Efficiency with Linear Prices? A Theoretical and Experimental Analysis of the Combinatorial Clock Auction.

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Combinatorial auctions have been suggested as a mean to raise efficiency in multi-item negotiations with complementarities as they can be found in procurement, in energy markets, in transportation, and for the sale of spectrum auctions. The Combinatorial Clock (CC) auction (Porter et al. 2003) has become very popular in these markets for its simplicity and as it "produces highly usable price discovery, because of the item prices (linear pricing)" (Ausubel et al. 2006). Unfortunately, the CC auction fails to lead always to efficient outcomes, and there is no theory on equilibrium bidding strategies in such auctions. Given the importance of the CC auction in the field, it is desirable to better understand the CC auction. In this paper, we first show that the worst-case efficiency in the CC auction can actually be as low as 0%, if bidders follow the straightforward strategy. We identify demand-masking valuations, i.e., the characteristics of valuation functions, which can cause such low efficiencies. We then introduce the CC+ auction, an extension of the CC auction with a modified price update rule and a VCG payment rule, and show that powerset bidding leads to efficient outcomes and an ex-post equilibrium for general valuations, while maintaining linear ask prices. While a powerset strategy might only be possible in very small scenarios, computational experiments show that both, the CC and the CC+ auctions achieve high levels of efficiency, even if bidders are restricted to submit only a small number of bids in each round. This result indicates that the CC+ auction is robust against some of the strong assumptions in the theory and it also explains high levels of efficiency of the CC auction that have been observed in the lab.

Key words: combinatorial clock auction, allocative efficiency, core-selecting auctions

1. Introduction

The single-item ascending Clock auction (aka. Japanese auction) has a number of desirable economic properties. It is individually rational, efficient, strategy-proof, and the payoff vector is in the core. When all bidders know their private valuations, truthfully revealing one's demand is a dominant strategy. It would be desirable to achieve such properties for *combinatorial auction* (CA) designs as well. Unfortunately the Vickrey-Clarke-Groves (VCG) design is the unique mechanism that satisfies individual rationality, efficiency, and strategy-proofness (Ausubel and Milgrom 2006b). However, its results can be outside the core, which leads to a number of problems in practical settings (Ausubel and Milgrom 2006b, Rothkopf 2007).

1.1. Iterative Combinatorial Auctions

Researchers have been trying to find generalizations of the single-item Clock auction for selling multiple items. For situations with multiple items, but unit demand (Demange et al. 1986) and for multiple homogeneous goods with marginal decreasing value (Green and Laffont 1979, Holmstrom 1979), it has been shown that there are generalizations, which can be used to implement efficient, strategy-proof mechanisms. Finding efficient auctions with strong incentive properties turned out to be much harder for CAs with general valuations.

The Ascending Proxy Auction (Ausubel and Milgrom 2006a), iBundle(3) (Parkes and Ungar 2000), and the dVSV auction (de Vries et al. 2007) achieve full efficiency with straightforward bidders, i.e., bidders only bid on those packages that maximize their payoff in each round. If the coalitional value function satisfies the buyer submodularity condition, straightforward bidding is a best-response strategy, which leads to an ex-post equilibrium and the auction results in the VCG outcome (Ausubel and Milgrom 2002). These auction formats are based on non-linear and personalized prices and can be modeled as an algorithm (primal-dual or subgradient) to solve the corresponding linear program. We refer to these auction formats as non-linear personalized price auctions (NLPPAs) in the following.

If the bidders' valuations in an NLPPA are not buyer submodular, bidders have an incentive to shade their bids and not follow the straightforward strategy. However, buyer submodularity is mostly not given for realistic value models. Even, if bidders knew that their valuations are buyer submodular and they would not need to speculate about other bidders' types, it is not obvious that other bidders are able to follow their best-response strategy in such an environment. Both computational and lab experiments have also illustrated the large number of auction rounds necessary for these NLPPAs (Schneider et al. 2010), in which nearly all valuations have to be elicited to achieve efficiency.

As an alternative, linear-price CAs have been suggested resembling the fictitious Walrasian tâtonnement. Linear prices are desirable for their simplicity and the reduced communication complexity in real world applications. One line of research is based on a restricted dual of the relaxed winner determination problem, in which the pseudo-dual variables are used as ask prices in the auction (Rassenti et al. 1982, Kwasnica et al. 2005, Bichler et al. 2009). Fluctuations of the ask prices and the complexity of the ask price calculation are problems in some applications. In contrast, Porter et al. (2003) suggested a simple mechanism with ascending linear ask prices, called the Combinatorial Clock (CC) auction. Initially prices for all items are zero. In every round bidders identify a package of items, or several packages, which they offer to buy at current prices. If two or more bidders demand an item then its price will be increased by a fixed bid increment in the next round. This process iterates. The bids which correspond to the current ask prices are called *standing*, and a bidder is standing if he has at least one standing bid. In a trivial case when at some point supply equals demand the auction terminates and the items are allocated according to the standing bids. If at some point there is excess supply for at least one item and no item is over-demanded, the auctioneer determines the winners to find an allocation of items that would maximize his revenue considering all submitted bids. If the solution displaces a standing bidder, then prices of items in the corresponding standing bids rise by the bid increment and the auction continues. The auction ends when no prices are increased and bidders finally pay their bid prices for winning packages. We analyze a version that uses an XOR bidding language.

The mechanism has achieved high levels of efficiency in the lab (Porter et al. 2003) and has a number of obvious advantages. It maintains strictly ascending, linear ask prices, and limits the computational burden on the auctioneer as he only has to solve the NP-hard winner determination problem in the last rounds. Also, the information revelation between rounds makes it quite robust against collusion and limits the bidder's possibilities for signaling.

For these reasons, the Netherlands and the UK have recently started to use a CC auction for price discovery in the sale of spectrum licenses followed by a sealed bid auction (Cramton 2009). It is also being used in electricity markets, in which anonymous linear prices are often an important requirement (Cramton et al. 2006). Unfortunately, no equilibrium strategy is known, and it is unclear for real bidders, which strategy they should follow. Apart from a few lab experiments (Porter et al. 2003, Kagel et al. 2009, Scheffel et al. 2010) little theoretical research has focused on the CC auction as of yet.

1.2. Contribution and Composition of this Paper

Achieving efficiency when economic agents strategically pursue their individual self-interest is a fundamental problem in Economics. General equilibrium models showed that in classical convex economies with multiple products, the Walrasian price mechanism verifies the efficiency of a proposed allocation Arrow and Debreu (1954) while communicating as few real variables as possible (see Mount and Reiter (1974) and Hurwicz (1977)). Furthermore, Jordan (1982) showed that the Walrasian mechanism is a unique voluntary mechanism with this property. However, these results assume that all production sets and preferences are convex and do not apply to non-convex economies with indivisible goods, such as combinatorial auctions. Such neoclassical general equilibrium models have often been criticized for their strong assumptions (Georgescu-Roegen 1979).

Bikhchandani and Mamer (1997) showed that without convexity assumptions full efficiency cannot be achieved with linear *competitive equilibrium* (CE) prices for general valuations (see Nisan and Segal (2006) for an overview). Later Gul and Stacchetti (1999) proved that for all bidders it is almost necessary that *goods are substitutes* to ensure efficiency with linear CE prices. So far, only *iterative combinatorial auction* (ICAs) designs with non-linear and personalized prices have been shown to be fully efficient.

In most practical applications of ICAs, linear and anonymous ask prices are essential. For example, day-ahead markets for electricity sacrifice efficiency for the sake of having linear prices (Meeus et al. 2009). Also, the main auction formats, which have been tested for selling spectrum in the US used linear prices (Brunner et al. 2009). The CC auction is probably the most wide-spread ICA format, but the negative results by Gul and Stacchetti (1999) seem to indicate that there is no hope to make the CC auction fully efficient for general valuations.

A notable difference between the CC auction and auctions with pseudo-dual prices, however, is that bidders might not need to pay the ask prices of the final round. The winner determination in the final round can select a bid and the corresponding ask price from a previous round, so that there is a distinction between ask prices and payments. This distinction opens up the possibility to achieve efficiency with linear ask prices and a strong game-theoretical solution concept for general valuations. The latter is important, as any restriction on the valuations is typically unknown and makes even a strong solution concept weak. In this paper, we show conditions, under which the CC auction with linear ask prices satisfies an ex-post equilibrium and provide sensitivity analysis to understand, how robust the CC auction is against violations of these conditions.

In section 2 we summarize related literature on linear CE prices. In section 3 we present analytical results on the worst case efficiency of the CC auction assuming simple bidding strategies. This is useful, as due to the distinction between final ask prices and payments, the CC auction is not covered by the negative results outlined by (Gul and Stacchetti 1999). First we assume straightforward bidding, i.e., truthful revelation of the payoff-maximizing packages in response to ask prices (Parkes 2006), since this strategy is easy to follow and limits the amount of information that needs to be revealed in each round. We introduce a *demand-masking set of valuations* and show that the efficiency of the CC auction can be as low as 0%. While the example that leads to 0% efficiency can be considered a degenerate case, we also discuss situations that we found regularly in numerical experiments with realistic value models, and which can also lead to efficiencies as low as 50% with straightforward bidding. As an alternative, we evaluate a powerset strategy, in which bidders bid on all possible packages with positive payoff in each round. We show, however, that even if bidders reveal as much information in each round, efficiency of the CC auction can also decrease to 0%. This analysis helps to understand situations, in which the CC auction is inefficient and propose improvements.

Based on these results in Section 4 we identify properties of the auction mechanism that satisfy full efficiency with a strong game-theoretical solution concept. We suggest a variation of the CC auction, the CC+ auction, which leads to 100% efficiency with powerset bidding. First, we modify the price update rule to allow for efficency. Second, we introduce the VCG payment rule to assure incentives for truthful bidding. We show that powerset bidding becomes an ex-post equilibrium strategy for general valuations in the CC+ auction. From the revelation principle, we know that for each mechanism, we can construct a truthful direct-revelation mechanism whose performance is identical (Gibbard 1973). Therefore, while the CC+ auction is an indirect mechanism, the result is in line with Green and Laffont (1977), who proved that an efficient revelation mechanism in which honest revelation is a dominant strategy for each agent is necessarily a Groves mechanism.

Mathematical models of auctions and markets have been criticized as unrealistic (Rothkopf and Harstad 1994, Georgescu-Roegen 1979), as some of the assumptions are too strong and do not hold in practical applications. For example, auction formats such as the Ascending Proxy Auction (Ausubel and Milgrom 2006a), iBundle(3) (Parkes and Ungar 2000) or dVSV (de Vries et al. 2007) require the buyer submodularity condition to hold in order to achieve an ex-post equilibrium strategy in straightforward bidding. Clearly, these are significant contributions to the literature, not necessarily for their immediate practical applicability with human bidders, but as they show under which conditions full efficiency with a strong solution concept can be achieved in a CA. Based on an efficient auction design, researchers can then analyze the robustness of different designs with respect to deviations from some of the assumptions (Schneider et al. 2010).

Some assumptions of the CC+ auction are also strong and might not be given in practical applications. For example, the price-update rule in the CC+ auction reveals almost no information to the bidders. This rule is necessary for a provable ex-post equilibrium in the full information setting. Typically auctions are not used in a full information setting. In Section 5, we discuss a weaker price update rule, which makes it very hard for bidders to speculate on other bidders' types, but provides higher market transparency.

A powerset bidding equilibrium strategy is another limitation, as it will only be viable for small instances with a few items. In Section 6, we provide an experimental analysis of the CC and the CC+ auction and restrict bidders in the number of package bids they can submit in each round. We show that both the CC and CC+ auction are surprisingly robust against deviations of a powerset strategy. In lab experiments, bidders typically submit a set of bids with a large expected payoff (not necessarily only those, which maximize their payoff). This robustness against deviations of powerset bidding also explains, why the CC auction performed so well in the lab (Porter et al. 2003, Scheffel et al. 2010, Kagel et al. 2009). Section 7 provides a summary and conclusions.

2. Related Definitions and Theory

First we introduce the necessary notation and review the relevant theory on linear-price CAs. There is a set $\mathcal{K}$ of m indivisible items indexed with k or l, which are auctioned among n bidders. Let $i,j \in \mathcal{I}$ denote the bidders and $v_i : S \rightarrow \mathbb{R}$ denotes a value function of bidder i, which assigns a real value to every subset $S \subseteq \mathcal{K}$ of items. An allocation $X \in \Gamma$ of the m items among bidders is $X = \{X_1, ..., X_n\}$, with $X_i \cap X_j = 0$ for every $i \neq j$. X_i is the package of items assigned to bidder i. The social welfare of an allocation X is $\sum_{i\in\mathcal{I}} v_i(X_i)$, and the efficient allocation X^* maximizes social welfare among all allocations X, such that $\forall v \forall X, \sum_{i\in\mathcal{I}} v_i(X_i^*) \geq \sum_{i\in\mathcal{I}} v_i(X_i)$.

We focus on linear-price CAs, in which an ask price β_k for each of the m items is available, and the price of a package S is the sum of the prices of the items in this package. We assume that the *demand* of each bidder are the packages which maximize his utility.

Definition 1. (Blumrosen and Nisan 2007) For a given bidder valuation v_i and given item prices $\beta_1, ..., \beta_m$, a package $R \subseteq \mathcal{K}$ is called a *demand* of bidder i, if for every other package $S \subseteq \mathcal{K}$ we have that $v_i(S) - \sum_{k\in S} \beta_k \leq v_i(R) - \sum_{k\in R} \beta_k$.

A feasible allocation X and a price vector β_k are in competitive equilibrium (CE), when the allocation maximizes the payoff of every bidder and the seller given the prices. A *Walrasian equilibrium* can then be described as a vector of linear or item prices.

Definition 2. A Walrasian equilibrium is a set of nonnegative prices $\beta_1, ..., \beta_m$ and an allocation X, if for every player i, X_i is the demand of bidder i at those prices and for any item k that is not allocated $\beta_k = 0$.

Simple examples illustrate that Walrasian equilibria do not exist for a general valuations in CAs if goods are indivisible; in other words, for certain types of bidder valuations it is impossible to find linear CE prices which support the efficient allocation X^* (Blumrosen and Nisan 2007).

The economic goods are substitutes property is a sufficient condition for the existence of Walrasian equilibrium prices (Kelso and Crawford 1982). Intuitively this property implies that every bidder will continue to demand the items which do not change in price, even if the prices on other items increase. Overall, the goods are substitutes condition is very restrictive as most known practical applications of CAs rather deal with complementary goods.

Actually, Gul and Stacchetti (2000) show that even if bidders' valuation functions satisfy the goods are substitutes condition, there exists no ascending CA that uses anonymous linear prices and arrives at the VCG solution. This means that bidders may have an incentive to demand smaller packages of items, in order to lower their payments.

Bikhchandani and Ostroy (2002) prove that only with personalized non-linear prices a CA always achieves a CE. The Ascending Proxy Auction, iBundle(3) and the dVSV auction are designs using non-linear personalized prices at the expense of an exponential (in m) number of auction rounds. Also, the final ask prices generated are not VCG prices for general valuations and thus bidders still might be incentivized to deflect from the assumed straightforward bidding strategy. Straightforward bidding is only an ex-post equilibrium in these NLPPAs, if bidder valuations are submodular (de Vries et al. 2007).

Definition 3. *Final ask prices* are the ask prices β_k of the last round of an iterative auction.

Definition 4. A *payment* is the amount of money a bidder has to pay for his winning items.

In the efficient ICAs (Ascending Proxy Auction, iBundle, dVSV), there is no difference between final ask prices and payments. They charge bidders the final ask prices to pay for their winning packages. This means ask prices need to be non-linear and personalized to guarantee efficiency. In contrast to NLPPAs the CC auction differentiates between final ask prices and payments. This opens up the possibility of maintaining linear ask prices, but achieve efficient solutions with a strong solution concept by implementing non-linear and personalized payments.

3. Efficiency of the CC Auction

Game-theoretical analyses of ICAs typically assume the straightforward bidding strategy. In this section, we analyze the worst-case efficiency of the CC auction with bidders following the straightforward strategy. We also evaluate a powerset strategy, which describes the situation, in which bidders reveal all packages with positive valuation at the current prices. We draw on this strategy in subsequent sections.

Definition 5. The *straightforward* bidder bids only for his *demand* in each round at the current ask prices $\beta_1, ..., \beta_m$.

Definition 6. The *powerset* bidder bids on all packages S with a positive value $v_i(S) - \sum_{k\in S}\beta_k \geq 0$ at the current set of ask prices $\beta_1, ..., \beta_m$.

We first show that if all bidders follow the straightforward strategy, the efficiency of the CC auction can be as low as 0%. For this, we refer to a recent Theorem by Kagel et al. (2009) on the efficiency of auctions which maximize the seller's revenue based on bid prices.

A *standard* package auction is defined such that it selects an allocation $\overline{X}$ to maximize the auctioneer's revenue $\overline{X} \in \arg\max_X \sum_{i\in\mathcal{I}} \beta_i(X_i)$ and has bidder i pay $\beta_i(\overline{X}_i)$. $\beta_i(X_i)$ denotes the highest price that i bids for a package X_i during the course of the auction.

A standard package auction can be modeled as a cooperative game with transferable utility, in which the payoff vector or imputation π is given by the auctioneer's revenue $\pi_0 = \sum_{i\in\mathcal{I}} \beta_i(\overline{X}_i)$, and bidder i's payoff $\pi_i = v_i(X_i) - \beta_i(X_i)$. The value of a coalition of the seller and the bidders in $T \subseteq \mathcal{I}$ is $w(T) = \sum_{i\in T} v_i(X_i^*|_T)$.

A feasible allocation X with prices β and a corresponding imputation π is a *core* allocation if, for every set of bidders $T \subseteq \mathcal{I}$, the imputation satisfies $\pi_0 + \sum_{i \in T} \pi_i \geq w(T)$. A set of bidders T is *relevant*, if there is some imputation such that $\pi_0 + \sum_{i \in T} \pi_i = w(T)$. The package X_i is the respective efficiency-relevant package.

Theorem 1. *(Kagel et al. 2009) In a standard package auction, if for some relevant allocation $X \in \arg\max_X \sum_{i \in \mathcal{I}} v_i(X_i)$ and for all bidders i, $v_i(X_i) - \beta_i(X_i) \leq \overline{\pi}_i$, then the allocation $\overline{X}$ is efficient: $\overline{\pi}_0 + \sum_{i \in \mathcal{I}} \overline{\pi}_i \geq w(\mathcal{I})$. If the efficient allocation is unique, then the auction outcome $(\overline{X}$ is efficient only if for every bidder i, $v_i(X_i) - \beta_i(X_i) \leq \overline{\pi}_i$.*

To promote these results, the auction mechanism must encourage bidders to bid aggressively all the way up to their full values ($\beta_i(X_i) = v_i(X_i)$) for *efficiency-relevant packages*, i.e., packages that may become winning.

3.1. Worst-case Efficiency of the CC Auction with Straightforward Bidders

	$\beta_{(1)}$	$\beta_{(2)}$	$\beta_{(3)}$	...	(1)	(2)	(3)	...	(1, 2)	(1, 3)	...
v_1					10*						
v_{2_a}						4*			10		
v_{2_b}									10		
v_{3_a}							4*			10	
v_{3_b}										10	
...								...			...
$t=1$	1	1	1	...	1_1				$2_{2_a,2_b}$	$2_{3_a,3_b}$	...
$t=2$	2	2	2	...	2_1				$4_{2_a,2_b}$	$4_{3_a,3_b}$	...
$t=3$	3	3	3	...	3_1				$6_{2_a,2_b}$	$6_{3_a,3_b}$	...
$t=4$	4	4	4	...	4_1				$8_{2_a,2_b}$	$8_{3_a,3_b}$	...
$t=5$	5	5	5	...	5_1				$10_{2_a,2_b}$	$10_{3_a,3_b}$	...
$t=6$	6	6	6	...	6_1						
$t=7$	7	6	6	...	7_1						
...											
$t=10$	10	6	6	...	10_1						

Table 1 Example of a demand masking set of bidder valuations and auction progress.

	R	$\{S_h\}$	$\{R\dot{\cup}S_h\}$
v_1	ξ	0	ξ
$\{v_{h_a}\}$	0	ν_h	μ
$\{v_{h_b}\}$	0	0	μ

Table 2 Demand masking set of bidder valuations.

If a bidder follows the straightforward strategy in the CC auction, he does not bid on all relevant packages in the course of the auction. The example in Table 1 illustrates a characteristic situation that we refer to as *demand masking set*. The upper part of the table describes valuations of $2m-1$ bidders for m items, while the lower part shows both ask prices for items and bid prices for packages in individual rounds t. The indices of the bid prices for different packages indicate, which straightforward bidder submits the bid on the respective package. There is one bidder called bidder 1 and for each $h \in \{2, ..., m\}$, there are two bidders h_a and h_b. Bidder 1 values item (1) at a value of 10 and does not value any other item. For $h = 2, ..., m$, bidders h_a and h_b value the package (1,h) at 10 and bidders h_a the item (h) at 4, and are not interested in any other package. WLOG we assume a bid increment of 1. Straightforward bidders h_a and h_b demand the package (1,h) until round 6 at which point they demand nothing. At round 7 there is excess supply and the auctioneer solves the winner determination problem, which displaces the sole remaining standing bid on item (1) of bidder 1. Thus the price on item (1) further increases until bidder 1 wins item (1) in round 10, and the auction terminates with a social surplus of 10. However, the efficient allocation gives item (1) to bidder 1, and item (h) to bidder h_a for a social welfare of $10 + 4(m-1)$. $10/(10 + 4(m-1))$ converges to 0 as m approaches infinity.

Now we provide a formal definition of a demand masking set and derive a worst-case bound for these situations as a function of m.

Definition 7. A *demand masking* set of bidder valuations is given, if the following properties are fulfilled. There is a set of bidders $\mathcal{I}$ with $|\mathcal{I}| \geq 3$, a set of items $\mathcal{K} = \{1, ..., m\}$ with $R \subseteq \mathcal{K}$ and a partition $\mathcal{H}$ of $\mathcal{K} \backslash R$. Let S_h be the elements of $\mathcal{H}$ with $h \in \{2, ..., |\mathcal{H}| + 1 = g\}$. For each S_h there are two bidders h_a and h_b. Bidder 1 values package R with ξ. For $h \in \{2, ..., g\}$ bidders h_a value the packages S_h with ν_h and $R \dot{\cup} S_h$ with μ and bidders h_b value only package $R \dot{\cup} S_h$ with μ. All bidders do not care for other packages, i.e., the marginal value of winning any additional item to the positive valued packages is zero.

Note that the valuations of zero as shown in the Table 2 need not to be strictly zero but rather sufficiently small not to influence the economy.

Theorem 2. *If bidder valuations are demand masking and all bidders follow the straightforward strategy in the CC auction, then the efficiency converges to* $\frac{2}{m+1}$ *in the worst case.*

Note proofs to theorems, propositions and corollaries are presented in the Appendix A.

In the example in Table 1 ν_h is smaller than 5 for all h. With $m = 3$ and $\nu_h = \nu = 5 - \rho$ for all h we would get approximately $50\% = 10/(10 + \nu(m-1))$ efficiency, which is equal $2/(m+1)$ in the worst case. Obviously if the number of items m and the corresponding number of bidders increases to fulfill the requirements of a demand masking set, efficiency converges to 0% in the worst case. While such a situation that leads to 0% efficiency can be considered a degenerated case that will not happen too often in practice. We found regulary situations in simulations with realistic value models, in which the case of $m = 2$ or $m = 3$ occured, which still leads to efficiencies of 67% or 50% in the worst case. Note that these are not necessarily the only characterizations of value models in which such low efficiency can occur.

3.2. Worst-Case Efficiency of the CC Auction with Powerset Bidders

One of the reasons for the popularity of ascending auctions is that they require only partial revelation of the private information (Blumrosen and Nisan 2007). In a CA this might be less of an advantage, as it is still necessary to elicit all valuations, except those of the winning bids in the efficient allocation in the worst case. This means that if there are z winning package bids in an efficient allocation, $n2^m - z$ valuations need to be elicited by the auctioneer to guarantee full efficiency. For example, ascending auctions with non-linear personalized prices such as iBundle (Parkes and Ungar 2000), the Ascending Proxy Auction (Ausubel and Milgrom 2002), or dVSV (de Vries et al. 2007) are protocols that in each round elicit the demand set of each bidder and provably find an efficient solution at the expense of an exponential number (in m) of auction rounds (Blumrosen and Nisan 2007). In such a NLPPA with straightforward bidders at least all valuations of all losing bidders get elicited.

As an alternative to straightforward bidding, the auctioneer can try to encourage bidders to bid on many packages from the start. In the best case, bidders would reveal all packages with positive payoff, i.e., they would follow a powerset strategy. Unfortunately, even if bidders follow the powerset strategy, the CC auction does not necessarily terminate with an efficient solution.

Proposition 1. *If all bidders follow the powerset strategy, the efficiency of the CC auction converges to 0% in the worst case.*

The inefficiency without free disposal described in the example in Table 3 of the Appendix A can only happen, if there are two overlapping packages by the winning bidder, and there is only competition on the package with the lower valuation. This drives up the prices on the latter one and this package gets sold, although the bidder had a much higher valuation for the first package.

3.3. Modifications of the CC Auction

The analysis in Section 3.2 shows that even if bidders reveal all profitable packages in each round, the CC auction can be inefficient. However, a small change in the price update rule allows to elicit all losing package valuations and makes the CC auction fully efficient with powerset bidders.

Definition 8. A *partial revelation price update rule* in the CC auction also increases prices for each overdemanded item and in addition for each item of a standing bid which is displaced by the winner determination.

Corollary 1. *If all bidders follow the powerset strategy, the CC auction with the partial revelation price update rule terminates with an efficient outcome.*

4. The CC+ Auction

Even if the powerset strategy leads to full efficiency in a modified CC auction with linear ask prices, it is not obvious, why a bidder should follow the powerset strategy. In the following, we show that the powerset strategy can be an ex-post equilibrium, but that it requires an even stronger price-update rule and a VCG payment rule (Ausubel and Milgrom 2006b). We refer to this auction design as CC+ auction.

Definition 9. A *full revelation price update rule* in the CC+ auction increases prices on items as long as at least a single bidder bids on the item.

We aim for a strong game-theoretical solution concept. A desirable property would be a profile of strategies with an ex-post equilibrium, in which a bidder would not regret his bid even when he is told what everyone's type was after the auction. Note that we are not attempting to achieve a dominant strategy equilibrium, as preference elicitation in an indirect mechanism can invalidate dominant strategy equilibria existing in a single-step version of a mechanism (Conitzer and Sandholm 2002). We discuss the type of speculation that would be possible in a CC+ auction with full information in Appendix C. This illustrates that ex-post equilibria are not as robust as dominant strategy equilibria, but they are much more robust than Bayesian Nash equilibria. When iterative preference elicitation is used to implement a mechanism which would be a dominant-strategy direct-revelation mechanism in a sealed-bid version, then each agent's best (even in hindsight) strategy is to act truthfully if the other agents act truthfully (Conen and Sandholm 2001).

Definition 10. Truthful bidding in every round of an auction is an *ex-post equilibrium* if, for every bidder $i \in \mathcal{I}$, if bidders in $\mathcal{I}_{-i}$ follow the truthful bidding strategy, then bidder i maximizes his payoff in the auction by following the truthful bidding strategy (Mishra and Parkes 2007).

A description as a pseudo code of the CC and the CC+ auction is provided in Appendix B. Changes to the original CC auction are underlined.

The CC+ auction can suffer from small inefficiencies due to the minimal bid increment. *Last-and-final bids* have been suggested as a mean to get rid of these inefficiencies (Parkes 2006). They allow bidders to submit a final bid on a package, which is above the ask price of the previous round, but below the current ask price for a package. For the sake of clarity, we omit this rule in our analysis.

4.1. Properties of the CC+ Auction

We show that the CC+ auction maintains linear ask prices and powerset bidding is an ex-post equilibrium strategy leading to an efficient solution. Note that we do not need to make any restrictive assumptions on the bidders' valuations. To prove the efficiency already the slightly weaker partial revelation price update rule is sufficient (cf. proof to Corollary 1).

Corollary 2. *A powerset strategy is an ex-post equilibrium in the CC+ auction.*

As all bidders reveal all valuations, a bidder cannot improve his payoff by unilaterally deviating from the truthful powerset strategy in a respective CC+ auction, or influence whether the other bidders reveal their valuations truthfully. Therefore, the bidder's truthful powerset strategy is independent of the other bidders' types. This result shows, what types of price update and payment rules are sufficient for a powerset strategy to satisfy an ex-post equilibrium.

While the partial revelation price update rule is sufficient for efficiency, when all bidders follow a powerset strategy, a full revelation price update rule is necessary to achieve an ex-post equilibrium.

PROPOSITION 2. *Powerset bidding does not satisfy an ex-post equilibrium in the CC+ auction with only a partial revelation price update rule.*

Nisan and Segal (2006) have shown that it will require an exponential amount of queries from the auctioneer to the bidders, in order to determine the optimal allocation. There are subtle differences, however, in the amount of information that is elicited by different auction formats. A VCG auction and a CC+ auction asks bidders to reveal all $n2^m$ valuations to the full extent. In a CC+ auction a bidder sees the price clock increase on various items and learns at which prices nobody demands a particular item any more. In a VCG auction, bidders only know that a bid on a particular package was losing. In both cases, the auctioneer learns all valuations of all bidders. Using the partial revelation price update rule in the CC+ auction with z winning bids, only $n2^m - z$ losing valuations get elicited.

In NLPPAs such as the Ascending Proxy Auction, iBundle(3), or dVSV the auctioneer elicits $n2^m - z$ preferences in the worst case. It might also be, that the winners do not need to reveal all valuations on losing packages. However, a strong solution concept is only satisfied, if buyer submodularity is given. Clearly, communication complexity will always remain a stumbling block for any of the theoretical models in situations with more than a few items only. The assumption of following a straightforward strategy in exponentially many auction rounds will only hold in automated settings with proxy agents. The same is true for the powerset strategy, even if the number of auction rounds is much lower. We address this issue and the robustness of the efficiency results with respect to deviations from the powerset strategy in the next section.

Similar to work on NLPPAs, the CC+ auction is, however, of theoretical value as it shows sufficient rules and assumptions to design an ascending CA that uses linear ask prices and achieves an efficient outcome with a strong solution concept for general valuations.

4.2. Alternative Payment Rules

Of course the CC+ action suffers from some of the problems of the VCG design, in particular that the outcome might not be in the core (Ausubel and Milgrom 2006b). In other words, there are some bidders who could make a counteroffer to the seller that both sides would prefer to the VCG outcome. In such situations, the auctioneer can increase his sales revenue by excluding certain bidders, which is also referred to as revenue non-monotonicity. The bidders could also increase their payoff through shill bidding. These vulnerabilities of VCG outcomes are considered as serious problems for applications in the field. In some settings, it might be sufficient to have a mechanism, which is in the core, but which is as close to incentive compatibility as possible.

Day and Raghavan (2007) have recently suggested bidder-Pareto-optimal prices in the core as an alternative to VCG prices. An outcome of an auction is bidder-Pareto-optimal in the core if no Pareto improvement is possible within the core. This means, if we lower one bidder's payment, some other bidder's payment must increase to remain in the core. Such an outcome minimizes the total payments within the core.

DEFINITION 11. (Day and Raghavan 2007) An outcome is *bidder-Pareto-optimal* if there is no other core outcome weakly preferred by every bidder and strictly preferred by at least one bidder in the winning coalition.

This has also been referred to as minimal CE prices.

Definition 12. (Parkes 2006) Minimal CE prices minimize the auctioneer's total revenue on the efficient allocation across all CE prices.

Note that if items are complements, core prices may need to strictly exceed VCG prices. The Ascending Proxy Auction by Ausubel and Milgrom (2002) results in bidder-Pareto-optimal outcomes. Day and Milgrom (2007) show that a core-selecting auction provides minimal incentives for bidders to deviate from truthful reporting, if it chooses a bidder-Pareto-optimal outcome. Day and Raghavan (2007) also describe a constraint generation approach that generates bidder-Pareto-optimal core prices rapidly for sealed bid auctions. The payment scheme minimizes the total availability of gains from unilateral strategic manipulation. The final bids of each bidder on all packages in a CC+ auction can also be used to calculate bidder-Pareto-optimal core prices.

Corollary 3. *The CC+ auction with powerset bidders terminates with a core outcome, if it charges bidder-Pareto-optimal prices as payments.*

Note that even with the weaker partial revelation price update rule Corollary 3 holds. In contrast to the Clock-Proxy auction (Ausubel et al. (2006)) bidders in the CC+ auction do not need to type in valuations to a proxy agent after the CC auction has finished, and the bidder-Pareto-optimal prices get calculated right away.

5. Computational Experiments

In the previous section, we have seen that a full powerset strategy leads to efficiency, but is not viable except for very small CAs. So far, only a few papers provide results on individual bidding behavior in CAs. Scheffel et al. (2010) report around 10 to 12 bids per round in linear-price auctions independent of the number of packages with positive valuation. Kagel et al. (2009) report that bidders bid only on a fraction of the profitable packages in the CC auction. Global bidders bid between 12-14% of the profitable packages in one treatment with 6 items and 21-28% in the a treatment with 4 items. Both papers report that the packages attracting most attention were the most profitable ones.

This section describes results of computational experiments and analyzes efficiency and the number of auction rounds with artificial bidders in the CC and the CC+ auction with respect to deviations from a full powerset strategy. Our bidders follow either the straightforward or the powerset strategy, plus we also implement agents with restrictions on the number of packages submitted in each round. The experiments should help us understand, how restricted communication impacts efficiency in the CC+ auction. In contrast to the worst-case analysis that we provide in the first sections of this paper, this section provides more of an average case analysis for different bidder types, based on realistic value models.

5.1. Experimental Setup

The experimental setup is based on two treatment variables, the bidding strategies and the types of valuations.

5.1.1. Bidder Valuations. Since there are hardly any real-world CA data sets available, we have based our experiments on synthetic valuations generated with the Combinatorial Auctions Test Suite (CATS) (Leyton-Brown et al. 2000) and some that have been proposed by An et al. (2005). A detailed description is presented in the Appendix E.

5.1.2. Bidding Agents. In our theoretical analysis, we have already introduced straightforward and powerset strategies. The ***Powerset*** bidder evaluates all possible packages in each round, and submits bids for all packages which are profitable given current prices. In addition to the Powerset bidder, we tested limited versions of this bidder who bid only on the best six or best ten packages in each round, i.e., those six or ten packages with the highest payoff. This restriction is motivated by observations in lab experiments described above. In contrast, the ***Straightforward*** bidder only bids on his demand in each round, i.e., on those package(s) that maximize his payoff given current prices.

Inspired by observations in the lab, we also modeled a ***Heuristic 5of20*** bidder. This agent randomly selects 5 out of his 20 best packages based on his payoff in a round. This bidder allows to evaluate the robustness of the auction against imprecise bidding strategies.

Finally, we also analyzed the ***Preselect 10*** bidder. If a bidder is restricted in time during the auction, he might select his most valuable packages a priori, and stick to this selection throughout the auction. This might be a strategy by some bidders in auctions with a large number of items. The Preselect agent selects his 10 most valuable packages before the auction. During the auction, the bidder follows the straightforward strategy but bids only on the preselected packages.

5.2. Experimental Results

We used a $5 * 6$ factorial design, in which 5 value models are analyzed with 6 different bidding strategies. Each treatment was repeated 50 times with different random seeds for value models and bidder strategies (if applicable), and the XOR bidding language. The auctions use the minimum increment of 1. All value models are tailored to have the value of the efficient allocation around 200. For comparison with the CC design which does not use last-and-final bids, we test two versions of the CC+ auction, with and without last-and-final bids.

Tables 7 to 11 and Figures 1 in the Appendix F provide an overview of the efficiency and the number of auction rounds of the CC auction and both versions of the CC+ auction. The efficiency of the CC+ auction with the Powerset bidder was almost always 100% even without last-and-final bids. The worst efficiency of 95.29% with Powerset bidders was obtained in one of the Transportation Large samples. The lack of full efficiency in some cases is due to the minimum bid increment. With an ϵ bid increment and m items, the outcome of a CC+ auction without last-and-final bids can be $(m-1)\epsilon$ away from full efficiency. With last-and-final bids the CC+ auction is always 100% efficient with Powerset bidders. They also bring a marginal improvement in efficiency for powerset bidders with limits on a number of bids per round.

Another possibility to address the remaining inefficiencies of the CC+ auction without last-and-final bids is to reduce the minimum increment, but at the expense of an increase of auction rounds. For example, with minimum increment of 0.1 Powerset bidders achieve 100% efficiency in 49 samples of the Transportation Large value model, and 99,38% efficiency in the last sample. Dynamic bid increments that adapt to the level of competition might provide a middle ground.

The efficiency with Powerset6, Powerset10, and Heuristic bidders was very high in the Pairwise Synergy value model. In contrast, these strategies led to efficiencies as low as 75% in the Real Estate value model in the CC+ auction. All variants of the powerset bidder outperform the straightforward bidders in the CC+ auction in terms of efficiency.

Interestingly, the average efficiency of the CC+ auction with the Poweset10 bidder, restricted to 10 package bids in each round was 99.72% in the Pairwise Synergy and 98.01% in the Real Estate value model. Even a Powerset6 bidder restricted to 6 bids in each round led to very high levels of efficiency, which suggests that the CC+ auction is fairly robust against restrictions in the number of bids submitted in each round. Throughout, also Heuristic (5 of 20 best bids) bidders achieved very high levels of efficiency in the CC+ auction.

Note that there was only a modest increase in the number of auction rounds in the CC+ auction compared to the CC auction. Apparently, the increased efficiency in the CC+ auction over the CC auction comes at almost no cost in terms of an increase in the number of auction rounds.

As already discussed, the exponential communication complexity remains a stumbling block (Nisan and Segal 2001). While in NLPPAs this leads to a huge number of auction rounds, the

CC+ auctions require bidders to submit a large number of bids in each round. However, while we show that winners need to reveal more information in the CC+ auction as in NLPPAs, the number of actual bids submitted by bidders in CC+ is much lower. This is due to the bid increments of packages. If the prices for 5 items increase by ϵ, then the price for the package of these 5 items increases by $5 * \epsilon$. For example, in our Real Estate (3x3) value model a bidder had 130 valuations. In the CC+ auction (with last-and-final bids) 4,128 bids were submitted in 32 rounds by powerset bidders, and only 419 bids were submitted by Powerset10 bidders. In contrast, in the same setting iBundle(3) (Parkes and Ungar 2000) elicited 7,741 bids per bidder in 150 rounds, and in the Credit-Debit auction even 14,895 bids in 266 rounds.

6. Conclusions

Combinatorial auctions have led to a substantial amount of research and found a number of applications in high-stakes auctions for industrial procurement, logistics, energy trading, and the sale of spectrum licenses. Anonymous linear ask prices are very desirable and sometimes even essential for many of these applications (Meeus et al. 2009). Unfortunately, Walrasian equilibria with linear prices were only found for restricted settings. Already, Kelso and Crawford (1982) showed that the goods are substitutes property (aka gross substitutes) is a sufficient and an almost necessary condition for the existence of linear competitive equilibrium prices. Later, Gul and Stacchetti (2000) found that even if bidders' valuation functions satisfy the restrictive goods are substitutes condition, there exists no ascending VCG auction that uses anonymous linear prices. Bikhchandani and Ostroy (2002) showed that there always exist personalized non-linear competitive equilibrium prices. Several auction designs are based on these theoretical assumptions and use non-linear personalized prices. While these NLPPAs achieve efficiency, they only satisfy an ex-post equilibrium if the valuations meet buyer submodularity conditions, and they lead to a very large number of auction rounds requiring bidders to follow the straightforward strategy throughout.

A way out of this dilemma is to differentiate between ask prices in the final rounds and payments. We propose an extension of the CC auction, the CC+ auction design, which achieves full efficiency with bidders following a powerset strategy. This design modifies the price update rule of the CC auction and adds a VCG payment rule. We show that with such a VCG payment rule, a powerset strategy leads even to an ex-post equilibrium. Note that there are no restrictions on the type of valuations of bidders. This is important for any application. The discussion also provides a number of worst-case bounds on the efficiency of the CC auction.

Clearly, a powerset strategy is prohibitive for any but small combinatorial auctions and some other auction rules of the CC+ auction are impractical for real world applications. Actually, the CC+ auction is almost equivalent to a VCG auction, except that bidders learn the highest valuations of items throughout the auction, which they do not in a sealed-bid auction. Since the CC+ auction is iterative, however, we give up on dominant strategies and limit ourselves to an ex-post equilibrium. This is in line with previous results of the uniqueness of the VCG auction. Results by Green and Laffont (1979) and Holmstrom (1979) imply that any efficient mechanism with the dominant strategy property are equivalent to the VCG mechanism, always leading to identical equilibrium outcomes. Later, Williams (1999) found that all Bayesian mechanisms that yield efficient equilibrium outcomes and in which losers have zero payoffs lead to the same expected equilibrium payments as the VCG mechanism.

Now, however, as we understand under which assumptions the CC auction can be fully efficient, we can run sensitivity analyses to understand how robust the CC auction is against deviations of these assumptions. Arguably, powerset bidding is the strongest assumption. Interestingly, even if the number of bids submitted in each round is severely restricted or bidders heuristically select some of their "best" bids in each round, both the CC and the CC+ auction achieve very high efficiency levels. The results might also explain some of the high efficiency and robustness results of the CC auctions in the lab. The volume of ask prices that need to be communicated by the auctioneer, as well as the number of bids required by bidders is significantly lower than in NLPPAs. The model of the CC+ auction is of theoretical interest, as it shows under which conditions full efficiency with a strong solution concept for general valuations is possible. The discussion, however, also provides insights for the design of linear-price combinatorial auction formats, which achieve high levels of efficiency in the lab and in the field.

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Appendix A: Proofs

Theorem 2: If bidder valuations are demand masking and all bidders follow the straightforward strategy in the CC auction, then the efficiency converges to $\frac{2}{m+1}$ in the worst case.

Proof: The following proof is provided for two or more items for sale and $2m-1$ bidders. With less than $2m-1$ bidders and XOR bidding efficiency can only increase. Without loss of generality, we assume item-level bid increments of $\epsilon = 1$ in each round $t \in \mathcal{T} \subset \mathbb{N}$. We consider the value μ as given and determine ξ and ν_h such that efficiency decreases to the worst case of 0%.

case a) $\mu \geq \xi + \sum_h \nu_h$:

The efficient solution is to sell $R \cup S_h$ to one of the bidders h_a or h_b. The CC auction terminates with the efficient outcome in this case.

case b) $\mu < \xi + \sum_h \nu_h \wedge \xi = \mu$:

The proof is by showing that a straightforward bidder h_a cannot bid on S_h throughout the auction in a demand masking set of valuations. For this, the payoff $\pi_{h_a}(R\dot{\cup}S_k)$ must be higher than $\pi_{h_a}(S_h)$ for each bidder h_a in each round of the auction $t \in \mathcal{T}$:

$$v_{h_a}(R\dot{\cup}S_h) - \beta_{h_a,t}(R\dot{\cup}S_h) > v_{h_a}(S_h) - \beta_{h_a,t}(S_h) \quad \forall h \in \{2,...,g\}, \forall t \in \mathcal{T} \tag{1}$$

Since we know that $v_{h_a}(R\dot{\cup}S_h) = v_{h_b}(R\dot{\cup}S_h) = \mu$, and all bidders bid straightforward, we know that the price for all the items in $\mathcal{K}$ rise in each round by ϵ. Therefore, inequality (1) can be rewritten as

$$\mu - |R\dot{\cup}S_h|t\epsilon > \nu_h - |S_h|t\epsilon \overset{\epsilon=1}{\Longrightarrow} t < \frac{\mu - \nu_h}{|R|} \quad \forall h \in \{2,...,g\}, \forall t \in \mathcal{T} \tag{2}$$

Inequality (2) shows that as long as t is smaller than the right-hand side, a straightforward bidder always bids on the package $R\dot{\cup}S_h$. We can now determine a round $t_{min} = \min\{t | t \geq \frac{\mu - \nu_h}{|R|}, \forall h_a\}$, in which the payoff $\pi_{h_a}(R\dot{\cup}S_h)$ is for the first time smaller or equal to the payoff $\pi_{h_a}(S_h)$. We call t_{min} the *decisive round*. If either the right side or both sides of inequality (1) become negative in round t_{min}, bidder h_a cannot bid on S_h or the auction ends for bidder h_a as also the ask price for $R\dot{\cup}S_h$ is higher than $v_{h_a}(R\dot{\cup}S_h)$. If straightforward bidder h_a does not reveal his preferences for S_h throughout the auction, then the auctioneer in a class A auction selects any of the other bids with a revenue of μ, resulting in an efficiency of $\mu/(\xi + \sum_h \nu_h)$.

We determine maximal ν_h such that in round t_{min} the payoff of bidder h_a on package S_h is negative, which minimizes efficiency. We know that as long as bidder h_a's payoff is negative in the decisive round t_{min}, i.e., $\nu_h - |S_h|t_{min} < 0$, then bidder h_a does not bid on S_h. We also know that $t_{min} = \lceil (\mu - \nu_h)/|R| \rceil$ is the decisive round. We can now maximize ν_h such that $\nu_h - |S_h|\lceil(\mu - \nu_h)/|R|\rceil < 0$, resulting in $\nu_{h_{max}} = \max\{\nu_h | \nu_h < |S_h|\mu/(|R| + |S_h|)\}$. In order to maximize $\sum_h \nu_h$ and so minimize the efficiency $\mu/(\xi + \sum_h \nu_h)$ we set $|R| = 1$ and $|S_h| = 1$ for all $h \in \{2,...,g\}$. This results in an efficiency of $E(X) = \mu/(\xi + \sum_h (\frac{\mu}{2} - \rho))$ with $\rho > 0$. With $\rho \to 0$ and $\xi = \mu$ efficiency decreases to $2/(g+1)$ which is $2/(m+1)$ in the worst case. Note that it does not matter if ξ is smaller or larger than $\sum_h \nu_h$.

case c) $\mu < \xi + \sum_h \nu_h \wedge \mu \neq \xi$:

We show that efficiency can only increase compared to case b) considering the worst case. Either the enumerator of $E(X) = \frac{\max\{\xi,\mu\}}{\max\{\xi + \sum_h \nu_h, \mu + \sum_h \nu_h\}}$ increases or the denominator decreases.

- $\xi > \mu$: $\Rightarrow E(X) = \frac{\xi}{\xi + \sum_h \nu_h} = \frac{\mu + \delta}{\mu + \delta + \sum_h \nu_h}$ with $\delta > 0$ is always greater than the efficiency $E(X)$ in case b).
- $\xi < \mu$: $\Rightarrow$
  - —either $E(X) = \frac{\mu}{\xi + \sum_h \nu_h}$ which is certainly greater than $E(X) = \frac{\mu}{\mu + \sum_h \nu_h}$ the efficiency of case b).
  - —or $E(X) = \frac{\mu}{\mu + \sum_{h=2}^{g-1} \nu_h}$ which is also greater than $E(X) = \frac{\mu}{\mu + \sum_{h=2}^{g} \nu_h}$ the efficiency of case b).

□

Proposition 1: If all bidders follow the powerset strategy, the efficiency of the CC auction converges to 0% in the worst case.

Proof: Since efficiency cannot be negative it is sufficient to present an example, in which the efficiency is almost 0%. In the following example, we have two bidders and three items for sale. The two bidders have valuations for packages as shown in Table 3. They value all other packages with zero. The final ask prices would be $\beta_{(1)} = 2$, $\beta_{(2)} = 2$ and $\beta_{(3)} = 1$, and the final allocation would be to sell package $(1, 2)$ to bidder 1, which is inefficient if $\mu > 4$. Efficiency decreases to 0% if $\mu \to \infty$.

□

	(1, 2)	(2, 3)
v_1	4	μ
v_2	2	0

Table 3 Example of a CC auction with a powerset strategy.

We assume no free disposal in this example. Otherwise, bidder 1 would have a valuation of μ also for package (1, 2, 3), and this would get sold to bidder 1 for a price of 5. The payoff for bidder 1 in this allocation would be $\mu - 5$, which would be efficient, as the sum of the bidders' payoffs and the auctioneer revenue gets maximized. Free disposal can lead to situations, in which powerset bidding drives up prices to very high levels and reduces bidders' utility. It can also lead to high inefficiency (see Appendix D). Consequently, powerset bidding is an unlikely strategy in a CC auction with free disposal.

Corollary 1: If all bidders follow the powerset strategy, the CC auction with the partial revelation price update rule terminates with an efficient outcome.

Proof: Based on the statement of Theorem 1, we only need to show that the valuations of relevant packages get revealed with powerset bidders in the modified CC auction. By construction of the partial revelation price update rule powerset bidders, who are not part of the efficient allocation, reveal all their valuations. But the rule also ensures that all the bidders in the efficient allocation reveal their valuations on all packages except the ones that are in the winning allocation. As long as a bidder bids on more than one package the auction continues as each bidder can only win one package. As long as a bidder bids on a package that is not winning, prices increase and he can keep bidding. Thus the CC auction with the partial price update rule elicits all valuations except the ones of winning packages and terminates with an efficient allocation. □

Corollary 2: A powerset strategy is an ex-post equilibrium in the CC+ auction.

Proof: The proof for the ex-post equilibrium strategy is from the VCG mechanism. Let t_j denote the type of bidder j. We look at the bidder j and assume all other bidders follow the truth revealing powerset strategy. Bidder j receives a payment of $\sum_{i \neq j} u_i(t'_i, X) - \sum_{i \neq j} u_i(t'_i, X_{-j})$ from the center. The final payoff to bidder j reporting type t' and an allocation X and a VCG payment rule is $u_j(t_j, X) + \sum_{i \neq j} u_i(t'_i, X) - \sum_{i \neq j} u_i(t'_i, X_{-j})$. A bidder in this payment rule cannot affect the choice of X_{-j}. Hence, j can focus on maximizing $u_j(t_j, X) + \sum_{i \neq j} u_i(t'_i, X)$, i.e., his utility and the sum of the other's utilities. As the auction will maximize $\sum_i u_i(t'_i, X)$, j's utility will be maximized, if $t'_j = t_j$. □

Proposition 2: Powerset bidding does not satisfy an ex-post equilibrium in the CC+ auction with only a partial revelation price update rule.

Proof: In the example in Table 4 the CC+ auction with a partial revelation price update rule ends up with final ask prices of $\beta_{(1)} = 3$ and $\beta_{(2)} = 4$, before the VCG prices are calculated. If the auctioneer calculates VCG prices based on the submitted bids, then bidder 2 pays $3 - (7 - 5) = 1$ for the item (1). If bidder 2 knew $v_3(2)$, he could have bid up to 6 on item (2). This would increase the final ask price for (2) to 7, and lead to a new VCG price of $3 - (10 - 7) = 0$ for (1) for bidder 2. In a VCG mechanism, bidder 2 could not influence the bid submission of bidder 3 in a similar way, which is why the VCG mechanism has a dominant strategy. Therefore, in the CC+ auction with a partial revelation price update rule, the strategy of bidder 2 is not independent of other bidders' types. Even if the other bidders bid truthfully, a bidder could improve his payoff by deviating from a truth revealing powerset strategy, if he knew the other bidders' types and the other bidders truthfully follow the powerset strategy.

	(1)	(2)
v_1	0	3
v_2	3*	0
v_3	2	7*

Table 4 Counter-example.

□

Corollary 3: The CC+ auction with powerset bidders terminates with a core outcome, if it charges bidder-Pareto-optimal prices as payments.

Proof: Since the CC+ auction elicit all valuations from all bidders and the algorithm from Day and Raghavan (2007) calculates core prices upon the submitted bids the statement is shown. □

Appendix B: The CC and CC+ auction

```
Data: package bids βi(S)
Result: allocation X̄ and prices βi(X̄i)
initialization
  for k=1 to m do βk ← 0
  for i=1 to n do Xi ← ∅
repeat
  overdemand ← FALSE; undersupply ← FALSE
  for i=1 to n do
    bidders submit bids βi(S)
  for k=1 to m do
    if ≥ 2 bidders i ≠ j demand item k then
      βk ← βk + ε
      overdemand ← TRUE
    end
    if item k is not part of a bid βi(S) then
      undersupply ← TRUE
    end
  if overdemand = TRUE then exit iteration
  else if undersupply = FALSE then exit loop
  else
    for k=1 to m do
      Assign βi(S) with k ∈ S to the set of standing bids B
      Calculate X̄ based on all bids submitted in the auction
        if a bidder holding a bid in B is displaced and not in X̄ then
          foreach item k, which was displaced: do
          βk ← βk + ε
          end
        else X̄ is the final allocation
    end
until stop
```

Algorithm 1: CC auction

```
Data: package bids β_i(S)
Result: allocation X* and prices β_i(X_i*)
initialization
  for k=1 to m do β_k ← 0
  for i=1 to n do X_i ← ∅
repeat
  overdemand ← FALSE; undersupply ← FALSE
  for i=1 to n do
    submit a bid β_i(S) on each package S, which applies to v_i(S) − Σ_{k∈S}(β_k) ≥ 0
  for k=1 to m do
    if ≥ 1 bidders demand item k then
      β_k ← β_k + ϵ
      overdemand ← TRUE
    end
    if item k is not part of a bid β_i(S) then
      undersupply ← TRUE
    end
  if overdemand = TRUE then exit iteration
  else if undersupply = FALSE then exit loop
  else
  Calculate the final efficient allocation X* based on all submitted bids
  exit loop
until true
Calculate VCG prices β*_VCG based on all submitted bids
```

Algorithm 2: CC+ auction with powerset bidding

Appendix C: Ex-Post Equilibrium of the CC+ Auction

Does the CC+ auction satisfy a dominant strategy or an ex-post equilibrium? In the single-unit case, there has been an interesting recent discussion on the types of ascending auctions that actually satisfy a dominant strategy equilibrium. Isaac et al. (2007) have shown that while the clock version of an ascending single-item auction has a dominant strategy, the wide-spread English auction, which allows for jump bids, has not.

The CC+ auction can be seen as a multi-item generalization of the ascending clock auction. Also, the VCG auction can be thought of a single-round version of the CC+ auction, in which the bidder's dominant strategy is to bid truthfully on all possible packages, similar to a powerset strategy. Both auctions satisfy a dominant strategy equilibrium. Does also the CC+ auction satisfy a dominant strategy, or is it restricted to an ex-post equilibrium? In the following, we provide an example, in which signals revealed throughout the CC+ auction can make it beneficial for a bidder to deviate from his truth telling powerset strategy, when also others deviate from this strategy.

	(1)	(2)	(1, 2)
v_1	2*	0	0
v_2	0	3*	0
v_3	0	0	4

Table 5 **Example.**

The valuations for three bidders and two items are given in Table 5. The VCG price of bidder 1 is $2-(5-4)=1$ for item (1), and his payoff is 1. Now, lets assume that bidder 1 knows that bidder 2 will increase his bid on (2) to 4, if the ask price for (1) was 3. At round 2 the price clock ticks to 2 for each item and all three bidders signal demand at these prices. At round 3 prices are 3 for both items and again bidders 1 and 2 will signal demand. This will encourage bidder 2 to signal demand even in round 4 for item (2), when bidder 1 drops out. Now, bidder 1 gets a VCG price of $3+(7-4)=0$ and consequently increased his true payoff from 1 to 2. Bidder 2 learns through the course of the CC+ auction that there is a demand for (1) at a price of 3, which would not be possible in a direct revelation VCG auction.

This cannot happen in a clock auction with only a single item, as the bidders can only drop out or continue to signal demand on a single item. This illustrates that the dominant strategy equilibrium does not extend from the single-item clock auction to its multi-item generalization. The powerset strategy in a multi-item CC+ auction is therefore an ex-post equilibrium and no dominant strategy equilibrium.

Appendix D: Powerset Strategies in a CC Auction with Free Disposal

In the following, we describe an economy with powerset bidders and free disposal. We show that the CC auction leads to very high prices, thus reducing the bidders' utility, even in cases, where there is no competition. The example shows that the inefficiency in these situations can be almost as low as 50%.

	(1)	(2)	...	(m)
v_1	μ	0	...	0
v_2	0	$(\mu/m)-\epsilon$	...	0
...	...	...	...	...
v_m	0	0	...	$(\mu/m)-\epsilon$

Table 6 Valuations in an economy with powerset bidders and free disposal.

Given the valuations in Table 6 and an economy without free disposal, the bidders would all bid on a single item only, and the CC auction would stop after the first round at a price of the minimum bid increment ϵ. In an example, let's assume $m=n=100$, $\epsilon=1$, and $\mu\geq 200$. The allocation assigning bidder i item (i) is efficient and would maximize overall welfare. Bidder 1 would get a payoff of $\pi-1$, while all other bidders achieve a payoff of $(\mu/100)-1$. With an auctioneer revenue of 100, the social welfare is $199\pi/100$. If we assume m is the number of items, then the social welfare would be maximized at $(2m-1)\mu/m$.

Now, with free disposal bidder 1 would bid on all $2^{(m-1)}$ packages that enfold the item (1) in each round until a price of μ/m is reached and he wins all items. His payoff would be 0 and the auctioneer would make a revenue of μ, which is inefficient. With $m\to\infty$ efficiency converges to 50%.

Appendix E: Bidder Valuations

The ***Transportation*** value model uses the *Paths in Space* model from the CATS. It models a nearly planar transportation graph in Cartesian coordinates, in which each bidder is interested in securing a path between two randomly selected vertices (cities). The items traded are edges (routes) of the graph. Parameters for the Transportation value model are the number of items (edges) m and graph density η, which defines an average number of edges per city, and is used to calculate the number of vertices as $(2m)/\eta$. The bidder's valuation for a path is defined by the Euclidean distance between two nodes multiplied by a random number, drawn from a uniform distribution. Consequently only a limited number of packages, which represent paths between both selected cities, are valuable for the bidder. This allows to consider even larger transportation networks in a reasonable time. In this work we use the ***Transportation Small*** value model with 25 items and 15 bidders and the ***Transportation Large*** value model with 50 items and 30 bidders. Every bidder has interest in 16 different packages on average.

The ***Real Estate 3x3*** value model is based on the *Proximity in Space* model from the CATS. Items sold in the auction are the real estate lots k, which have valuations $v(k)$ drawn from the same normal distribution for each bidder. Adjacency relationships between two pieces of land p and q (e_{pq}) are created randomly for all bidders. Edge weights $r_{pq}\in[0,1]$ are then generated for each bidder, and they are used to determine package valuations of adjacent pieces of land $v(S)=(1+\sum_{e_{pq}:p,q\in S} r_{pq})\sum_{k\in S} v(k)$. In this work we use the *Real Estate 3x3* value model with 9 lots for sale. Individual item valuations have a normal distribution with a mean of 10 and a variance of 2. There is a 90% probability of a vertical or horizontal edge, and an 80% probability of a diagonal edge. Edge weights have a mean of 0.5 and a variance of 0.3. All experiments with the Real Estate value model are conducted with 5 bidders.

The ***Airports*** value model is an implementation of the *matching* scenario from CATS. It models the four largest USA airports, each having a predefined number of departure and arrival time slots. For simplicity there is only one slot for each time unit and airport available. Each bidder is interested in obtaining one departure and one arrival slot (i.e., item) in two randomly selected airports. His valuation is proportional to the distance between the airports and reaches maximum when the arrival time matches a certain randomly selected value. The valuation is reduced if the arrival time deviates from this ideal value, or if the time between departure and arrival slots is longer than necessary.

The ***Pairwise Synergy*** value model from An et al. (2005) is defined by a set of valuations of individual items $v(k)$ with $k \in \mathcal{K}$ and a matrix of pairwise item synergies $\{syn_{k,l} : k, l \in \mathcal{K}, syn_{k,l} = syn_{l,k}, syn_{k,k} = 0\}$. The valuation of a package S is then calculated as $v(S) = \sum_{k=1}^{|S|} v(k) + \frac{1}{|S|-1} \sum_{k=1}^{|S|} \sum_{l=k+1}^{|S|} syn_{k,l}(v(k)+v(l))$. A synergy value of 0 corresponds to completely independent items, and the synergy value of 1 means that the package valuation is twice as high as the sum of the individual item valuations. The model is very generic, as it allows different types of synergistic valuations, but it was also used to model valuations in transportation auctions (An et al. 2005). We use the Pairwise Synergy value model with 7 items, item valuations are drawn for each auction independently from a uniform distribution between 4 and 12. The synergy values are drawn from a uniform distribution between 1.5 and 2.0. The auctions with the Pairwise Synergy value model have 5 bidders.

In the *Real Estate* and *Pairwise Synergy* value models bidders were interested in a maximum package size of 3, because in these value models large packages are always valued over small ones. This is also motivated by real-world observations An et al. (2005), in which bidders typically have an upper limit on the number of items they are interested in. Without this limitation, the auction easily degenerates into a scenario with a single winner for the package containing all items.

Appendix F: Results of Computational Experiments

Format \ Bidder Type		Best-Response	Preselect10	5of20	Powerset6	Powerset10	Powerset
CC	Mean Efficiency in %	99.48	99.43	97.02	97.39	96.96	96.83
	Min. Efficiency in %	94.81	94.81	84.65	86.71	83.22	83.15
	Mean Rounds	29.10	29.42	25.36	25.28	25.10	24.96
CC+ no L&F Bids	Mean Efficiency in %	99.48	99.43	99.87	99.87	99.86	99.93
	Min. Efficiency in %	94.81	94.81	96.69	96.69	96.69	98.60
	Mean Rounds	29.94	30.02	31.78	31.94	31.66	31.50
CC+ with L&F Bids	Mean Efficiency in %	99.78	99.64	100.00	99.90	99.93	100.00
	Min. Efficiency in %	94.81	94.81	100.00	96.69	96.69	100.00
	Mean Rounds	30.50	30.16	32.02	32.02	31.52	31.32

Table 7 Transportation Small with 25 items and 15 bidders

Format \ Bidder Type		Best-Response	Preselect10	5of20	Powerset6	Powerset10	Powerset
CC	Mean Efficiency in %	98.48	97.60	97.94	97.85	97.86	98.07
	Min. Efficiency in %	90.74	90.09	85.00	85.00	85.00	85.00
	Mean Rounds	17.80	17.08	14.14	14.08	13.82	13.62
CC+ no L&F Bids	Mean Efficiency in %	98.44	97.60	99.12	99.17	99.25	99.34
	Min. Efficiency in %	90.74	90.09	95.29	95.29	95.35	95.29
	Mean Rounds	18.18	17.44	15.82	15.78	15.46	15.22
CC+ with L&F Bids	Mean Efficiency in %	98.95	98.17	99.87	99.87	99.91	100.00
	Min. Efficiency in %	93.21	90.09	97.78	97.78	98.38	100.00
	Mean Rounds	15.94	16.32	15.70	15.72	15.64	15.38

Table 8 Transportation Large with 50 items and 30 bidders

Format \ Bidder Type		Best-Response	Preselect10	5of20	Powerset6	Powerset10	Powerset
CC	Mean Efficiency in %	98.60	98.60	97.02	97.26	97.20	97.28
	Min. Efficiency in %	95.39	95.74	93.16	93.17	93.97	93.86
	Mean Rounds	10.88	10.86	8.50	8.32	8.22	8.20
CC+ no L&F Bids	Mean Efficiency in %	98.57	98.58	98.84	98.84	98.81	98.78
	Min. Efficiency in %	95.39	95.74	96.57	96.45	97.01	97.03
	Mean Rounds	13.72	13.70	11.98	11.62	11.46	11.48
CC+ with L&F Bids	Mean Efficiency in %	99.22	99.20	99.88	99.76	99.97	100.00
	Min. Efficiency in %	96.49	96.49	98.94	98.58	99.47	100.00
	Mean Rounds	15.46	15.56	12.48	12.04	11.90	11.82

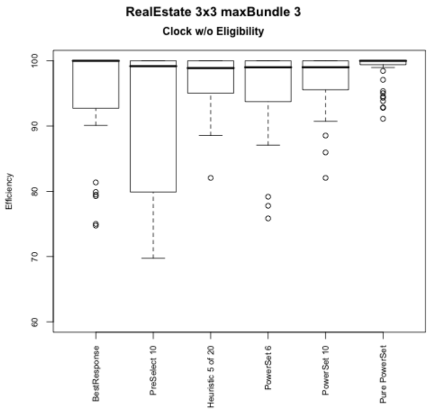
Table 9 Airports with 84 items and 40 bidders

Format \ Bidder Type		Best-Response	Preselect10	5of20	Powerset6	Powerset10	Powerset
CC	Mean Efficiency in %	95.63	92.47	97.18	95.95	97.19	98.74
	Min. Efficiency in %	74.74	69.73	82.05	75.84	82.05	91.11
	Mean Rounds	29.54	29.02	26.10	26.08	25.84	25.70
CC+ no L&F Bids	Mean Efficiency in %	95.52	92.47	97.79	96.74	98.00	99.85
	Min. Efficiency in %	74.74	69.73	82.05	75.84	82.05	98.45
	Mean Rounds	30.62	29.62	29.32	29.20	29.10	28.54
CC+ with L&F Bids	Mean Efficiency in %	94.82	92.48	98.10	96.99	98.01	100.00
	Min. Efficiency in %	70.81	72.51	82.05	75.84	82.05	100.00
	Mean Rounds	30.82	29.82	29.90	29.80	29.70	29.02

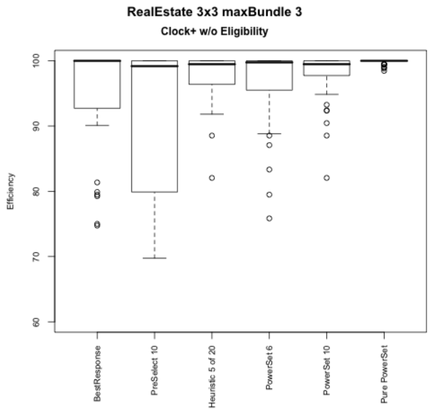
Table 10 Real Estate 3x3 with 9 items and 5 bidders

Format \ Bidder Type		Best-Response	Preselect10	5of20	Powerset6	Powerset10	Powerset
CC	Mean Efficiency in %	99.62	82.53	99.00	99.10	99.20	99.26
	Min. Efficiency in %	91.48	43.98	94.32	91.48	91.48	94.51
	Mean Rounds	31.08	30.00	30.92	31.18	30.96	30.72
CC+ no L&F Bids	Mean Efficiency in %	99.21	82.53	99.51	99.37	99.56	99.84
	Min. Efficiency in %	91.48	43.98	94.32	91.48	91.48	98.45
	Mean Rounds	35.14	30.40	34.08	34.36	33.96	33.70
CC+ with L&F Bids	Mean Efficiency in %	98.84	83.64	99.81	99.38	99.72	100.00
	Min. Efficiency in %	89.58	44.90	96.70	91.48	91.48	100.00
	Mean Rounds	35.94	30.88	34.34	34.86	34.30	34.10

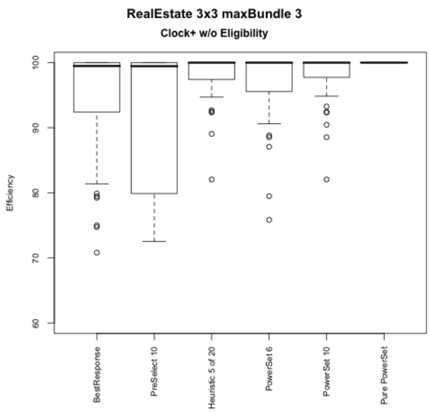
Table 11 Pairwise Synergy with 7 items and 5 bidders



(a)CC auction



(b)CC+ auction without L&F bids



(c)CC+ auction with L&F bids

Figure 1 Efficiency in the Real Estate value model (9 items, 5 bidders).

Appendix G: List of Symbols

$\mathcal{K}$	set of items
$R, S \subseteq \mathcal{K}$	subset of items
k, l	item index $k, l \in \{1, \ldots, m\}$
$\mathcal{I}$	set of bidders
$T \subseteq \mathcal{I}$	subset of bidders
i, j	bidder index $i, j \in \{1, \ldots, n\}$
$\mathcal{T}$	set of auction rounds
t	round index $t \in \{1, \ldots, r\}$
Γ	set of allocations
X	allocation $X = (X_1, \ldots, X_n)$ with package X_i assigned to bidder i
X^*	efficient allocation $X^* = (X_1^*, \ldots, X_n^*)$
$\overline{X}$	allocation that maximizes the auctioneers revenue/payoff
$E(X) \in [0, 1]$	efficiency of the allocation X
$v_i(S)$	private valuation of bidder i for package S
$w(T)$	coalitional value of coalition T
β_k	ask price for item k (optional $\beta_{k,t}$: in round t)
$\beta_i(S)$	(highest) bid price of bidder i for package S
$\mathcal{B}$	set of standing bids (optional $\mathcal{B}_t$: in round t)
$\pi_i(S)$	bidder i's payoff/revenue if he were to win package S
π_0	auctioneers' payoff/revenue
z	number of winning package bids
ϵ	minimum increment
ρ	marginal value
t_i	type of bidder i
η	graph density in the Transportation value model
$e_{p,q}$	adjacency relationship of item p to item q
$r_{p,q}$	edge weight of item p to item q
$syn_{k,l}$	synergy value of item k to item l

Appendix H: List of Abbreviations

(I)CA	(Iterative) Combinatorial Auction
CC	Combinatorial Clock
VCG	Vickrey-Clarke-Groves
NLPPA	Non-Linear-Personalized-Prize Auctions
CE	Competitive Equilibrium