestricted bidder valuations. It has been shown that a CE always exists in combinatorial auctions, but it possibly requires non-linear and non-anonymous prices [30, 31]. Prices are *non-linear* if the price of a bundle is not equal to the sum of prices of its items, and prices are *non-anonymous* or *personalized* if prices for the same item or bundle differ across bidders. This leads to the following classification of CE prices:

1. linear anonymous prices $\mathcal{P} = \{p(k)\}$
2. linear personalized prices $\mathcal{P} = \{p_i(k)\}$
3. non-linear anonymous prices $\mathcal{P} = \{p(S)\}$
4. non-linear personalized prices $\mathcal{P} = \{p_i(S)\}$

Indeed, there have been proposals for *ascending* combinatorial auctions with non-linear and personalized prices which have been shown to be fully efficient if bidders follow a straightforward bidding strategy [32]. Such a strategy assumes that bidders bid only on those bundles, which maximize their payoff in each round. Unfortunately, straightforward bidding is only a best response for bidders in pure ascending combinatorial auction formats, if bidders' valuations are restricted (see Section 4).

More generally, it is known shown that the only efficient mechanisms in which honest revelation is a dominant strategy for each agent is the Vickrey-Clarke-Groves (VCG) mechanism [33]. VCG mechanisms, however, exhibit significant problems in practical applications [34, 35]. Among others, the VCG mechanism can lead to low seller revenues, non-monotonicity of the seller's revenues in the set of bidders, and is susceptible to collusion. Apart from this, all bidders would need to submit all their valuations for an exponential number of bundles, which is not practical for all but very small auctions with only a few items.

In summary, designing strategy-proof and practical combinatorial auction formats turns out to be a formidable task. The VCG auction does not seem practical in most applications and iterative forms of combinatorial auctions are bound to non-linear and personalized competitive equilibrium prices for full efficiency.

4 Combinatorial Auction Formats

In the following, we will provide an overview of well-known combinatorial auction formats and discuss some of the concepts from the overview in the previous section in more detail.

4.1 The Vickrey-Clarke-Groves Auction

Vickrey-Clarke-Groves (VCG) mechanisms describe a class of strategy-proof economic mechanisms [36, 37], where sealed bids are submitted to the auctioneer. The winners are also determined by the WDP. However, rather than paying the bid prices, the winners pay a discounted price. This price is calculated in the following manner.

$$p_i^{VCG} = v_i(X^*) - [w(\mathcal{I}) - w(\mathcal{I}_{-i})]$$

Here p_i^{VCG} describes the Vickrey price, while $w(\mathcal{I})$ is the objective value the WDP with the valuations of all bidders, and $w(\mathcal{I}_{-i})$ is the objective value to the WDP with all bidders except the winning bidder i. If the auction is modeled as a coalitional game, $w(\bullet)$ can also be referred to as the coalitional value function, i.e., the outcome of the auction game with a certain set of bidders. In a combinatorial auction, this means, a bidder needs to submit bids on all possible bundles, a number which is exponential in the number of items. Each winning bidder receives a Vickrey payment, which is the amount that he has contributed to increasing the total value of the auctioneer.

Let's take an example with two items x and y which are to be sold in a combinatorial auction. The bids of bidder 1 and 2 are described in Table 2. The total value will be maximized at \$34, while selling $\{x\}$ to bidder 1 and $\{y\}$ to bidder 2. Bidder 1 bids \$20 for $\{x\}$, but he receives a Vickrey payment of \$34 - \$29 = \$5, since without his participation the total value would be \$29. In other words, the net payment or Vickerey price p_1^{VCG} bidder 1 has to pay to the auctioneer is (\$20 (bid price) - \$5 (Vickrey payment) =) \$15.

	Bids		
Items	{x}	{y}	{x,y}
Bidder 1	20*	11	33
Bidder 2	14	14*	29

Table 2: Bids submitted in a VCG auction

Bidder 2 bids \$14 on y, but receives a Vickrey payment of \$34 - \$33 = \$1, because without his participation the total valuation of this auction would be \$33. Auctioneer revenue would then be \$15 + \$13 = \$28 in this auction.

In this auction bidders have a dominant strategy of reporting their true valuations $b_i(S) = v_i(S)$ on all bundles S to the auctioneer, who then determines the allocation and respective Vickrey prices. As already introduced in the previous section, the VCG design suffers from a number of practical problems.

The decisive fault of the VCG is best understood if the auction is modeled as a coalitional game [34]. (N, w) is the coalitional game derived from trade between the seller and bidders. Let N denote the set of all bidders $\mathcal{I}$ plus the auctioneer with $i \in N$, and $M \subseteq N$ be a coalition of bidders with the auctioneer. Let $w(M)$ denote the coalitional value for a subset M, equal to the objective value of the WDP with all bidders $i \in M$ involved. A core payoff vector Π, i.e., payoffs of the bidders in this auction, is then defined as follows

$$Core(N, w) = \{\Pi \geq 0 | \sum_{i \in N} \pi_i = w(N), \sum_{i \in M} \pi_i \geq w(M) \quad \forall M \subset N\}$$

This means, there should be no coalition $M \subset N$, which can make a counteroffer that leaves themselves and the seller at least as well off as the currently winning coalition. Unfortunately, in the VCG auction there can be outcomes which are not in the core. To see this, assume again a combinatorial sales auction with three bidders and two items (see Table 3).

Bidder 1 bids $b_1(x) = \$0$, $b_1(y) = \$2$ and $b_1(x, y) = \$2$. Bidder 2 bids $b_2(x) = \$2$, $b_2(y) = \$0$ and $b_2(x, y) = \$2$. Finally, bidder 3 only has a bid of $b_1(x, y) =\$2$, but no valuation for the individual items. In this situation the net payments of the winners (bidder 2 and 3) are zero, and bidder 3 could find a solution with the auctioneer that makes both better off. It has been shown that there is an equivalence between the core of the coalitional game

	Bids		
Items	{x}	{y}	{x,y}
Bidder 1	0	2	2
Bidder 2	2	0	2
Bidder 3	0	0	2

Table 3: Bids submitted in a VCG auction

and the competitive equilibrium for single-sided auctions [31]. Outcomes, which are not in the core lead to a number of problems, such as low seller revenues or non-monotonicity of the seller's revenues in the set of bidders and the amounts bid. To see this, just omit bidder 1 from the auction. Also, such auction results are vulnerable to collusion by a coalition of losing bidders. Therefore, it has been argued that the outcomes of combinatorial auctions should be in the core [38].

The *bidders are substitutes condition (BSC)* is necessary and sufficient to support VCG payments in competitive equilibrium [31]. A bidder's payment in the VCG mechanism is always less than or equal to the payment by a bidder at any other CE.

Definition 4 (Bidders are Substitutes Condition, BSC)**.** *The BSC condition requires*

$$w(N) - w(N \setminus M) \geq \sum_{i \in M} [w(N) - w(N \setminus i)], \forall M \subseteq N$$

In words, BSC holds where the incremental value of a subset of bidders to the grand coalition is at least as great as the sum of the incremental contributions of each of its members. When at least one bidder has a non-substitutes valuation an ascending CA cannot implement the VCG outcome [39].

4.2 Non-Linear Personalized Price Auctions

In this section, we will discuss relevant theory with respect to ascending combinatorial auctions using non-linear and personalized prices (NLPPAs). We have seen that the WDP is a non-convex optimization problem. By adding constraints for each set partition of items and each bidder to the WDP the formulation can be strengthened, so that the integrality constraints on all variables can be omitted but the solution is still always integral [31, 39]. Such a formulation describes every feasible solution to an integer problem, and is solvable with linear programming. We will refer to this formulation as NLPPA WDP.

$$
\begin{array}{llll}
\max & \sum_{i\in\mathcal{I}}\sum_{S\subseteq\mathcal{K}} v_i(S)x_i(S) & & \\
\text{s.t.} & & & \\
& x_i(S)=\sum_{X:x_i=S}\delta_X & \forall i\in\mathcal{I},\forall S\subseteq\mathcal{K} & (p_i(S)) \\
& \sum_{S\subseteq\mathcal{K}} x_i(S)\le 1 & \forall i\in\mathcal{I} & (\pi_i) \\
& \sum_{X\in\Gamma}\delta_X = 1 & & (\pi^s) \\
& 0\le x_i(S) & \forall S\subseteq\mathcal{K},\forall i\in\mathcal{I} & \\
& 0\le \delta_X & \forall X\in\Gamma &
\end{array}
\qquad \textbf{(NLPPA WDP)}
$$

Personalized non-linear CE prices can now be derived from the dual of the NLPPA WDP. In the first side constraint, $x_i(S)$ is equal to the sum of weights δ_X over all allocations X where bidder i gets bundle S. The dual variables of this constraint are the personalized prices $p_i(S)$. The second side constraint makes sure that each bidder i receives at most one bundle, and the dual variable π_i describes bidder i's payoff. Finally, the total weight of all selected allocations $X \in \Gamma$ equals 1, such that only one allocation can be selected. Here, Γ describes the set of all possible allocations. The dual variable (π^s) for this side constraint describes the seller's payoff.

From duality theory follows that the *complementary slackness conditions* must hold in the case of optimality. This is equivalent to the CE, where every buyer receives a bundle out of his demand set or demand correspondence $D_i(\mathcal{P})$, i.e., the bundles maximizing his payoff at the prices, and the auctioneer selects the revenue maximizing allocation at these prices.

Definition 5 (Demand Set)**.** *The demand set $D_i(\mathcal{P})$ of a bidder i includes all bundles which maximize a bidder's payoff π_i at the given prices $\mathcal{P}$:*

$$D_i(\mathcal{P}) = \{S : \pi_i(S,\mathcal{P}) \ge \max_{T\subseteq\mathcal{K}} \pi_i(T,\mathcal{P}), \pi_i(S,\mathcal{P}) \ge 0, S \subseteq \mathcal{K}\}$$

Complementary slackness provides us with an optimality condition, which also serves as a termination rule for NLPPAs. If bidders follow the straightforward strategy then terminating the auction when each active bidder receives a bundle in his demand set will result in the efficient outcome. Note that a demand set can include the empty bundle. Additionally, the starting prices must represent a feasible dual solution. A trivial solution is to use zero prices for all bundles.

Although such non-linear personalized prices always exist, the NLPPA WDP is huge since one must enumerate all possible feasible coalitions. Nevertheless, it has provided a guideline to a number of practical auction designs using non-linear personalized prices. Individual NLPPA formats discussed in the following such as the Ascending Proxy Auction, iBundle, and the dVSV auction have different rules for determining the prices provided to the bidders and for determining how bidders submit new bids based on these announced prices.

iBundle [40] calculates a provisional revenue maximizing allocation at the end of every round and increases the prices based on the bids of non-winning bidders. Three different versions of iBundle have been suggested [40]: iBundle(2) with anonymous prices, iBundle(3) with personalized prices, and iBundle(d) wich starts with anonymous prices and switches to personalized prices for agents which submit bids for disjoint bundles. The ***Ascending Proxy Auction*** [32] is similar to iBundle(3), but the use of proxy agents is mandatory, which essentially leads to a sealed-bid auction format.

The ***dVSV*** auction [39] design differs from iBundle in that it does not compute a provisional allocation in every round but increases prices for one *minimally undersupplied set of bidders*. A set of bidders is minimally undersupplied if each bidder in this set receives a bundle from his demand set, and removing only one of the bidders from the set forfeits this property. Similar to iBundle(3), it maintains non-linear personalized prices and increases the prices for all agents in a minimally undersupplied set based on their bids of the last round. While the Ascending Proxy Auction can be interpreted as a subgradient algorithm, the dVSV auction can be interpreted as a primal-dual algorithm for the NLPPA WDP [39].

Even though the BSC condition is sufficient for VCG prices to be supported in CE, the slightly stronger *bidder submodularity condition (BSM)* is required for a pure ascending combinatorial auction to implement VCG payments [39].

Definition 6 (Bidder Submodularity Condition, BSM)**.** *BSM requires that for all* $M \subseteq M' \subseteq N$ *and all* $i \in N$ *there is*

$$w(M \cup \{i\}) - w(M) \geq w(M' \cup \{i\}) - w(M')$$

Here bidders are more valuable, when added to a smaller coalition. Under BSM the NLPPAs yield VCG payments and straightforward bidding is an ex-post equilibrium. An ex-post equilibrium is stronger than a Bayes-Nash equilibrium, but weaker than a dominant strategy equilibrium. It does not require bidders to speculate about other bidders' types, but requires assumptions about their strategies. When the BSM condition does not hold, the property breaks down and a straightforward strategy is likely to lead a bidder to pay more than the VCG price for the winning bundle, and bidders have an incentive to shade their bids and deviate from straightforward bidding. In case of non-straightforward bidding the outcome of NLPPAs can deviate significantly from the efficient solution [41].

The restriction to BSM valuations is mainly due the definition of ascending auctions, such that prices can only increase and no payments from the auctioneer are allowed. The ***Credit-Debit auction*** is an extension to the dVSV design which achieves the VCG outcome for general valuations by determining payments or discounts from the auctioneer to the bidders at the end. Similarly, ***iBEA*** is described as an extension of iBundle. Both approaches are based on universal competitive equilibrium (UCE) prices, which are CE prices for the main economy as well as for every marginal economy, where a single buyer is excluded [42]. These auctions terminate as soon as UCE prices are reached and VCG payments are determined as one-time discounts dynamically during the auction. Truthful bidding is an ex-post equilibrium in the Credit-Debit auction and iBEA. The auctions are an important contribution to the literature, because they describe fully efficient iterative combinatorial auctions where straightforward bidding is an ex-post equilibrium for general valuations. However, they share a central problem of the VCG auction: if buyer submodularity does not hold, the outcomes might not be in the core.

4.3 Linear Price Auctions

In many applications of ICAs, linear and anonymous ask prices are essential. For example, day-ahead markets for electricity sacrifice efficiency for the sake of having linear prices [43]. Also, the main auction formats, which have been tested for selling spectrum in the US used linear ask prices [44]. Simple examples illustrate that linear anonymous CE prices do not exist for general valuations. It has been shown that the *goods are substitutes* property is a sufficient condition for the existence of the exact linear CE prices [20], as the

LP-relaxation of the WDP is integral (see Section 2) and dual variables can be interpreted as prices. The substitutes condition is, however, very restrictive and not satisfied in most combinatorial auctions. Inspite of these negative results, some combinatorial auction designs with linear prices achieved high levels of efficiency in the lab.

The ***CCA*** (Combinatorial Clock Auction) [45] utilizes anonymous linear ask prices called *item clock prices*. In each round bidders express the quantities desired on the bundles at the current prices. As long as demand exceeds supply for at least one item (each item is counted only once for each bidder) the price clock "ticks" upwards for those items (the item prices are increased by a fixed price increment), and the auction moves on to the next round. If there is no excess demand and no excess supply, the items are allocated corresponding to the last round bids and the auction terminates. If there is no excess demand but there is excess supply (all active bidders on some item did not resubmit their bids in the last round), the auctioneer solves the winner determination problem considering all bids submitted during the auction runtime. If the computed allocation does not displace any bids from the last round, the auction terminates with this allocation, otherwise the prices of the respective items are increased and the auction continues. Note that due to the winner determination the final payments can deviate from the ask prices.

The ***RAD*** (Resource Allocation Design) proposed in [46] uses anonymous linear ask prices. However, instead of increasing the prices in case of overdemand, the auction lets the bidders submit priced bids and calculates so called pseudo-dual prices based on a restricted dual of the LP relaxation of the WDP [7]. The dual price of each item measures the cost of not awarding the item to whom it has been allocated in the last round. In each round the losing bidders have to bid more than the sum of ask prices for a desired bundle plus a fixed minimum increment. RAD suggests an OR bidding language and only winning bids remain in the auction in its original design. The ***ALPS*** (Approximate Linear PriceS) design [47] is also based on the ideas in [7], but improves termination rules and the ask price calculation to better balance prices across items and have the auction avoid cycles. Note that in RAD and ALPS prices can also decrease if the competition shifts to different items.

HPB (Hierarchical Package Bidding) imposes a hierarchical structure of allowed package bids. This hierarchy and an OR bidding language reduce the winner determination problem to a computationally simple problem that can be solved in linear time [15]. If the hierarchy meets the bidders preferences, the auction is likely to achieve efficient outcomes, and reduces the strategic complexity for bidders. HPB provides a simple and transparent pricing mechanism [48]. It uses a recursive algorithm to determine new ask prices which starts with the highest bids on every single item as a lower bound, adding a tax if the next level package received a bid higher than the sum of the single item bids contained in the package. The difference is distributed uniformly upon the respective item prices. The algorithm ends evaluating the package(s) of the top level, resulting in new ask prices for each item.

A few of other combinatorial auction designs have been suggested, which use linear and non-linear prices. For example, in the Clock-Proxy auction a clock auction is followed by a best-and-final Ascending Proxy Auction [49]. The approach combines the simple and transparent price discovery of the clock auction with the efficiency of the Ascending Proxy Auction. PAUSE combines the simultaneous multi-round auction with bidding on bundles in later stages. Here, the burden of evaluating a combinatorial bid is transferred to the bidder [50]. Also alternative ways of pricing and bidder support have shown promising results [51].

Interestingly, experimental research has shown that iterative auction designs with linear prices achieved very high levels of efficiency, even for auctions with up to 18 items [44, 48, 52]. While linear competitive equilibrium prices do not always exist, linear ask prices used in the combinatorial clock auction [45], HPB [48], or ALPS [47] have shown to be a good guideline to bidders in finding the efficient solution, even though no formal equilibrium analysis is available for any of these auction formats.

5 Conclusions

Many theoretical results on combinatorial auctions are negative in the sense that it seems quite unlikely that practical applications would satisfy the assumptions, which would lead to efficiency with a strong game-theoretical solution concept. Nevertheless, experimental results have yielded very high levels of efficiency in the lab. These results suggest that even if full efficiency is not always possible, combinatorial auction designs can achieve very high levels of efficiency, higher than what would be possible in simultaneous or sequential auctions in the presence of complementarities. The results of this research can have significant impact on the design and the efficiency of real-world markets. Further development of practical combinatorial auction designs will probably remain an active and rewarding area of both theoretical, experimental, and applied research for the foreseeable future.

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