# Finding and Recognizing Popular Coalition Structures


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## ABSTRACT

An important aspect of multi-agent systems concerns the formation of coalitions that are stable or optimal in some well-defined way. The notion of *popularity* has recently received a lot of attention in this context. A partition is popular if there is no other partition in which more agents are better off than worse off. In 2019, a long-standing open problem concerning popularity was solved by proving that computing popular partitions in roommate games is NP-hard, even when preferences are strict. We show that this result breaks down when allowing for randomization: *mixed* popular partitions can be found efficiently via linear programming and a separation oracle. Mixed popular partitions are particularly attractive because they are guaranteed to exist in any coalition formation game. Our result implies that one can efficiently verify whether a given partition in a roommate game is popular and that strongly popular partitions can be found in polynomial time (resolving an open problem). By contrast, we prove that both problems become computationally intractable when moving from coalitions of size 2 to coalitions of size 3, even when preferences are strict and globally ranked. Moreover, we give elaborate proofs showing the NP-hardness of finding popular, strongly popular, and mixed popular partitions in additively separable hedonic games and finding popular partitions in fractional hedonic games.



## KEYWORDS

Coalition Formation; Social Choice Theory

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## 1 INTRODUCTION

Coalitions and coalition formation have been a central concern of game theory, ever since the publication of von Neumann and Morgenstern's *Theory of Games and Economic Behavior* in 1944. The traditional models of coalitional game theory, in particular TU (transferable utility) and NTU (non-transferable utility) coalitional games, involve a formal specification of what each group of agents can achieve on their own. Drèze and Greenberg [20] noted that in many situations this is not feasible, possible, or even relevant to the coalition formation process, as, e.g., in the formation of social clubs, teams, or societies. Instead, in coalition formation games, the agents' preferences are defined directly over the coalition structures, i.e., partitions of the agents in disjoint coalitions. Formally, coalition formation can thus be considered as a special case of the general voting setting, where the agents entertain preferences over a special type of alternatives, namely coalition partitions of themselves, from which one or more need to be selected. In most situations it is natural to assume that an agent's appreciation of a partition only depends on the coalition he is a member of and not on how the remaining agents are grouped. Popularized by Bogomolnaia and Jackson [10], much of the work on coalition formation now concentrates on these so-called *hedonic games*.

The main focus in hedonic games has been on finding and recognizing partitions that satisfy various notions of *stability*—such as Nash stability, individual stability, or core stability—or optimality—such as Pareto optimality, utilitarian welfare maximality, or egalitarian welfare maximality [see 7, for an overview]. In this paper, we focus on the notion of *popularity* [24], which has the flavor of both stability and optimality. A partition is popular if there is no other partition that is preferred by a majority of the agents. Moreover, a partition is strongly popular if it is preferred to every other partition by some majority of agents. Popularity thus corresponds to the notion of weak and strong Condorcet winners in voting theory, i.e., candidates that are at least as good as any other candidate in pairwise majority comparisons. Just like stability notions, popularity is based on the idea that a subset of agents breaks off in order to increase their well-being. However, since the new partition has to make at least as many agents better off than worse off, popularity also has the flavor of optimality. In the standard reference *Algorithmics of Matching Under Preferences*, Manlove [36, p. 333] writes that "popular matchings […] have been an exciting area of research in the last few years." A recent survey on popular matchings is provided by Cseh [17].

In contrast to Pareto optimal partitions, popular partitions are not guaranteed to exist. We therefore also consider *mixed* popular partitions, as proposed by Kavitha et al. [31] and whose existence follows from the Minimax Theorem. A mixed popular partition is a probability distribution over partitions $p$ such that there is no other mixed partition $q$ such that the expected number of agents who prefer the partition returned by $p$ to $q$ is at least as large as the other way round. Mixed popular partitions are a special case of *maximal lotteries*, a randomized voting rule that has recently gathered increased attention in social choice theory [11, 13, 14, 23].

We study the computational complexity of popular, strongly popular, and mixed popular partitions in a variety of hedonic coalition formation settings including additively separable hedonic games, fractional hedonic games as well as hedonic games where the coalition size is bounded. The latter includes flatmate games (which only allow coalitions of up to three agents) and roommate games (which only allow coalitions of up to two agents). Our main findings are as follows.


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- Generalizing earlier results by Kavitha et al. [31], we show how mixed popular partitions in roommate games can be computed in polynomial time via linear programming and a separation oracle on a subpolytope of the matching polytope for non-bipartite graphs.[1] This stands in contrast to a recent result showing that computing popular partitions in roommate games is NP-hard [22, 26].
- As corollaries we obtain that verifying popular partitions [9], finding Pareto optimal partitions [4], and finding strongly popular partitions can all be done in polynomial time in roommate games, even when preferences admit ties. The latter statement resolves an acknowledged open problem.[2]
- We provide the first negative computational results for mixed popular partitions and strongly popular partitions by showing that finding these partitions in flatmate games is NP-hard. Moreover, it turns out, that verifying whether a given partition is popular, strongly popular, or mixed popular in flatmate games is coNP-complete. All of these results hold for strict and globally ranked preferences, i.e., coalitions appear in the same order in each individual preference ranking. This is interesting because finding popular partitions in roommate games becomes tractable under the same restrictions.
- We prove that computing popular, strongly popular, and mixed popular partitions is NP-hard in symmetric additively separable hedonic games and that computing popular partitions is NP-hard in symmetric fractional hedonic games. Furthermore, we show coNP-completeness of all corresponding verification problems.

## 2 RELATED WORK

Gärdenfors [24] first proposed the notions of popularity and strong popularity in the context of marriage games. He showed that popular matchings (or "majority assignments" in his terminology) need not exist when preferences are weak, but that existence is guaranteed for strict preferences because every stable matching is popular. As a consequence, the Gale-Shapley algorithm efficiently identifies popular matchings in marriage games with strict preferences. Kavitha and Nasre [32], Huang and Kavitha [27], and Kavitha [30] provide efficient algorithms for computing popular matchings that satisfy additional properties such as rank maximality or maximum cardinality. For weak preferences, computing popular matchings is NP-hard, even when all agents belonging to one side have strict preferences [9, 18].

In the more restricted setting of house allocation (henceforth housing games), Abraham et al. [2] proposed efficient algorithms for finding popular allocations of maximum cardinality for both weak and strict preferences. Mahdian [35] proved an interesting threshold for the existence of popular allocations: if there are $n$ agents and the number of houses exceeds $\alpha n$ with $\alpha \approx 1.42$, then the probability that there is a popular allocation converges to 1 as $n$ goes to infinity.

For roommate games with weak preferences, NP-hardness of computing popular matchings follows from the above-mentioned hardness results for marriage games. It was recently shown that this problem is still NP-hard when preferences are strict [19, 22, 26]. Also, finding a maximum-cardinality popular matching in instances where popular matchings are guaranteed to exist is NP-hard [15].

There are less results on *strongly* popular matchings. It is known from Gärdenfors [24] that a strongly popular matching is a unique popular matching and that every strongly popular matching is stable in roommate and marriage games. Based on these insights, Biró et al. [9] showed that strongly popular matchings in roommate games and marriage games with strict preferences can be found efficiently by first computing an arbitrary stable matching and then checking whether it is strongly popular. The case of weak preferences was left open and little progress has been made since then. Király and Mészáros-Karkus [34] recently gave an algorithm for finding strongly popular matchings in marriage games where preferences are strict, except that agents belonging to one side may be completely indifferent. In housing games, a matching is strongly popular if and only if it is a unique perfect matching. Hence, strongly popular matchings in housing games can be found in polynomial time. All of the above mentioned results on strong popularity, including the open problem, follow from Corollary 4.8.

*Mixed* popular matchings were introduced by Kavitha et al. [31] who also showed how to compute a *fractional* popular matching in housing games and marriage games, which can then be translated into a mixed popular matching via a Birkhoff-von Neumann decomposition. This is possible in these bipartite settings because every fractional matching is implementable as a probability distribution over deterministic matchings. When moving from marriage markets to roommate markets, this does not hold anymore. For example, a matching involving three agents where every pair of agents is matched with probability $1/2$ is not implementable. Huang and Kavitha [28] have shown that in marriage games with strict preferences, the popular matching polytope is half-integral and that half-integral mixed popular matchings can be computed in polynomial time. No such matchings are guaranteed to exist when preferences are weak. They also apply the same techniques to roommate games in order to compute an optimal half-integral solution over the bipartite matching polytope in the case of strict preferences. However, the resulting solutions may again fail to be implementable. Apart from that, their methods heavily rely on computing stable matchings, which may be intractable when preferences are weak. By contrast, our results in Section 4.2.1 are based on the matching polytope for non-bipartite graphs via odd-set constraints and allow both to deal with ties and to efficiently compute a solution that is implementable using LP methods (Proposition 4.2). The axiomatic properties of mixed popular matchings such as efficiency and strategyproofness were investigated by Aziz et al. [6], Brandt et al. [16], and Brandl et al. [12].

To the best of our knowledge, popularity, strong popularity, and mixed popularity have not been studied for coalition formation settings that go beyond coalitions of size 2 except for a theorem by Aziz et al. [5, Th. 15] who claimed that checking whether a partition is popular in ASGHs is NP-hard and that verifying whether

[1] The results by Kavitha et al. [31] only hold for house allocation and marriage markets and cannot be straightforwardly extended to roommate markets. See Section 2 for more details.

[2] See, for example, Biró et al. [9] and Manlove [36]: "A third open problem is the complexity of finding a strongly popular matching (or reporting that none exists), for an instance of RPT [Roommate Problem with Ties]" [9, p. 107]; "Our last open problem concerns the complexity of the problem of finding a strongly popular matching, or reporting that none exists, given an instance of SRTI [Stable Roommates with Ties and Incomplete lists], which is unknown at the time of writing" [36, p. 380].

a partition is popular is coNP-complete. However, the proof of the first statement is incorrect.[3] We substantially modified the reduction to prove a stronger statement and independently proved a stronger statement for the verification problem.

# 3 PRELIMINARIES

Let $N$ be a finite set of agents. A *coalition* is a non-empty subset of $N$. By $\mathcal{N}_i$ we denote the set of coalitions agent $i$ belongs to, i.e., $\mathcal{N}_i = \{S \subseteq N : i \in S\}$. A *coalition structure*, or simply a *partition*, is a partition $\pi$ of the agents $N$ into coalitions, where $\pi(i)$ is the coalition agent $i$ belongs to. A *hedonic game* is a pair $(N, \succsim)$, where $\succsim = (\succsim_i)_{i \in N}$ is a *preference profile* specifying the preferences of each agent $i$ as a complete and transitive *preference relation* $\succsim_i$ over $\mathcal{N}_i$. If $\succsim_i$ is also anti-symmetric we say that $i$'s preferences are *strict*. Otherwise, we say that preferences are *weak*. $S \succ_i T$ if $S \succsim_i T$ but not $T \succsim_i S$—i.e., *i strictly prefers S* to $T$—and $S \sim_i T$ if both $S \succsim_i T$ and $T \succsim_i S$—i.e., $i$ is *indifferent* between $S$ and $T$. In hedonic games, agents are only concerned about their own coalition. Accordingly, preferences over coalitions naturally extend to preferences over partitions as follows: $\pi \succsim_i \pi'$ if and only if $\pi(i) \succsim_i \pi'(i)$.

Two basic properties of partitions are Pareto optimality and individual rationality. Given a hedonic game $(N, \succsim)$, a partition $\pi$ is *Pareto optimal* if there is no partition $\pi'$ such that $\pi' \succsim_j \pi$ for all agents $j$ and $\pi' \succ_i \pi$ for at least one agent $i$. Partition $\pi$ is *individually rational* if $\pi(i) \succsim_i \{i\}$ for all $i \in N$, i.e., each agent $i$ prefers $\pi(i)$ to staying alone. The rationale behind individual rationality is that agents cannot be forced into a coalition.

Individual rationality is also the crucial ingredient of a succinct representation of hedonic games where only the preferences over individual rational coalitions are considered [8]. A hedonic game $(N, \succsim)$ is represented by *Individually Rational Lists of Coalitions* (IRLC) via the game $(N, \succsim')$ where $\succsim'$ is a preference profile such that $\succsim'_i$ is the restriction of $\succsim_i$ to individually rational sets in $\mathcal{N}_i$. In this case, $(N, \succsim)$ is called a *completion* of $(N, \succsim')$. This representation of games is useful to obtain meaningful hardness results because the size of the naive representation of a hedonic game is exponential in the number of agents while the IRLC representation may only require polynomial space if the number of individually rational coalitions is small enough.

In order to define popularity and strong popularity, let $N(\pi, \pi')$ be the set of agents who prefer $\pi$ over $\pi'$, i.e., $N(\pi, \pi') = \{i \in N : \pi(i) \succ_i \pi'(i)\}$, where $\pi, \pi'$ are two partitions of $N$. On top of that, we define the *popularity margin* of $\pi$ and $\pi'$ as $\phi(\pi, \pi') = |N(\pi, \pi')| - |N(\pi', \pi)|$. Then, $\pi$ is called *more popular* than $\pi'$ if $\phi(\pi, \pi') > 0$. Furthermore, $\pi$ is called *popular* if, for all partitions $\pi'$, $\phi(\pi, \pi') \geq 0$, i.e., no partition is more popular than $\pi$. $\pi$ is called *strongly popular* if, for all partitions $\pi'$, $\phi(\pi, \pi') > 0$, i.e., $\pi$ is more popular than every other partition. Note that there can be at most one strongly popular partition in any hedonic game.

For a hedonic game $(N, \succsim)$ in IRLC representation, a partition $\pi$ is called *popular* if it is popular in the completion of $(N, \succsim)$ where, for each agent, all coalitions that are not individually rational are gathered in a single indifference class that is less preferred than the singleton coalition. This definition of popularity generalizes the definition of popularity that is used for marriage games by Kavitha et al. [31], and adds the appropriate perspective on individual rationality.[4] Note that a popular partition need not be individually rational.

Many hedonic games do not admit a popular partition. However, existence can be guaranteed by introducing randomization via mixed partitions, i.e., probability distributions over partitions. Let two mixed partitions $p = \{(\pi_1, p_1), \ldots, (\pi_k, p_k)\}$ and $q = \{(\sigma_1, q_1), \ldots, (\sigma_l, q_l)\}$ be given, where $(p_1, \ldots, p_k), (q_1, \ldots .q_l)$ are probability distributions. We define the popularity margin of $p$ and $q$ as their expected popularity margin, i.e.,

$$\phi(p, q) = \sum_{i=1}^{k} \sum_{j=1}^{l} p_i q_j \phi(\pi_i, \sigma_j).$$

Clearly, the definition of popularity carries over to the extension of $\phi$. As first observed by Kavitha et al. [31], mixed popular partitions always exist, because they can be interpreted as maximin strategies of a symmetric zero-sum game [see, also 6, 23].

Proposition 3.1. *Every hedonic game admits a mixed popular partition.*

Proof. Every hedonic game can be viewed as a two-player symmetric zero-sum game where the rows and columns of the two players are indexed by all possible partitions $\pi_1, \ldots, \pi_{B_{|N|}}$ and the entry at position $(i, j)$ of the game matrix is $\phi(\pi_i, \pi_j)$. By the Minimax Theorem [38], the value of this game is 0 and therefore, any maximin strategy, whose existence is guaranteed, is popular. □

# 4 RESULTS

## 4.1 Basic Relationships

Clearly, a strongly popular partition is also popular and a popular partition, interpreted as a probability distribution with singleton support, is mixed popular. Furthermore, every coalition structure in the support of a mixed popular partition is Pareto optimal. This already follows from a more general statement by Fishburn [23, Prop. 3]. We give a simple proof for completeness.

Proposition 4.1. *Let $p = \{(\pi_1, p_1), \ldots, (\pi_k, p_k)\}$ be a mixed popular partition. Then, for every $i = 1, \ldots, k$ with $p_i > 0$, $\pi_i$ is Pareto optimal.*

Proof. Let $p = \{(\pi_1, p_1), \ldots, (\pi_k, p_k)\}$ be a mixed popular partition and fix $i \in \{1, \ldots, k\}$ such that $p_i > 0$. Assume for contradiction that $\pi'_i$ is a Pareto improvement over $\pi_i$. Define $p' = \{(\pi_1, p_1), \ldots, (\pi_{i-1}, p_{i-1}), (\pi'_i, p_i), (\pi_{i+1}, p_{i+1}), \ldots, (\pi_k, p_k)\}$. Note that $\phi(\pi'_i, p) = \sum_{j=1, j \neq i}^{k} p_j \phi(\pi'_i, \pi_j) + p_i \phi(\pi'_i, \pi_i) \geq \sum_{j=1, j \neq i}^{k} p_j \phi(\pi_i, \pi_j) + p_i \phi(\pi'_i, \pi_i) > \sum_{j=1, j \neq i}^{k} p_j \phi(\pi_i, \pi_j) + p_i \phi(\pi_i, \pi_i) = \phi(\pi_i, p)$.

[3] The reduction fails because for a 'yes'-instance of Exact 3-Cover, the partition $\pi$ claimed to be popular for the ASHG it maps to is not popular: the partition $\pi' = \{\{y^s, z_1^s, z_2^s\} : s \in S\} \cup \{\{b_1^r, a_2^r\} : r \in R\} \cup \{\{b_2^r, a_1^r, a_3^r\} : r \in R\}$ is more popular.

[4] The IRLC representation ignores preferences over coalitions that are not individually rational. However, in contrast to core stability or Nash stability, these preferences can affect whether a partition is popular or not. In order to circumvent this problem one could strengthen the definition of popularity by requiring that a coalition needs to be popular for *all* extensions of the IRLC represented preferences. All our results also hold for this notion, because we construct individually rational partitions for which the two notions of popularity coincide.

Then, $\phi(p', p) = \sum_{j=1, j\neq i}^{k} p_j \phi(\pi_j, p) + p_i \phi(\pi_i', p) > \sum_{j=1, j\neq i}^{k} p_j \phi(\pi_j, p) + p_i \phi(\pi_i, p) = \phi(p, p) = 0$.

Hence, $p$ is not mixed popular, a contradiction. □

We thus have the following relationships between strong popularity (sPop), popularity (Pop), partitions in the support of any mixed popular partition (supp(mPop)), and Pareto optimality (PO):

$$\text{sPop} \implies \text{Pop} \implies \textbf{supp(mPop)} \implies \textbf{PO}.$$

The concepts printed in boldface are guaranteed to exist. As a consequence, hardness results for Pareto optimality imply hardness of mixed popular partitions (though not for popular partitions since they need not exist). The existence problems for popular and strongly popular partitions are naturally contained in the complexity class $\Sigma_2^p$. The verification problems are contained in coNP. The coNP-hardness of the verification problem of popular partitions implies coNP-completeness of the verification of mixed popular partitions. This is because every popular partition is a degenerate mixed popular partition and because a mixed popular partition is less popular than another mixed partition if and only if it is less popular than a deterministic partition. Conversely, polynomial-time algorithms for mixed popularity can be used to efficiently verify whether a partition is popular.

## 4.2 Ordinal Hedonic Games

In this section we investigate hedonic games in IRLC representation. Important subclasses of these games are defined by restricting the size of individually rational coalitions using a global constant. We thus obtain *flatmate games* as games in which only coalitions of up to three agents are individually rational and *roommate games* as games in which only coalitions of size 2 are individually rational. Further restrictions are obtained by bipartitioning the set of agents, say, into males and females and additionally demanding that one group of agents is completely indifferent. A *marriage game* is a roommate game where the agents can be partitioned in two sets such that the only individually rational partitions are formed with agents from the other set. A *housing game* is a marriage game where all agents belonging to one set of the partition are completely indifferent. All of these classes permit polynomially bounded IRLC representations and form the following inclusion relationship when preferences are weak:

$$\text{Housing} \subset \text{Marriage} \subset \text{Roommates} \subset \text{Flatmates} \subset \text{IRLC}.$$

In roommate games (and their subclasses), partitions are referred to as *matchings*.

*4.2.1* ***Roommate Games****.* We start by investigating mixed popularity in roommate games, which will turn out to have important consequences for popular and strongly popular matchings.

Kavitha et al. [31] showed that mixed popular matchings in housing games and marriage games can be found in polynomial time. However, as explained in Section 2, their algorithm cannot be applied to roommate games. In this section, we show how to obtain an algorithm for the more general class of roommate games.

To introduce our matching notation, we fix a graph $G = (N, E)$ where the vertex set is the set of agents and there is an edge between two vertices if the corresponding coalition of size 2 is individually rational for both agents. For technical reasons, it is useful to restrict attention to the case of perfect matchings. Similarly to the construction by Kavitha et al. [31], this can be achieved by introducing worst-case partners $w_a$ for every agent $a$ with $\{a, w_a\} \sim_a \{a\}$. These worst-case partners are not individually rational for all other original agents, and are indifferent among all other agents themselves. They mimic the case that an agent remains unmatched and do not affect the popularity of a partition. We now establish a relationship between mixed matchings and fractional matchings, where the latter are defined as points in the matching polytope $P_{Mat} \subseteq [0,1]^E$, defined as follows [21].

$$\begin{aligned} P_{Mat} = \{x \in \mathbb{R}^E \colon & \sum_{e \in E, v \in e} x(e) = 1 \ \forall v \in N, \\ & \sum_{e \in \{\{v,w\} \in E \colon v, w \in C\}} x(e) \leq \frac{|C|-1}{2} \ \forall C \subseteq N, |C| \text{ odd}, \\ & x(e) \geq 0 \ \forall e \in E\} \end{aligned}$$

The main constraint ensures that for every odd set of agents $C$, the weight of the fractional matching restricted to these agents is at most $(|C|-1)/2$, where this fraction denotes the maximum cardinality that any matching on the set $C$ may have.

Given a matching $M$, denote by $\chi_M \in P_{Mat}$ its incidence vector. We obtain a correspondence of mixed matchings and fractional matchings by mapping a mixed matching $p = \{(M_1, p_1), \dots, (M_k, p_k)\}$ to the fractional matching $x_p := \sum_{i=1}^{k} p_i \chi_{M_i}$. Note that $x_p \in P_{Mat}$ by convexity. Since we only want to operate on the more concise matching polytope, we need to ensure that we can recover a mixed matching efficiently. The following proposition, which is based on general LP theory, can be seen as an extension of the Birkhoff-von Neumann theorem to non-bipartite graphs.

Proposition 4.2. *Let $G = (N, E)$ be a graph and $x \in P_{Mat}$ a vector in the associated matching polytope. Then, a mixed matching $p = \{(M_1, p_1), \dots, (M_k, p_k)\}$ such that $x_p = x$ can be found in polynomial time.*

Proof. The separation problem for the matching polytope $P_{Mat}$ can be solved in polynomial time, i.e., the class of matching polytopes is solvable. Therefore, given a graph $G = (N, E)$ and a vector $x \in P_{Mat}$ we can find a convex combination of extreme points of $P_{Mat}$ that yield $x$ in polynomial time [25, Th. 3.9].

Since the extreme points of the matching polytope are the incidence vectors of matchings [21], this is a mixed matching whose corresponding fractional matching is $x$. □

It thus suffices to define popularity of fractional matchings equivalently to popularity of mixed matchings that induce them. Popular fractional matchings will be described as feasible points of a (nonempty) subpolytope of the matching polytope. The separation problem for the subpolytope will be tractable by a modification of the algorithm that determines the unpopularity margin of a matching given by McCutchen [37].

To this end, we need to define the popularity margin for fractional matchings. Given $x, y \in P_{Mat}$, we define their *popularity* *margin* as

$$\phi(x,y) = \sum_{a \in N} \sum_{i,j \in N_G(a)} x(a,i)y(a,j)\phi_a(i,j)$$

where $N_G(a) = \{v \in N \colon \{v,a\} \in E\}$ is the neighborhood of $a$ in $G$ and

$$\phi_a(i,j) = \begin{cases} 1 & \text{if } i \succ_a j \\ -1 & \text{if } i \prec_a j \\ 0 & \text{if } i \sim_a j \end{cases}.$$

The proof of the next property is identical to the corresponding statement for marriage games by Kavitha et al. [31].

Proposition 4.3. *Let $p$ and $q$ be mixed matchings. Then,*

$$\phi(p,q) = \phi(x_p, x_q).$$

*In particular, $p$ is popular if and only if for all matchings $M$, $\phi(x_p, x_M) \geq 0$, where $x_M := \chi_M$.*

As a consequence, mixed popular matchings correspond precisely to the feasible points of the following polytope.

$$P_{Pop} = \{x \in P_{Mat} : \phi(x, x_M) \geq 0 \ \text{ for all matchings } M\}$$

It remains to find a feasible point of the popularity polytope $P_{Pop}$. By adopting the auxiliary graph in McCutchen's algorithm for non-bipartite graphs, we can find a matching $M$ minimizing $\phi(x, x_M)$ by solving a maximum weight matching problem. This solves the separation problem for $P_{Pop}$.

Proposition 4.4. *The separation problem for $P_{Pop}$ can be solved in polynomial time.*

The proofs of this and other statements are omitted because of space restrictions.

We are now ready to prove the following theorem.

Theorem 4.5. *Mixed popular matchings in roommate games with weak preferences can be found in polynomial time.*

Proof. By Proposition 4.4 and by means of the Ellipsoid method [33], we can find a fractional popular matching in polynomial time. This can be translated into a mixed popular matching by Proposition 4.2. □

Theorem 4.5 has a number of interesting consequences. Since every mixed popular matching is Pareto optimal, we now have an LP-based algorithm to find Pareto optimal matchings for weak preferences as an alternative to combinatorial algorithms like the Preference Refinement Algorithm by Aziz et al. [4].

Corollary 4.6. *Pareto optimal matchings in roommate games with weak preferences can be found in polynomial time.*

Biró et al. [9] provided a sophisticated algorithm for verifying whether a given matching is popular. An efficient LP-based algorithm for this problem follows from Theorem 4.5.

Corollary 4.7. *It can be efficiently verified whether a given matching in a roommate game is popular.*

Finally, the linear programming approach allows us to resolve the open problem of finding strongly popular matchings when preferences are weak.

Corollary 4.8. *Finding a strongly popular matching or deciding that no such matching exists in roommate games with weak preferences can be done in polynomial time.*

Proof. If a strongly popular matching exists, it is unique. In particular, it is the unique mixed popular matching. Given a (deterministic) matching $M$, we can check in polynomial time if it is strongly popular. Simply apply the reduction of Proposition 4.4 and check whether the maximum weight matching amongst the matchings different to $M$ on the auxiliary graph has negative weight (in which case the matching is strongly popular) or not. To this end, we compute a maximum weight matching for every (incomplete) graph that is obtained by deleting exactly one edge from the auxiliary graph. The maximum weight matching amongst these matchings has the highest weight amongst matchings different from $M$.

The algorithm to compute a strongly popular matching if one exists first computes a fractional popular matching. If it does not correspond to a deterministic matching, there exists no strongly popular matching. Otherwise, it is deterministic and, as described above, we can check if it is strongly popular. If this is the case, we return it. If not, there exists no strongly popular matching. □

Since there can be at most one strongly popular matching, the verification problem for strongly popular matchings in roommate games can also be solved efficiently.

*4.2.2* ***Flatmate Games.*** It turns out that moving from coalitions of size 2 to size 3 renders all search problems related to popular partitions intractable. For mixed popular partitions, we can leverage the relationship to Pareto optimal partitions. Aziz et al. [4, Th. 5] have shown that finding Pareto optimal partitions in flatmate games with weak preferences is NP-hard. Since mixed popular partitions are guaranteed to exist (Proposition 3.1) and satisfy Pareto optimality (Proposition 4.1), this immediately implies the NP-hardness of mixed popular partitions by means of a Turing reduction.[5]

Theorem 4.9. *Computing a partition in the support of a mixed popular partition in flatmate games with weak preferences is NP-hard.*

For strict preferences, the same method does not work. Pareto optimal partitions can always be found efficiently by serial dictatorship. Therefore, we will give direct reductions that yield hardness for strong popularity and mixed popularity in flatmate games with strict preferences. These reductions are based on related graphs that correspond to instances of the NP-complete problem X3C [29]. An instance $(R,S)$ of *Exact* 3*-Cover* (X3C) consists of a ground set $R$ together with a set $S$ of 3-element subsets of $R$. A 'yes'-instance is an instance such that there exists a subset $S' \subseteq S$ that partitions $R$. We will first describe the graph underlying our hardness constructions, then prove a key property of this graph, and finally give the actual reduction.

To this end, consider an instance $(R,S)$ of X3C. Let $k = \min\{k \in \mathbb{N} \colon 2^k \geq |R|\}$ be the smallest power of 2 that is larger than the cardinality of $R$. We define a flatmate game on vertex set $N = \bigcup_{j=0}^{k} N_j$, where $N_j = \bigcup_{i=1}^{2^j} A_j^i$ consists of $2^j$ sets of agents $A_j^i$.

We define the sets of agents as

[5]Using the same argument, one can transfer further results on Pareto optimality [4], e.g., for room-roommate games or three-cyclic matching games.

- $A_k^i = \{a_k^i, b_k^i, c_k^i\}$ for $i = 1, \dots, 2^k$, and
- $A_j^i = \{a_j^i, b_j^i, c_j^i, \alpha_j^i, \beta_j^i, \gamma_j^i, \delta_j^i\}$ for $j = 0, \dots, k-1, i = 1, \dots, 2^j$.

Similar names of agents suggest that these agents are going to play the same role in the reduction. The preferences are designed in a way such that if there exists no 3-partition of $R$ through sets in $S$, then there exists a unique best partition that assigns more than half of the agents a top-ranked coalition. Otherwise, there exists a partition that puts exactly all the other agents in one of their top coalitions. For the sets in the definition of the preferences, an arbitrary tie-breaking can be used to obtain strict preferences. We order the set $R$ in an arbitrary but fixed way, say $R = \{r^1, \dots, r^{|R|}\}$ and for a better understanding of the proof and the preferences, we label the agents $b_k^i = r^i$ for $i = 1, \dots, |R|$. If we view the set of agents $N$ as $k+1$ levels of agents, then the ground set $R$ of the instance of X3C is identified with some specific agents in the top level $k$. Preferences of the agents are as follows.

- $\{a_k^i, b_k^i, c_k^i\} \succ_{a_k^i} \{a_k^i\}, i = 1, \dots, 2^k$
- $\{a_j^i, \beta_j^i, \gamma_j^i\} \succ_{a_j^i} \{a_j^i, b_j^i, c_j^i\} \succ_{a_j^i} \{a_j^i\}, j = 0, \dots, k-1, i = 1, \dots, 2^j$
- $\{\{b_k^i, b_k^v, b_k^w\} : \{r^i, r^v, r^w\} \in S \text{ for some } 1 \le v, w \le |R|\} \succ_{b_k^i} \{a_k^i, b_k^i, c_k^i\} \succ_{b_k^i} \{b_k^i\}, i = 1, \dots, |R|$
- $\{b_k^i\}, i = |R|+1, \dots, 2^k$
- $\{b_j^i, c_{j+1}^{2i-1}, c_{j+1}^{2i}\} \succ_{b_j^i} \{a_j^i, b_j^i, c_j^i\} \succ_{b_j^i} \{b_j^i\}, j = 0, \dots, k-1, i = 1, \dots, 2^j$
- $\{a_j^i, b_j^i, c_j^i\} \succ_{c_j^i} \{c_j^i\}, j = 0, \dots, k, i = 1, \dots, 2^j$
- $\{\alpha_j^i, \beta_j^i\} \succ_{\alpha_j^i} \{\alpha_j^i\}, j = 0, \dots, k-1, i = 1, \dots, 2^j$
- $\{\beta_j^i, \gamma_j^i, a_j^i\} \succ_{\beta_j^i} \{\beta_j^i, \alpha_j^i\} \succ_{\beta_j^i} \{\beta_j^i\}, j = 0, \dots, k-1, i = 1, \dots, 2^j$
- $\{\gamma_j^i, \delta_j^i\} \succ_{\gamma_j^i} \{\gamma_j^i\}, j = 0, \dots, k-1, i = 1, \dots, 2^j$
- $\{\delta_j^i, \alpha_{j+1}^{2i-1}, \alpha_{j+1}^{2i}\} \succ_{\delta_j^i} \{\delta_j^i, \gamma_j^i\} \succ_{\delta_j^i} \{\delta_j^i\}, j = 0, \dots, k-1, i = 1, \dots, 2^j$

The structure of the flatmate game is illustrated in Figure 1 for the case $k = 3$. We will be particularly interested in coalitions of the types $\{a_j^i, b_j^i, c_j^i\}$, $\{\alpha_j^i, \beta_j^i\}$, and $\{\gamma_j^i, \delta_j^i\}$, which are indicated by undirected edges. These coalitions form the partition $\pi^*$ of Lemma 4.10 that we need later to investigate for strong and mixed popularity in the respective reductions. The directed edges indicate that an agent at the tail of the arrow needs to form a coalition with the agent at the tip of the arrow in order to improve from her coalition of the above type. The set of agents consists of a binary tree of triangles depicted by the circular-shaped vertices. The important property of this tree is that whenever a coalition of the type $\{a_j^i, b_j^i, c_j^i\}$ gets dissolved, there can only be an improvement in popularity for the agents in $A_j^i$ if they propagate changes in the partition upwards within this tree. This is achieved for agents $b_j^i$ directly through the binary tree and for agents $a_j^i$ with help of the auxiliary agents $\{\alpha_j^i, \beta_j^i, \gamma_j^i, \delta_j^i\}$ that are depicted as diamond-shaped vertices.

In the following lemma and theorem, we denote for any subset $M \subseteq N$ of agents and partitions $\pi, \pi'$ of $N$, $\phi_M(\pi, \pi') = |N(\pi, \pi') \cap M| - |N(\pi', \pi) \cap M|$, that is, the popularity margin on a the subset $M$ with respect to $\pi$ and $\pi'$.

Lemma 4.10. *Let an instance $(R, S)$ of X3C be given and define the corresponding flatmate game as above. Consider the partition $\pi^* = \{\{a_j^i, b_j^i, c_j^i\} : j = 0, \dots, k, i = 1, \dots, 2^j\} \cup \{\{\alpha_j^i, \beta_j^i\}, \{\gamma_j^i, \delta_j^i\} : j = 0, \dots, k-1, i = 1, \dots, 2^j\}$. Let $\pi \neq \pi^*$ be an arbitrary partition of agents distinct from $\pi^*$. Then $\phi(\pi^*, \pi) \ge 1$. In addition, if $c_0^1 \in N(\pi^*, \pi)$, then $\phi(\pi^*, \pi) \ge 3$ or $\{b_k^i : i = 1, \dots, 2^k\} \subseteq N(\pi, \pi^*)$.*

Proof sketch. Let an instance $(R, S)$ of X3C be given and define the corresponding flatmate game as above. Let $\pi^*$ be defined as in the lemma and $\pi \neq \pi^*$ an other partition. We recursively define the following sets of agents: for $i = 1, \dots, 2^k$, $T_k^i = A_k^i$ and for $j = k-1, \dots, 0, i = 1, \dots, 2^j$, $T_j^i = A_j^i \cup T_{j+1}^{2i-1} \cup T_{j+1}^{2i}$. The core of the proof is the following claim that can be proved by induction over $j = k, \dots, 0$.

For every $i = 1, \dots, 2^j$ holds: Assume there exists an agent $x \in T_j^i$ with $\pi(x) \neq \pi^*(x)$. Then $\phi_{T_j^i}(\pi^*, \pi) \ge 1$. If even $\pi(a_j^i) \neq \pi^*(a_j^i)$, then $\phi_{T_j^i}(\pi^*, \pi) \ge 3$ or $\{b_k^i : i = 1, \dots, 2^k\} \cap T_j^i \subseteq N(\pi, \pi^*)$.

For the induction step, one essentially proves that changing the coalitions in $A_j^i$ causes severe loss in popularity, unless we propagate changes to substructures via $b_j^i$ or $\beta_j^i$. Clearly, the assertion of the lemma follows from the case $j = 0$. □

We are now ready to apply the lemma for the desired reductions.

Theorem 4.11. *Deciding whether there exists a strongly popular partition in flatmate games is coNP-hard, even if preferences are strict.*

Proof. The reduction is from X3C. Given an instance $(R, S)$ of X3C, we define a hedonic game on agent set $N' = N \cup \{z\}$ where the agents $N$ are as in the above construction with the identical preferences and $\{c_0^1, z\} \succ_z \{z\}$. Note that $|N'| = 3\sum_{j=0}^{k} 2^j + 4\sum_{j=0}^{k-1} 2^j + 1 = 10 \cdot 2^k - 6 = O(|R|)$ and the reduction is in polynomial time.

Consider the partition $\pi^* = \{\{a_j^i, b_j^i, c_j^i\} : j = 0, \dots, k, i = 1, \dots, 2^j\} \cup \{\{\alpha_j^i, \beta_j^i\}, \{\gamma_j^i, \delta_j^i\} : j = 0, \dots, k-1, i = 1, \dots, 2^j\} \cup \{\{z\}\}$. It follows directly from Lemma 4.10 that $\pi^*$ is popular and hence there exists a strongly popular partition if and only if $\pi^*$ is strongly popular. We will prove that this is the case if and only if the instance of X3C is a 'no'-instance.

Assume that there exists no 3-partition of $R$ through sets in $S$ and let an arbitrary partition $\pi \neq \pi^*$ be given. Then there exists an agent $x \in N$ with $\pi(x) \neq \pi^*(x)$ and it follows from Lemma 4.10 that $\phi(\pi^*, \pi) \ge \phi_N(\pi^*, \pi) - 1 \ge 3 - 1 = 2$. Hence, $\pi^*$ is strongly popular.

Conversely, assume that there exists a 3-partition $S' \subseteq S$ of $R$. Define $\pi = \{\{b_k^v, b_k^w, b_k^x\} : \{v, w, x\} \in S'\} \cup \{\{b_k^i\} : i = |R|+1 \dots, 2^k\} \cup \{\{\delta_{k-1}^i, a_k^{2i-1}, a_k^{2i}\} : i = 1, \dots, 2^{k-1}\} \cup \{\{b_j^i, c_{j+1}^{2i-1}, c_{j+1}^{2i}\}, \{\delta_j^i, \alpha_{j+1}^{2i-1}, \alpha_{j+1}^{2i}\}, \{a_j^i, \beta_j^i, \gamma_j^i\} : j = 1, \dots, k-1, i = 1, \dots, 2^j\} \cup \{\{\alpha_0^1\}, \{z, c_0^1\}\}$. It is easily checked that $\phi(\pi, \pi^*) = 0$.

Indeed, $N(\pi, \pi^*) = \{b_k^i : i = 1, \dots, 2^k\} \cup \{\beta_j^i, \delta_j^i, a_j^i : j = 0, \dots, k-1, i = 1, \dots, 2^j\} \cup \{z\}$. Therefore, $|N(\pi, \pi^*)| = 2^k + 4\sum_{j=1}^{k-1} 2^j + 1 = 5 \cdot 2^k - 3 = \frac{1}{2}|N'|$. Hence, $\phi(\pi, \pi^*) \ge 0$ and equality follows from popularity of $\pi^*$. Therefore, there exists no strongly popular partition. □

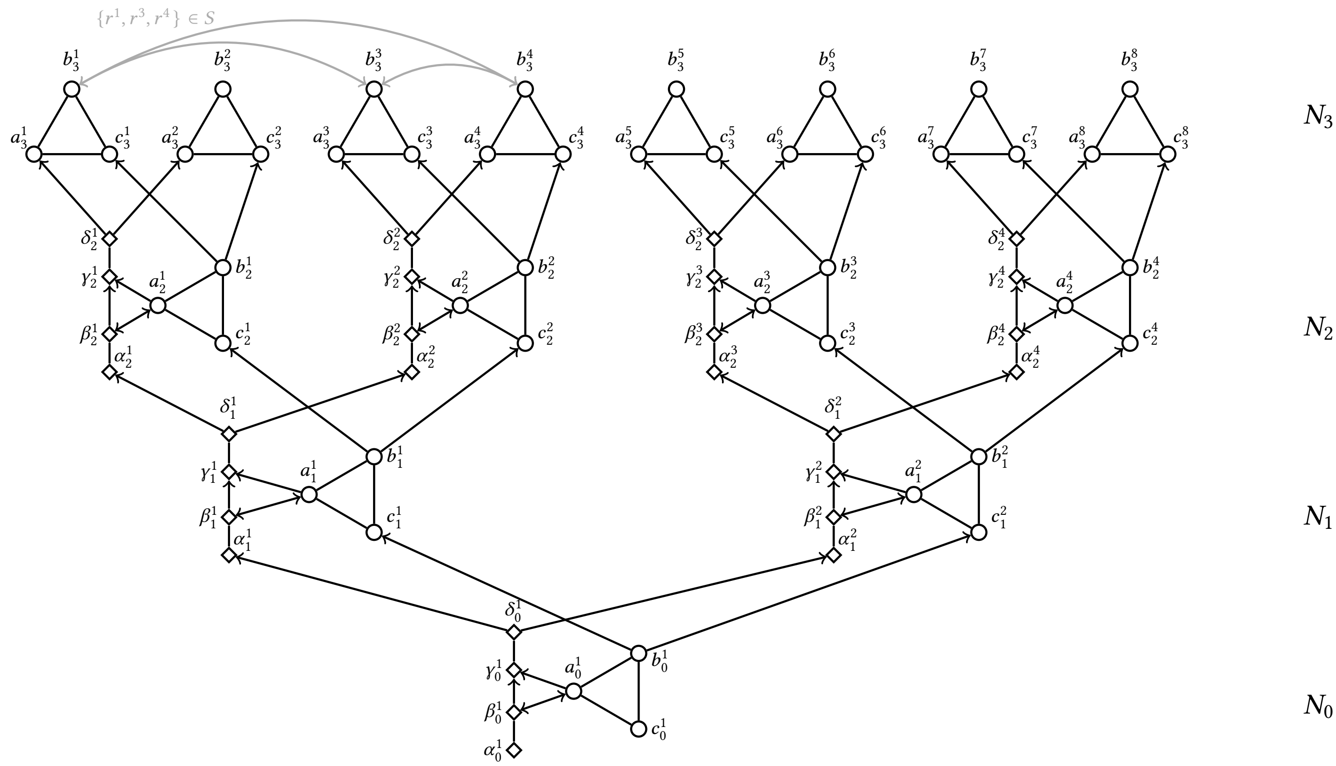


**Figure 1: Schematic of the reduction for flatmate games with strict preferences. There is an edge between two agents if they are in the coalition $\pi^*$ defined in Lemma 4.10. Directed edges indicate improvements from $\pi^*$. The gray edges suggest a 3-elementary set in $S$.**

A similar reduction as in Theorem 4.11 works also for mixed popularity. However, we need two auxiliary agents to control the switch between a strongly popular and non-popular partition.

Theorem 4.12. *Computing a mixed popular partition in flatmate games is NP-hard, even if preferences are strict.*

To conclude the section, we deal with the problem of verifying whether a given partition is popular or strongly popular. Hardness of verifying popular partitions in flatmate games is shown by a complicated reduction from E3C. We have gadgets for elements in $S$ and control the switch between 'yes' and 'no' instances by means of a binary tree. For a simpler, yet weaker hardness result, this tree could be contracted into a single vertex, but only at the expense of having to allow for coalitions of more than three agents.

Theorem 4.13. *Verifying whether a given partition in a flatmate game with strict preferences is popular is coNP-complete.*

For strong popularity, we obtain the same result.

Theorem 4.14. *Verifying whether a given partition in a flatmate game is strongly popular is coNP-complete, even if preferences are strict.*

Proof. In the proof of Theorem 4.11, the partition $\pi^*$ is strongly popular if, and only if, $(R, S)$ is a 'no'-instance of X3C. □

*4.2.3* ***Globally Ranked Preferences***. A natural question that arises after hardness results have been established is whether there are meaningful preference restrictions under which these results do not hold. In many cases, hardness breaks down when assuming that preference profiles adhere to certain structural restrictions. One such preference restriction that has been considered in the domain of coalition formation is that there exists one common global ranking $\succsim$ of all coalitions in $2^N \setminus \{\emptyset\}$ and each individual preference relation $\succsim_i$ is the restriction of $\succsim$ to $\mathcal{N}_i$. It is known that under these *globally ranked preferences*, every roommate game admits a stable matching, which can furthermore be efficiently computed [1]. Since every stable matching also happens to be popular (see Section 2), this implies that computing popular matchings in roommates games, which was recently shown to be NP-hard [19, 22, 26], becomes tractable under globally ranked preferences.

By contrast, all hardness results shown in Section 4.2.2 hold even when preferences are globally ranked. This confirms the robustness of these results and underlines the crucial difference between settings with coalitions of size 2 and coalitions of size 3.

## 4.3 Cardinal Hedonic Games

Important subclasses of hedonic games that admit succinct representations are based on cardinal utility functions. For one, there are *additively separable hedonic games* [10], where the utility that an

| | *weak preferences* | | | | *strict preferences* | | | |
|---|---|---|---|---|---|---|---|---|
| | PO | mPop | sPop | Pop | PO | mPop | sPop | Pop |
| IRLC | ↑ | ↑ | ↑ | ↑ | in P | ↑ | ↑ | ↑ |
| Flatmates | NP-h.[a] | NP-h. (Th. 4.9) | NP-h. (Th. 4.11) | ↑ | ↓ | NP-h. (Th. 4.12) | NP-h. (Th. 4.11) | ↑ |
| Roommates | in P[b] | in P (Th. 4.5) | in P (Cor. 4.8) | ↑ | ↓ | in P (Th. 4.5) | in P[d] | NP-h.[g] |
| Marriage | ↓ | ↓ | ↓ | NP-h.[e] | ↓ | ↓ | ↓ | in P[f] |
| Housing | ↓ | ↓ | ↓ | in P[c] | in P | in P[h] | in P | in P[c] |

**Table 1: Complexity of finding partitions in ordinal hedonic games. New results are highlighted in gray and implications are marked by gray arrows. NP-hardness of computing a popular or strongly popular partition always follows by a Turing reduction from the existence problem. Whenever computing a mixed popular partition is NP-hard, then verifying a deterministic partition is coNP-complete.**

[a]: Aziz et al. [4, Th. 5], [b]: Aziz et al. [4, Th. 7], [c]: Abraham et al. [2, Th. 3.9], [d]: Biró et al. [9, Th. 6], [e]: Biró et al. [9, Th. 11], Cseh et al. [18, Th. 2], [f]: Gärdenfors [24, Th. 3], [g]: Gupta et al. [26, Th. 1.1], Faenza et al. [22, Th. 4.6], Cseh and Kavitha [19, Th. 2], [h]: Kavitha et al. [31, Th. 2]

agent associates with a coalition is the sum of utilities he ascribes to each member of the coalition. On the other hand, there are *fractional hedonic games* [3], where the sum of utilities is divided by the number of agents contained in the coalition.

In the following, let $v_i(j)$ denote the utility that agent $i$ associates with agent $j$. A hedonic game $(N, \succsim)$ is an *additively separable hedonic game* (ASHG) if there is $(v_i(j))_{i,j \in N}$ that for every agent $i$, the preferences $\succsim_i$ are induced by the cardinal utilities given by $v(S) = \sum_{j \in S} v_i(j)$, for $S \subseteq N$. The hedonic game $(N, \succsim)$ is a *fractional hedonic game* (FHG) if there exists $(v_i(j))_{i,j \in N}$ such that for every agent $i$, the preferences $\succsim_i$ are induced by the cardinal utilities given by $v(S) = (\sum_{j \in S} v_i(j))/|S|$, for $S \subseteq N$. We focus on *symmetric* ASHGs and FHGs, i.e., games for which $v_i(j) = v_j(i)$ for all $i, j \in N$.

All hardness results in this section are obtained by rather involved reductions from E3C.

Theorem 4.15. *Checking whether there exists a popular partition in a symmetric ASHG is NP-hard.*

The verification problem for ASHGs turns out to be coNP-complete. The proof of Theorem 4.16 is simpler and holds for a more restricted class of games than the proof by Aziz et al. [5].

Theorem 4.16. *Checking whether a given partition in a symmetric ASHG is popular is coNP-complete.*

The reductions for mixed and strong popularity on ASHGs rely on a similar idea as for flatmate games with strict preferences. We find a graph that satisfies similar properties as the flatmate game considered in Lemma 4.10. This graph is used for the next four results.

Theorem 4.17. *Checking whether there exists a strongly popular partition in a symmetric ASHG is coNP-hard.*

Theorem 4.18. *Verifying whether a given partition in a symmetric ASHG is strongly popular is coNP-complete.*

Theorem 4.19. *Computing a mixed popular partition in a symmetric ASHG is NP-hard.*

We even obtain coNP-hardness of the existence of popular partitions.

Theorem 4.20. *Checking whether there exists a popular partition in a symmetric ASHG is coNP-hard.*

Theorem 4.21. *Checking whether there exists a popular partition in a symmetric FHG is NP-hard, even if all weights are non-negative.*

The hardness proof for the verification problem for FHGs is a more involved version of the proof for ASHGs.

Theorem 4.22. *Checking whether a given partition in a symmetric FHG is popular is coNP-complete, even if all weights are non-negative and the underlying graph is bipartite.*

The graphs used in the proof of Theorem 4.22 have girth 6. This is in contrast to the polynomial-time algorithm by Aziz et al. [3] for computing the core on FHGs with girth at least 5.

## 5 CONCLUSION

We have investigated the computational complexity of finding and recognizing popular, strongly popular, and mixed popular partitions in various types of ordinal hedonic games (see Table 1) and cardinal hedonic games. Two important factors that govern the complexity of computing these partitions in ordinal hedonic games are whether preferences may contain ties and whether coalitions of size 3 are allowed. When preferences are weak, computing mixed popular and strongly popular partitions is only difficult for representations for which we cannot even compute Pareto optimal partitions efficiently. For strict preferences, however, Pareto optimal partitions can be found efficiently while computing mixed popular and strongly popular partitions remains intractable, even when preferences are globally ranked. Our positive results are obtained via a single linear programming approach that unifies a number of existing results and exploits the relationships between the different types of popularity. Finally, we complete the picture by providing a variety of results showing the intractability of popular, strongly popular, and mixed popular partitions in ASHGs and FHGs.


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