Equilibria of Graphical Games with Symmetries[☆]

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Abstract

We study graphical games where the payoff function of each player satisfies one of four types of symmetry in the actions of his neighbors. We establish that deciding the existence of a pure Nash equilibrium is NP-hard in general for all four types. Using a characterization of games with pure equilibria in terms of even cycles in the neighborhood graph, as well as a connection to a generalized satisfiability problem, we identify tractable subclasses of the games satisfying the most restrictive type of symmetry. Hardness for a different subclass leads us to identify a satisfiability problem that remains NP-hard in the presence of a matching, a result that may be of independent interest. Finally, games with symmetries of two of the four types are shown to possess a symmetric *mixed* equilibrium which can be computed in polynomial time. We thus obtain a natural class of games where the pure equilibrium problem is computationally harder than the mixed equilibrium problem, unless P=NP.

Key words: Algorithmic game theory; Computational complexity; Nash equilibria; Graphical games; Symmetries
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1. Introduction

The idea underlying *graphical games* [1] is that in games with a large number of players, the payoff of any particular player will often depend only on the actions of a small number of other players in a local neighborhood. More formally, a graphical game is given by a (directed or undirected) graph on the set of players of a normal-form game, such that the payoff of each player depends only on the actions of his neighbors in this graph. If neighborhoods are bounded, graphical games can be represented using space polynomial in the number of players. *Symmetric games* constitute another natural and well-studied class of games, characterized by the fact that players can not, or need not, distinguish between other players. In this paper, we consider graphical games where the payoff function of each player is symmetric in the actions of his neighbors. For instance, consider a setting where each player is faced with the decision of producing one of two types of complementary goods within a regional neighborhood. Players are not only producers but also consumers and thus happier when both products are available within their neighborhood. We will see in Section 3.3 that deciding the existence of a pure Nash equilibrium, i.e., a profile of mutual best responses, in such a setting is highly nontrivial yet computationally tractable.

Related Work. The computational problem of finding Nash equilibria in graphical games with degree bounded by 3 has recently been shown equivalent to the same problem for general n-player games with $n \geq 4$ [2], and thus complete for the complexity class PPAD [3]. It is not surprising that the structure of the neighborhood graph greatly influences the complexity of the equilibrium problem. PPAD-hardness holds even if the underlying graph has constant pathwidth, but becomes tractable for graphs of degree 2, i.e., for *paths* [4]. All known algorithms for the more general case of *trees* have exponential worst-case running time even on trees with bounded degree and pathwidth 2, but equilibria satisfying various fairness criteria can be computed in polynomial time if additionally there are only two actions per player and the best response policy, a data structure representing *all* Nash equilibria of a game, has polynomial size [5].

A different line of research has investigated the problem of deciding the existence of *pure* Nash equilibria, i.e., equilibria where the support of each strategy contains only a single action. Unlike Nash equilibria in *mixed* strategies, i.e., probabilistic combinations of actions, pure equilibria are not guaranteed to exist. If they exist, however, pure equilibria have two distinct advantages over mixed ones. For one, requiring randomization in order to reach a stable outcome has been criticized on various grounds. In multi-player games, where action probabilities in equilibrium can be irrational numbers, randomization is particularly questionable. Secondly, pure equilibria as computational objects are usually much smaller in size than mixed ones. The pure equilibrium problem is NP-complete for graphical games on directed graphs with outdegree bounded by 2 and with only two actions for each player and two different payoffs, and tractable for graphs with bounded treewidth [6, 7].

Brandt et al. [8] analyze four classes of symmetric games, and show that the pure equilibrium problem is tractable if the number of actions is a constant, and complete for NP or PLS, respectively, if the number of actions grows in the number of players. One of the classes, in which all players have identical payoff functions, is guaranteed to possess a symmetric equilibrium, i.e., one where all players play the same strategy. This equilibrium is not necessarily pure, but can be found efficiently if the number of actions is not too large compared to the number of players [9]. A larger class, allowing different payoff functions for different players, admits an approximation by a factor depending on the Lipschitz constant of the payoff function and on the square of the number of actions, and a polynomial-time approximation scheme for the case of two actions [10].

These results fuel hope that tractability results can be obtained for larger classes of games satisfying some kind of symmetry. In this regard, Daskalakis and Papadimitriou [11] consider games on a d-dimensional undirected torus or grid with payoff functions that are identical for all players and symmetric in the actions of the players in the neighborhood, a condition that will be called *symmetry* in this paper. In particular, they show that deciding the existence of a pure Nash equilibrium in such a game is NL-complete when $d = 1$ and NEXP-complete for $d \geq 2$. In this paper, we investigate the pure equilibrium problem in graphical games satisfying the kinds of symmetries considered by Brandt et al. [8]. Our work can thus be seen as a refinement of the work of Gottlob et al. [6] and of Daskalakis and Papadimitriou [11].

Paper Structure and Results. We begin by introducing the necessary game-theoretic concepts. In Section 3, we then investigate the computational complexity of the pure equilibrium problem in graphical games satisfying four different types of symmetries. The question of tractable classes of graphical games is answered mostly in the negative. For three of the four symmetry classes, deciding the existence of a pure equilibrium is NP-hard already for the case of two actions, two payoffs, and neighborhoods of size two. Assuming the most restricted type of symmetry, the problem becomes NP-hard when there are three different payoffs, or neighborhoods of size four. On the other hand, we use interesting connections of the latter class to even cycles in directed graphs and to generalized satisfiability to identify tractable classes of games. As a corollary, we exhibit a satisfiability problem that remains NP-hard in the presence of a matching. We present this result, which may be of independent interest, in Section 4. Finally, in Section 5, we show that mixed equilibria in games with two of the above symmetry types can be found in polynomial time if the number of actions grows only slowly in the neighborhood size. Quite interestingly, there exists a class of games where deciding the existence of a pure equilibrium problem is likely to be harder than finding a mixed equilibrium. We assume the reader to be familiar with the complexity classes P, NP, and #P, and the notion of polynomial-time reducibility (see, e.g., [12]).

2. Preliminaries

An accepted way to model situations of strategic interaction is by means of a normal-form game (see, e.g., [13]).

Definition 1 (normal-form game)**.** A *game in normal-form* is a tuple $\Gamma = (N, (A_i)_{i \in N}, (p_i)_{i \in N})$ where N is a set of *players* and for each player $i \in N$, A_i is a nonempty set of *actions* available to i, and $p_i : (\prod_{i \in N} A_i) \to \mathbb{R}$ is a function mapping each action profile of the game, i.e., combination of actions, to a real-valued *payoff* for i.

A vector $s \in \prod_{i \in N} A_i$ of actions is also called a profile of *pure strategies*. This concept can be generalized to *(mixed) strategy profiles* $s \in S = \prod_{i \in N} S_i$, by letting players randomize over their actions. Here, we have S_i denote the set of probability distributions over player i's actions, or *(mixed) strategies* available to player i. We further write $n = |N|$ for the number of players in a game, s_i for the ith strategy in profile s, and s_C for the vector of strategies for all players in a subset $C \subseteq N$.

A graphical game is given by a graph on the set of players, such that the payoff of a player only depends only on his own action, and on the actions of his neighbors in the graph. In the following definition, the underlying graph is directed, corresponding to a neighborhood relation that is not necessarily symmetric.

Definition 2 (graphical game)**.** Let $\Gamma = (N, (A_i)_{i\in N}, (p_i)_{i\in N})$ be a normal-form game, $\nu : N \to 2^N$. Γ is a *graphical game* with neighborhood ν if for all $i \in N$ and $s, s' \in A^N$, $p_i(s) = p_i(s')$ whenever $s_{\hat{\nu}(i)} = s'_{\hat{\nu}(i)}$, where $\hat{\nu}(i) = \nu(i) \cup \{i\}$.

A game Γ has *k-bounded neighborhoods* if there exists $\nu : N \to 2^N$ such that Γ is a graphical game with neighborhood ν and for all $i \in N$, $|\nu(i)| \leq k$.

We assume throughout the paper that graphical games are encoded by listing the payoffs of each player as a function of the actions of his neighbors.

Symmetry as a property of a mathematical object refers to its invariance under a certain type of transformation. Symmetries of games usually mean invariance of the payoffs under automorphisms of the set of action profiles induced by some group of permutations of the set of players. *Anonymous* games as considered by Daskalakis and Papadimitriou [10], for example, require the set of available actions to be the same for all players, and the payoff of a particular player to remain the same under any permutation of the elements of an action profile. This imposes constraints on individual payoff functions only and can therefore directly be applied to graphical games as well. In general, however, it does not make much sense from a computational point of view to consider symmetries of the payoff functions without requiring the neighborhood graph to be "symmetric" in an appropriate way as well. Consider, for example, the class of all graphical games whose payoff functions are invariant under automorphisms in the automorphism group of the neighborhood graph. While this class of games is very natural, it does not impose meaningful computational restrictions. Indeed, it is not too hard to see that any graphical game can be encoded by a game in the above class that has a neighborhood graph with a trivial automorphism group. Hardness results for both pure and mixed equilibria thus carry over immediately.

In general, different types of restrictions on the neighborhood structure will be required for different kinds of symmetries of the payoff functions. In this paper, we take a slightly different approach. We consider properties found in anonymous and symmetric games, and study graphical games that possess these properties. A characteristic feature of symmetries in games is the inability to distinguish between other players. Following Daskalakis and Papadimitriou [10], the most general class of games with this property will be called *anonymous*. Four different classes of games are obtained by considering two additional characteristics: *identical payoff functions* for all players[1] and the ability to *distinguish oneself* from the other players. The games obtained by adding the former property will be called *symmetric*, and presence of the latter will be indicated by the prefix "*self*." For the obvious reason, we will henceforth talk about games where the set of actions is the same for all players and write $A = A_1 = \cdots = A_n$ and $k = |A|$, respectively, to denote this set and its cardinality.

An intuitive way to describe anonymous games is in terms of equivalence classes of the aforementioned automorphism group, using a notion introduced by Parikh [14] in the context of context-free languages. Given a set A of actions, the *commutative image* of an action profile $s \in A^N$ is given by $\#(s) = (\#(a, s))_{a \in A}$ where $\#(a, s) = |\{\, i \in N : s_i = a \,\}|$. In other words, $\#(a, s)$ denotes the number of players playing action a in action profile s, and $\#(s)$ is the vector of these numbers for all the different actions. This definition naturally extends to action profiles for subsets of players.

Definition 3 (symmetries)**.** Let $\Gamma = (N, (A_i)_{i \in N}, (p_i)_{i \in N})$ be a graphical game with neighborhood ν, A a set of actions such that for all $i \in N$, $A_i = A$. Γ is called

- *anonymous* if for all $i \in N$ and all $s, s' \in A^N$, $p_i(s) = p_i(s')$ whenever $s_i = s'_i$ and for all $a \in A$, $\#(a, s_{\nu(i)}) = \#(a, s'_{\nu(i)})$;
- *symmetric* if for all $i, j \in N$ and all $s, s' \in A^N$, $|\nu(i)| = |\nu(j)|$ and $p_i(s) = p_j(s')$ whenever $s_i = s'_j$ and for all $a \in A$, $\#(a, s_{\nu(i)}) = \#(a, s'_{\nu(j)})$;
- *self-anonymous* if for all $i \in N$ and all $s, s' \in A^N$, $p_i(s) = p_i(s')$ whenever for all $a \in A$, $\#(a, s_{\hat{\nu}(i)}) = \#(a, s'_{\hat{\nu}(i)})$; and
- *self-symmetric* if for all $i, j \in N$ and all $s, s' \in A^N$, $|\nu(i)| = |\nu(j)|$ and $p_i(s) = p_j(s')$ whenever for all $a \in A$, $\#(a, s_{\hat{\nu}(i)}) = \#(a, s'_{\hat{\nu}(j)})$.

In other words, a (graphical) game is anonymous if the payoff of each player depends only on his own action and the number of his neighbors

[1] We assume the set of actions and the payoff function to be the same for *all* players rather than just those with intersecting neighborhoods. This is only done for ease of exposition.

playing each of the actions. In a self-anonymous game the payoff of each player depends only on the "observed" number of players for each action, where a player observes his own action and those of his neighbors. A player in the latter class of games thus essentially does not differentiate his neighbors from himself. A game is symmetric if it is anonymous and if the neighborhood size and payoff function are identical for all players. Similarly, a game is self-symmetric if it is self-anonymous and if neighborhood size and payoff function are identical for all players.

It should be noted that a *graphical* game in one of the four classes does not necessarily belong to the corresponding class of general normal-form games as defined by Brandt et al. [8], unless the neighborhood of every player contains all other players. When talking about self-anonymous and self-symmetric games with two actions, we write $\mathbf{p}_i(m) = p_i(s)$ where $\#(1, s_{\hat{\nu}(i)}) = m$ for the payoff of player i when m players in his neighborhood, including i himself, play action 1, and $\mathbf{p}_i = (\mathbf{p}_i(m))_{0 \leq m \leq |\hat{\nu}(i)|}$ for the vector of payoffs for the possible values of m.

One of the best-known solution concepts for strategic games is Nash equilibrium [15]. In Nash equilibrium, no player is able to increase his payoff by *unilaterally* changing his strategy.

Definition 4 (Nash equilibrium)**.** Let $\Gamma = (N, (A_i)_{i \in N}, (p_i)_{i \in N})$ be a normal-form game. A strategy profile $s \in S$ of Γ is called *Nash equilibrium* if for each player $i \in N$ and each strategy $s'_i \in S_i$, $p_i(s) \geq p_i((s_{N \setminus \{i\}}, s'_i))$. A Nash equilibrium is called *pure* if it is a pure strategy profile.

3. Complexity of the Pure Equilibrium Problem

For graphical games with neighborhoods of size one, the pure equilibrium problem can be decided in polynomial time even without further restrictions on the payoff functions (see, e.g., [7]). On the other hand, the game used by Schoenebeck and Vadhan [16] to show NP-completeness of the pure equilibrium problem in general graphical games with neighborhoods of size two is anonymous. We thus have the following initial result.

Theorem 1 (Schoenebeck and Vadhan [16])**.** *Deciding whether a graphical game has a pure Nash equilibrium is NP-complete, even if every player has only two neighbors, two actions, and two different payoffs, and when restricted to anonymous games.*

$\#(1, s_{\nu(i)})$	0	1	2
0	0	0	1
1	1	1	0

$\#(1, s_{\hat{\nu}(i)})$	0	1	2	3
	0	1	2	0

Figure 1: NAND payoffs $p_i(s)$ for the symmetric and the self-symmetric case. Columns correspond to the different values of the commutative image of s with respect to $\nu(i)$ and $\hat{\nu}(i)$. In the symmetric case, rows correspond to the different actions of player i.

3.1. Symmetry and Self-Symmetry

We now turn to more restrictive kinds of symmetry. The following theorem concerns games where the utility functions of all players are identical. The proof of this theorem is similar to a construction used by Schoenebeck and Vadhan [16] where each gate of a Boolean circuit corresponds to a player in a graphical game, and two additional players play a game with or without a pure equilibrium, depending on the output of the circuit. The main difficulty is to model these two steps using only a single payoff function.

Theorem 2. *Deciding whether a graphical game has a pure Nash equilibrium is NP-complete, even if every player has only two neighbors, two actions, and when restricted to symmetric games with two different payoffs or to self-symmetric games with three different payoffs.*

Proof. Membership in NP is obvious. We can simply guess an action profile and verify that the action of each player is a best response to the actions of the players in his neighborhood.

For *hardness*, we provide a reduction from the NP-complete problem circuit satisfiability (CSAT, see, e.g., [12]). All hardness proofs in this paper are based on games that simulate Boolean circuits, in the sense that players of the game are associated with gates of the circuit, and the actions played by some of them in any pure equilibrium mirror the outputs of the corresponding gates given a satisfying assignment of the circuit. Each of these games is constructed inductively from smaller games simulating different Boolean operators. While it is not always easy to distinguish between players corresponding to inputs and outputs of a gate, and "auxiliary" players corresponding to neither of the two, we nevertheless try to label players consistently: we mostly use letters from the beginning of the alphabet for players corresponding to inputs and for auxiliary players, and letters from the end of the alphabet for players corresponding to outputs.

For a set N of players with appropriately defined neighborhoods ν, let $\Gamma(N) = (N, \{0,1\}^N, (p_i)_{i \in N})$ be a graphical game with payoffs satisfying symmetry or self-symmetry as given in Figure 1.[2] We observe the following properties:

1. Let N be a set of players with $|N| = 3$. For each $i \in N$, let $\hat{\nu}(i) = N$. Then, an action profile s of $\Gamma(N)$ is a pure equilibrium if and only if $\#(1, s) = 2$. In particular, for every $i \in N$, there exists a pure equilibrium where player i plays action 0 and a pure equilibrium where he plays action 1.
2. Let N and N' be two disjoint sets of players with neighborhoods such that for all $i \in N$, $\nu(i) \subseteq N$, and for all $i \in N'$, $\nu(i) \subseteq N'$. Then, s is a pure equilibrium of $\Gamma(N \cup N')$ if and only if s_N and $s_{N'}$ are pure equilibria of $\Gamma(N)$ and $\Gamma(N')$, respectively.
3. Let N be a set of players such that $\Gamma(N)$ has a pure equilibrium and consider two players $a, b \in N$. Further consider an additional player $x \notin N$ with $\nu(x) = \{a, b\}$. Then the game $\Gamma(N \cup \{x\})$ has a pure equilibrium, and in every pure equilibrium s of $\Gamma(N \cup \{x\})$, $s_x = 0$ if $s_a = s_b = 1$ and $s_x = 1$ otherwise. In other words, such a strategy profile always satisfies $s_x = s_a \,\text{NAND}\, s_b$.
4. Let N be a set of players and consider a particular player $x \in N$. Further consider five additional players $a, b, c, d, e \notin N$ with neighborhoods according to Figure 2, and denote $N' = N \cup \{a, b, c, d, e\}$. Then, $\Gamma(N')$ has a pure equilibrium if and only if $\Gamma(N)$ has a pure equilibrium s where $s_x = 0$. For the direction from right to left, assume that $\Gamma(N)$ has a pure equilibrium s where $s_x = 0$ and extend it to an action profile for $\Gamma(N')$ by letting $s_a = 0$ and $s_b = s_c = s_d = s_e = 1$. It is easily verified that s is a Nash equilibrium of $\Gamma(N')$. For the direction from left to right, consider an action profile s for $\Gamma(N')$ where $s_x = 1$. If $s_a = 0$, then action 1 is the unique best response for players c and d, after which action 0 is the unique best response for players b and e. In this case, player a can change his action to 1 to get a higher payoff. If $s_a = 1$, then the unique best response for players c and d is 0, and the unique best response for players b and e is 1. Again, player a can

[2] It was shown by Brandt et al. [8] that every anonymous or symmetric game with two actions per player can respectively be transformed into a self-anonymous or self-symmetric game, while preserving pure equilibria.

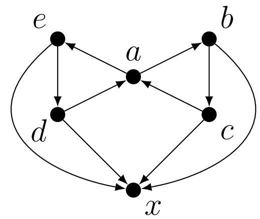


Figure 2: Output gadget. All players have payoffs as in Figure 1. Player x must play action 0 in every pure equilibrium of the game.

change his action to get a higher payoff.

5. Let $N_1 = \{b, c, d\}$ be an instance of N in Property 1, and N_2 an instance of N' in Property 4 with $N = \{b\}$. Let N be any set of players such that $\Gamma(N)$ has a pure equilibrium, let $a \in N$, and denote $N' = N_1 \cup N_2 \cup N$. Further consider an additional player $x \notin N'$ with $\nu(x) = \{a, c\}$. Then, $\Gamma(N' \cup \{x\})$ has a pure equilibrium, and in every pure equilibrium s of $\Gamma(N' \cup \{x\})$, $s_x = 1 - s_a$. To see this, observe that by Property 1 exactly two players in N_1 must play action 1, which, by Property 4, have to be players c and d. By Property 3, and since $\varphi\,\mathrm{NAND}\,\mathit{true} = \neg\varphi$, the claim follows.

Now consider an instance $\mathcal{C}$ of CSAT, and assume without loss of generality that $\mathcal{C}$ consists exclusively of NAND gates and that no variable appears more than once as the input to the same gate. The latter assumption can be made by Property 5. We construct a game $\Gamma = \Gamma(N)$ as follows. For every input of $\mathcal{C}$ we augment N by three players according to Property 1. We then inductively define Γ by adding, for a gate with inputs corresponding to players $a, b \in N$, a player x as described in Property 3. Finally, we construct a player according to Property 5 who plays the opposite action as the one corresponding to the output of $\mathcal{C}$, and identify this player with x in a new instance of Property 4. It is now easily verified that a pure equilibrium of Γ corresponds to a computation of $\mathcal{C}$ which outputs *true*, and that such an equilibrium exists if and only if $\mathcal{C}$ has a satisfying assignment. □

3.2. Self-Anonymity and Two Different Payoffs

Since self-symmetric games form a subset of self-anonymous games, Theorem 2 also implies NP-hardness of the self-anonymous case. However, the result is not tight in that three different payoffs are required for hardness. A natural question is what happens for self-anonymous games with only two different payoffs. In this section we will prove a tight result for the most restricted version of self-anonymity, i.e., the case with only two different payoff *functions*.

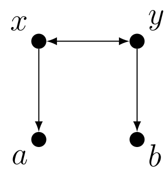
The problem with anonymity and the construction used in the proof of Theorem 2 is that two different payoffs are not enough to make a player care about his own action no matter which actions are played by his neighbors. With four different values for $\#(1, s_{\hat{\nu}(i)})$, there will either be an equilibrium where all players play the same action, or a situation where a player is indifferent between both of his actions. When we want to use games to compute a function, such indifference is clearly undesirable. The key idea that will enable us to prove the following theorem is to isolate pure equilibria that are themselves *symmetric* in the actions of a subset of the players, i.e., equilibria in which these players all play the same action. To enforce that two particular players play the same action in every equilibrium, we will add two additional players, each of which observes the other as well as one of the original players. Depending on the actions of the original players, the new players will either play a game with a unique pure equilibrium, or a game that is prototypical both for self-anonymous games and for games without pure equilibria, namely Matching Pennies.

Theorem 3. *Deciding whether a graphical game has a pure Nash equilibrium is NP-complete, even if every player has only two neighbors, two actions, and two different payoffs, and when restricted to self-anonymous games with two different payoff functions.*

Proof. Membership in NP is obvious.

For *hardness*, we again provide a reduction from CSAT. Let $\Gamma(N) = (N, \{0,1\}^N, (p_i)_{i \in N})$ denote a graphical game for a set N of players with neighborhood ν and payoff functions p_i satisfying self-anonymity. We observe the following properties:

1. Let N be a set of players, $a, b \in N$, and consider two additional players $x, y \notin N$ with neighborhoods and payoffs according to Figure 3. Then, $\Gamma(N \cup \{x, y\})$ has a pure equilibrium if and only if $\Gamma(N)$ has a pure equilibrium s where $s_a = s_b$. For the direction from right to left, assume that $\Gamma(N)$ has a pure equilibrium s where $s_a = s_b$ and extend this to an action profile for $\Gamma(N')$ by letting $s_x = s_a$ and $s_y = 1$. It is easily verified that under this action profile players x and y both receive the maximum payoff of 1, such that the equilibrium condition

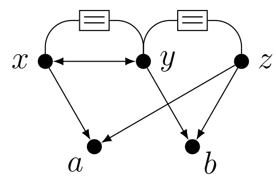


$\#(1, s_{\hat{\nu}(i)})$	0	1	2	3
$p_i(s)$	0	1	0	1

Figure 3: Equality gadget. A pure equilibrium exists if and only if players a and b play the same action.

is trivially satisfied. For the direction from left to right, assume that one of the players x and y observes action 0 being played by player a or b, while the other one observes action 1. Then players x and y effectively play the well-known Matching Pennies game. More precisely, the player observing 0 receives a payoff of 1 if and only if $\#(s_{\{x,y\}})$ is odd, while the same is true for the player observing 1 if and only if this number is even. Since both players can change between the two outcomes by changing their own action, there is no pure equilibrium.

2. Let N be a set of players with $|N| = 3$. For each $i \in N$, let $\hat{\nu}(i) = N$ and let p_i be defined according to Figure 3. Then, any action profile s satisfying $\#(1, s) = 1$ or $\#(1, s) = 3$ is a pure equilibrium of $\Gamma(N)$. In particular, for each $i \in N$, there exist equilibria s and s' with $s_a = 0$ and $s'_a = 1$.
3. Let N and N' be two disjoint sets of players with neighborhoods such that for all $i \in N$, $\nu(i) \subseteq N$, and for all $i \in N'$, $\nu(i) \subseteq N'$. Again, s is a pure equilibrium of $\Gamma(N \cup N')$ if and only if s_N and $s_{N'}$ are pure equilibria of $\Gamma(N)$ and $\Gamma(N')$, respectively.
4. Let $N = \{a, b, c\}$ with neighborhoods and payoffs as in Property 2, and assume by Property 1 that every pure equilibrium s of $\Gamma(N)$ is symmetric, i.e., satisfies $s_a = s_b = s_c$. Then, s with $s_a = s_b = s_c = 1$ is the unique pure equilibrium of $\Gamma(N)$. Clearly, s is an equilibrium of $\Gamma(N)$, since all players receive the maximum payoff of 1. In the only other symmetric action profile, all players play action 0 and receive a payoff of 0. Either one of them can change his action to 1 to receive a higher payoff.
5. Let N be a set of players such that $\Gamma(N)$ has a pure equilibrium. Let $a, b \in N$, and consider three additional players $x, y, z \notin N$ with neighborhoods and payoffs according to Figure 4. Then, $\Gamma(N \cup \{x, y, z\})$ has a pure equilibrium, and for every pure equilibrium s of $\Gamma(N \cup \{x, y, z\})$, $s_x = 0$ if $s_a = s_b = 1$, and $s_x = 1$ otherwise. It is easily verified



$\#(1, s_{\hat{\nu}(i)})$	0	1	2	3
$p_i(s)$	0	1	1	0

Figure 4: NAND gadget. The construction of Figure 3 is used to ensure that players connected by "=" play the same action in every pure equilibrium.

that players x, y, and z get the maximum payoff of 1, and thus will not deviate, under any action profile s where $s_x = s_y = s_z = 1$ and $\#(s_{\{a,b\},1}) \leq 1$ or where $s_x = s_y = s_z = 0$ and $s_a = s_b = 1$. On the other hand, let s be an arbitrary action profile of $\Gamma(N \cup \{x, y, z\})$. By Property 1, s cannot be an equilibrium unless $s_x = s_y = s_z$. If $s_a = s_b = s_z = 0$ or $s_a = s_b = s_z = 1$, then player z can change his action to receive a higher payoff. If otherwise $s_a \neq s_z$ and $s_x = s_y = 0$, then there exists $i \in \{x, y\}$ such that $\#(1, s_{\hat{\nu}(i)}) = 0$, and player i will deviate.

6. Let N be a set of players, $o \in N$. Let $N' = \{a, b, c\}$ with neighborhoods as in Property 4, $N'' = \{d, e\}$ with $\nu(d) = \{a, e\}$ and $\nu(e) = \{x, d\}$. Then, $\Gamma(N \cup N' \cup N'')$ has a pure equilibrium if and only if $\Gamma(N)$ has a pure equilibrium s with $s_x = 1$. Clearly, an action profile that is not an equilibrium of $\Gamma(N)$ cannot be extended to an equilibrium of $\Gamma(N \cup N' \cup N'')$. On the other hand, assume that s is an equilibrium of $\Gamma(N \cup N' \cup N'')$. Then, by Property 4, $s_a = 1$. Furthermore, by Property 1, $s_a = s_x$, and thus $s_x = 1$.

Now consider an instance $\mathcal{C}$ of CSAT, and assume without loss of generality that $\mathcal{C}$ consists exclusively of NAND gates. Since $\varphi\,\text{NAND}\,\mathit{true} = \neg\varphi$, and using Property 4, we can further assume that no variable appears more than once as an input to the same gate. We construct a game $\Gamma = \Gamma(N)$ as follows: For every input of $\mathcal{C}$, we add three players according to Property 2. For every gate of $\mathcal{C}$ with inputs corresponding to players $a, b \in N$, we add three players according to Property 5. Finally, we add five players according to Property 6, where x is the player corresponding to the output of $\mathcal{C}$. It is now readily appreciated that Γ has a pure equilibrium if and only if $\mathcal{C}$ is satisfiable. $\square$

3.3. Self-Symmetry and Two Different Payoffs

Let us return to self-symmetric graphical games. Self-symmetric games as studied by Brandt et al. [8] always possess a pure Nash equilibrium due to the fact that they are common-payoff games. This is not the case for self-symmetric graphical games, even when there are only two different payoffs. In particular, there exists a seven-player game in the latter class that does not have a pure equilibrium, and in which each player has exactly two actions and two neighbors. It will be instructive to view a graphical game as a hypergraph, with each vertex corresponding to a player and each edge to the set of players in the neighborhood of one particular player *including* the player himself. Corresponding to the set of games with m-neighborhood is the set of $(m+1)$-uniform hypergraphs that possess a matching in the sense of Seymour [17], i.e., a bijection from the set of vertices to the set of edges that maps every vertex to an edge containing it. Then, a self-symmetric game with two actions and payoffs $\mathbf{p}_i = (0, 1, 1, 0)$ for all $i \in N$ has a pure Nash equilibrium if and only if the corresponding hypergraph is vertex two-colorable. Given a two-coloring, every player observes either one or two players in his neighborhood, including himself, who play action 1, and thus obtains the maximum payoff of 1. If on the other hand there is no two-coloring, then there is at least one player for every action profile who plays the same action as all of his neighbors and can deviate to obtain a higher payoff. Figure 5 shows the neighborhood of a graphical game with seven players and two neighbors for each player. This graph induces the 3-uniform square hypergraph corresponding to the lines of the Fano plane, which in turn cannot be two-colored (see, e.g., [17]). We leave it to the avid reader to verify that there is no game with the above properties and less than seven players.

An interesting property of the neighborhood graph on the left of Figure 5 is that it does not have any cycles of even length. We will begin our investigation of the pure equilibrium problem in self-symmetric games by generalizing this observation to games with arbitrary neighborhoods and $\mathbf{p}_i = (0, 1, 1, \dots, 1, 0)$ for all $i \in N$. The following lemma characterizes games with pure equilibria in the above subclass in terms of cycles in the neighborhood graph. Seymour [17] provides a similar characterization of the minimal uniform square hypergraphs that do not have a two-coloring.

Lemma 1. *Let Γ be a self-symmetric graphical game with two actions per player and payoffs p_i such that for all $i \in N$, $\mathbf{p}_i = (0, 1, 1, \dots, 1, 0)$. Then, Γ*

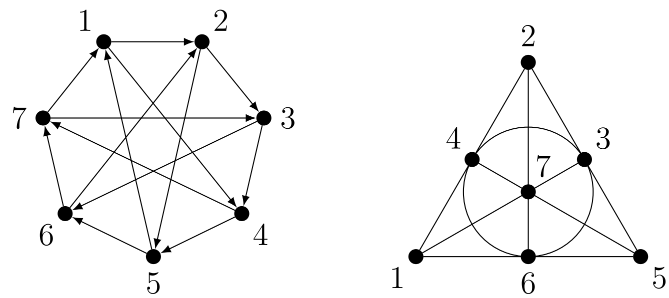


Figure 5: Neighborhood graph of a graphical game with seven players (left), corresponding to the three-uniform square hypergraph given by the lines of the Fano plane (right). A directed edge from vertex i to vertex j of the neighborhood graph denotes that $j \in \nu(i)$.

has a pure Nash equilibrium if and only if for all $i \in N$, there exists $j \in N$ reachable from i such that j lies on a cycle of even length.

Proof. For the implication from left to right, assume that there exists a pure equilibrium, i.e., a two-coloring $c : N \to \{0,1\}$ of the neighborhood graph such that for every $i \in N$, there exist $j, j' \in \hat{\nu}(i)$ with $c(j) = 0$ and $c(j') = 1$. Now consider an arbitrary player $v_1 \in N$. Using the above property of c, we can construct a path $v_1, v_2, \ldots, v_{|N|+1}$, $v_i \in N$, such that for all i, $1 \leq i \leq |N|$, $c(v_i) = 1 - c(v_{i+1})$. By the pigeonhole principle, there must exist i, j, $1 \leq i < j \leq |N| + 1$, such that $v_i = v_j$ and for all j', $i < j' < j$, $v_{j'} \neq v_i$. Then, $v_i, v_{i+1}, \ldots, v_j$ is a cycle of even length reachable from v_1.

For the implication from right to left, let $N' \subseteq N$ be a set of players such that for every $i \in N$ there exists a directed path to some $j \in N'$, and such that N' induces a set of vertex-disjoint cycles of even length. We construct a two-coloring $c : N \to \{0,1\}$, corresponding to an assignment of actions to players, as follows. First color the members of N' such that for all $i \in N'$ and $j \in \nu(i) \cap N'$, $c(i) = 1 - c(j)$. While there are uncolored vertices left, find $i, j \in N$ such that $j \in \nu(i)$, i is uncolored, and j is colored. Such a pair of vertices must always exist, since for every member of N there is a directed path to some member of N', and thus to a vertex that has already been colored. Color i such that $c(i) = 1 - c(j)$. It is now easily verified that at any given time, and for all $i \in N$ that have already been colored, there exist $j, j' \in \hat{\nu}(i)$ with $c(j) = 0$ and $c(j') = 1$. If all vertices have been colored, then every neighborhood will contain at least one player playing action 0, and at least one player playing action 1. The corresponding action profile is a pure Nash equilibrium. □

Thomassen [18] has shown that for every k, there exists a directed graph without even cycles where every vertex has outdegree k. Together with Lemma 1, this means that the pure equilibrium problem for the considered class of games is nontrivial.

Corollary 1. *For every $m \in \mathbb{N}$, $m > 0$, there exist self-symmetric graphical games Γ and Γ' with two actions where for all $i \in N$, $|\nu(i)| = m$ and $\mathbf{p}_i = (0, 1, 1, \ldots, 1, 0)$, such that Γ has a pure Nash equilibrium and Γ' does not.*

We are now ready to identify several classes of graphical games where the existence of a pure equilibrium can be decided in polynomial time.

Theorem 4. *Let $\Gamma = (N, (A_i)_{i \in N}, (p_i)_{i \in N})$ be a self-symmetric graphical game with two actions and two different payoffs. Then, the pure equilibrium problem for Γ can be decided in polynomial time if one of the following properties holds:*

(*i*) *for all $i \in N$, $\mathbf{p}_i(0) \geq \mathbf{p}_i(1)$ or for all $i \in N$, $\mathbf{p}_i(|\hat{\nu}(i)|) \geq \mathbf{p}_i(|\hat{\nu}(i)| - 1)$;*

(*ii*) *for all $i \in N$ and all m, $1 \leq m \leq |\nu(i)|$, $\mathbf{p}_i(m-1) > \mathbf{p}_i(m)$ and $\mathbf{p}_i(m+1) > \mathbf{p}_i(m)$, or $\mathbf{p}_i(m-1) < \mathbf{p}_i(m)$ and $\mathbf{p}_i(m+1) < \mathbf{p}_i(m)$;*

(*iii*) *for all $i \in N$ and all m, $1 \leq m < |\nu(i)|$, $\mathbf{p}_i(m) = \mathbf{p}_i(m+1)$.*

Proof. It is easy to see that a game Γ satisfying (i) possesses a pure equilibrium s in which $\#(0, s) = 0$ or $\#(1, s) = 0$.

For a game Γ satisfying (ii), we observe that in every equilibrium s, $p_i(s) = 1$ for all $i \in N$. The pure equilibrium problem for Γ thus corresponds to a variant of generalized satisfiability, with clauses induced by neighborhoods of Γ. The constraints associated with this particular variant require that the number of variables in each clause set to true is odd, and can be written as a system of linear equations over $GF(2)$. Tractability of the pure equilibrium problem for Γ then follows from Theorem 2.1 of Schaefer [19].

Finally, a game satisfying (iii) but not (i) can be transformed into a best response equivalent one that satisfies the conditions of Lemma 1. We further claim that we can check in polynomial time whether for every $i \in N$, there exists $j \in N$ on a cycle of even length and reachable from i. For a particular $i \in N$, this problem is equivalent to checking whether the subgraph induced by the vertices reachable from i contains an even cycle. The latter problem has long been open, but was recently shown to be solvable in polynomial time [20]. □

It turns out that Theorem 4 specifically applies to every self-symmetric graphical game with two different payoffs and neighborhoods of size at most three. We thus have the following.

Corollary 2. *The problem of deciding whether a self-symmetric graphical game with two different payoffs and three-bounded neighborhood has a pure equilibrium is in P.*

Proof. Consider a game Γ as in the statement of the corollary, and assume without loss of generality that the two payoffs of Γ are 0 and 1. Since Γ has neighborhoods of size at most three, the payoff of player i can be described by a function $\mathbf{p}_i : \{0,1\} \to \{0,1\}$, $\mathbf{p}_i : \{0,1,2\} \to \{0,1\}$, or $\mathbf{p}_i : \{0,1,2,3\} \to \{0,1\}$. This function must actually be the same for all players because Γ is self-symmetric. It is now easy, if somewhat tedious, to verify that each of the possible functions satisfies one of the conditions of Lemma 4. In fact, two of the functions, namely those where $\mathbf{p}_i = (0,1,0)$ or $\mathbf{p}_i = (0,1,1,0)$, satisfy condition (*iii*), while all others satisfy condition (*i*). □

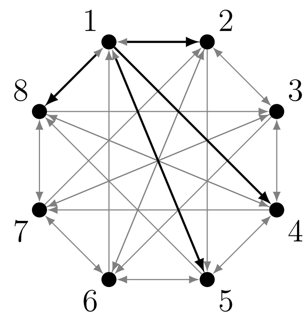
3.4. Self-Symmetry and Larger Neighborhoods

The question that remains is whether the pure equilibrium problem can be solved in polynomial time for all self-symmetric graphical games with two payoffs, or whether there is some bound on the neighborhood size where this problem again becomes hard. We will show in this section that the latter is true, and that the correct bound is indeed four, as suggested by Corollary 2.

We will essentially use the same tools as in Section 3.2, but will extract the necessary complexity from only a single payoff function. The additional insight required for this extraction will be that "constant" players, i.e., players who play the same action in every pure equilibrium of a game, can be used to prune a larger payoff table and effectively obtain different payoff functions for smaller neighborhoods that can then be used to proceed with the original proof. Constructing such players will prove a rather difficult task in its own right.

Theorem 5. *Deciding whether a self-symmetric graphical game with two actions and two different payoffs has a pure Nash equilibrium is NP-complete, even if every player has exactly four neighbors.*

Proof. Membership in NP is obvious. We can simply guess an action for each player and then verify that no player can increase his payoff by playing a different action instead.



$\#(1, s_{\hat{\nu}(i)})$	0	1	2	3	4	5
$p_i(s)$	0	1	0	1	1	0

Figure 6: Neighborhood graph and payoffs of a graphical game with eight players and neighborhoods of size four used in the proof of Theorem 5. The neighborhood graph satisfies rotational symmetry, the neighborhood of player 1 is highlighted.

For *hardness*, we again give a reduction from CSAT to the problem at hand. The central idea of this proof will be to guarantee that some players in a neighborhood only play certain well-defined actions in equilibrium. By this, the original payoff table is effectively "pruned" to a smaller one that can then be used, like in earlier proofs, to model the behavior of gates in a Boolean circuit.

As a first step, we will show how to construct "constants," i.e., players who play action 0 or action 1, respectively, in *every* equilibrium of a game. To achieve this, we will construct a set of four players, such that in every equilibrium two of these players play action 0 and two of them play action 1. A player observing these four players can determine if the number of players in his neighborhood, including himself, who play action 1 is two or three. Clearly, such a player will play action 1 in every equilibrium. By a similar argument, a player who observes four players who play action 1 in every equilibrium will himself play action 0 in every equilibrium.

Consider the graphical game Γ with eight players and neighborhood of size four given in Figure 6. We will argue that in every pure equilibrium of this game, *exactly* two players $i, j \in N$ play action 0 and $i - j = 2$ $(mod\ 8)$. We exploit the following properties of the neighborhood graph:

1. For any $N' \subseteq N$, $|N'| = 3$, there exists a player $i \in N$ such that $N' \subseteq \hat{\nu}(i)$. Due to the rotational symmetry of the neighborhood graph, we can assume without loss of generality that $1 \in N'$. The property then follows by a straightforward if somewhat tedious case analysis.
2. For any $N' \subseteq N$, $|N'| = 3$, there exists a player $i \in N$ such that $|N' \cap \hat{\nu}(i)| = 2$. Showing this property is again straightforward by

assuming without loss of generality that $1 \in N'$ and showing that for any pair of distinct players, there exists a player $i \in N$ such that either $\hat{\nu}(i)$ contains player 1 and exactly one element of the pair, or both elements of the pair but not player 1.

3. For any $N' \subseteq N$, $|N'| = 4$, there exists a player $i \in N$ such that $|N' \cap \hat{\nu}(i)| = 3$. To show this property, we can again assume without loss of generality that $1 \in N'$, and distinguish neighborhoods that contain player 1 from neighborhoods that do not. The analysis is again straightforward.

Now consider an equilibrium s of Γ, and observe that, due to the structure of the payoffs, it must be the case that $p_i(s) = 1$ for all $i \in N$. If $\#(0, s) < 2$ or $\#(1, s) < 2$, then there exists a player $i \in N$ such that $\#(0, s_{\hat{\nu}(i)}) = 0$ or $\#(1, s_{\hat{\nu}(i)}) = 0$. If $\#(0, s) = 2$, assume without loss of generality that $s_1 = 0$, and further assume for contradiction that there exists $i \in N \setminus \{1, 3, 7\}$ such that $s_i = 0$. Inspection of the neighborhood graph reveals that in this case there exists a player $j \in N$ such that $\#(0, s_{\nu(j)}) = 0$. If $\#(0, s) = 3$, then by Property 1 there must exist a player $i \in N$ such that $\#(0, s_{\hat{\nu}(i)}) = 3$ and thus $\#(1, s_{\hat{\nu}(i)}) = 2$, contradicting the assumption that s is an equilibrium. By Property 3, the same holds if $\#(0, s) = 4$. If $\#(0, s) = 5$ and thus $\#(1, s) = 3$, then by Property 2 there must yet again exists a player $i \in N$ such that $\#(1, s) = 2$, a contradiction. The same trivially holds if $\#(1, s) = 2$.

Now we augment Γ by a set $\{9, 10, \ldots, 13\}$ of five additional players such that

$$\nu(i) = \begin{cases} \{1, 3, 5, 7\} & \text{if } i \in \{9, 10\} \\ \{2, 4, 6, 8\} & \text{if } i \in \{11, 12\} \\ \{9, 10, 11, 12\} & \text{if } i = 13. \end{cases}$$

By construction of the original game with eight players, every pure equilibrium has either two or four players in the common neighborhood of players 9 and 10 play action 1. Furthermore, if players 9 and 10 observe two players who play action 1, then players 11 and 12 will observe four players who play action 1, and *vice versa*. As a consequence, either players 9 and 10 will play action 0, and players 11 and 12 will play action 1, or the other way round. In any case, exactly two players in the neighborhood of player 13 will play action 1 in every equilibrium of the augmented game, and player 13 himself will therefore play action 1.

In the following, we will denote by $\mathbf{0}_1, \mathbf{0}_2, \mathbf{0}_3 \in N$ three players who play action 0 in every equilibrium, and by $\mathbf{1}_1, \mathbf{1}_2 \in N$ two players that constantly play action 1. Using these players to prune the payoff table, we will proceed to design games that simulate Boolean circuits. These games will satisfy self-symmetry, and the payoff of all players will therefore be determined by the table already used above and shown in Figure 6. As for the inputs of the circuit, it is easily verified that a game with players N, $|N| = 5$, such that for all $i \in N$, $\hat{\nu}(i) = N$, has pure equilibria s and s' such that for an arbitrary $i \in N$, $s_i = 0$ and $s'_i = 1$.

As before, we will now construct a subgame that simulates a functionally complete Boolean gate, in this case NOR, and a subgame that has a pure equilibrium if and only if a particular player plays action 1. For a set N of players with appropriately defined neighborhoods ν, let $\Gamma(N) = (N, \{0,1\}^N, (p_i)_{i \in N})$ be a graphical game with payoff functions p_i satisfying self-symmetry as in Figure 6. We observe the following properties:

1. Let N and N' be two disjoint sets of players with neighborhoods such that for all $i \in N$, $\nu(i) \subseteq N$, and for all $i \in N'$, $\nu(i) \subseteq N'$. Again, s is a pure equilibrium of $\Gamma(N \cup N')$ if and only if s_N and $s_{N'}$ are pure equilibria of $\Gamma(N)$ and $\Gamma(N')$, respectively.
2. Let N be a set of players such that $\Gamma(N)$ has a pure equilibrium, let $a, b \in N$, and consider two additional players $x, y \notin N$ with $\nu(x) = \{\mathbf{0}_1, \mathbf{0}_2, a, y\}$, and $\nu(y) = \{\mathbf{0}_1, \mathbf{0}_2, b, x\}$. Then every pure equilibrium of $\Gamma(N \cup \{x, y\})$ satisfies $s_a = s_b$.
3. Letting $b = \mathbf{1}_1$ in the previous construction, we have that $\Gamma(N \cup \{x, y\})$ has a pure equilibrium if and only if $s_a = 1$ in some pure equilibrium of Γ.
4. Let N be a set of players such that $\Gamma(N)$ has a pure equilibrium, let $a, b \in N$, and consider two additional players $x, y \notin N$ with neighborhoods given by $\nu(x) = \{\mathbf{0}_1, \mathbf{0}_2, \mathbf{0}_3, y\}$ and $\nu(y) = \{\mathbf{0}_1, \mathbf{0}_2, a, b\}$. Then $\Gamma(N \cup \{x, y\})$ has a pure equilibrium, and every pure equilibrium s of $\Gamma(N \cup \{x, y\})$ satisfies $s_x = 1$ whenever $s_a = s_b = 0$, and $s_x = 0$ whenever $s_a \neq s_b$. For every pure equilibrium s with $s_a = s_b = 1$, there exists a pure equilibrium s' such that $s_x \neq s'_x$, and $s_i = s'_i$ for all $i \in N$.
5. Consider an additional player $z \notin N \cup \{x, y\}$, and let $\nu(z) = \{\mathbf{1}_1, \mathbf{1}_2, a, b\}$. Then $\Gamma(N \cup \{x, y, z\})$ has a pure equilibrium, and every pure equilibrium s of $\Gamma(N \cup \{x, y, z\})$ satisfies $s_z = 1$ whenever $s_a = s_b = 0$, and $s_z = 0$ whenever $s_a = s_b = 1$. For every pure equilibrium s with $s_a \neq s_b$, there exists a pure equilibrium s' such that $s_z \neq s'_z$, and $s_i = s'_i$ for all $i \in N$.

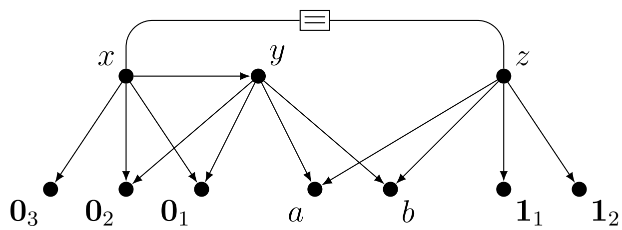


Figure 7: NOR gadget. Payoffs are identical to those in Figure 6. A construction analogous to the one shown in Figure 3 is used to ensure that players x and z play the same action in every pure equilibrium.

6. By Property 2, we can assume that every equilibrium s of $\Gamma(N \cup \{x, y, z\})$ satisfies $s_x = s_z$, and thus that $s_x = 1$ if and only if $s_a = s_b = 0$.

Steps 4 through 6 are illustrated in Figure 7.

Now consider an instance $\mathcal{C}$ of CSAT, and assume without loss of generality that $\mathcal{C}$ consist exclusively of NOR gates and that no variable appears more than once as an input to the same gate. The latter assumption can be made since $\varphi\,\mathrm{NOR}\,\mathit{false} = \neg\varphi$, and since there exists a self-symmetric game and a player in this game who plays action 0 in every pure equilibrium. As before, we construct a game Γ by simulating every gate of $\mathcal{C}$ according to Property 6 and identifying the player that corresponds to the output of the circuit with a in Property 3. It is now readily appreciated that Γ has a pure equilibrium if and only if $\mathcal{C}$ is satisfiable. □

Observing that in the constructions used in the proofs of Theorems 2, 3, and 5 there is a one-to-one correspondence between satisfying assignments of a Boolean circuit and pure equilibria of a game, we have that counting the number of pure equilibria in the respective games is as hard as computing the permanent of a matrix.

Corollary 3. *For graphical games with neighborhoods of size two, counting the number of pure Nash equilibria is #P-hard, even when restricted to symmetric games with two different payoffs, to self-anonymous games with two different payoffs and two different payoff functions, or to self-symmetric games with three different payoffs. The same holds for self-symmetric graphical games with neighborhoods of size four and two different payoffs.*

4. Interlude: Generalized Satisfiability in the Presence of a Matching

The analysis at the end of the previous section allows us to derive a corollary that may be of independent interest. Schaefer [19] completely characterizes which variants of the generalized satisfiability problem are in P and which are NP-complete. Some of the variants become tractable if there exists a matching, i.e., a bijection from variables to clauses that maps every variable to a clause it appears in. For not-all-equal 3SAT, for example, this follows from equivalence with two-colorability of three-uniform hypergraphs and from the work of Robertson et al. [20]. On the other hand, the proof of Theorem 5 identifies a variant that is NP-complete and remains so in the presence of a matching. We thus obtain the following result, a complete characterization is left for future work.

Corollary 4. *Generalized satisfiability is NP-complete, even if there exists a matching and all clauses have size five.*

Proof. Given a game Γ as constructed in the proof of Theorem 5, define a satisfiability problem with variables N and clauses $C = \{\, (i, \hat{\nu}(i)) : i \in N \,\}$, where N is the set of players of Γ and $\hat{\nu}(i)$ is the set containing the neighbors of a player $i \in N$ and i itself. Call an instance of this problem satisfiable if there exists an assignment that sets 1, 3, or 4 variables of each clause to 1, i.e., a function $v : N \to \{0, 1\}$ such that for each $(i, c) \in C$, $|\{\, j \in c : v(j) = 1 \,\}$.

It is now easy to see that a particular instance is satisfiable if and only if the corresponding game has a pure Nash equilibrium, which together with the proof of Theorem 5 implies that deciding the former is NP-hard. It is furthermore obvious from the definition of C that the satisfiability problem has a matching, i.e., a bijection between variables and clauses, and that each clause has size five. □

5. Mixed Equilibria

Let us now briefly look at the problem of finding a mixed equilibrium. The following theorem states that this problem is tractable in symmetric graphical games if the number of actions grows slowly in the neighborhood size. The proof relies on the fact that such games always have a symmetric equilibrium.

Theorem 6. *Let $\Gamma = (N, A^N, (p_i)_{i \in N})$ be a symmetric graphical game with neighborhoods of size k and $|A| = O(\log k / \log \log k)$. Then, a Nash equilibrium of Γ can be computed in polynomial time.*

Proof. We show that Γ possesses a symmetric equilibrium, i.e., one where all players play the same (mixed) strategy, and that this equilibrium can be computed efficiently. For this, choose an arbitrary player $i \in N$ and construct a game $\Gamma_i = (N_i, A^{N_i}, (p_{i,j})_{j \in N})$ with players $N_i = \hat{\nu}(i)$, and for all $j \in N_i$, $p_{i,j}(s') = p_i(s)$ if $s'_j = s_i$ and for all $a \in A$, $\#(a, s') = \#(a, s_{\nu(i)})$. Now, since Γ is a symmetric graphical game, it is easily verified that Γ_i is a symmetric game, and must therefore possess a symmetric equilibrium, i.e., one where all the players in N_i play the same strategy. By a result of Papadimitriou and Roughgarden [9], one such equilibrium s' can be computed in polynomial time if $|A| = O(\log |N_i| / \log \log |N_i|)$. This is the case because $|N_i| = k$ and $|A| = O(\log |k| / \log \log |k|)$. Moreover, due to the symmetry of Γ, all the games Γ_i for $i \in N$ are isomorphic, and thus s' is an equilibrium in each of them.

Now define a strategy profile s of Γ by letting, for each $i \in N$, $s_i = s'_1$, and assume for contradiction that s is *not* an equilibrium. Then there exists a player $i \in N$ and some strategy $t \in \Delta(A)$ for this player such that $p_i(s_{N \setminus \{i\}}, t) > p_i(s)$. Then, by definition of $p_{i,j}$, $p_{i,i}(s'_{N_i \setminus \{i\}}, t) > p_{i,i}(s)$, contradicting the assumption that s' is an equilibrium of Γ_i. □

This result applies in particular to the case where both the number of actions and the neighborhood size are bounded. Since the pure equilibrium problem in symmetric graphical games is NP-complete even in the case of two actions, we have identified a class of games where computing a mixed equilibrium is computationally easier than deciding the existence of a pure one, unless P=NP. A different class of games with the same property is implicit in Theorem 3.4 of Daskalakis and Papadimitriou [11]. On the other hand, existence of a symmetric equilibrium does not in general extend to games that are not anonymous or in which players have different payoff functions.

6. Open Problems

In this paper we have mainly considered neighborhoods of constant size. The construction used in the proof of Theorem 5 can be generalized to arbitrary neighborhoods of even size, but it is unclear what happens for odd-sized neighborhoods. The extreme case when the neighborhood of every player consists of all other players yields ordinary symmetric games, and it is known that the pure equilibrium problem in these games is in P when the number of actions is bounded [8]. It is an open problem at which neighborhood size the transition between membership in P and NP-hardness occurs. Another open question concerns the complexity of the mixed equilibrium problem in anonymous graphical games. A promising direction for proving hardness would be to make the construction of Goldberg and Papadimitriou [2] anonymous. Finally, as suggested in Section 4, it would be interesting to study the complexity of generalized satisfiability problems in the presence of matchings.

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